

Chapter 13

PME and the International Community of Mathematics Education

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Abstract The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 in Karlsruhe (Germany), during the ICME-3 Congress. Since 1977, the PME group has met every year somewhere in the world, since then, and has developed into one of the most interesting international groups in the field of educational research. In this paper, after a short introduction, we draw some main features of the unique essence of the PME as a research group. We focus on and analyse the change and development of the group's research over the past 40 years, and exemplify these changes and developments by tracing on a few main research lines. Based on specifics of PME research, we describe the more comprehensive lines of PME research, its change and progress in the past four decades.

Keywords The International Group for the Psychology of Mathematics Education History of mathematics education • Research trends in IGPME • Theory in mathematics education research • Methods of mathematics education research

13.1 Introduction

13.1.1 *Some General Features of PME*

In the introduction to the “Handbook of Research on the Psychology of Mathematics Education” (PME 1976–2006) which was published in 2006 for the

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G. Kaiser et al. (eds.), *Invited Lectures from the 13th International Congress on Mathematical Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-72170-5_13

30 years of PME, the editors Gutierrez and Boero (2006), mentioned two reasons for the success of the PME group.

Their first reason is the human and scientific quality of the founding members —“The fathers”: E. Fischbein, H. Freudenthal & R. Skemp and many others who shared both the important decision itself and bringing the PME into the existence.

We feel that this is the place to say some words in the memory of Efraim Fischbein (Fig. 13.1) who was in a sense the “inspiration spirit” of the PME organization. We borrowed these words from Tall (1998), who wrote after Fischbein’s death:

The 23rd meeting of the International Group for the Psychology of Learning Mathematics in Israel is touchingly the first in which we cannot be joined by our Founder President, Professor Efraim Fischbein, who left us on July 22nd 1998. It is a time of sadness, yes, but it is also a time for celebrating the achievements of this gentle man who is responsible for the existence of our organization. In particular, it is to him that we owe our focus on the psychology of learning mathematics. (Tall 1998)

Gutierrez and Boero see the second reason for the success of the PME group in ‘the fact that the growth of “PME”... happened during the full development of mathematics education as a research domain, contributing to that development, but also profiting from it’, (2006, p. 1).

We may conclude from the above that the editors of the book see a kind of symmetrical relationships between mathematics education research community in general and the PME group activity. We may also conclude that the main contribution of PME to mathematics education research and practice is being a—not necessarily always coherent—core group in the domain in which mathematics education researchers and practitioners, psychologists, and mathematicians from different countries and cultures may meet on an annual basis within a well-organized group and work together. The group created a constitution which has been adopted in the PME 1980 conference, and went through a few democratic changes from then.

Fig. 13.1 Efraim Fischbein,
first PME president



13.1.2 PME Spirit Through the Lens of Its Goals, Conferences, Proceedings and Books

The major formal goals of PME, as they appeared in the PME constitution (1980–2016), frames its activity. They are:

(i) to promote international contacts and exchange of scientific information in the field of mathematics education; (ii) to promote and stimulate interdisciplinary research in the afore-said area; and (iii) to further a deeper and more correct understanding of the **psychological** and other aspects of teaching and learning mathematics and the implications thereof.

In spite of the diversity of research included in the PME, PME as a scientific organization and especially the annual conferences and the proceedings volumes, in which the annual conference contributions are published, have the following unifying characteristics:

- Democratization and freedom spirit is one of the characteristics of the International Group for the Psychology of Mathematics Education from its very beginning.
- This spirit encourages members of PME to present (orally and in writing) their empirical studies, as well as their hypothetical new thoughts and theories (partial or more complete), in a very early stage of their development. These presentations enable the researchers to get the critical and often wise feedback from their peers in PME, and to interweave what they had learned from it in the longitudinal thread of their work and also publications in the PME proceedings without the highly demanding procedures of scientific journals. This way, young researchers are able to “gain time and help” on their way to become mature researchers in their domain, working as members in a community and/or cooperate with others on individual or small group basis, rather than in isolation. Members of PME are quite aware to this advantage. E.g., the dedication which has been written on the new handbook for the 40 years of PME says (Gutierrez et al. 2016):

To the young researchers, throughout the world, who are the future of mathematics education research and of the PME community

- An additional characteristic of the PME activity is the fact that there are and always were mathematics education researchers who are active in both the PME community, as well as in other international frameworks in mathematics education and educational research beyond. These people are “interactive pipes” among the various mathematical activity frameworks.

13.2 First Views on the Research Presented at PME

One focus of research in PME, among many others, from the very beginning has been conceptual understanding of the number system. When having a superficial glance at the papers from PMEs in the late 70s and early 80s, one might sometimes wonder if anything has changed at all since then. One illustrative example is the following: In the PME38 proceedings, we find a neatly designed, experimental study with pre-, post- and follow-up test by Heemsoth and Heinze (2014), titled “How should students reflect upon their own errors with respect to fraction problems?” It investigates, flatly speaking, if it is better for students to reflect on their errors in exercises or if it is better if they study the correct solutions of the exercises. In the proceedings of PME7, one finds a neatly designed, experimental study with pre-, post- and follow-up test by Swan (1983) titled “Teaching decimal place value. A comparative study of ‘conflict’ and ‘positive only’” approaches. The study investigates, flatly speaking, if it is better in instruction to focus on the errors students’ make in exercises, when learning decimals, or if it is better to focus on the correct solution of these exercises. Both studies come to the conclusion that, particularly in the long run, focusing on errors is more effective than an exclusive focus on the correct solution. So, nothing has changed? Well, of course important things have changed—when looking at the research in both periods in more detail; we see continuity, but also substantial development.

13.2.1 *The Theoretical Basis That Is Used to Frame Findings*

When we consider these two studies, we see important differences. The Swan study (1983) starts out from the point that “traditional courses” do not remedy students’ misconceptions on fractions, and then proposes the two teaching styles as potential solutions to this problem. Afterwards, it presents a mathematical analysis of potential student errors in decimal arithmetic—which is actually mostly in line with what we can read in the literature today. Next, the sample, intervention methodology, and results are presented. At the end, the author hypothesizes that the “conflict” approach should be particularly effective to foster *conceptual knowledge* in contrast to *procedural knowledge*, connecting his study to a model of cognition that is not further specified. The Heemsoth and Heinze (2014) study can refer to the then more extensive literature on students’ misconceptions about fractions and on learning from errors, but it also refers explicitly to a developmental learning model that includes conceptual and procedural knowledge. The authors interpret their results in view of this model and see some peculiarities, e.g. that the “error centred” (conflict) intervention showed immediate effects on procedural knowledge, while an effect on conceptual knowledge occurred only by the follow-up test. Our main point here is not if the results are conclusive. What is visible in the newer

contribution explicitly—even though it might also have been done in the older one—is how theoretical models can be challenged by empirical data, and how questions regarding their power to describe mathematical learning processes can be derived from well-planned studies. In the sequel, we will contrast the accounts of numerical cognition in earlier and newer times of PME. However, the discussion of the theoretical basis for and the role of theories in mathematics education in general cannot be tackled here in detail (see English and Sriraman 2009).

In his PME2 paper, Noelting (1978) discusses a developmental model for proportional reasoning, based on Piaget and Inhelders' developmental stage models. He presents 23 tasks that required students to compare which of two mixtures of orange juice and water with respect to the "relative orange taste". He categorizes the answers of 321 students aged 6–16 into four developmental stages statistically, and provides a qualitative description of each stage. This study is one example of a number of studies from "early PME" that closely connected to general theories of cognitive development and learning going back to the Piagetian tradition. We see the criticism of these theoretical accounts reflected in later PME proceedings, for example in an analysis of students' conceptions of multiplication by Herscovics et al. (1983) in PME7, who indicate that some of Piaget and Szeminska's original findings might be due to the specific tasks used in their experiment. Even though no empirical data is presented in their contribution, the more critical stance towards the established psychological theories is visible. One direction this discussion took was to take into account the socio-cultural context in which learning and mathematical thinking take place (see Sect. 13.3.4).

Beyond this, descriptions of tasks and students' individual understanding, based on analyses of the underlying mathematical structures, have a tradition in and beyond PME (e.g., Carpenter et al. 2012). In the recent years, different perspectives, based on specific theories of numeric processing like Dehaene's triple code model (Dehaene 1992), and assumptions about how human process numbers are studied in detail. Under the umbrella term "natural number bias", for example, several studies follow the question if and under which conditions humans process fractions as one holistic magnitude, instead of processing their denominator and numerator separately (e.g. van Dooren 2016). These accounts use psychological theories of number processing (e.g., Dehaene 1992), as well as dual process theories of cognition which differentiate between quick default heuristics and more demanding analytic strategies. Of course, strategies which students use when dealing with fractions have been described in studies long before, primarily using self-reports. One possible focus of PME in the future might be to study the link between the existing descriptions of mathematical thinking and these specific models of number processing. This cannot only enrich our understanding of mathematical cognition, but also help interpreting students' strategy choice and provide means which advance students' use of mathematics.

13.2.2 Methods Used to Approach Questions

As the theoretical perspectives of research shift, this often has an impact on the research methods required or deemed relevant to study mathematical cognition and learning. With the strong basis on the ideas from Piaget's school, research in the early phases of PME focused primarily on individual thinking and learning processes, often in well-controlled settings such as clinical interviews or paper-and-pencil tests. Two developments can be distinguished from that point. Firstly, together with a perspective on learning which focused on the social and cultural embedding of mathematical cognition and learning, methods have been developed to study these phenomena in authentic, realistic settings, taking into account not only psychological, but also social phenomena that influenced mathematical learning and thinking (e.g., Cobb and Yackel 1996). Secondly, new theoretical models, such as those about number processing mentioned above, drove the application of methods recently, that had not been used before a lot. These "new" research methods, like eye-tracking and reaction time analysis or, less frequently applied, brain imaging methods, have been discussed in several recent group activities in PME, including a Research Forum on the role of neurocognitive research for mathematics education in PME39 (Tzur and Leikin 2015), and a working session on the use of eye-tracking technologies organized by Barmby et al. (2014) in PME38. A good example for a new method that found its way into PME this way is the choice/no-choice method to study students' strategy choice in different kinds of tasks (Luwel et al. 2009). Originally, it builds on the assumption that students chose their calculation strategies from a pre-defined set of strategies, and it allows to study strategy choice within this theoretical frame, well beyond the restrictions of usual self-report methods. It is also a good example that a new method need not be based on innovative technology—sometimes creative but systematic thinking is a good first step.

13.3 Development and Changes in PME Research on Mathematics Learning

13.3.1 General Features of Trends in This Research

We use the thread of the research on processes of mathematics learning as a second example for a more detailed demonstration of the development, the theoretical and methodological changes and milestones, in PME research.

While starting to search in PME's proceedings and books we realized that the reality is quite complex and it is not easy to describe how the focus changed, raised and fell along the 40 years of PME; paradigms, trends and fashions were changing in an evolutionary way, were in use, and would almost vanish after a short time, or would become at least partially nesting in the following one or vice versa. In our

view, the main reason for such an interweaving emerges from the following research situation:

...it seems that more than in the past, researchers today do not feel obliged to and/or satisfied with sticking to one methodological paradigm. Research trends in our area are nowadays characterized by flexibility and creativity in combining research methods and methodological tools, which fit the researchers' theoretical framework and meet their goals and needs to explain and answer some 'big questions' emerging from their explorations. (Hershkowitz 2009, p. 273)

In the following we will try to discuss a few research trends that became milestones in certain points during the PME 40 years. Those trends seemed to be the most dominant in the area of investigating processes of learning mathematics. This discussion is not a statistical survey. Our words express our view which is supported by our knowledge, memory and rereading in PME proceedings and in other publications. On our way we were helped by others who wrote on similar topics—"we were standing on the shoulders of giants" (Sreen 1990).

13.3.2 Learning as It Is Expressed in the Accumulation of Learners' Responses (as Individuals) to Purposeful Tasks in Tests and Questionnaires (Quantitative Research)

In this trend of research, the responses of each individual are analysed separately. However the accumulation of all responses draws a picture of the collective knowledge as a product at one point of time. But, it does not allow direct inferences on the processes of the knowledge construction; neither on the individual, nor on the collective knowledge construction. Yet, valuable information on achieved knowledge of the collective at one point of time is collected.

While searching in one of the first proceedings of PME, (e.g., the first part in the proceedings of the 1980 PME conference, edited by Robert Karplus), it can be seen that many of the contributions use a very popular quantitative methodological tool: The questionnaire. If we will examine the first part of the 1980 proceedings, there are mainly two patterns of research methodology of making use in questionnaires:

The first one can be described by the following pattern of research elements:

- Research question;
- Hypothesis;
- The rationale for the hypothesis;
- The methodology and methodological tool—the questionnaire;
- Big heterogeneous sample which fits the research question and the hypothesis;
- Results presentations and analysis;
- Comparison between the findings and the hypothesis;
- Answer to the research question;

- Interpretation and explanation;
- Some ideas for a follow-up research and/or for activities of practice.

The above is a kind of *Top-Down Research*, and many quite valuable studies were done according to such a pattern. A classic example is the seminal study done by Efraim Fischbein and Irith Kedem, which was published in the PME (1982) proceedings. In this study the researchers investigated the question:

Does the high-school student, normally involved in courses of mathematics, physics etc. ... clearly understand that a formal proof of a mathematical statement confers on it the attribute of a priori, universal validity – and thus excludes the need for any further check?

The second pattern of research in which its main research tool is the questionnaire, is a pattern of *bottom-up research*, whose main line might be described as follows:

- Focusing on the mathematics learning topic to be investigated;
- A priori epistemic analysis of the knowledge students are expected to develop by learning the above topic;
- Constructing questionnaire;
- Defining the research sample and circulating the questionnaire;
- Analyses of the finding; Interpretations of the above, which on one side is often supported by previous well known studies and learning theories, but on the other side provides opportunities for the emergence of new theories (either partial or not);

This pattern is a *Bottom-up research*, as the findings and also the conclusions are concluded directly from the data, without the need to confirm or to refute a pre-given theory or hypothesis. As an example of this pattern we may read the contribution by Haseman (1980), which investigates difficulties of 7th graders in addition of fractions.

13.3.3 *Theory in the Center*

It is quite amazing that only three years after the 1980 Conference in Berkeley, we may find so many theoretical contributions in the Proceedings of the 1983 PME Conference in Israel. As for the 1980 conference, we should mention the theory concerning the differentiation between the *Concept Definition* and the *Concept Image*, suggested at PME 1980 by Vinner and Hershkowitz (1980) (and later by Tall and Vinner 1981).

Section B in the above proceedings with the title: “Learning Theories” (pp. 52–122) includes 12 contributions, with almost each of them discussing a theme in mathematics learning from a theoretical point of view, rather than having a discussion based on empirical data. Often some mathematical-pedagogical examples do appear, mostly as illustrations for the main ideas. If empirical data are mentioned (for example interviews with learners), they are summarized in a “meta way”

Fig. 13.2 Richard Skemp,
second PME president



without real examples. We assume that the lack of a written space (six pages per a contribution) is also a reason for this style of representing findings. At any case it leaves the impression that for these researchers theoretical ideas are much more important than the findings or their interpretations. Also if empirical findings appear, they mostly emphasize the end-products and less the observation and analysis of the learning processes.

A very typical example is the dominant theory in the late seventies and eighties by Richard Skemp (Fig. 13.2), the second PME President. Richard Skemp was a mathematician who later studied psychology (Skemp 1986) and drew on both these disciplines to explain understanding in mathematics. The main ‘thrust’ of his argument is that learners construct *schemata* to link what they already know with new learning. According to Skemp, mathematics involves an extensive hierarchy of concepts—we cannot form any particular concept until we have formed all the subsidiary ones upon which it is depends. Skemp stressed that instrumental and relational understandings are both ways of understanding. This is a distinction in a theory of understanding: ***Instrumental understanding***: a rule/method/algorithm’ kind of understanding, which gives quicker results in the short term. ***Relational understanding***: a more meaningful understanding in which the pupil is able to understand the links and relationships which give mathematics its structure (which is considered more beneficial in the long term and aids motivation). Both are deemed important for mathematics learning (Skemp 1977).

13.3.4 Constructivism and Socio-cultural Approaches, as Catalysts for Classroom Research or Vice-Versa

One of the main theories which raised an intensive theoretical, empirical and practical interest as well as intensive debates at PME’s community is *constructivism*, which might be considered as one of the PME’s milestones in the late eighties. Looking at PME proceedings from 1987 in Montreal, we may learn about

the place of the constructivism theory within the thinking, discussions and debates at the PME's eleventh conference. Confrey and Kazak (2006) described clearly this milestone, with its peak at the PME conference, in their chapter in the PME 30 book. The authors start from mentioning PME organization's main goal which expresses the need to integrate together *Mathematics Education and Psychology* (see the third goal of PME above). There is no doubt that the presentations at the 1987 Conference (especially the four plenaries) demonstrate the advantages as well as the difficulties in interweaving together the "P", "M" and "E" at the PME community. Confrey and Kazak describe main features of constructivism and explain how it became so popular. They wrote:

As a grand theory constructivism served as a means of prying mathematics education from its sole identification with the formal structure of mathematics as the sole guide to curricular scope and sequence. (p. 306)

An extreme example of the above trend of mathematics education is the sequence of the SMSG text books (e.g., SGMG 1961) in which the "curriculum developers" were mostly mathematicians, who seemed to believe that if only the text book will be mathematically correct and the order of the various chapters will be constructed according to a mathematical logic, students will be able to learn the mathematical subject, no matter how abstract it is. Confrey and Kazak continued and said:

The constructivism created means to examine mathematics from a new perspective, the eyes, mind and hand of the child... Constructivism evolved as researchers interests' in the child's reasoning went beyond a simple diagnostic view of errors, to understanding the richness of student strategy and approach. (p. 306)

Gerard Vergnaud, the PME third president, explained in his presentation at the 1987 PME Conference what he considered as the main goal of constructivism:

As a matter of fact, our job, as researchers, is to understand better the processes by which students learn, construct or discover mathematics and help teachers, curriculum and test devisors and other actors in mathematics education to make better decisions. (p. 43)

This approach, which is related to the investigation of the learning process of the student as an individual, and was based on the belief that the learner has to construct his/her mathematics by him/herself, had its own research methodologies. For example, Steffe and his colleagues describe the methodologies they used for teaching experiments and for clinical interviews with an individual child, through which they built models of children's mathematics, meaning the mathematics which was created by the student. In these models they suggest ways to consider the role of interactions between the interviewer and the student and/or between the teacher and the student (Confrey and Kazak, p. 313).

Confrey and Kazak also explained why the constructivist approach held for quite a long time:

It took hold in practice because it addressed the two primary concerns of teachers: (1) students' weak conceptual understanding with over-developed procedures (relational vs. instrumental in Skemp's language) and (2) students' demonstrated difficulties with recall and transfer to new tasks.... (p. 306)

When mathematics education researchers' and practitioners' interest moved on to focus on processes of the collective's mathematics learning (mostly classroom research), constructivist psychological approach was not enough. The socio-cultural approaches, which were established and developed at the beginning mainly in the Soviet countries (Vygotsky 1978 and many others), were adopted by theoreticians and researchers in many areas all over the world and also by the mathematics education people as well as the PME community. E.g., "*The materialist psychology by Vygotsky*", was mentioned by Lerman (2006) in his chapter in the 30 PME handbook. Lerman (2006) wrote that the main elements of the theory are: "That development led by learning (Vygotsky 1986); that concepts appear first on the social plane and only subsequently on the individual plane; that the individual plane is formed through the process of internalization; Psychological phenomena are social events; Learning takes place in the *zone of proximal development* and pulls the child into their tomorrow; and motives are integral to all actions" (p. 350).

Coordination between the constructivist and the socio-cultural approaches (theories and methodologies), led to a deep investigation of what is going on in the mathematical classroom. As an example we cite Cobb and Yackel (2011):

...we differentiated between what we termed the social aspects of the classroom which included classroom social norms and the cognitive aspects which included students' mathematical reasoning.... We instead came to the view that any aspect of the classroom can be analysed from either social or a cognitive perspective. (p. 38)

They called this new theoretical framework *The Emergent Perspective*. The emergent perspective framework is a powerful framework for describing socio-cognitive development within a classroom and was established upon the need to better understand and interpret what was observed in the mathematics classroom.

13.3.5 Research in the Mathematics Classroom and the Mathematics that is Taught and Learned in the Classroom

For more than 20 years now mathematics educators and researchers have been discussing intensively teaching and learning practices like cooperative learning, interactions and argumentation in the various classroom settings, social norms, socio-mathematical norms and more. The discourse about these practices is often general and does not always relate much to what contents and structures of mathematics are the most appropriate for teaching and learning so that the above practices will be activated by needed mathematical contents and mathematical means. Yet, there were and still are projects and innovative curricula in different countries which are enriched by the vision of such practices and vice versa.

Realistic Mathematics Education (RME), which is a teaching and learning theory in mathematics education, was the vision and curriculum development of the Freudenthal Institute in the Netherlands. This theory influenced and has been adopted by a large number of countries all over the world (de Lange 1996). The vision of RME was led mostly by Freudenthal's view on mathematics learning (Freudenthal 1991). Two of his important points of views are: "mathematics must be connected to reality" and "mathematics as human activity". First, mathematics must be close to children and be relevant to everyday life situations. However, the word 'realistic', refers not just to the connection with the real-world, but also refers to problem situations which are real in students' minds. Second, the idea of mathematics as a human activity is stressed. Mathematics education organized as *a process of Guided Reinvention*, where students can experience a similar process compared to the process by which mathematics is being invented. The meaning of *invention* is steps in learning processes while the meaning of *guided* is the instructional environment of the learning process. The *reinvention process* can use concepts of mathematization (Freudenthal 1991, p. 41) as a guide.

13.3.6 Networking—Connecting Theoretical Approaches for Better Interpretation of Empirical Findings

In Sect. 13.3.4 we discussed the coordination between the constructivist and the socio-cultural approaches (theories and methodologies), for a deep investigation of what is going on in the mathematical classroom. In the current years researchers discuss *networking* which is engaged in connecting different theoretical/methodological approaches, where each of them is a framework underlying some trend/s of research. Coordinating them together in planning the study and in analysing the findings enables higher levels of interpreting the results, innovative understanding and insights (Prediger et al. 2008). We see the interest in networking an additional evidence for the trend of flexibility in choosing both, theoretical and methodological basis for research with the aim of better understanding and interpreting the meaning of research.

The work of the researchers (In alphabetic order—Dreyfus, Hershkowitz, Rasmussen and Tabach) is a good example representing the above trend. First, the researchers coordinated together two theoretical/methodological frameworks: The AiC (Abstraction in Context) framework which analysed construction processes of mathematical knowledge of individuals as well as small groups within an inquiry classroom. The second framework is the DCA (Documenting Collective Activity), whose aim is investigating processes of constructing mathematical knowledge within the whole class community. By the above coordination, the researchers were able to trace processes of knowledge constructing and knowledge shifts between and within the different settings in the working mathematics classroom along a whole lesson and more. While doing so they were able to reveal the active role of some students and the teacher in the shifts of knowledge. Currently (Hershkowitz et al. 2017)

the group started to characterize the shifts of knowledge by additional step of networking. The group raised the question: In what way is the shifted knowledge creative? For searching the authors used the work of Lithner (2008) on creativity. This new study represents progress in terms of what Prediger et al. (2008) refer to as the local integration of different theoretical/methodological approaches.

13.4 Factors Influencing PME's Development—Examples from Research on Mathematics Teachers

Besides the perspectives on developments within PME, we will outline factors influencing PME from the outside, exemplified by research on mathematics teachers. This research area went through an extraordinary development in the last 30–40 years. Some researchers speak of an “explosion”. Thus it is not possible to aggregate all the many individual single contributions within and outside PME into a consistent account of the development of the field and derive influencing factors. However, we will highlight some aspects we regard as most important.

13.4.1 The Development of Research on Teachers and Teaching in PME

Several papers have summarized the development of research on teachers and teaching in PME in the past (e.g., Hoyles 1992; da Ponte and Chapman 2006; Llinares and Krainer 2006; Jaworski 2011; Lin and Rowland 2016). In most contributions, the development of the field is described by three phases.

1. *Teachers getting recognized*: All these accounts agree that the teacher was not in the focus of PME research until the end of the 80s. Even though research from this phase has been criticized as simplistic and deficit-oriented, it has certainly played an important role in the formation of the research area.
2. *Towards a research area*: In her 1992 plenary lecture, Hoyles (1992) diagnoses two trends in PME's work: A quantitative increase in contributions that focus on the “teacher as an integral – and crucial – facet of mathematics learning and a series of qualitative shifts as to how the teachers’ role is conceptualized”. If the teachers occurred in research contributions before, they seem to have played a side-role as the facilitator for students’ development, while the student was at the centre of researchers’ attention. At the time, teachers were increasingly recognized as a possible focus of research, initially with a restriction to teachers’ beliefs, later on the relation of these beliefs to teachers’ classroom practice. Studies addressed the question, if and how teachers’ attitudes could be changed so that curricular innovations would be taken up. For the late 80s, Hoyles observed that research increasingly addressed how teachers’ beliefs and actions

were connected to a specific classroom context as well as its broader cultural (e.g., national) embedding. Finally, she described the first developments of a research area focusing on teachers, including a reflection on theoretical perspectives and methodologies used.

3. *Differentiation of the field*: As Jaworski (2011) outlines in her historical account, the 1990s started a very active phase of teacher and teaching research in PME. In particular, she mentions three working groups, which met regularly for five years during PME conferences during this phase (Psychology of In-Service Education of Mathematics-Teachers, Research on the Psychology of Mathematics Teacher Development, Teachers as Researchers in Mathematics Education). Each of these groups produced a book volume by 1999, there were two PME plenaries on the topic (Hoyles 1992), and the Journal of Mathematics Teacher Education was founded under the lead of Tom Cooney in 1998. The topic was also taken with a special survey for ICME-10 in 2004 and the ICMI Study 15 *The professional Education and Development of Teachers of Mathematics*.

In their summary of the field for the 30 years PME volume in 2006, da Ponte and Chapman (2006) identify four main objects of study in teacher-related research on PME: Teachers' mathematical knowledge, Teachers' knowledge of mathematics teaching, Teachers' beliefs and conceptions, Teachers' practices. While all of these topics are still in place, Lin and Rowland (2016) put teacher knowledge in the centre of their contribution for the 40 year handbook ten years later, highlighting its role in PME research.

13.4.2 Trends Impacting the Development of Research on Teachers and Teaching

The development described above was influenced by different other trends in mathematics education and related fields.

Advent of so-called “socio-cultural approaches”: Several authors offer explanations for the increased focus on the teacher in the late 80s and in the 90s. Lin and Rowland (2016) note that this development coincided with the so-called “social turn”, meaning an increased focus on social context, in which students' mathematical thinking and development takes place (see Sect. 13.3.4). Mathematical thinking and learning cannot be considered as something that happens in the students' isolated mind, independently of external influences. This idea that the environment—most prominently the classroom and the teacher—have an influence is already visible in many early PME papers. As soon as these ideas spread, it was only natural to pay more attention to the role of the teacher.

Discussion on the situatedness of cognition: This trend relates to the discussion, to what extent cognition in general, or knowledge specifically is connected to specific situational contexts, or to what extent it may be considered as a more

general disposition that can be activated in a variety of situations. A famous part of this discussion is the so-called Anderson-Greeno-debate, which went on over several papers in the *“Educational Researcher”*. In a joint paper (Anderson et al. 2000), the opposing groups propose a research agenda that pursues both approaches “vigorously”, and argue for attempts to integrate the different understandings of learning and knowing into a comprehensive account in the future.

Within PME, researchers favouring the “situated approach” have often focused on teachers’ practices in realistic situations, research initiatives from the “cognitive side” have tried to build up models that describe the knowledge that is necessary for the professional work of a teacher, and studied them often using paper-and-pencil tests. Apart from the different conceptualizations of knowledge in both approaches, each perspective has developed and often stuck to a specific set of research methods. The discussion seems to have split the research tradition into two parts. Even (2009) asks “Are the two perspectives compatible? Do they complement each other?” Based on a model of assessment proposed by Blömeke et al. (2015), Gabriele Kaiser illustrated one approach towards an integration of both views in her plenary lecture on PME38 (Kaiser et al. 2014). They propose to study teachers’ knowledge not only with methods traditionally applied in the cognitive tradition, as paper and pencil tests, but also using complex, authentic assessment situations to observe teachers’ practices systematically. However, the path towards the integration of both perspectives seems to be rocky. Lin and Rowland (2016) state after discussing these and related ideas: “The paradigmatic differences in conceptualizations of mathematics teacher knowledge [...] remain intact”.

Parallel developments in teacher research in other areas: Krauss (2011) reviews the history of teacher related research in education in general, mostly focusing on developments that were not explicitly connected to PME. He describes four phases of teacher research in the past on the quest for identifying characteristics of “good teachers”, in the sense of teachers who support their students’ development successfully. While the first phase did not yield strong results on the effectiveness of teachers, the second and the third phase brought up substantial knowledge that can today be found in typical texts on instructional psychology, for example on the role of instructional clarity, prevention of disturbances, and adequate speed of instruction. Since 1985, Krauss (2011) describes an increasing interest in the teacher again, now with a focus on teacher characteristics that can be connected to teacher practices and student development theoretically and empirically. The historical narrative in the mathematics education tradition is that developments within mathematics education itself led to this “discovery of the teacher” at this time, and that the developments in general educational research on the role of the teacher were viewed with scepticism at that time, due to different theoretical perspectives on classroom learning. However, some PME members surely had contacts to the general education community. Given that the trend to focus on the teacher developed about the same time in both communities, it cannot be excluded that they were connected to a certain extent—be it with the goal of integrating or of delineating the different approaches in both research communities.

13.4.3 *What Can We Learn from Research on Teachers and Teaching?*

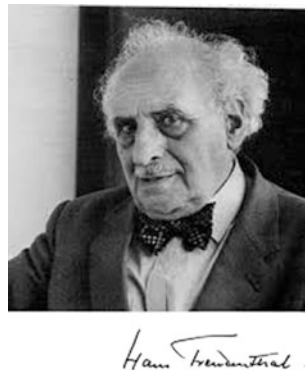
Discussing frameworks: Mathematics education is a quite young science, and thus it is not clear yet in many fields, which notions are best suited to describe the phenomena and problems we observe in mathematics teaching and learning. Diverse models of teacher knowledge have been discussed in the past (see Lin and Rowland 2016 for an overview). Lin and Rowland (2016) describe attempts to find relations between these frameworks and study their unique characteristics. Whether a “Mainstream Theory”, as Lin and Rowland (2016) call it, is a realistic goal of research or merely a guiding ideal is still under debate. Ideas to compare and combine different theories have been proposed to deal with these multiple perspectives in design research (Artigue and Mariotti 2014). However, it will be very interesting to follow the development of our understandings of teacher knowledge and its effects towards increasing coherence, since this field has proven productive in generating a variety of frameworks in the past (Lin and Rowland 2016).

Struggling to integrate opposing views: As indicated above, the discussion about the nature of knowledge and cognition, as well as different methodological approaches, pose a major challenge for a field of research. Along them comes the danger of the field splitting into separate subfields, but also the chance to reach a deeper understanding of the field of study, be it teachers and teaching, or students’ mathematical cognition. Of course we can admit that each perspective has something to contribute, but does this really increase our understanding? Sometimes it increases our joint confusion, since we arrive at different conclusions, even from the same data. If our goal is to further our joint understanding of mathematics learning and teaching, we will not get around trying to find a common basis to talk to each other.

Talk to your neighbour: Research on mathematics teachers and teaching has developed parallel, and more or less independently, with trends in neighbouring disciplines like psychology, education, and sociology. Research on teachers is perfect for such an intense discourse with these disciplines, since effective teaching is not a purely subject-related matter. Many PME contributions are discussing not only problems that are closely connected to the specifics of mathematics teaching and learning. They address also general aspects of learning, knowing, and cognition. The increased and ongoing contact with disciplines that deal with the same issues will stimulate the discussion and scientific progress on both sides and prevent divergent developments.

13.5 Epilog

We would like to end this paper, with a few sentences from Hans Freudenthal (Mathematician and educator, who left his deep traces on the mathematics education community, one of the PME’s “fathers”). The following sentences are borrowed from his plenary at PME 1983.

Fig. 13.3 Hans Freudenthal

Freudenthal (Fig. 13.3) claimed that for him an education is a human activity, which is about learning and teaching as processes, taking place in a more or less organized way". He complains that in the late 70th and early 80th, *education* meant for many people *education research*, and many publications were concerned with **states** (his words) rather than **processes**. Many of the publications were in the style of: "before the treatment and after and what happened in between was indeed a treatment rather than a teaching-learning process." Freudenthal was also happy to tell the PME 1983 conference that: about a third of the contributions in the few PME proceedings which existed at 1983, "were concerned in what I (he) like to call education, that is learning and teaching as a process" (1983, p. 46). Over the years, PME members have taken Freudenthal's insightful comments to heart. It is our wish that the community continue to follow his inspirational path.

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