Chapter 20
Low Numeracy: From Brain to Education

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20.1 Introduction

Leopold Kronecker is quoted famously as making the ontological claim that 'God made the integers, all else is the work of man.'[1] This is not a testable hypothesis. Kronecker may or may not have been a believer in the supernatural when he made this statement. He was born a Jew but converted to Christianity a year before his death. He apparently believed that only integers and objects constructed from them actually existed. This included rational numbers but excluded the reals, π, transcendental numbers more generally, and infinities, all of which may be mathematically useful, but didn't really exist.

If God did make the integers, how did we come to know them? This is a problem that has exercised the best philosophical minds since the time of Plato. However, if we take his apothegm more metaphorically, he may be arguing that our *knowledge* of maths depends on our *knowledge* of integers. That is, we recast his ontological claim as an epistemological one. We can go further, and recast God as evolution. That is to say, is there an evolutionary basis for our knowledge of integers? Here we need to step back from the term 'integer', which includes negative numbers, and restrict ourselves to positive whole numbers, the so-called 'natural numbers'.

It is now widely acknowledged that the typical human brain is endowed by evolution with a mechanism for representing and discriminating numbers. It is important to be clear right at the outset, that when I talk about numbers I do not mean just our familiar symbols – counting words and 'Arabic' numerals, I include any representation of the number of items in a collection, more formally the cardinality of the set, including unnamed mental representations. Evidence comes from a variety of sources.

[1] http://www-history.mcs.st-andrews.ac.uk/Biographies/Kronecker.html

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Human infants notice changes in the number of objects they can see, when other dimensions of the objects are controlled. In the first study of this kind, infants of 5–6 months noticed when successive displays of two dots were followed by a display of three dots and when successive displays of three dots were followed by a display of two dots. However, they did not notice a change from four to six dots or from six to four dots (Starkey and Cooper 1980). With larger numbers of dots, infants need a ratio of 2:1 to notice a change in the number of dots (Xu and Spelke 2000). Recently, studies have shown that infants notice the matches between the number of sounds and the number of objects on the screen (Izard et al. 2009; Jordan and Brannon 2006), suggesting that the infant's mental representation of number is relatively abstract – that is, independent of modality of presentation.

There is also evidence for individual differences in various measures of this ability, at least in older children (Geary et al. 2009; Piazza et al. 2010; Reeve et al. 2012). Twin studies suggest that differences appear to be at least partly genetic (Geary et al. 2009; Piazza et al. 2010; Reeve et al. 2012). The genetic factor is reinforced by the finding that certain kinds of genetic anomaly, such as Turner's Syndrome, affects numerical abilities, including very basic abilities such as selecting the larger of two numbers or giving the number of dots in an array, even when general cognitive ability is normal or even superior (Bruandet et al. 2004; Butterworth et al. 1999; Temple and Marriott 1998).

Another line of evidence comes from the studies of other species. Many of those in which numerical abilities have been tested show performance comparable with or significantly better than human infants. Chimpanzees are able to match the correct digit to a random display of dots up to at least ten (Matsuzawa 1985; Tomonaga and Matsuzawa 2002). Monkeys are able to select the larger numerosity of two displays even when the elements in the display are novel. Moreover, they show a very similar 'distance effect' to humans – that is, the more different the numbers, the more likely they are to select the larger correctly (Brannon and Terrace 1998). Birds have been known to be good at number tasks for 80 years or more. Numerical abilities have been demonstrated in elephants, cats, rats, salamanders and even fish (Agrillo et al. 2012).

Neuropsychological studies of patients with brain damage reveal a complex network in the brain that supports arithmetical processes. Damage to the frontal lobes affects the ability to solve novel problems, while damage to the parietal lobes, usually the left parietal lobe, affects the ability to do routine tasks or to recall previously learned facts (Cipolotti and van Harskamp 2001; see Butterworth 1999, Chap. 4 for reviews). Neuroimaging shows that the parietal lobes are activated by very simple tasks, such as selecting the larger of two numbers or the display with more dots (Dehaene et al. 2003; Pinel et al. 2001). In fact, small regions in the left and right parietal lobes (the intraparietal sulci) are specific for processing the numerosity of displays (Castelli et al. 2006). These regions are part of a brain network involving both the parietal and frontal lobes that are activated almost every time we carry out a numerical calculation, routine or novel (Andres et al. 2011). These findings link numerosity processing and arithmetical calculation in the brain. See Butterworth and Walsh (2011) for a review of the neural basis of mathematics. I will return to the question of whether individual differences in brain structure and functioning can be linked to individual differences in arithmetical competence.

Various environmental factors can all be associated with lower mathematics attainment, including socioeconomic status and minority ethnic status, as well as gender, which should perhaps be considered a social rather than genetic factor in this context (Royer and Walles 2007). Although it is difficult to assess the role of poor or inappropriate teaching, the fact that the introduction of a detailed new national primary school strategy for numeracy in the UK had only a minor and possibly nonsignificant effect on numeracy for the group studied is indicative (Gross et al. 2009). It should also be noted that there are wide individual differences on even very simple tasks that depend relatively little on the quality of educational experience, such as comparison of the magnitude of two single-digit numbers or enumerating a small array of objects (Reigosa-Crespo et al. 2012; Wilson and Dehaene 2007).

Taken together, the evidence presented here suggests that factors specific to the domain of numbers and arithmetic make a major independent contribution to low arithmetic attainment. This is supported by findings from studies that have found low attainment in learners matched for IQ and working memory. In a longitudinal study by Geary and colleagues, tests on understanding the numerosity of sets and on estimating the position of a number on a number line were two important predictors of low achievement in mathematics, affecting some 50% of the sample, and of mathematics learning disability, affecting approximately 7% of the sample (Geary et al. 2009). In a sample of 1500 pairs of monozygotic (MZ) and 1375 pairs of dizygotic (DZ) 7-year-old twins, Kovas and colleagues found that approximately 30% of the genetic variance was specific to mathematics (Kovas et al. 2007). In another genetic study, this time of first-degree relatives of dyslexic probands, it was found that numerical abilities constituted a separate factor (Schulte-Körne et al. 2007). In fact, recent reviews have proposed that developmental dyscalculia follows from a core deficit in this domain-specific capacity (Butterworth 2005; Rubinsten and Henik 2009; Wilson and Dehaene 2007).

One obvious question arises: how do our numerical innate capacities relate to the learner's ability to acquire arithmetic?

20.2 Innate Capacities

Now it will come as no surprise to teachers of the first 3 years of school that children's numerical competence begins with whole numbers. However, recent research on the innate mechanisms available to humans (and many other species) propose two foundational 'core systems' that do not involve whole numbers. Deficiencies in these core systems – it has been argued – could contribute to low numeracy.

1. A mechanism for keeping track of the objects of attention. This is sometimes referred to as the 'object-tracking system' (OTS) and has limit of three or four objects. It is thought to underlie the phenomenon of 'subitising' – making an

accurate estimate of one to four objects without serial enumeration (Feigenson et al. 2004). It is proposed that the objects to be enumerated are held in working memory and that they constitute a representation with 'numerical content' (Carey 2009; Le Corre and Carey 2007).

2. A mechanism for the analogue representation of the approximate number objects in a display. This is referred to as the 'analogue number system' (ANS). The internal representations of different numerical magnitudes can be thought of as Gaussian distributions of activation on a 'mental number line'. It is typically tested by tasks involving clouds of dots (or other objects), typically too numerous to enumerate exactly in the time available. One common task is to compare two clouds of dots. (Addition and subtraction tasks for which the solution is compared with a third cloud of dots are also used.) Individual differences are described in terms of a psychometric function, such as the Weber fraction, the smallest proportional difference between two clouds that can be reliably distinguished by the individual (Feigenson et al. 2004).

There has been considerable interest, indeed excitement, in many studies that show the performance on tasks designed to measure competence in the approximate number system correlates significantly with arithmetical performance in both children and adults (Barth et al. 2006; Gilmore et al. 2010; Halberda et al. 2008, 2012). But as we all know, correlation is not cause, and no plausible mechanism for the relationship has been proposed and accepted.

Now there are various problems with both core systems from the point of view of learning arithmetic. In the case of 1, there is an upper limit of 4. Now one key property of the number system is that a valid operation on its elements always yields another element in the same system. If one such operation is addition and if 3 is an element, then 3 + 3 should yield an element in the system, but it cannot, since the limit is 4. To get round this, it has been proposed that noticing the number of objects being tracked can be linked to the number words a child hears and that they will be able to generalise – 'bootstrap' – from these experiences to numbers above the limit (Carey 2009; Le Corre and Carey 2007). The problem is that the object-tracking system is designed to keep track of particular objects with as much detail as is required by the task, not abstract away from them (Bays and Husain 2008).

The problem with 2 is that it deals only in approximate quantities, whereas ordinary school arithmetic deals with exact quantities, and the transition from approximations to exact whole number arithmetic is still mysterious. These problems are well known.

While we do not doubt that these systems exist in the brains of human infants and other species, we have argued that a quite different core system underlies the development of arithmetic. We and others have proposed a mechanism that can represent the 'numerosity' of a collection of objects, that is the number of objects exactly, not approximately, up to a limit imposed by the developing brain. In a pioneering exploration, Gelman and Gallistel called these representations 'numerons' and argued that learning to count is a process of learning how to map number words consistently onto numerons (Gelman and Gallistel 1978). I have argued, following

Gelman and Gallistel, that humans inherit a 'number module' to deal with sets and their numerosity and that some developmental weaknesses in arithmetical development can be traced to deficiencies in the module (Butterworth 1999, 2005).

We have shown that a neural network computer simulation of the number module using what we have called a 'numerosity code' accurately models the 'size effect' in addition. This is where accuracy and speed are a function of the addends – that is, the larger the addends or their sum, the longer it takes to retrieve or calculate the answer (Butterworth et al. 2001; Zorzi et al. 2005).

In the next section, I describe briefly some studies we have carried out that stress the importance of whole number competence in the subsequent development of arithmetic, using a very simple test: how quickly and accurately the child can enumerate a display of dots and say the answer.

20.3 Longitudinal Study of Arithmetical Development from Kindergarten to Grade 5

This is a study carried out in Melbourne, Australia, led by Robert Reeve and his lab. The sample comprised 159 5.5–6.5-year-olds (95 boys). The children attended one of seven independent schools in middle-class suburbs of a large Australian city and, at the beginning of the study, were halfway through their first year of formal schooling. The children were interviewed individually on seven occasions over a 6-year period as part of a larger study. On each occasion they completed a series of tests, including those reported here. The mean ages for the test times were (a) 6 years (5.5–6.5 years) kindergarten, (b) 7 years (6.5–7.5 years), (c) 8.5 years (8–9 years), (d) 9 years (8.5–9.5 years), (e) 9.5 years (9–10 years), (f) 10 years (9.5–10.5 years) and (g) 11 years (10.5–11.5 years). For full details, see Reeve et al. (2012). Here, I will focus on two aspects of the study: competence in numerosity processing as measured by the speed and accuracy of dot enumeration and age-appropriate arithmetic accuracy.

Using cluster analysis, dot enumeration competence revealed three clusters at each age, which we labelled fast (31% of the children), medium (50%) and slow (19%). These were relatively stable on retesting over the period of the study. That is, although children in each cluster improved with age, each tended to stay in the same cluster.

It turns out that the cluster established in kindergarten predicts age-appropriate arithmetic up to the age of 11 at least. I give below the results for three-digit calculations at ages 10–11 years (Table 20.1).

Our recent analyses show that from kindergarten to Year 2, the clusters are the main predictors of the strategies used in single-digit addition, with fast clusters more likely to recall answers from memory and use decomposition for sums over 10 in kindergarten, whereas the slow cluster children are only recalling the answers and decomposing in Year 2 and then less than 30% of the time.

Table 20.1 Three-digit subtraction, three-digit multiplication and three-digit division accuracy at age 10–11 years

	Dot enumeration cluster established in kindergarten					
	Slow		Medium		Fast	
	M	*SD*	*M*	*SD*	*M*	*SD*
Subtraction	46.67	7.38	81.25	2.90	90.65	2.58
Multiplication	60.56	6.53	85.10	2.15	87.07	3.57
Division	41.67	7.02	75.62	2.88	84.86	2.97

20.4 The Neural and Genetic Basis of Low Numeracy

This is a study of 104 monozygotic twins and 56 same-sex dizygotic twins aged 8–14 years. (Zygosity was assessed using molecular genetic methods.) For more further details, see Ranpura et al. (2013, submitted). All the twins in the study had brain scans and carried out a battery of 40 cognitive and numerical tests. Using factor analysis, we extracted four factors, with numerical processing accounting for 24% of the variance and having the highest loading. It comprised three timed arithmetic scores (addition, subtraction, multiplication), together with dot enumeration speed and the standardised WOND Numerical Operations (Wechsler 1996) score. Thus, a second factor (19% of the variance) included measures of general intelligence and working memory; a third factor (12%) included processing speed and performance IQ; while the fourth factor (9%) included tests of motor praxis and finger gnosis. Thus, the factor analysis reveals that that core number skills and arithmetic correlate well with each other and segregate from general cognitive and performance measures.

We replicated other research in finding a difference in grey matter in the brains of children with low numeracy or dyscalculia in the brain region of interest for numerosity processing (Isaacs et al. 2001). See Fig. 20.1.

We were also able to establish the heritability of both competence and grey matter density by comparing MZ with DZ twins: if the concordance between pairs of MZ twins is significantly higher than between pairs of DZ twins, this indicates a genetic factor.

1. Grey matter density is moderately heritable ($h^2 = 0.28$), but common environmental and unique environmental factors are also significant. Shared environment (c^2) is usually thought of as home background and schooling, which applies to both twins; unique environment (e^2) is thought of as factors specific to one of the twins.
2. Arithmetical competence and dot enumeration are both heritable. See Table 20.2.
3. The link between dot enumeration and both arithmetical competence and the region of interest is heritable. Using a different way of analysing the heritability data, called 'cross-twin, cross-trait correlation', we found that the correlation of dot enumeration with timed addition was substantially heritable, with over 50% of that correlation attributable to genetic factors ($h^2h^2r_G = 0.54$, $rho = 0.76$, $p < 0.05$). Moreover, the links between the region of interest and dot enumeration, as well as arithmetical competence, were also heritable.

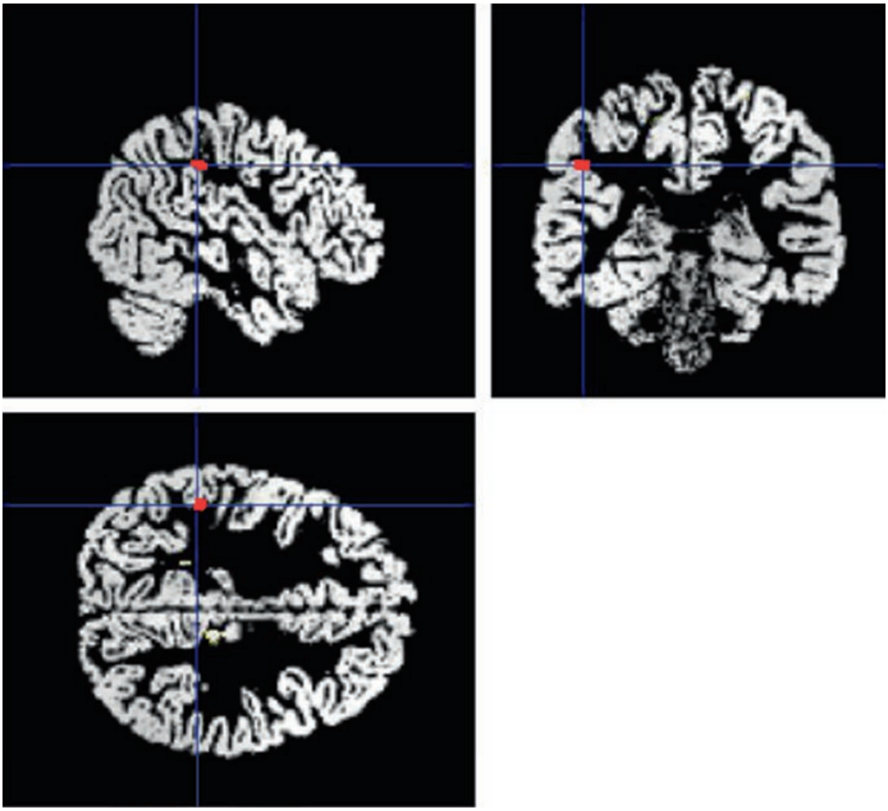

Fig. 20.1 Voxel-based morphometry (structural brain imaging) identifies a *left* parietal cluster that correlates with core number skill (35 voxels with a peak at MNI –48, −36, 34, pFWE-corrected <0.05)

Table 20.2 Heritability of arithmetic and dot enumeration

	h^2	c^2	e^2
	Genetic factor	Shared environment	Unique environment
Timed addition	0.54	0.28	0.17
Timed subtraction	0.44	0.38	0.18
Timed multiplication	0.55	0.31	0.15
Dot enumeration	0.47	0.15	0.38

20.5 Implications for Mathematics Education

The starting point for intervention should be a recognition that some children begin with a disadvantage and that their disadvantage lies in their capacity to deal with sets and their numerosities. This, of course, is the basis of arithmetic both from a logical and a developmental point of view. As we show here, low numeracy has a heritable component, which confirms recent genetic studies as noted above (e.g. Kovas et al. 2007).

We can use dot enumeration in diagnostic assessments. Because these numerosity-based assessments depend much less on educational experience than tests of arithmetic, they minimise noise from instructional and motivational factors, not to mention family and environmental stressors that can also lead to low math attainment scores. Getting the correct assessment is fundamental to selecting the appropriate intervention.

Early attempts to develop new instructional interventions were based on neuroscience findings and the best practices of skilled teachers (e.g. Butterworth and Yeo 2004; Griffin et al. 1994). An important limitation of these interventions is that they required detailed instructional schemes and one-to-one teaching. It is difficult to implement these interventions in the typical math classroom, which has a whole-class age-related curriculum that makes little allowance for atypically developing children who require more attention and practice. In theory, remediation requires an approach personalised to individual learners. In practice, it is difficult to afford such instruction for even a small proportion of pupils in publicly funded education. In the UK, it has been estimated that effective intervention for 5–7-year-olds in the lowest 10th percentile, using one-to-one teaching, would cost about £2600 per learner.

The result is that many learners are still struggling with basic arithmetic in secondary school (Shalev et al. 2005). And yet effective early remediation is critical for reducing the later impact on poor numeracy skills. Although very expensive, it promises to repay 12–19 times the investment (Gross et al. 2009).

As I have argued elsewhere, one approach to the problem of delivering personalised instruction to individual learners is to make use of technology. Personalised adaptive learning technology solutions emulate the guidance of the special educational needs teacher, focusing on manipulation of numerosities (Butterworth and Yeo 2004; Räsänen et al. 2009; Wilson et al. 2006). These solutions go far beyond the educational software currently in use for numeracy teaching, which mainly targets mainstream learners. Commercial software does little more than rehearse students in what they already know, perhaps building automaticity and efficiency, but it does not foster understanding, and it does not address the numerosity processing deficit in many learners and, especially, in dyscalculics. Rarely are commercial games founded on good pedagogy.

Of course, there is no clear logical pathway from assessment to educational remedy, so our software seeks to use ideas from the best practitioners, such as Dorian Yeo (Butterworth and Yeo 2004), and established pedagogical principles, including:

1. Constructionism – construct an action to achieve goal (Harel and Papert 1991).
2. Informative feedback (Dayan and Niv 2008).
3. Concept learning through contrasting instances and generalising concepts through attention to invariant properties (Marton and Pong 2007).
4. Direct attention to salient properties (Frith 2007). This entails ensuring that everything on the screen is relevant to the task in hand.
5. The zone of proximal development – adapt each task to be just challenging enough (Vygotsky 1978).
6. Use intrinsic rather than extrinsic reinforcement (Laurillard 2012).

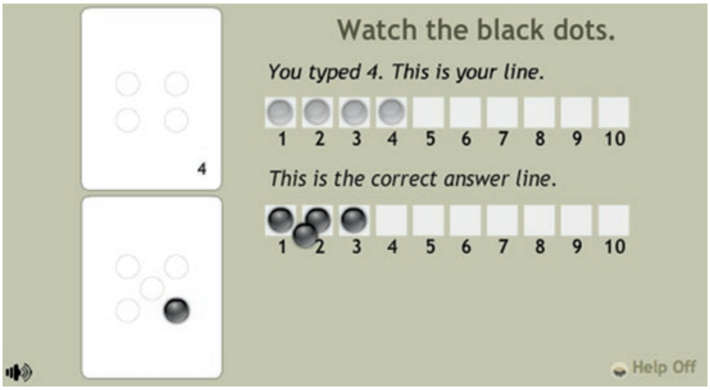

Fig. 20.2 Dots2Track (for an explanation see text)

Examples of the games following these principles have been developed by Diana Laurillard and Baajour Hassan and can be found at http://number-sense.co.uk (see Fig. 20.2).

Their Dots2Track game exemplifies these principles. The task is to type the number of dots in a display. At level 1, these are arranged as in dominoes. In the case of an error, learner's dots are counted onto a line above it and the correct number of dots on the line below it, exploiting principles 2 and 3. There is an opportunity to construct the correct answer by increasing or decreasing the number the learner chose (1). Everything on the screen is relevant (4), and game is adaptive, becoming more difficult depending on the accuracy and speed of the responses (5). The only reward is getting the right answer (6). There is preliminary data on the effectiveness of these games (Butterworth and Laurillard 2010).

Even if a learner has an inherited deficiency in the number module that is reflected in brain structure and functioning, this does not mean a life sentence of low numeracy. It may be that the right interventions over sufficient time can strengthen the number competence to a typical level and indeed modify the brain to a more typical structure, as has been shown in the case of phonological training for dyslexic learners (Eden et al. 2004). This will require a longitudinal study that has not yet been carried out.

20.6 Conclusions

I have argued here that the genetic research is supported by neurobehavioural research identifying the representation of numerosities – the number of objects in a set – as a *foundational capacity* in the development of arithmetic. Where this capacity is weak, education should seek to strengthen this capacity using sets of real or virtual objects and linking the sets to the spoken and written numbers until the learner can use numbers fluently and confidently. This will provide a sound basis for developing arithmetic.

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