Chapter 18
The Theory of School Arithmetic: Whole Numbers

Liping Ma and Cathy Kessel

18.1 Introduction

There are at least two different perspectives on whole number arithmetic in primary school. In the USA, the tendency is to consider it as only learning to compute the four basic operations with whole numbers (e.g. asking students 1 + 1 = ?). In China, however, whole number arithmetic involves much more. For example, it is expected that students explore the quantitative relationships among the operations (e.g. given that 1 + 1 = 2, then 2 – 1 = ?) and represent these (sometimes quite sophisticated) relationships with (sometimes quite complicated) numerical equations.

As mentioned in the article 'A critique of the structure of U.S. elementary school mathematics' (Ma 2013), part of this difference in perspectives is due to a theory that underlies school arithmetic in China and several other countries.[1] Although this theory underlies present-day school arithmetic in China, an important stage of its development occurred in Europe and the USA, initiated by the spread of mass education in the middle of the nineteenth century. This significant social

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[1] Textbook analyses point out specific aspects of this general difference (Ding and Li 2010; Ding et al. 2013).

L. Ma (✉)
Independent Scholar, Shanghai, China, and Palo Alto, CA, USA
e-mail: maliping51@gmail.com

C. Kessel (✉)
Consultant, Berkeley, CA, USA
e-mail: cbkessel@earthlink.net

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change initiated a significant change in arithmetic, which we briefly outline here.[2] Mid-nineteenth-century primary school textbooks such as *Warren Colburn's First Lessons in Intellectual Arithmetic* included Arabic numerals and notation for whole numbers, fractions and operations on them, inherited from commercial arithmetic textbooks for adults such as *Cocker's Arithmetick* (first published in 1677), which focused on efficient computation.

From this school arithmetic, mathematical scholars began to forge an academic subject more closely connected to the rest of mathematics. They introduced two important new features:

- Horizontal expressions. These allowed significantly more sophisticated quantitative relationships to be expressed than did the vertical columns used for the calculations of commercial arithmetic.[3]
- A system of definitions and axioms modelled on that of Euclid's *Elements*.[4] These included the definition of a number as a collection of units. Most included 'rules of likeness' such as the rule that 'only like numbers can be added'. Some included compensation principles or the commutative, associative and distributive properties, but not necessarily both.[5]

The significance of this system in connecting arithmetic with the rest of mathematics is hard to underestimate. In assessing the impact of the *Elements*, the mathematician Bartel van der Waerden (1978/2015) wrote:

> Almost from the time of its writing, the *Elements* exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, other than the Bible, the *Elements* is the most translated, published, and studied of all the books produced in the Western world. Euclid may not have been a first-class mathematician, but he set a standard for *deductive reasoning* and geometric instruction that persisted, practically unchanged, for more than 2,000 years. (emphasis added)

At the beginning of the twentieth century, the system of definitions and axioms was almost complete, as can be seen by examining US textbooks. Its development in the USA did not continue, possibly due to decreased emphasis on 'mental discipline' and increased concern about high failure rates (Stanic 1986; Stanic and Kilpatrick 1992). However, as evidenced in textbooks of other countries, development of the system continued outside the USA.[6]

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[2] Ma (in preparation) gives a detailed account.

[3] The prolific textbook author and translator Charles Davies seems to have initiated this change in US primary mathematics textbooks; see his *Common School Arithmetic* (1834, pp. 17, 33). Use of horizontal expressions was further developed in later textbooks such as *Robinson's Progressive Practical Arithmetic* (1875) and *Sheldons' Complete Arithmetic* (1886).

[4] The first instance in US arithmetic textbooks may be in *School Arithmetic: Analytical and Practical* (Davies 1857). Further developments can be seen in *Sheldons' Complete Arithmetic*.

[5] For example, *The Normal Elementary Arithmetic* (1877) states, 'The sum is the same in whatever order the numbers are added' (p. 208) and 'If the multiplicand be multiplied by all the parts of the multiplier, the sum of all the partial products will be the true product' (p. 223).

[6] Xu notes that, in the first major period of textbook development after 1950, China was 'translating and modifying textbooks from the Soviet Union' (Xu 2013, p. 725). Before 1950, China's school

During the twentieth century, school arithmetic evolved in three ways:

- The system of definitions and rules was augmented by the commutative, associative and distributive properties.
- Prototypical word problems with variants were added, e.g. pursuit, cistern, or work problems (see Ma 2013, Appendix).
- Instructional approaches advanced (see Ma n.d.).

This chapter discusses the first item in this list. In it, we present the central pieces of the theory – the definition system and axioms for whole numbers – distilled from the textbooks of the nineteenth-century USA and twentieth-century China listed in the references. (Details of this development are discussed by Ma in preparation.) The theory built around these central pieces explains all the computational algorithms in whole number arithmetic. Moreover, it can foster primary students' ability to deal with quite sophisticated quantitative relationships.

18.2 Characteristics of the Theory

Like the *Elements*, the theory has definitions, postulates and theorems. It presents a small number of fundamental definitions and shows how other definitions can be derived from those in order to avoid circularity. Its analogue to the postulates of the *Elements* is 'basic rules and basic laws'. Its analogue for theorems is rationales for computational algorithms. The theory differs from the *Elements* in not giving explicit analogues to Euclid's 'common notions' (e.g. 'Things which are equal to the same thing are also equal to each other'). As will be illustrated in this chapter, the common notions were implicitly assumed and used.

The theory differs from modern mathematical theories in several other ways.

First, it follows the *Elements* in style, using only words and diagrams. The advantage of this formulation is its closeness to everyday life. Pedagogical instantiations of this theory, i.e. textbooks, can act as a bridge between lay experiences and the abstractions of formal mathematics.

Second, like the *Elements*, this theory is less precise than modern approaches. Instances of this lack of precision are noted and discussed in this article.

A third difference is that the theory is not intended to be entirely parsimonious. It is parsimonious in giving a small number of fundamental definitions; however, some of the basic laws are redundant. In particular, the laws of compensation can be derived from other basic laws.

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mathematics textbooks were influenced by those of other foreign countries. For example, *The Arithmetic Series* by the Japanese mathematician Tsuruichi Hayashi (1926/1933) was translated into Chinese and used in schools during the 1920s and 1930s. There were also Chinese textbooks that were strongly influenced by US 'progressive education', for example, *The New Ideology Arithmetic Series* (Yang and Tang 1931) and *The New Curriculum Standard Arithmetic Series* (Zhao and Qian 1933). In all of these textbooks with various foreign impacts, however, important features of the theory of school arithmetic, such as emphasis on the relationships among the four operations, can be identified.

18.3 Content and Organisation of This Chapter

We present the definitions and basic laws for whole numbers. The definitions are presented in order, that is derived definitions appear after those on which they depend, followed by a list of the basic laws in the Appendix. The general definitions are numbered.

Definitions like those presented here were given in nineteenth-century US textbooks. After the pedagogical advances of the twentieth century, however, explicit (and sometimes complicated) definitions like these were not presented to children. The numbered pedagogical remarks that follow each definition note ways in which it may be presented to children. Historical remarks that discuss sources and variants are given in the footnotes.

18.4 The Arena of Primary School Arithmetic

18.4.1 Units

Definition 1 A single thing, or one, is called a *unit* or *unit one*.

A group of things or a group of units, if considered as a single thing or one, is also called a *unit*, a *unit one* or a *one* (Fig. 18.1).

One or one thing is a primitive conception that we are born with. The definition of unit is abstracted from this conception. This is the starting point of the definition system.

In this definition, we see two types of unit. The first type we call 'one-as-one unit' and the second 'many-as-one unit'.

Although the concept is called 'unit', use of the terms 'unit one' and 'one' in teaching helps to connect 'unit' with students' conception of 'one'.

Students' understanding of the concept of unit deepens step by step through arithmetic learning. They shouldn't be expected to read or know abstract definitions such as the definition above.

As students progress through primary mathematics, their concept of unit becomes more abstract. Although this deepening of the concept of unit occurs throughout primary mathematics, the term 'unit' is generally not used until middle and upper

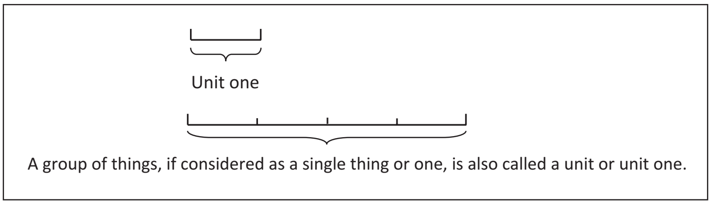

Fig. 18.1 The definition of unit

primary grades. In those grades, students may need to use the terms ‘unit’ and ‘unit one’ when solving certain kinds of word problems, some of them involving multiplication and division of fractions.

18.4.2 Numbers

Definition 2 A *number* is a unit (one) or a collection of units (ones).

This definition of number is given in terms of the definition of unit. It generates one set of numbers, the natural numbers (1, 2, 3, etc.). This chapter does not discuss how the definition of unit will expand to generate a second set of numbers, expanding the number system of primary mathematics. Together, the two sets of numbers, natural numbers and positive rational numbers, form the arena of school arithmetic.

The symbol 0 has two features: as a digit in the notation system and as a number. As a digit, it plays an important role in the notation system. But, as a number, 0 is not part of the arena of school arithmetic.[7]

This definition generates the natural numbers, the set of numbers already familiar to students. Primary students are not expected to learn a separate definition for ‘number’.

Definitions 3 and 4 An *abstract number* is a number whose units are not named.

A *concrete number* is a number whose units are named.

To classify numbers as concrete and abstract is a need specific to school arithmetic. The terms *abstract number* and *concrete number* were created after a long-term effort of primary teachers with the assistance of mathematical scholars.[8]

When they begin school, most primary school students do not have conceptions of abstract numbers such as ‘five’, ‘six’ or ‘seven’. Instead, their conceptions are concrete numbers such as ‘five friends’, ‘six books’ and ‘seven apples’. An important task of primary mathematics is to lead students to complete their transition from concrete number to abstract number and be able to compute with abstract numbers. During this process, students’ original conception of concrete number serves as an

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[7] How many aspects of the number zero should be taught in primary school is an issue which needs further discussion. Consider Alfred North Whitehead’s remark: ‘The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought’ (1948, p. 43).

[8] Smith wrote: “The distinction between abstract and concrete numbers is modern. The Greek arithmeticians [who studied number theory] were concerned only with the former, while the writers on logistic [arithmetic] naturally paid no attention to such fine distinctions. It was not until the two streams of ancient number joined to form our modern elementary arithmetic that it was thought worth while to make this classification, and then only in the elementary school. […] The terms ‘abstract’ and ‘concrete’ were slow in establishing themselves. The mathematicians did not need them, and the elementary teachers had not enough authority to standardize them” (1925/1953, pp. 11–12).

important resource for instruction.[9] The concept of concrete number can also serve as cornerstone in learning to analyse quantitative relationships.

The terms 'abstract number' and 'concrete number' are not terms that students should be expected to know. However, they denote concepts that are important for teachers, curriculum designers, and textbook authors in describing students' mathematical development and in designing instruction to help students develop more abstract thinking.

Definition 5 Like numbers

If two concrete numbers have units with the same name, they are called *like numbers*.

The concept of like numbers is a useful support for students as they learn to analyse quantitative relationships.

18.5 Notation: Base-Ten Positional Numeral System

18.5.1 Digits and Numerals

Digits are symbols used to represent numbers. There are nine significant digits and one non-significant digit.

Each of the nine significant digits represents a different number of units:

1	2	3	4	5	6	7	8	9
One	Two	Three	Four	Five	Six	Seven	Eight	Nine

The non-significant digit is 0. It represents no units.

A sequence of digits is called a *numeral*.

A numeral can have one or more digits. A number represented by a numeral with only one digit is called a one-digit number. A number represented by a numeral with two digits is called a two-digit number. A number represented by a numeral with three digits is called a three-digit number, and so on.

Because there are only nine significant digits, one digit cannot represent more than nine units.

[9] For example, the book *First Lessons in Intellectual Arithmetic*, by Warren Colburn (1793–1833), a Harvard mathematics baccalaureate, gave examples of how this resource could be used. It was published in 1821 and was in 'almost universal use' for several decades (Monroe 1912, p. 424). By 1890, 3,500,000 copies had been sold in the USA (Cajori 1890). Ninety years after its publication in 1912, it was still being used in the USA (Monroe 1912, p. 424). The impact of *First Lessons* was not confined to the USA. *First Lessons* was translated into several European languages and distributed in Europe (*Scientific American Supplement*, No. 455, September 20, 1884). Missionaries translated it into Asian languages and distributed it in some Asian countries. During the mid-nineteenth century, the book sold 50,000 copies per year in England (Monroe 1912, p. 424).

Although there are only ten different digits, every natural number can be represented as a numeral.

18.5.2 *Place of a Digit, the Unit Value of a Digit, the Name of a Place and Place Value*

The position of a digit in a numeral is called the *place* of a digit. The largest digit in any place represents nine units. Each ten units is written as one unit in one place to the left.

Digits at different positions have different unit values. For numerals with two or more digits, the unit value of a place is ten times the unit value of the place immediately to its right.

The places are named according to the value of the unit they represent. From right to left: ones place, tens place, hundreds place, etc.

The unit value determined by the position of the digit is also called the value of the place or place value. In arithmetic with natural numbers, these values are powers of ten: 1, 10, 100, 1000, etc.

The digits in a numeral are named according to their positions: ones digit, tens digit, hundreds digit, etc.

Positional notation is one of several kinds of notation for numbers.[10] A key feature of positional notation is that the place of a digit determines the unit value represented by the digit. In school arithmetic, only one kind of positional notation is taught, base-ten notation. Concepts of positional notation are introduced in the specific context of this notation rather than in a general way (Fig. 18.2).

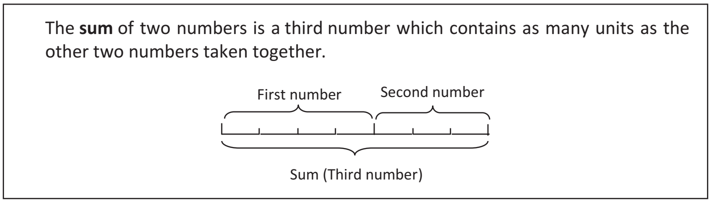

Fig. 18.2 The definition of the sum of two numbers

[10] In addition to positional notation, there are other types of notation systems for numbers such as Roman numerals and Chinese notation.

18.6 Addition and Subtraction

18.6.1 Addition

Definition 6 The *sum* of two numbers[11] is a third number which contains as many units as the other two numbers taken together.

The operation of finding the sum of two numbers is called *addition*.[12]

The definition of 'sum', one of the two basic quantitative relationships in school arithmetic, is given in terms of the definitions of 'unit' and 'number'.

The quantitative relationship formed by three numbers has the following feature. If two of the three numbers are known, the third is determined. Because of this, it is possible to define addition and subtraction in terms of this quantitative relationship.

Although the definition of the sum of two numbers may seem obscure, it reveals the key relationship that underlies addition and subtraction in school arithmetic. The line segment diagram in Fig. 18.2 represents this definition in a form that is suitable for teaching.

After the quantitative relationship of sum is defined, then addition can be defined in terms of this relationship. In a similar way, subtraction can be defined. In this way, the connection of sum, addition and subtraction is given explicitly, using a small collection of fundamental concepts.

It is very likely that the concept of addition is closely related to a primitive conception that we are born with. A contemporary cognitive science researcher Karen Wynn (1992, 1995) has published research to demonstrate that several weeks after birth, infants can recognise the quantities 1, 2 and 3 and do computations such as 1 + 1 and 2 – 1 (see also, National Research Council 2009, p. 65). By the time primary children come to school, they may have developed a variety of strategies for addition: counting all, counting on or using known sums (National Research Council 2001, p. 169). Often, they find it easier to convert subtraction computations to addition computations by counting on, e.g. to compute 8 – 5, count up from 5, '6, 7, 8. So 3 left' (National Research Council 2001, p. 190). The nineteenth-century textbook author Warren Colburn may have been noting this phenomenon when he wrote, 'It is remarkable that a child, although he is able to perform a variety of examples which involve addition, subtraction, multiplication, and division, recognizes no operation but addition' (1821/1863, p. 9).

The task of teaching is to make a bridge from the inborn conception to the abstract quantitative relationship sum of two numbers.

[11] Natural numbers do not include zero.

[12] Addition and subtraction are binary operations. A binary operation is a calculation involving two input quantities. While the sum of three and more numbers can be computed, it needs to be found step by step, at each step computing the sum of two numbers.

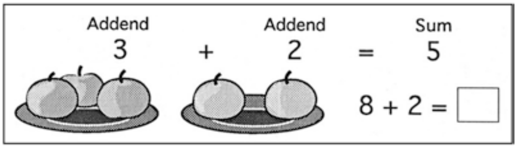

Fig. 18.3 Example of introducing definition and terms of addition to first-grade students (Moro et al. 1992, p. 38) (Reprinted with the permission of the University of Chicago School Mathematics Project)

18.6.1.1 Addends

Definition 7 The two numbers summed are called *addends*.[13]

The two terms 'addend' and 'sum' are important thinking tools to understand and work with the quantitative relationship 'sum of two numbers'. Students should be exposed to them at the beginning of their addition and subtraction learning. Figure 18.3 is an example from a Grade 1 Russian mathematics textbook (the addition on the right (8+2) is a new problem to be solved). It introduces the definition of the sum of two numbers, the definition of addition and the definitions of 'addend' and 'sum' in a form suitable for young children. (For more details of how this may occur in teaching, see Ma n.d., pp. 15–16.)

Some early primary teachers tell their students that because the sum is greater than the addends, if we see that the result of a word problem will be greater than the known numbers in the problem, we use addition to solve the problem. This approach, when compared with the approach of looking for keywords such as 'left', 'together', 'more' or 'less', is more conceptual. However, it needs to be noted that this approach is only useful for one-step word problems.[14] Therefore, at an appropriate point, we need to lead students to notice the limitations of this statement. By noticing these limitations, students gain the experience of developing new knowledge by understanding limitations of knowledge developed earlier.

18.6.1.2 The Rule of Like Numbers for Addition

When two addends are concrete numbers, they must be like numbers. Their sum and the two addends are like numbers.

[13] Terms of the definition system such as 'like numbers' in section I, in this section, 'sum' and 'addends', and in the next section 'product', 'multiplicand' and 'multiplier' all first appeared in arithmetic textbooks during the past 400 years. Over the years, various definitions have been given for these terms. These definitions were not always given as part of a system in which definitions depend on a few fundamental definitions but instead as definitions that were independent of each other.

[14] When we solve multi-step word problems, when the result is larger than the known numbers, we may need to use operations other than addition.

There are two rules of likeness: rule of like number and rule of like unit. Of these two, the rule of like numbers is more closely connected with quantitative relationships.

The rule of like numbers for addition seems very simple. It is easily ignored. Its importance is more noticeable from the perspective of the entire theory.

In teaching, this rule can be said as ‘addends must be like numbers’ or ‘only like numbers can be added’.

18.6.1.3 The Rule of Like Unit Value for Addition

In computing the sum of two numbers, their representations as numerals are used. Only digits of like unit value can be added.

This rule is part of the explanation of the algorithm for multi-digit addition. For example, the digit 5 in the ones place and the digit 3 in the ones place have the same unit value so they can be added. The digit 5 in the ones place and the digit 3 in the tens place do not have the same unit value and cannot be added.

This rule is also an important part of the explanation of the algorithm for multi-digit multiplication.

In teaching, we can say ‘only digits with the same units can be added’ or ‘only the same units can be added’, omitting the word ‘value’ which is not relevant to students and omitting the distinction between number and numeral.

18.6.2 Subtraction

Definition 8 If a sum and one addend are known, the operation of finding the unknown addend is called *subtraction*.

Subtraction is the inverse of addition in the sense that it ‘undoes’ addition.

Defining subtraction and addition in terms of ‘the sum of two numbers’ connects the two operations of subtraction and addition with one quantitative relationship. However, there is a difference between the conceptions of subtraction that students already have developed on their own and this definition of subtraction. Teaching needs to start from these conceptions and gradually lead students to see the quantitative relationship that underlies the operations of addition and subtraction.

It is possible to introduce this definition of subtraction early in arithmetic learning in a form suitable for students (see Ma, n. d., pp. 18–20). Figure 18.4 illustrates two stages of the path between students’ conceptions of subtraction and the quantitative relationship in first grade (Moro et al. 1992/1982, pp. 15, 55).

At early stages of learning subtraction, there are two instructional approaches: learning the operation of subtraction and also leading students to pay attention to the relationship between addition and subtraction. These different approaches will have different impacts on students’ later learning.

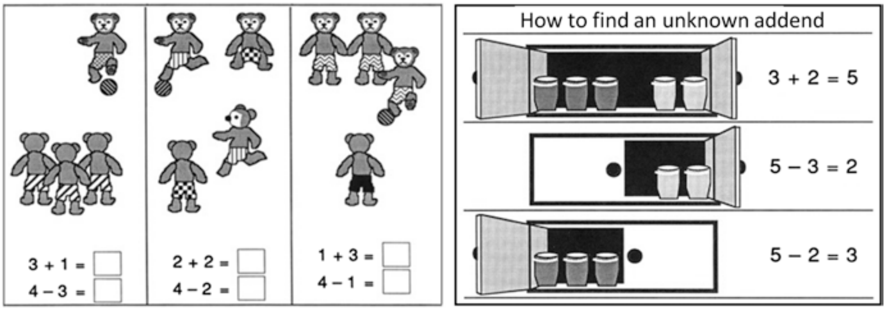

Fig. 18.4 Two stages in first-grade subtraction (Reprinted with the permission of the University of Chicago School Mathematics Project)

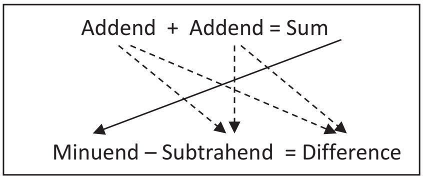

Fig. 18.5 The correspondence between terms in addition and subtraction

18.6.2.1 Minuend, Subtrahend, Difference

Definition 9 The known sum in subtraction is called the *minuend*. The known addend is called the *subtrahend*. The unknown addend, which is the result of the operation of subtraction, is called the *difference*.

Like the concepts 'addend' and 'sum', the concepts 'minuend', 'subtrahend' and 'difference' are important thinking tools for understanding and working with the quantitative relationship 'sum of two numbers'. Students do not need to memorise the definitions, but they need to have terms to use for the things described in the definitions, allowing them to describe how the terms are related. For example, the sum in an addition equation corresponds to the minuend in a subtraction equation[15] (Fig. 18.5).

Because a minuend is greater than a corresponding difference, early primary teachers tend to tell students that if we see that the result of a word problem will be smaller than the known numbers in the problem, we use subtraction to solve the problem. However, as with addition, it needs to be noted that this approach is only useful for one-step word problems.

[15] This is an example of two definitions that depend on a more fundamental definition. Rather than being defined independently, subtraction and addition are both defined in terms of the relationship 'sum of two numbers'. Figure 18.5 illustrates one consequence: terms for parts of an addition equation have an explicit correspondence with terms for parts of a subtraction equation.

18.6.2.2 The Rule of Like Numbers and the Rule of Like Units for Addition Applied to Subtraction

When minuend and subtrahend are concrete numbers, they must be like numbers. Their difference, the minuend and the subtrahend are also like numbers.

This is the rule for subtraction that corresponds to the rule of like numbers for addition. In teaching, this rule can be said as 'minuend and subtrahend must be like numbers or only like numbers can be subtracted'.

When computing a difference, only digits of like unit value can be subtracted.[16]

This is the rule for subtraction that corresponds to the rule of like unit value for addition.

This rule is part of the explanation of the algorithms for multi-digit subtraction and for long division. In teaching, we can say 'only digits with the same units can be subtracted' or 'only the same units can be subtracted'.

18.6.3 The Three Cases for Unknown Number in the Relationship 'Sum of Two Numbers'

The quantitative relationship 'sum of two numbers' concerns three numbers. When two are known, the third can be found. The three cases are:

A. The two addends are known, to find the sum. (In terms of subtraction: the subtrahend and difference are known, to find the minuend.)
B. The sum and the first addend are known, to find the second addend. (In terms of subtraction: the minuend and subtrahend are known, to find the difference.)
C. The sum and the second addend are known, to find the first addend. (In terms of subtraction: the minuend and subtrahend are known, to find the difference.)

In Fig. 18.6, the diagram on the left represents the relationship 'sum of two numbers'. On the right are the three possible cases for one number to be unknown in this relationship. All addition and subtraction word problems in school arithmetic, whether single- or multi-step, that ask students to find one unknown can be built from these three forms.

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[16] Because the Arabic numeral system can only represent one ten but not ten ones, the explanation of the rationale for subtraction with borrowing cannot be represented completely in Arabic numerals. For example, 235 − 117 the five ones in the ones place of the minuend are insufficient for conducting the operation. The first step is to convert one unit at the tens place of the minuend into ten ones. The next step can occur in two ways. One is to subtract the seven ones from the ten ones, resulting in three ones, and add the five ones, to find the digit (8) in the ones place of the difference. The second way is to combine the ten ones and the five ones, making fifteen ones, and then subtracting the seven ones from fifteen ones. The ten ones and fifteen ones cannot be represented with Arabic numerals without additional conventions or notation.

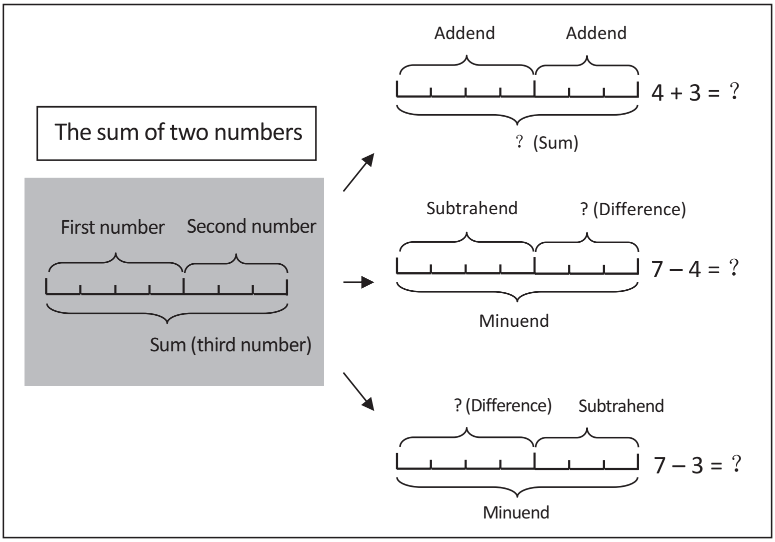


Fig. 18.6 Addition and subtraction derived from the quantitative relationship 'sum of two numbers'; terms used in addition and subtraction

According to the Common Core State Standards (2010), there are four main categories of one-step addition and subtraction word problems. In China, there are five categories for such problems.[17] No matter how word problems are categorised or named, each kind is a direct or indirect representation of one of the three forms. Those represented indirectly use the approach of 'equivalent substitution' (Fig. 18.7), illustrated by the following example:

> James caught three fish, Henry caught five fish, how many more fish did Henry catch?

At first glance, this problem does not correspond to any of the three cases in Fig. 18.6. But, if we analyse the quantitative relationship in the problem, we will find that the problem corresponds to the second case. (Here Euclid's first common notion, 'Things which equal the same thing also equal one another', is implicitly used.)

[17] The five categories are finding the sum, finding an amount that remains, finding an unknown which is a given amount larger than a known number, finding an unknown which is a given amount smaller than a known number and finding a difference. (See *Research and Practice in Teaching Elementary Arithmetic Word Problems*, 1994.) The Common Core State Standards list four main categories, each with three subcategories that depend on the position of the unknown. The categories are Add To (result unknown, change unknown and start unknown); Take From (result unknown, change unknown and start unknown); Put Together/Take Apart (total unknown, addend unknown, both addends unknown); and Compare (difference unknown, bigger unknown and smaller unknown, each with two language variants: how many more? vs how many fewer?).

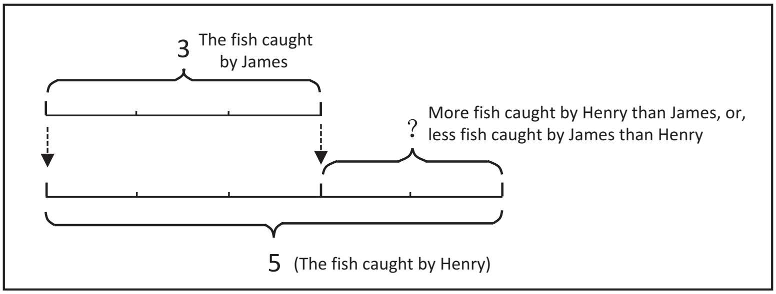


Fig. 18.7 Example of 'equivalent substitution' in 'sum of two numbers'

18.7 Multiplication and Division

18.7.1 Multiplication

Definition 10 The *product* of two numbers is a third number which contains as many units as one number taken as many times as the units in the other.

The operation of finding the product of two numbers is called *multiplication*. (For example, how much is three taken four times?)

This quantitative relationship is obviously more sophisticated than that of the sum of two numbers. First of all, there is a new type of unit in this relationship: in Fig. 18.8, each copy of the first number is a new 'many-as-one' unit created by considering a group of units as a single thing.[18]

Second, unlike the relationship 'sum of two numbers', the product of two numbers involves two types of units: 'one-as-one' and 'many-as-one' units. This is illustrated by Fig. 18.8.

Third, in Fig. 18.8, the units of the second number determine the number of copies of the first number. The 'one-as-one' units in the collection of copies form the third number, which is the product.

The definition of product of two numbers above is not equivalent to considering the product as the result of repeated addition. This definition is also not equivalent to the definition of Cartesian product because it involves the creation of a new type of unit rather than a collection of pairs of units.

In teaching, multiplication is often introduced with repeated addition. When students see 4 + 4 + 4 as '4 added to 4, added to 4', they are using the concept of addition. When students can recognise 4 + 4 + 4 as three 4s, they start to develop the concept of multiplication. We should help students to accomplish this transition as soon as possible.

Although it is likely that the concept of addition is closely related to an inborn primitive conception, the concept of multiplication is not. In forming the concept of

[18] This 'many-as-one' unit is a unit, due to Definition 1: 'A group of things, if considered as a single thing or one, is also called a *unit*'.

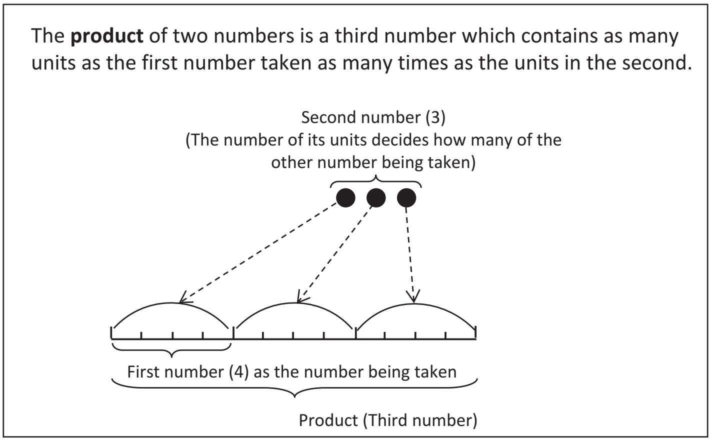


Fig. 18.8 The definition of product of two numbers

multiplication, there are three stages of learning. First, being able to consider many as one, such as one group, one class and one basket of things. Second, being able to imagine several many-as-one units, such as several groups, classes or baskets of things (all of the same size). Third, when analysing quantitative relationships, being able to manage the two types of units at the same time.

In human history, there is a long gap between the development of addition and the development of multiplication. For students, there is also a gap. One task of school arithmetic is to give students a pathway across that gap.

18.7.1.1 Multiplicand, Multiplier and Factors

Definition 11 *Multiplicand* is the number to be taken. The *multiplier* is the number that indicates how many times the multiplicand is taken.

Multiplicand is the number represented by the first term in a multiplication expression. It is represented by the term at the left of the multiplication sign.

The multiplicand is the number being taken. Its copies are the newly formed many-as-one units. For students, this is their earliest use of many-as-one units.

For the past few hundred years, the multiplicand has traditionally been represented as the first term in multiplication.[19] This tradition of giving the multiplicand

[19] According to the *Oxford English Dictionary*, the terms 'multiplicand' and 'multiplier' first appeared in 1592 and 1542. In early arithmetics, multiplication was often written vertically with the multiplicand above the multiplier. One very widespread textbook, *Cocker's Arithmetic*, first published in 1677, said, 'Multiplication has three parts. First, the multiplicand. . . . Second, the multiplier. . . . And thirdly, the product' (1677, p. 32). In the nineteenth century, this description was repeated by Charles Davies in his textbook (Davies 1857, p. 45). Davies used horizontal expressions, writing the multiplicand to left of the multiplication sign (Davies 1834, p. 33).

first is consistent with its role in commercial arithmetic.[20] In the definition system, giving the multiplicand first is consistent with the emphasis on the concept of unit.

When introducing multiplication with repeated addition, students should notice that the multiplicand is the addend.

The multiplier is represented by the third term in a multiplication expression. It is represented by the term at the right of the multiplication sign.

The multiplier is always an abstract number.

A multiplication expression is read as 'multiplicand multiplied by multiplier' or 'multiplier times multiplicand' (e.g. 5×3 read as '5 multiplied by 3' or '3 times 5').

The multiplier indicates how many copies of the multiplicand are in the product.

Some think that distinguishing between multiplicand and multiplier or reading the expression as described above burdens students with unnecessary detail. But this temporary complication is the price of a simpler future.

When both multiplicand and multiplier are abstract numbers, they are also called *factors*.

In school arithmetic, there are two situations where the distinction between multiplicand and multiplier is irrelevant. First, when factoring. Second, in formulas such as area of a rectangle or triangle, volume of a cube. The latter is the last step of a process that begins by depending on the distinction between multiplier and multiplicand.

Although the distinction between multiplicand and multiplier does not remain throughout primary mathematics, it is important because it helps students to be aware of the new type of unit, thus helping to expand their conception of unit.

18.7.1.2 The Rule of Like Numbers for Multiplication

When the multiplicand is a concrete number, the multiplicand and the multiplier are not like numbers. In that case, the product and the multiplicand are like numbers.

Analysing quantitative relationships in school arithmetic is practised mainly by solving word problems. Most numbers in word problems are concrete numbers. For example:

A. There are 24 books on the shelf. Bill puts six more books on the shelf. How many books are there now? The solution is 30 books.
B. There are 24 books on a shelf. How many books are there on six shelves? The solution is 144 books.

Problem A is to find the sum. The solution and the addends are all like numbers.

Problem B is to find the product of two numbers. The two concrete numbers presented in the problem, 24 books and 6 shelves, are not like numbers. The former is the number being taken, the multiplicand. The latter, the multiplier, determines

[20] In commerce, the seller first sets the price per unit, then the total price of multiple units is computed each time a sale is made.

that there are six 24-book groups. The product, 144 books, and the multiplicand are like numbers. This is consistent with the like number rule for multiplication: When the multiplicand is a concrete number, the multiplicand and the multiplier are not like numbers. In that case, the product and the multiplicand are like numbers.

Although 6 shelves is a concrete number, as a multiplier, it is treated as an abstract number.

18.7.2 Division

Definition 12 If a product and one of the multiplicand or multiplier are known, the operation of finding the unknown multiplier or, respectively, multiplicand is called *division*.

Division is also the operation of finding the unknown factor when the product and one factor are known.

Division is the inverse of multiplication in the sense that it 'undoes' multiplication.

Defining multiplication and division in terms of 'product of two numbers' connects the two operations of division and multiplication with one quantitative relationship, in a way that is similar to the connection between subtraction and addition. However, because multiplicand and multiplier may be different types of numbers, there are several possible forms for the inverse operation.

Definition 13 To find an unknown multiplicand is called *partitive division*.

For example, 12 apples are equally shared among 3 children. How many does each child get? (Partition 12 into three pieces. How many in each piece?)

To find an unknown multiplier is *quotitive division*.

For example, there are 12 apples. Give each child 4 apples. How many children can get apples? (How many 4s are there in 12? 12 is how many times as many as 4?)

To find an unknown factor is neither quotitive nor partitive division.

For example, the area and length of a rectangle are known. Find the width.

We may say, 'For example, the product of two factors is 15. One factor is 5, what is the other factor?'

18.7.2.1 Dividend, Divisor, Quotient, Remainder

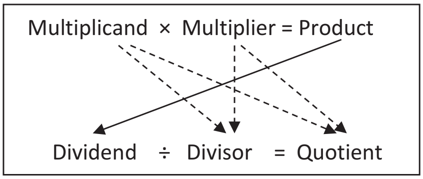
Definition 14 The known product in division is called the *dividend*.

A known multiplicand, multiplier or factor is called the *divisor*.

The unknown, which is the result of the operation of division, is called the *quotient*.

The correspondence between terms in multiplication and division is illustrated in Fig. 18.9.

Fig. 18.9 The correspondence between terms in multiplication and division



The dividend may be the sum of a product where one factor is the divisor and a number smaller than the divisor. The latter is called the *remainder*. In this case, the result of division has two parts: quotient and remainder.

Remainder is a temporary term in school arithmetic. After fractions are introduced, there is no longer a need for this term.

18.7.2.2 The Rule of Like Numbers for Multiplication Applied to Division

In partitive division, dividend and quotient are like numbers.

In quotitive division, dividend and divisor are like numbers.

The rule of like numbers can help students recognise quantitative relationships.

18.7.3 The Three Cases for Unknown Number in the Relationship 'Product of Two Numbers'

The quantitative relationship 'product of two numbers' concerns three numbers. When two are known, the third can be found (Fig. 18.10). The three cases are:

(a) The multiplicand and multiplier are known, to find the product. (In terms of division: the divisor and quotient are known, to find the dividend.)
(b) The product and the multiplicand are known, to find the unknown multiplier. (In terms of division: the dividend and quotient are known, to find the unknown divisor.)
(c) The product and the multiplier are known, to find the multiplicand. (In terms of division: the dividend and divisor are known, to find the unknown quotient.)

18.8 Concluding Remarks

The definition system and basic laws for whole numbers discussed above form the core of the theory of school arithmetic. The remaining content in this theory – the definition system for fractions and theorems in school arithmetic (analogous to the propositions in the *Elements*) – is built on this foundation.

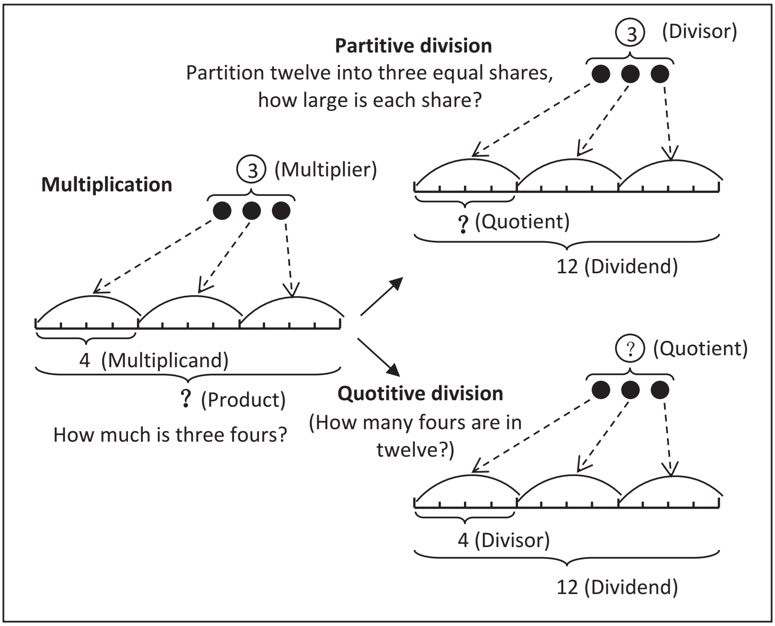


Fig. 18.10 Multiplication, partitive and quotitive divisions

Returning to the puzzle of the US and Chinese teachers' responses, we briefly sketch examples of connections – or lack thereof – with the theory.

The teachers responded to the question of what students needed to know about subtraction with regrouping in two ways. Nineteen of the 23 US teachers focused on the procedure of borrowing, speaking of taking one ten from the tens place and exchanging it for ten ones (Ma 2010, p. 2). Their explanations did not connect the procedure with a correct rationale and sometimes suggested that the digits representing ones and tens were two independent numbers rather than representations of two parts of a number. In contrast, the other four teachers noted that students should understand that exchanging one ten for ten ones did not alter the value of the minuend. The rationale for such exchanges relies on Definition 2 – 'a number is a collection of units' – and the notational conventions described in Section II about how these units are represented as tens and ones. Like their US counterparts, some Chinese teachers focused on the procedure of borrowing (p. 7). Most, however, focused on the idea of regrouping, describing the exchange of one ten for ten ones as 'decomposing a unit of higher value' (pp. 8–10). This description expresses a general feature of base ten notation and can be used not only for exchanges of 1 ten for 10 ones but many others, e.g. 1 hundred for 10 tens and 1 one for 10 tenths.

In discussing 123 × 645, many US and few Chinese teachers gave only a procedural account of the multiplication algorithm. Conceptual explanations from both countries fell into two categories: place value system and meaning of multiplication and – implicitly or explicitly – the distributive property. Two US teachers explained the rationale for the multiplication algorithm in terms of the meanings of base ten notation and multiplication. Five other US teachers noted that the problem of computing 123×645 could be reduced to the problem of computing the sum of 123×600, 123×40 and 123×5, but none justified this transformation in any way (Ma 2010, pp. 35–36). It may be that the US teachers had encountered the distributive property at some point, perhaps in an algebra course. However, it was not evident in their responses. In contrast, about one third of the Chinese teachers used a similar approach (pp. 39–42). A difference was that they presented the transformation in a more formal way, and over half referred to the distributive property. The other Chinese teachers gave explanations in terms of the place value system and the units of a number (pp. 42–45), echoing the definitions of unit value of a digit, place value and multiplication presented in this article. A few mentioned both approaches (p. 45).

This is consistent with findings of more recent studies. US primary textbooks and teachers guides published in 2004 and 2005 treat the distributive property in less depth than their Chinese counterparts (Ding and Li 2010). Prospective US primary teachers sometimes confuse the associative property with the commutative property, and the textbooks that they use in preparation and practice teaching provide little support in this matter (Ding et al. 2013).

More such connections could be traced, and more details could be given (Ma in preparation). However, we wish to end by emphasising the point that teachers' knowledge may reflect the substance of the school mathematics that they learned as students and teach as teachers. The theory presented in this article was distilled from textbooks of nineteenth-century USA and twentieth-century China (see the textbooks listed in the references). It is not surprising that we can recognise features of the theory in the responses of the Chinese teachers. In contrast, the US teachers' responses seem to reflect an absence of underlying theory in US school arithmetic. Given this absence, it is remarkable that *any* of the US teachers gave conceptual explanations and not surprising that their explanations were not as well elaborated as those of the Chinese teachers.

Appendix: Basic Laws

Commutative Property of Addition and Corresponding Property for Subtraction

Commutative property of addition: if two addends are exchanged, their sum is unchanged.

$$\text{Since } 5+3=8, \text{ then } 3+5=8; \text{ or } 5+3=3+5.$$

The corresponding property for subtraction is: the positions of subtrahend and difference can be exchanged.

$$\text{Since } 8-5=3, \text{ then } 8-3=5.$$

Associative Property of Addition and Corresponding Property for Subtraction

Associative property of addition: when three numbers are added, the sum of the first two added to the third is the same as the first number added to the sum of the last two. For example, 5 + 3 + 2:

$$(5+3)+2 = 5+(3+2).$$

The corresponding property for subtraction is: when two numbers are subtracted from a third number, the difference of the sum of the two numbers and the third is the same as the difference of the difference of the first number and the third and the second number. For example, 12 – 3 – 4:

$$12-(3+4) = (12-3)-4.$$

Compensation Property for Addition

If an addend is increased and the other addend is decreased by the same amount, their sum is unchanged. For example, 5 + 3:

$$5+3 = (5+2)+(3-2) = (5-2)+(3+2).$$

Therefore, if one addend increases (or decreases) by a given amount and the other addend is unchanged, then their sum increases (or decreases) by the same amount. For example, 5 + 3 = 8:

$$\text{Since } 5+3 = 8, \text{then } (5+2)+3 = 8+2 \text{ and } (5+3)+2 = 5+(3+2).$$

The corresponding property for subtraction is: if the minuend and subtrahend increase (or decrease) by a given amount, their difference is unchanged. For example, 12 – 7 = 5:

$$\text{Since } 12-7 = 5, \text{then } (12+2)-(7+2) = 5 \text{ and } (12-2)-(7-2) = 5.$$

If the minuend increases (or decreases) by a given amount and the subtrahend is unchanged, their difference increases (or decreases) by the same amount. For example, 12 – 7 = 5:

$$\text{Since } 12-7 = 5, \text{then } (12+2)-7 = 5+2 \text{ and } (12-2)-7 = 5-2.$$

If the minuend is unchanged and the subtrahend increases (or decreases) by a given amount, their difference decreases (or increases) by the same amount. For example, 12 – 7 = 5:

$$\text{Since } 12 - 7 = 5, \text{ then } 12 - (7 + 2) = 5 - 2 \text{ and } 12 - (7 - 2) = 5 + 2.$$

Commutative Property of Multiplication and Corresponding Property for Division

Commutative property of multiplication: if multiplier and multiplicand exchange positions, their product is unchanged. For example, 3×5:

$$\text{Since } 5 \times 3 = 15, \text{ then } 3 \times 5 = 15; \text{ or } 5 \times 3 = 3 \times 5.$$

The corresponding property for division is: if divisor and quotient exchange positions, their dividend is unchanged. For example, $15 \div 5 = 3$:

$$\text{Since } 15 \div 5 = 3, \text{ then } 15 \div 3 = 5.$$

Associative Property of Multiplication

Associative property of multiplication: when three numbers are multiplied, the product of the first number with the product of the last two numbers is the same as the product of the product of the first two numbers with the last number. For example, $5 \times 3 \times 2$:

$$\text{Since } (5 \times 3) \times 2 = 30, \text{ then } 5 \times (3 \times 2) = 30.$$

The corresponding property for division is: the result of division by one number then dividing by a second number is the same as the result of dividing by the product of the two numbers. For example, $(30 \div 3) \div 2$:

$$\text{Since } (30 \div 3) \div 2 = 5, \text{ then } 30 \div (3 \times 2) = 5.$$

Distributive Property

A number multiplied by a sum is the same as the sum of the products of the number with each addend. For example, $5 \times (4 + 3)$:

$$\text{Since } 5 \times (4 + 3) = 35, \text{ then } 5 \times 4 + 5 \times 3 = 35.$$

$$\text{Since } 5 \times (4 + 3 + 2) = 45, \text{ then } 5 \times 4 + 5 \times 3 + 5 \times 2 = 45.$$

There is no corresponding property for division.

Compensation Property of Multiplication

If the multiplicand is multiplied by and the multiplier is divided by the same amount, their product is unchanged. For example, 12×9:

$$\text{Since } 12 \times 9 = 108, \text{ then } (12 \times 3) \times (9 \div 3) = 108 \text{ and } (12 \div 3) \times (9 \times 3) = 108.$$

$$\text{Equivalently, } 12 \times 9 = (12 \times 3) \times (9 \div 3) = (12 \div 3) \times (9 \times 3).$$

Therefore, if the multiplicand is enlarged (or diminished) by a given amount and the multiplier is unchanged, then their product is enlarged (or diminished) by the same amount.

$$\text{Since } 12 \times 9 = 108, \text{ then:}$$

$$(12 \times 3) \times 9 = 108 \times 3$$

$$12 \times (9 \times 3) = 108 \times 3 \text{ and } (12 \div 3) \times 9 = 108 \div 3 \text{ and } 12 \times (9 \div 3) = 108 \div 3.$$

If both the dividend and the divisor are enlarged (or diminished) by a given amount, then their quotient is unchanged. For example, $36 \div 4$:

$$\text{Since } 36 \div 4 = 9, \text{ then } (36 \times 2) \div (4 \times 2) = 9 \text{ and } (36 \div 2) \div (4 \div 2) = 9.$$

Therefore, the dividend is enlarged (or diminished) by a given amount and the divisor is unchanged, and then their quotient is enlarged (or diminished) by the same amount.

If the dividend is unchanged and the divisor is enlarged (or diminished) by a given amount, their quotient is enlarged (or diminished) by the same amount.

$$\text{Since } 24 \div 6 = 4, \text{ then:}$$

$$(24 \times 2) \div 6 = 4 \times 2 \text{ and } (24 \div 2) \div 6 = 4 \div 2$$
$$\text{and } 24 \div (6 \times 2) = 4 \div 2 \text{ and } 24 \div (6 \div 2) = 4 \times 2.$$

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[21] Note that many of the nineteenth-century textbooks can be downloaded from the Internet.