Chapter 17
Professional Development Models for Whole Number Arithmetic in Primary Mathematics Teacher Education: A Cross-Cultural Overview

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17.1 General Introduction[1]

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This chapter proceeds from the plenary panel on teacher education presented at the ICMI Study 23 conference in Macao (Novotná 2015). The main goal of the panel, as well as of this chapter, was to explore approaches to, and within, primary mathematics teacher education in different parts of the world and to discuss commonalities and differences in relation to broader cultural and curricular traditions.

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The ICMI Study 23 was focused on the mathematical domain of WNA. There is broad agreement that deep understanding of school mathematics in general, and WNA in particular (for primary teachers), is critical. It follows from this that WNA provides a critical context for developing understandings and constructing arguments that adhere to the practices and norms of more advanced mathematics. One thrust of this chapter is to present and discuss examples from several parts of the world that approach this theme through a focus on primary mathematics teacher education on developing, discussing and applying mathematical models. But this thrust is set within the broader terms of different curricular approaches to WNA in different countries and regions and different cultures and structures regulating the ways in which primary mathematics teacher education (and primary teacher education more generally) is organised. In this broader curricular and cultural terrain, we take up two issues specifically. In the curricular approaches to WNA, we take up the issue of two ways of thinking about early number – in terms of length or location on a number line (where ordinality can be more emphasised, alongside cardinality) versus in terms of base ten structure (where place value structure and relationships can be more emphasised, alongside cardinality) – and use specific examples to point to number being presented in different ways in different countries. Secondly, we take up an issue that emerged from looking at the foci of different presentations at the ICMI Study 23 conference with antecedents in prior writing: of more individualist, decentralised and autonomous cultures and views of teacher learning versus more collective and centralised views of teacher learning. These latter differences have consequences for the models of teacher education that are possible in different contexts and for the ways in which teacher education is structured within the timetables of schooling.

In this chapter, we begin our discussion in the terrain of cultures associated with teacher education, before moving into curricular approaches to WNA. Key models for teacher education are then introduced and discussed to exemplify key contrasts. We then move into the detail of mathematical models that have been used within primary mathematics teacher education and use these examples to highlight attention to differences relating to length (i.e. models focusing on the ordinal aspects of number) and base ten structure (focusing on the cardinal aspects), as well as the

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broader organisation of teacher education across these contexts. The key questions that we address in this chapter can therefore be summarised thus:

- What broad similarities and differences can be seen in relation to cultural views of primary teacher learning in different parts of the world? How do these differences play out in models of primary teacher development?
- What are the key differences between curricular approaches that foreground more ordinal versus place value structural relations views of early number?
- What key mathematical models are promoted in primary mathematics teacher development in different parts of the world? How do cultural and curricular differences play out within these mathematical models and the professional development models of primary mathematics development that these mathematical models are couched within?

The panel consisted of six scholars representing different parts of the world, complemented by two discussants. All of them have rich experience with mathematics teacher education in their countries/regions. The panellists were (in alphabetical order) Maria G. Bartolini Bussi (University of Modena and Reggio Emilia, Italy), Sybilla Beckmann (University of Georgia, USA), Maitree Inprasitha (Khon Kaen University, Thailand), Berinderjeet Kaur (National Institute for Education, Singapore), Xu Hua Sun (University of Macau, China) and Hamsa Venkat (University of the Witwatersrand, South Africa). The discussants were Deborah Loewenberg Ball (University of Michigan, USA) and Mike Askew (University of the Witwatersrand, South Africa).

17.2 Cultural Views of Primary Teacher Learning and WNA

Jarmila Novotná and Hamsa Venkat

Alexander (2009) proposes a view of pedagogy as profoundly cultural, while at the same time, exhibiting some continuities that appear to transcend place and time:

> pedagogy does not begin and end in the classroom. It is comprehended only once one locates practice within the concentric circles of local and national, and of classroom, school, system and state, and only if one steers constantly back and forth between these, exploring the way that what teachers and students do in classrooms reflects the values of the wider society. (p. 924)

It follows from this that examining teacher development in the context of WNA also entails attention to broader curricular and cultural ideologies, in order to more completely understand their contents and approaches to supporting primary mathematics teaching.

Some broad differences in the emphasis of mathematics education research between the East and West have been noted in the papers presented in this ICMI study and in prior work. For example, Ma (1999) has noted the central role, and study by teachers, of carefully developed, standardised textbooks in Chinese primary mathematics education. This contrasts with a much more critical view of the standardisations inherent in textbook sequences, sometimes set within a broader culture that celebrates and emphasises the varied individualism of children's interests, understandings and learning pathways (Triandis and Trafimow 2001) and which was prevalent in much of the literature on the need for learner-centred education.

Moving into the terrain of WNA, Sun et al. (2013), comparing the ways in which addition and subtraction are presented in Chinese and Portuguese textbooks, noted the central role of textbooks in the Chinese professional education of teachers. They point to carefully varied sets of problems in which structural similarities of part-whole relations are at the fore and combinations of and with ten are collectively viewed as the key conceptual idea to be foregrounded in order for these ideas to become problem-solving tools for working with subsequent mathematical content. In base ten-oriented models of number, stick bundles or Dienes block-type manipulatives, triad diagrams representing a whole with its constituent parts and symbolic number sentences are commonly juxtaposed. Length- and location-based models using the number line predominate in early number tasks in the Portuguese textbooks in focus in this study, emphasising the direction and extent of motion associated with operations and quantities. There is less overt connection between addition and subtraction in the early examples shown in the Portuguese textbooks, with later work pointing to these as operations connected through the idea of inverse operations. Of interest is that while multiple methods are expected and promoted in the Chinese textbooks, all of these depend on ideas of decomposition, with specific selections often based around decimal structure. Beishuizen's (1993) emphasis on using number line models with compensatory steps as important higher-level efficiency moves, e.g. solving 65 – 38 with a jump backwards of 40 and a compensatory forward jump of 2, tends not to feature in the Chinese approach. The forward trajectories in the Chinese approach are seen as preparation for the focus on structure and relations needed for algebraic working, as well as for the decimal number structure. In contrast, the Realistic Mathematics Education approach emphasises the number line model as having strong associations backwards into counting and forwards into models and methods in the real number terrain (Anghileri 2006). These differences in goals and trajectories within the WNA context can be seen more broadly in the contrasts drawn between counting-based and structure-based approaches to early number (see for example Schmittau 2003).

Bartolini Bussi and Martignone (2013) have also pointed out the ways in which both mathematical models and structures had to be adapted in schooling in order to explore the 'transposition' of models beyond the ground of their cultural origins. The broader point we can make based on the ICMI study contributions is that some national traditions, more than others, foreground studies in which there is a broad focus on teaching and particular pedagogic tools and how such tools have both been developed and refined over the years. In these traditions, there are indications of greater homogeneity and 'taken as shared' views of both trajectories of mathematical content and approaches to support its learning. Backgrounded in these papers from the ICMI Study 23 Proceedings was how teachers subsequently work with and make sense of such tools, with this occurrence perhaps reflecting the expectation of buy-in to the theories and models being promulgated, rendering a study of differences in take up less culturally normal.

The foci in these papers stand in stark contrast to those written by authors with a more typically 'Western' sensibility. For example, Askew (2015) presented a case study of a teacher in South Africa, focusing on how, in a lesson on place value, the teacher's actions, representations and discourse achieved coherence and connections. Similarly, Tempier (2013), cited in Chambris (2015), explores overlaps and contrasts in three teachers' handling of place values ideas in France, and Venenciano et al. (2015) present detailed excerpts of a teaching approach designed to support children's appropriation of ideas of unit and structural relations between units. While cardinality and place-value relations are at the fore of these studies, we draw attention to these studies here because of their common focus on to the ways in which mathematical ideas are taken up within teaching and learning.

This brief overview of culture and WNA models and approaches feeds into the range of cases that we present below dealing both with models of teacher education and mathematical models of WNA in teacher education. In some of these contexts, the professional development and mathematical models have broad cultural and historical support and are often accompanied by systemic sanction in the form of structural support for their take up in teacher development. In other countries, the cases reflect take up of approaches and models in more local initiatives, often accompanied by the need to build structural affordances for supporting take up. In each of the cases that follow, we provide an overview of the national/provincial context in relation to the extent of standardisation of curricula/textbooks and the structure of teacher education, with this data gathered from contributing authors, prior studies and the context forms that were collected with submissions for the ICMI 23 Study conference.

All themes dealt with during the ICMI Study 23 conference and in the volume are deeply linked with teacher education and development. This was both intentional and a natural property of the whole WNA domain. In the discussion document for the study conference, one of the basic questions was: how can each of the themes be effectively addressed in teacher education and professional development? In order to teach elementary mathematics effectively, there is a need for sound professional knowledge, both in mathematics and in pedagogy. In Theme 1 ('The why and what of whole number arithmetic'), the topic was explored from the perspective of teachers' mathematical content knowledge. Theme 2 ('Whole number thinking, learning and development') addressed cognitive aspects of WNA with attention paid to, among other aspects, integrating different perspectives into a more coherent view with consequences for teacher education and development. In Theme 3 ('Aspects that affect whole number learning'), teacher education played a central role and included examples of impact of the topic in teacher education and development (Canada and Thailand). In Theme 4 ('How to teach and assess whole number arithmetic'), teacher education and development were present explicitly or implicitly in all contributions, including examples of various pedagogic approaches, textbook organisation and used artefacts and a broader study explicitly acknowledging teachers' didactical or pedagogical knowledge and noting that neither teachers' extent of teaching experience nor their teacher education accounted for observed differences. Theme 5 ('Whole numbers and connections with other parts of mathematics') included the study of avenues through which whole number arithmetic learning might be supported in teacher education.

17.3 Primary Teacher Education Across the World

17.3.1 US Experience[2]

Sybilla Beckmann

17.3.1.1 Organisation of Primary Teacher Education in the USA

General or local organisation: Primary teacher education is developed locally. Each state has its own guidelines or regulations for the preparation, certification and licensing of teachers. However, there are influential non-governmental accrediting bodies that operate nationally and issue standards for teacher education.

Teacher qualification: Primary teachers are usually generalists although there is growing interest in building specialist development.

Curriculum for primary mathematics: There is no national curriculum. Since 2010, the Common Core State Standards for Mathematics have been adopted by most states.

17.3.1.2 Key Questions About Teacher Knowledge in the USA

A key concern in the mathematical education of primary and middle-grade teachers in the USA is how teachers can further their own disciplinary knowledge of mathematics while also studying deeply the mathematics that they will teach. According to a report issued jointly by national mathematical societies in the USA (Conference Board of the Mathematical Sciences 2012), teachers should know the mathematical topics they will teach and how these topics connect to others in earlier and later grades. Teachers should also know the ways of reasoning and constructing arguments in mathematics, how these ways of reasoning and argumentation apply at the elementary level and how to teach these ways reasoning and argumentation to students.

[2] Research was supported by the National Science Foundation under Grant No. DRL-1420307. The opinions expressed are those of the author and do not necessarily reflect the views of the NSF.

In the USA, the *Common Core State Standards for Mathematics* document (Common Core State Standards Initiative 2010), which has been adopted by most states, describes standards for mathematical practice for students in kindergarten through Grade 12. In particular, according to these standards, mathematically proficient students should be able to understand and use stated assumptions, definitions and previously established results in constructing arguments, and they should try to use clear definitions in discussion with others and in their own reasoning. Therefore even students in elementary school are expected to understand suitable definitions and use them in explaining and arguing for the validity of mathematical statements.

Several aspects of definition have been studied in mathematics education research including the distinction for students and teachers between concept image and concept definition (Edwards and Ward 2004; Tall and Vinner 1981; Tsamir et al. 2015; Vinner 1991), students' and teachers' conceptions of definition and their understanding of definitions and alternate definitions for a given concept (Zaslavsky and Shir 2005; Zazkis and Leikin 2008), students' difficulties in using definitions the way mathematicians do (Edwards and Ward 2004) and how to lay a foundation for understanding definitions (Bartolini Bussi and Baccaglini-Frank 2015). Throughout, there is special concern about teachers' knowledge of definitions and the role of definitions in mathematics.

Research in mathematics education has considered definitions mainly within geometry (e.g. definitions of shapes, such as squares and rectangles) and topics related to functions (e.g. limits). However, there is a clear need to use definitions within other mathematical domains. For example, how could a student construct a mathematical argument demonstrating that 1/2 • 1/3 is equal to 1/6 without definitions for multiplication and fractions? Careful mathematical arguments should appeal to definitions of multiplication and fractions rather than leaving those definitions implicit.

Beckmann and Izsák (2015) defined multiplication, $M \bullet N = P$, for non-negative quantities M, N and P by interpreting the multiplier, M, as a number of equal groups; the multiplicand, N, as a number of units in 1 (or each) group; and the product, P, as the number of units in M groups. They argued that this quantitative definition of multiplication organises not only multiplication and division, but also proportional and inversely proportional relationships between covarying quantities. Thus, the multiplicative conceptual field (e.g. Vergnaud 1988), which encompasses multiplication, division, fraction, ratio and proportion, and is a foundation for such critical topics as linear functions, rates of change and slope, should be an excellent domain in which to hone the skill of arguing from a definition.

Preliminary results of Beckmann et al. (2015) indicate that future middle-grade teachers can construct viable arguments to devise solutions to missing-value proportion problems using the quantitative definition of multiplication. In their study, future teachers were also asked to generate equations in two variables to relate quantities covarying in a proportional relationship. On a written test, the teachers were given a scenario in which a type of fertiliser was made by mixing nitrogen and

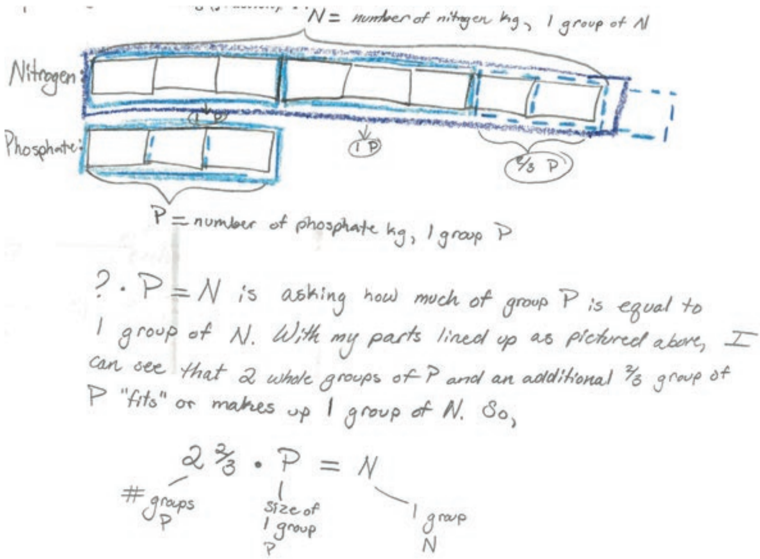

Fig. 17.1 Using a definition of multiplication and a strip diagram to explain an equation

phosphate in an 8 to 3 ratio and asked to use math drawings and the definition of multiplication to generate and explain an equation of the form:

$$(\text{fraction}) \bullet P = N$$

given that *N* and *P* are unspecified numbers of kilogrammes of nitrogen and phosphate, which could vary. Figure 17.1 shows part of one teacher's explanation. The drawing indicates that the teacher views the nitrogen as consisting of eight parts, which encompass a total of *N* kilogrammes or one group of *N*, and views the phosphate as consisting of three parts, which encompass a total of *P* kilogrammes or one group of *P*. The teacher formulates the equation:

$$? \bullet P = N$$

and interprets it as 'asking how much of group *P* is in 1 group of *N*'. From the drawing, the teacher deduces that the answer is 2 2/3 because '2 whole groups of *P* and an additional 2/3 group of *P* "fits" or makes up 1 group of *N*', thus concluding with the equation 2 2/3 • *P* = *N*.

Notice how the teacher uses the definition as an organisational framework for a coherent mathematical argument. In the given problem, the goal is to explain a linear equation in two variables. Because the coefficient (the multiplier) in that equation is not given, it must be found. The teacher uses the definition of multiplication as a vehicle for finding the coefficient. To do so requires that the teacher think about one quantity flexibly in multiple ways: she views the phosphate simultaneously as P kilogrammes, as one group of P and as three parts. Thus, the definition of multiplication may function not only as an organising framework, but also as a vehicle for thinking more deeply about coordinating the fixed and varying aspects of quantities.

The teacher's argument relies critically on her drawing in Fig. 17.1. This drawing is an example of a strip diagram, also known as a tape diagram, which is one kind of diagram or 'math drawing' that Beckmann and Izsák (2015) presented in explaining how a quantitative definition of multiplication applies to interpreting and reasoning about proportional relationships between covarying quantities. These math drawings represent quantities with lengths and show relationships between quantities. For example, a longer length implies a greater quantity; a length that is three times as long as another implies one quantity is three times the other. Strip diagrams are also the representation used in the model method, which has been used effectively in Singapore with primary students studying whole number arithmetic (Kaur 2015).

In mathematics, definitions have a scientific function rather than an everyday one. Definitions provide technical power – they 'have the potential of saving you from many traps which are set by the concept image' (Vinner 1991, p. 69). But to use definitions, they must be understandable to students while also being mathematically accurate. Representations may be a key tool that helps students and teachers use definitions in constructing mathematical arguments. Representations may be even more important in teacher education. As argued by Venkat (2015, p. 587), 'attention to representational competence can provide a bridge that allows for concurrent attention to teachers' learning of mathematics and their teaching of mathematics'.

17.3.2 Singapore: The Model Method

Berinderjeet Kaur

17.3.2.1 Organisation of Primary Teacher Education in Singapore

General or local organisation: There are common teacher education standards as all the teachers are trained in the sole teacher education institute in Singapore.

Teacher qualification: Generalists.

Curriculum for primary mathematics: There is a common national curriculum developed by mathematics curriculum specialists at the Ministry of Education and revised periodically so that it remains relevant.

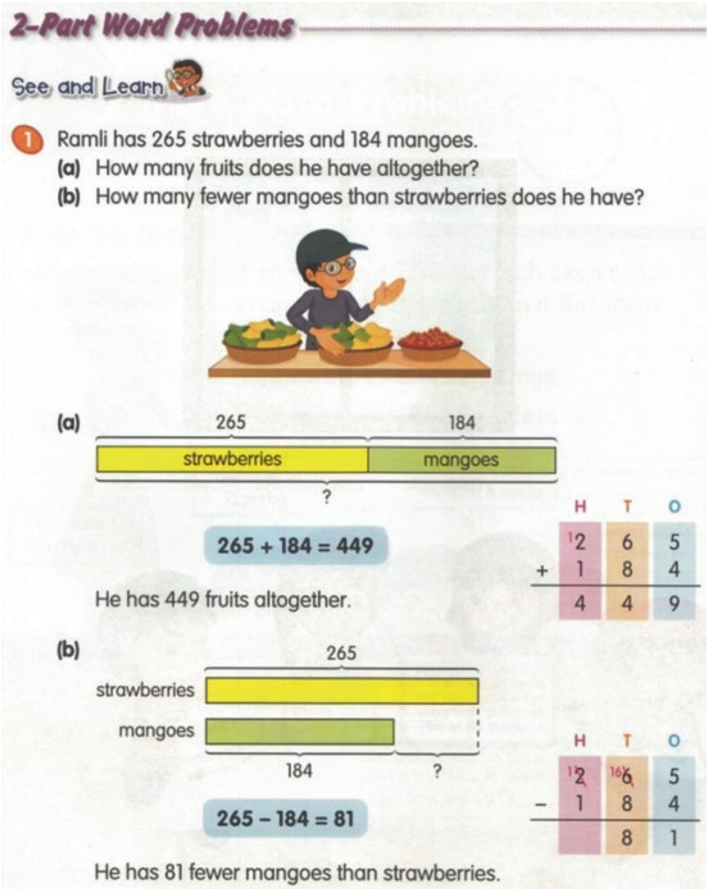

Fig. 17.2 Use of models to solve a two-part word problem (Chan and Cole 2013a, p. 4)

17.3.2.2 Examples

The primary school mathematics curriculum in Singapore places emphasis on quantitative relationships when students learn the concepts of number and the four operations. The model method (Kho 1987), an innovation in the teaching and learning of primary school mathematics, was developed by the primary school mathematics project team at the Curriculum Development Institute of Singapore in the 1980s. The method, a tool for representing and visualising relationships, is a key heuristic students' use for solving whole number arithmetic (WNA) word problems.

The concrete-pictorial-abstract (CPA) approach of the primary school mathematics curriculum in Singapore is congruent with the concepts of the part-whole and comparison models. In the CPA approach, students make use of concrete objects, while in the model approach they draw rectangular bars to represent the concrete

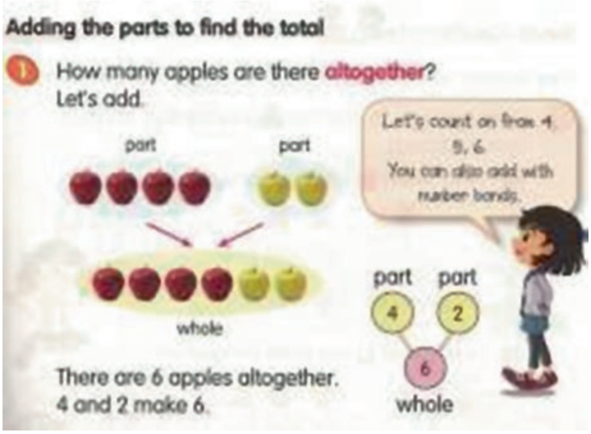

Fig. 17.3 Implicit introduction of the part-part-whole concept in Grade 1 (Chan and Cole 2013b, p. 27)

objects. The rationale for the choice of rectangular bars is that they are relatively easy to partition into smaller units when necessary compared with other shapes.

The part-whole model illustrates a situation where the whole is composed of two or more parts. When the parts are given, the students can determine the whole. Sometimes the whole and some parts are given and other parts are unknown. The comparison model demonstrates the relationship between two or more quantities when they are compared, contrasted or described in terms of differences.

When students make representations using the part-whole (shown in Fig. 17.2 – part a) and comparison models (shown in Fig. 17.2 – part b), the problem structure emerges and students are able to visualise the relationship between the known and unknown and determine what operation to use and solve the problem.

17.3.2.3 Primary School Mathematics Textbooks Used in Singapore Schools

Textbooks used in Singapore schools must be approved by the Ministry of Education. In addition, textbook writers work very closely with the Ministry of Education Mathematics specialists when writing the books. Therefore it may be said that textbooks are vehicles for the intended curriculum prescribed by the Ministry of Education. Teacher guides accompanying the textbooks make explicit the pedagogy the textbooks support. A significant pedagogical strategy is the model method (Ministry of Education 2009). It is introduced in the textbooks from Grade 1 onwards. Figures 17.3 and 17.4 illustrate how the idea of models is implicitly and explicitly introduced in Grades 1 and 2 respectively.

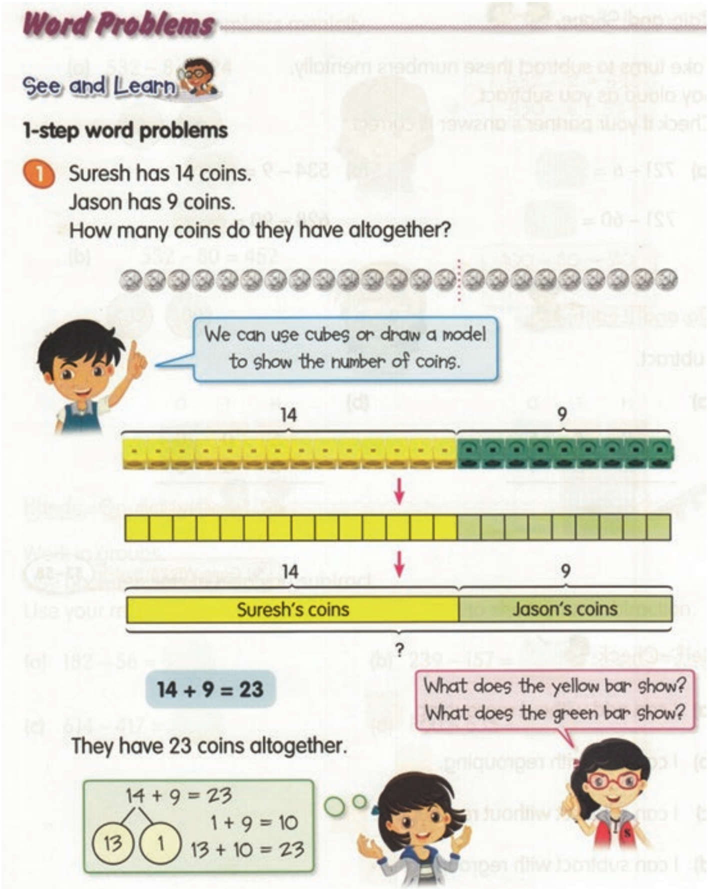

Fig. 17.4 Explicit introduction of part-part-whole models in Grade 2 (Chan and Cole 2013c, p. 52)

17.3.2.4 Preparation of Primary School Mathematics Teachers to Teach the Model Method

Prospective primary school mathematics teachers are introduced to the model method as part of their curriculum studies (mathematics) during their pre-service teacher education at the National Institute of Education in Singapore. As part of their pre-service course work, they use textbooks that are approved by the Ministry of Education which adopt the method of models as a pedagogical strategy for the learning of mathematics. Since the method has been used widely in Singapore schools from the 1980s, many of the prospective teachers since the late 1990s are very familiar with the strategy as they themselves used it to solve problems in their primary school days.

17.3.2.5 What Does Research Say About the Effectiveness of the Model Method?

The model method has proved to be effective for building number sense and solving arithmetic word problems in Singapore schools. A rigorous study by Ng and Lee (2009) of the model method clarified that the method engages students in capturing the inputs, the relationships between the inputs and the output of the problem. Once students have constructed a model, they use it 'to plan and develop a sequence of logical statements, which allows for the solution of the problem' (p. 291). Their study also noted that 'average ability children's solution of word problems involving whole numbers could be improved if they learn to exercise more care in the construction of related models' (p. 311).

However, when very challenging questions are posed, in Grades 5 and 6 such as the following:

> Mr Lim had a total of 540 long and short rulers. After selling an equal number of both types, he had $\frac{1}{3}$ of the long rulers and $\frac{1}{6}$ of the short ones left. What was the total number of rulers left? (Singapore Examinations and Assessment Board 2014)

students often have difficulties drawing models to work through the solution process. They have difficulty constructing the before-after models and also determining the basic unit (Goh 2009). This certainly has implications for teacher education.

17.3.3 South Africa: Models of Situations

Hamsa Venkat

17.3.3.1 Organisation of Primary Teacher Education in South Africa

General or local organisation: There are national standards for teacher education, but these are at the generic, rather than subject-specific, level.

Teacher qualification: Primary teachers are usually trained as generalists, although some higher education institutions include optional 'elective' courses for training to be a mathematics specialist.

Curriculum for primary mathematics: There is a national curriculum for the primary mathematics years and, indeed, for all years of school mathematics. In recent iterations, mathematics curricula have seen an increase in the extent of specification of content and prescription of sequencing and pacing.

17.3.3.2 Recent Studies

The post-apartheid context in South Africa is one that is marked by a range of concerns: some focused within education and others focused on the broader sociocultural terrain. Within education, there are ongoing concerns about mathematical performance and teachers' mathematical knowledge. In the broader society, praise for the extent of social mobility into the emerging and broadening 'middle class' has been tempered by alarm at the increasing extent of socio-economic inequality, which plays through into significant differences between the mathematical performance of students in elite schools and the rest of the population. In the early years of schooling, the predominance of unwieldy concrete counting-based methods has been widely reported (Schollar 2008; Ensor et al. 2009), with standardised curricula seeking increasingly to prescribe pacing and sequencing as one way of pushing progression to more sophisticated methods.

A recent study located in Intermediate Phase (Grades 4–6) pre-service teacher education has noted broad differences in the nature and extent of the mathematical content and pedagogic knowledge emphases within different courses, in spite of a broad national framework of standards for teacher education (Taylor 2011). In-service options for primary mathematics teaching development remain limited and piecemeal. In this context, the Wits Maths Connect-Primary project, a 5-year linked research and development project, established a 20-day in-service primary mathematics for teaching course. Responding to the need to encourage teachers to attend to structural relations between quantities rather than quantities solely as counted entities, the course focused particularly on key models of structural relations for additive and multiplicative situations. In other work, we have noted the ways in which teachers' representational and communicative repertoires have broadened through an approach in which recognition of the nature of relations between quantities and familiarity with key models that represent these relations can be used to simultaneously develop both mathematical and pedagogical competence (Venkat 2015; Venkat et al. 2016). Here, we note in relation to the WNA focus that we have discussed and used part-whole models similar to those promoted in the Singapore approach to represent a range of additive relations situations and used number lines as 'models for' calculating the relevant missing values. Similarly, double number lines, T-tables and area models have been discussed as models of a range of multiplicative situations. In relation to the earlier discussion of base ten (structural relation-oriented) and number line (ordinal) models, our use of both structural part-whole models and more operational number line models is directed at balancing the emphasis on counting and ordinal approaches that tend to be in the foreground of the early grades' curriculum (DBE 2011), by promoting attention to the relationships between quantities. This combination is purposively driven by evidence also of frequently algorithmic and error-prone 'columnwise' approaches seen in the Intermediate Phase years, showing limited carrying through of a quantified sense of number in increasing number ranges (Graven et al. 2013).

17.3.4 Thailand: Traditional vs Open Approach[3]

Maitree Inprasitha

17.3.4.1 Organisation of Primary Teacher Education in Thailand

General or local organisation: Following the Education Act in 1999, Thailand implemented national curricula for basic education in 2001 and national standards for teacher education in 2005. However, these standards focus primarily on knowledge with less emphasis on practical or professional competency.

Teacher qualification: Since 2004, K–12 teachers have been trained as specialists categorised into eight subjects such as mathematics and sciences, but there are also some optional programmes for those majoring in elementary education in a few institutions.

Curriculum for primary mathematics: There is a national curriculum for the primary mathematics years and for all years of school mathematics. These curricula focus on six domains: number and operation, measurement, geometry, algebra, data analysis and skills for mathematical processes.

17.3.4.2 Reforms in the New Century

While a new national agenda of 'Reforming Learning Process' of the 1999 Education Act was declared more than a decade ago, pre- and in-service mathematics teacher education programmes at most universities in Thailand have struggled to respond to this demand. Most teacher education programmes still emphasise subject matter (i.e. mathematics content at the university level in programmes for mathematics teachers) with little or no emphasis on courses associated with pedagogical content knowledge (Inprasitha 2015). This subject emphasis is viewed as exerting strong influence on the traditional teaching approach of school teachers, involving transmission of content-based approaches to the students.

Moreover, within these traditional teaching approaches, most Thai teachers heavily rely on using textbooks as key instructional media for classroom teaching practices. IEA results during the 1980s showed that more than 90% of Thai mathematics teachers used textbooks as a tool for teaching: they taught the content that appeared in the textbook and set student exercises from those textbooks. Currently, the exercises and the instruction guidelines in these textbooks still emphasise computation skills and techniques focused on rapid completion (Anderson et al. 1989). Mathematics teachers commonly use either the national textbook provided by the

[3] The study was supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand; the Students' Mathematical Higher Thinking Development Project in Northeastern Thailand; Centre for Research in Mathematics Education, Faculty of Education, Khon Kaen University.

Institute for the Promotion of Teaching Science and Technology (IPST) or private publishing company textbooks.

The traditional approach to teaching mathematics typically starts with teachers explaining new content through some examples related to rules or formula and then giving students a worksheet on some related examples and exercises (Kaewdang 2000; Khemmani 2005; Inprasitha 2011). As noted already, this teaching approach is teacher centred with the emphasis on teachers transmitting or transferring the contents to students (Inprasitha 2011).

Since 2004, the Faculty of Education at Khon Kaen University has run a new mathematics teacher education programme that has 57% of credits focused on PCK courses and 43% of credits for collegiate mathematics courses across the total 170 credits. The 'open approach', as a new mathematics teaching approach focused on teaching through problems, has been implemented both in the programme and also in the project schools collaborating with the Faculty of Education, Khon Kaen University, since 2006. This approach is described next.

17.3.4.3 An Exemplar of How Teachers Use the Approach

Teachers' understandings of school mathematics from the textbook have influenced the way they teach mathematics in their classrooms. As Dossey (1992) has mentioned, comprehension for mathematical conceptual understanding is extremely important in development and success in mathematics teaching and learning in school and research understanding in school mathematics. An understanding of mathematical knowledge as 'outside' the teachers and students (Plato 1952 cited in Dossey 1992) can be linked to Thai mathematics teachers' transmission orientation to knowledge to the students (Office of the Education Council 2013).

By contrast, in the Lesson Study project introduced by the Centre for Research in Mathematics Education (CRME) since 2006, the open approach (Inprasitha et al. 2003) has been introduced for developing rich mathematical activity based on open-ended problems (Nohda 1991; Inprasitha 1997). A group of student teachers did their practicum teaching in the 2002 academic year in seven secondary schools in Khon Kaen and found that these kinds of mathematical activities could change the way teachers interact with students and interacting among students themselves. Being engaged in the activity provides a chance for students to produce and generate various ways of thinking. This phenomenon was influential for teachers to become aware of their pedagogical beliefs about teaching mathematics (e.g. students cannot think by themselves unless the teachers provide the way for them to think first).

During 2003–2005, the open approach has been widely used by some 800 school teachers in Khon Kaen province through training by CRME. Between 2006 and 2009, four lesson study project schools implemented innovative mathematics teaching by incorporating three steps of lesson study into four steps of open approach (Fig. 17.5). In this project, the Japanese textbook of Gakko Tosho (Inprasitha and Isoda 2010, 2014) has been mainly used by the teachers in the project schools (Fig. 17.6).

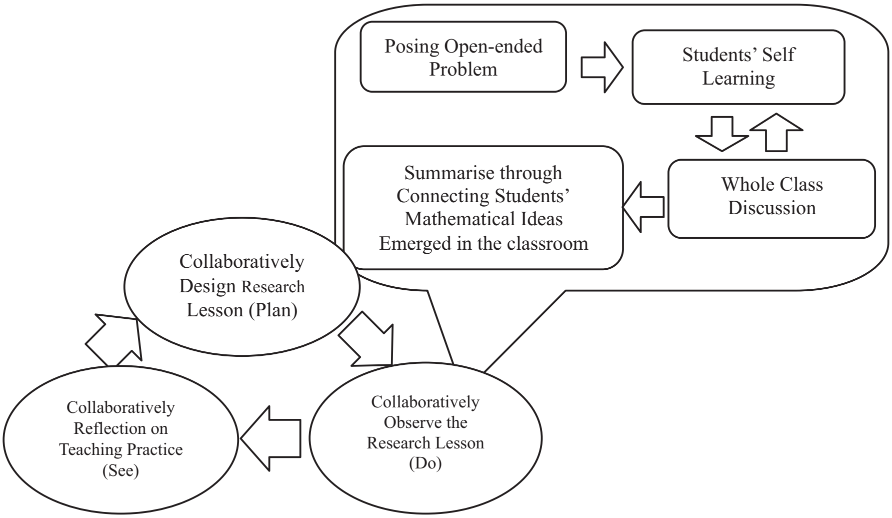

Fig. 17.5 Lesson study incorporating open approach (Inprasitha 2011)

17.3.4.4 An Exemplar of How Teachers Learn Through the Four Steps of Open Approach

Topic: Teaching 9 + 4.

The objective of this learning unit (12 periods): The students can understand addition in terms of 'together' and 'add up' and use 'base ten' for addition.

Content in textbook – see Fig. 17.6.

Typically, in traditional teaching approaches, most Thai teachers teach 9 + 4 using 'count all' or 'count on' techniques and focus on producing 13 as the answer to this situation. They tend to ignore the development of meaningful number sense through using students' real-world experiences to teach addition. From previous classes, students know how to write 9 + 4 as a number sentence for this situation. However, they might not know the meaning or number sense of 9 + 4. Since this is the first time students will consider 'addition in which the result is greater than 10', then encouraging them to discover whether '9 + 4' is greater or less than 10 is as important as simply getting the answer of 13.

In the project schools use the tasks and steps shown in the textbook in Fig. 17.6. According to step ①, after interpreting the real-world situation as 9 + 4, typically most school teachers focus on getting the answer, while in this textbook, the yellow cartoon provides a hint to the teacher to ask the question 'Is the answer greater than 10?' In order to answer this question, the students need to decompose 9 or 4 to making 10 according to step ②.

The textbook also provides models of students' ideas to help beginner teachers anticipate students' responses to the given questions as in Fig. 17.7.

Some beginner teachers raise questions about why we need to ask the question in step ① because they think students know earlier that 9 students together with 4 stu-

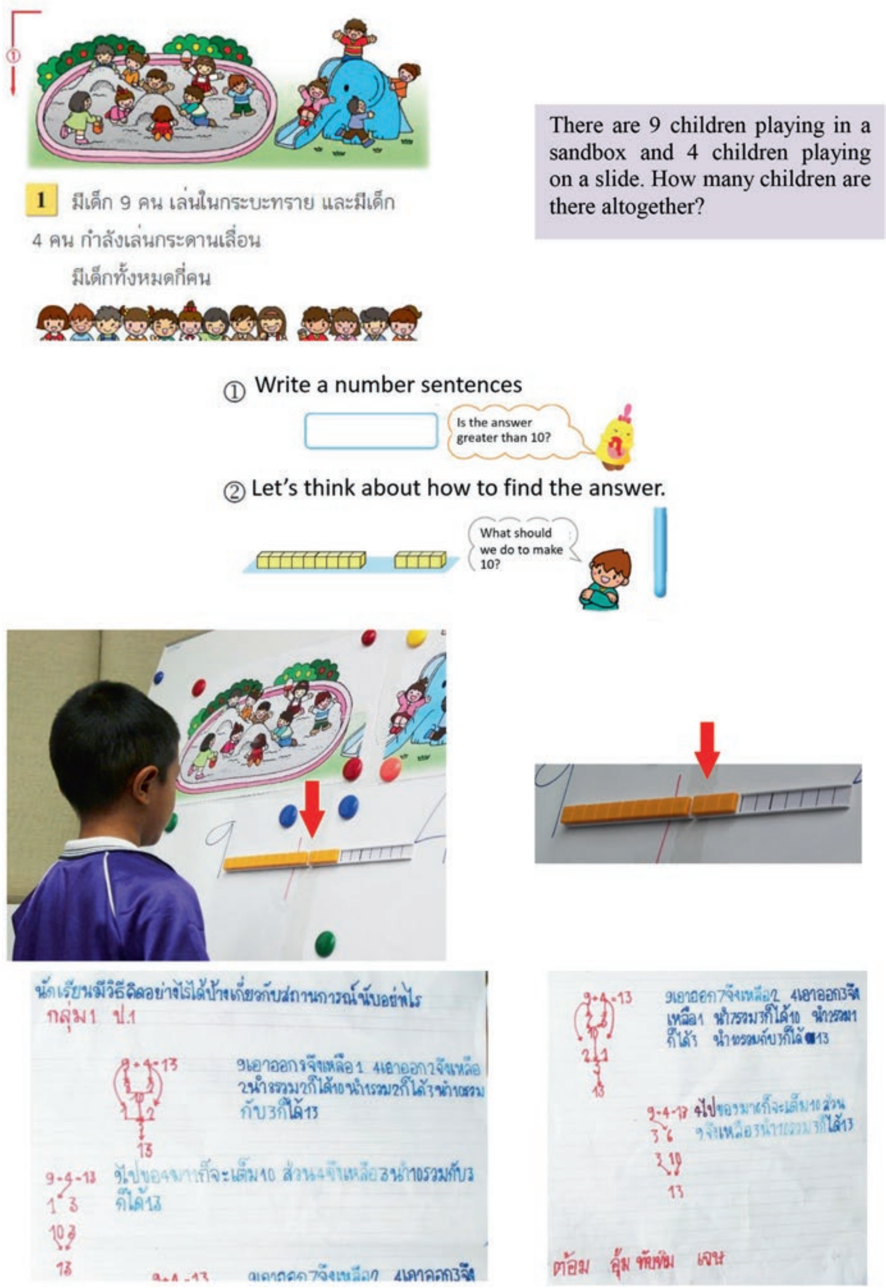

Fig. 17.6 Contents and steps for teaching 9 + 4 (Extracted from Study with Your Friends: Mathematics for Elementary School 1st grade (in Thai))

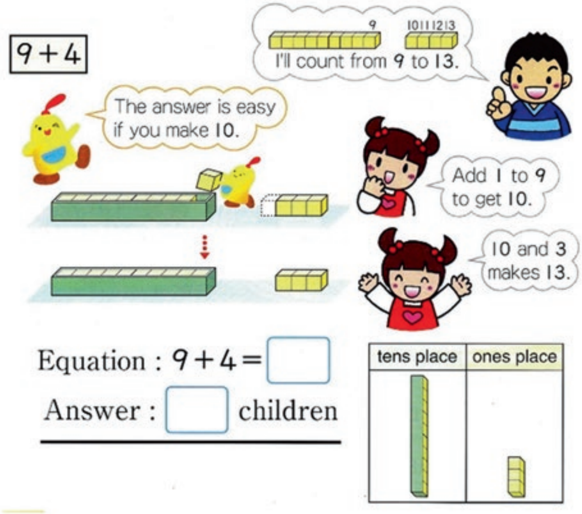

Fig. 17.7 Students' responses for 9 + 4 (Extracted from Study with Your Friends: Mathematics for Elementary School 1st grade (in Thai))

dents becomes 13 students. But this knowing is completely different from knowing 'how to know' what the 9 + 4 number sentence means. Once students have obtained insight into making a 10, they can see that 4 can be decomposed into 1 and 3 and that this 1 can be associated with the 9 to make a 10. Students then notice they still have 3 left (using blocks or other materials). This discovery of 'using decompose/compose and making ten strategy' as a tool to do further addition is meaningful and has broader mathematical value.

Thus, for teachers, teaching using open approach helps them to extend their understanding of students' ideas as useful for bridging the students' real-world understandings into the mathematical world. For students, learning to solve given real-world situations by themselves accumulates their 'how to learn', which is more important for engaging in new problem situations.

17.3.5 Chinese Open-Class Approach[4]

Xu Hua Sun

17.3.5.1 Organisation of Primary Teacher Education in Macao

General or local organisation: Common teacher education standards are locally developed, but there is broad overlap in standards as all the teachers are mainly trained in University of Macau and a small part by Great China area.

[4]This study was supported by Research Committee, University of Macau, Macao, China (MYRG2015-00203-FED). The opinions expressed in the article are those of the author.

Teacher qualification: This depends on school tradition. Some schools inheriting the mainland tradition require specialists, but the schools inheriting the Portuguese or British tradition require generalists.

Curriculum for primary mathematics: There are common Macao curriculum standards developed by the Ministry of Education of Macao, but schools have the freedom to develop school-based curricula based on their educational visions and students' abilities.

17.3.5.2 Some Findings

The international literature base continues to both emphasise and discuss the mathematics content knowledge, pedagogical content knowledge and curriculum knowledge needed for effective primary mathematics teaching (Shulman 1986). The unity of these knowledge bases in action tends to receive less attention, appearing – particularly in Western traditions – to be largely assumed as automatically realised. The need for unification is the missing gap addressed by lesson study (or learning study or open class) in the East and is represented in Chinese and Japanese traditions as the main concern and question to be solved in the field of teacher education. Open class is a typical Chinese approach for teacher professional development which uses a single teaching circle, in which teachers are required to teach for 40 mins with half an hour for discussion; the open-class approach is more flexible than Japanese lesson study in terms of organisation, budgeting and timetabling (Sun et al. 2015). Learning study is a collaborative action research approach which aims to improve the effectiveness of student learning by enhancing the professional competence of teachers through joint construction of pedagogical content knowledge to help students to learn specific objects of learning based on variation theory, which originated in Hong Kong, and is now also a significant mode of practice in countries such as Sweden and Brunei (Cheng and Lo 2013). Lesson study is a Japanese model of teacher-led research in which a triad of teachers work together to target an identified area for development in their students' learning: using existing evidence, participants collaboratively research, plan, teach and observe a series of lessons, using ongoing discussion, reflection and expert input to track and refine their interventions. This section focuses on the open-class approach, a key strength of the Chinese system of teacher education. Key principles underlying the open-class approach in China are described below:

- For primary teacher education, a typical belief in Asia is that mathematics teachers should possess specific subject knowledge, which might be not known by lay people.
- Professional knowledge must be public and communicated among colleagues through collaboration.
- Professional knowledge must be storable and shareable.
- Professional knowledge requires a mechanism for verification and improvement.

This is one of the reasons why primary mathematics teachers in Mainland China are required, commonly, to be specialists, not generalists. A key antecedent for this belief may be Confucius' establishment of a tradition of deep respect for teachers in China – similar to lawyers and doctors in the West – but that is not common in the West with respect to teachers. However, this belief could also relate to an exam-driven teaching and learning culture, in which teacher professional knowledge, like student's knowledge, is expected to be examined through given lessons. The exam-driven culture in China can be linked to its great population and more limited resources. Similar to student examinations, there is a strict professional assessment system for practising teachers as part of career ladders in China. There are four formal hierarchical grades for teachers that indicate professional status in China, and promotion is based on school-based evaluation of the open class. In this way, teacher professional ranks (three ranks) are clearly defined. Compared with other systems, Chinese teachers' working time (about ten classes in each week) is low, but their research time is strictly stipulated/controlled. Open class is one of the windows to reflect their research results. This professional development tool is currently underrepresented in international research (lesson study is more well-known in the global context than open class; so far, open class is well-known within China only). It plays critical and different roles in teacher recruitment, professional assessment and professional research and development in primary teacher education to meet the goals of specialists (not generalists). How this tool has been transposed and applied in Macao and Italy are presented next.

17.3.5.3 The Goal of Open Class

The open-class approach was established in the early 1950s by the Chinese Ministry of Education with the primary purpose of organising teacher study groups in schools. There are two general categories of open classes, those for outside audiences and those for internal audiences. The classes for outside audiences are divided into three categories: open classes to publicly demonstrate new education ideas (e.g. new curriculum/textbook use and expert-level classroom instruction demonstrations), open classes for research (e.g. research lessons for new thought 觀摩課, 'same content-different-approach' 同課異構) and open classes for evaluation purposes (e.g. recruitment, teacher promotion and teaching competitions). These open classes generally involve a single teaching cycle of planning-designing-teaching-reflecting either by the conducting teachers or a school-based research group.

The open classes for internal audiences include single-circle open classes for mentor-mentee training (師徒公開課) and multiple-circle open classes for mentor-mentee training (校本培訓公開課). The multiple-circle open classes, which involve co-planning, co-designing, co-teaching and co-reflecting, are supported by school-based mentor-mentee programmes. This more complex open-class system is similar to the Japanese lesson study approach. Compared to those for outside audiences, the internal training open classes are more effective for professional development, although they are also organisationally more demanding. These internal training open classes, which involve a series of different goals and are mainly organised by the teaching research group as a routine research activity, have played a role in securing incremental, accumulative and sustainable improvements in China (e.g. Huang et al. 2011). In these group classes, the teachers reflect on their teaching practice, design innovative activities based on class observations and engage in practitioner-driven research. These activities were originally aligned with the promotional criteria for teachers in Mainland China, where teachers are required to conduct research and publish practice-oriented papers in education journals or magazines and undergo regular teaching evaluations. This approach, which is also known as 'learning study' in Hong Kong (Sun 2007; Lo 2005) and 'public class' in China (Liang 2011; Shen et al. 2007), is a popular method for developing and enhancing the teaching profession in many countries. In practice, in an open class a group of teachers first observe a lesson taught by a colleague and then discuss its merits (Miyakawa and Winsløw 2013).

17.3.5.4 Open-Class Roles in Primary Teacher Professional Life

Open-class model:

- Has exploited the Chinese conception of teaching as a public activity with norms and structures that favour a *collaborative spirit*.
- Has exerted a major influence in the *professional development* of teachers in China for many years.
- Has played a major role in fostering *learning communities* within Chinese schools.
- Has proven to be an effective way to induct *new and inexperienced teachers* into the teaching profession.

This approach is simpler and more economical than the 'lesson study' method. When used in teacher education, the open-class approach enables pre-service teachers to observe more experienced teachers as part of their professional training. According to the literature, the open-class approach is used in many teacher preparation programmes to induct pre-service teachers into the practice of teaching (Wang and Paine 2003). The approach is also used for professional development and teacher evaluation in schools (Liang 2011).

In Mainland China, teachers are required to attend public classes (open classes) as part of the professional development activities in their school- or district-based research groups. This approach is largely shaped by Eastern traditions in which teaching is regarded as a public activity with norms and structures that favour collectivism (Liang 2011; Shen et al. 2007).

The open-class approach is an integral part of the educational audit culture in China. Accordingly, the open-class approach has been a major influence in the professional development of teachers in China for many years. Ecologically, the approach has also played a major role in fostering learning communities within Chinese schools. However, compared with Mainland China, the open-class approach is rarely employed in Macao.

17.3.5.5 The Difference: Open Class vs Lesson Study

These open classes generally require:

- Single cycles (planning-designing-teaching-reflecting) vs multiple cycles in lesson study.
- Involves within-school or research group teachers vs outside teacher/researcher in lesson study.
- More frequency daily practice vs non-daily practice in lesson study.
- Simpler/more economical vs complex procedure in daily practice.

17.3.5.6 Difficulties in transposing to Macao

A Portuguese colony for more than 400 years, Macao was returned to China in 1999. As part of the legacy of the decentralised governing style of the Portuguese era, 90% of the schools in present-day Macao are privately run and have their own diverse curricula. Given its colonial past, the culture of teaching in Macao schools has largely been shaped by Western traditions in which teaching is regarded as a private activity with norms and structures that favour individualism and autonomy (Li 2003). In addition, Macao's decentralised and fragmented education system does not have a common curriculum within and across grade levels.

Accordingly, there is no national framework to guide teachers and each school designs its own curriculum, assessment tasks and standards and grade progression criteria. Coupled with a heavy teaching load (more than 20 classes a week), it is almost impossible for teachers to engage in curricular improvement and participate in professional development. Consequently, there is little or no opportunity for experienced teachers to share their experience with beginning and new teachers. For example, in an interview, a teacher stated that without a class visit tradition, teachers regard their own classrooms as a private space and, therefore, tend to work alone: 'They are usually doing their own thing' (各忙各的).

Traditionally, pre-service teacher training in Macao is divided into 'theory' (subject matter, curriculum, educational theory and pedagogy) and 'practice'. The unity of theory and practice is considered unimportant, as it is supposed to be realised automatically in teaching practice. As a result, no pre-service teaching courses focus on the connections between theory and practice. None of the literature on teaching addresses the central questions on how the unity or integrity of theory and practice is realised. Similarly, no research has examined whether particular theories are appropriate when they are applied in a classroom context (even though this question has become the central issue of teacher education). The lack of research on the role of unity within the teacher education curriculum can be regarded as a 'missing paradigm' (Day 1991, 1997; Cuban 1992).

Taking on board the idea that pedagogy is a culturally specific enterprise, we acknowledge that transposition requires adaptation. Here we present and discuss an instance of transposition of the open-class approach to Italy.

17.3.6 Open Class in Italy

Maria G. Bartolini Bussi

17.3.6.1 Organisation of Primary Teacher Education in Italy

General or local organisation: General organisation within governmental regulations. Some limited choices may be made locally.
Teacher qualification: Usually generalists. Some limited experiments of specialist mathematics teacher preparation are carried out in some primary schools.
Curriculum for primary mathematics: Common, with national standards.

17.3.6.2 Some Features of the Italian School System

The Italian mathematics curriculum is national (centralised) with weak control on processes: there is independent school management with governmental control (national assessment) at the end of the 2nd and 5th grades. Teachers are usually generalists (although some limited testing of specialist teachers is ongoing drawing on school independent management) (see the Table 17.1). Primary school teachers usually spend 5 years (the whole period of primary school) with the same group of students. The school system is totally inclusive, as all students (including students with special needs) are in mainstream classrooms.

17.3.6.3 Primary School Teacher Education and Development

A university master's degree has been compulsory for primary school teachers since 1998. However, most current teachers in primary schools (especially the older ones) have only a secondary school degree, with, in general, little experience of in-service development activity (a law about compulsory in-service development activity was issued only in September 2015). Hence, in many schools, there are still teachers with limited preparation: relevant exceptions are represented by teachers who have been

Table 17.1 International comparison

	China	Macao	Italy
Language	Chinese (Mandarin/Cantonese)		Italian
Standard	National (centralised) standard strong control	Fragmented curriculum	National (centralised) standard weak control
Primary teachers' beliefs	Teaching as public activity	Teaching as private activity	
	Specialist	**Generalist**	

in touch with a university research group. This happened in Reggio Emilia, thanks to the efforts of the Department of Education and Human Studies and to a generally positive attitude towards education issues, realised by means of a programme of early childhood education that has gained international repute in the last quarter century (the so-called Reggio approach, http://www.reggiochildren.it/?lang=en).

17.3.6.4 Some Local Studies

In Italy (University of Modena and Reggio Emilia, Department of Education and Human Studies at Reggio Emilia) we started an experiment 4 years ago with some dozen schools, in order to implement a 'lesson study style' process. The principal investigator is Maria G. Bartolini Bussi, with the collaboration of Alessandro Ramploud and others (doctoral students, post-doc scholars from different universities, teachers, principals, educators coordinated by the branch of the Municipality of Reggio Emilia 'Officina Educativa', i.e. 'Educational Workshop').

17.3.6.5 The 'Open-Class Model' Programme

In this general situation, the principal investigator and collaborators launched a new programme for pre-service teacher education and in-service teacher development in 2012 that was inspired by the 'lesson study' model, as developed in Japan. The principal investigator had the opportunity to visit classrooms in Far East (Japan, China, Thailand). Hence, she came in contact with different implementations of similar (although not identical) models. The most important differences between the Italian schools and the Eastern schools may be summarised as follows:

- The presence of generalist vs specialist teachers.
- The permanence of the same teacher with the same group of students for many years vs for 1 year only.
- The attention to students with special needs in the mainstream classes (according to the totally inclusive model in Italy).
- The conception of the classroom as a private space vs. a public space where also critical observers are welcome.

The research group eventually chose the Chinese model of 'open classes', as it seemed more school centred than University centred. The intention, discussed also with teachers, educators and principals, was to disseminate a model that was under the responsibility of the education agencies of the zone, with no special role (except the starter one) for the University research group.

We chose the title (English acronym DOR):

to design – to observe – to redesign
a Mathematics lesson

in order to capture the meaning of the Chinese expression.

Guānmó kè 观摩课

Until now we have completed three summer schools and some dozen DOR pilot examples.

The three summer schools were held in September 2012, 2013, 2014 with primary school teachers and general educators (who support teachers with the organisation of classroom activities and afterschool workshops for students with special needs). A further public event was realised in December 2015, in order to disseminate the last outcomes, with a further event held in November 2016.

At the beginning (2012 and 2013), the summer schools aimed at introducing participants to some activities typical of other cultures, mainly from the Far East, in order to foster discussion about cultural factors that are beyond the choices that are considered 'universal'. We were inspired by Jullien's statement:

> This is not about comparative philosophy, about paralleling different conceptions, but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own unthought. (Jullien 2006, p. iii)

The approach was very successful: from the first to the second summer school, we had an increase of participants from about 80 to more than 200.

We then started a cooperation with some selected schools, discussing with principals the possibility of realising in their schools pilot examples of DOR for mathematics lessons with a structure that may be roughly summarised as follows:

- 3 hours: design (for a group of teachers, educators and student teachers).
- 1 hour: lesson (with observers, including educators in charge of video documentation of the lesson).
- 3 hours: analysis and redesign (for a group of teachers, educators and student teachers).

The trickiest issue from the beginning was to force teachers to focus on the limited time of a lesson (about 45/60 mins). Italian teachers are not accustomed to controlled, careful, short-term processes, tending instead to work as long as needed on a particular issue, without focusing on the time span of a single lesson. With these experiments we wished to introduce attention to the careful design of short-term processes.

Table 17.2 Summary of experiments

Number of DOR experiments	Number of schools involved	Number of teachers	Number of educators
67	41	205	8

Long-term processes, in our approach, are designed and controlled by the semiotic mediation framework (Bartolini Bussi and Mariotti 2008).

The first pilot examples were publicly presented and discussed in the 3rd summer school (September 2014) with the invitation of more teachers.

Proceedings and documentation of all the three summer schools have been published in Italian and partially discussed in the doctoral thesis of Alessandro Ramploud (2015). More experiments are currently ongoing (Table 17.2). A research study concerning one of the teaching experiments has been published (Bartolini Bussi et al. 2017).

Also, some student teachers are involved in the experiments, as part of their practicum (internship) and, in some cases, as part of their master's dissertations, and some new teachers are involved.

The strength of the programme may be summarised as follows:

- Links with international programmes, with attention to the cultural aspects.
- Participation of principals in the definition of aims and goals.
- Extensive spread over the province and beyond.
- Mixed experience group (expert teachers, new teachers, students teachers, educators).

17.3.7 Czech Republic: Critical Places in School Mathematics

Jarmila Novotná

17.3.7.1 Organisation of Primary Teacher Education in the Czech Republic

General or local organisation: Locally developed, each faculty providing teacher education has individual curricula accredited by the Ministry of Education, Youth and Sports.

Teacher qualification: Generalists, possibility to extend the qualification by one subject.

Curriculum for primary mathematics: A general national framework education programme, individual school education programmes.

17.3.7.2 Critical Places: The Case of Word Problems

Future teachers' ideas about school mathematics and its teaching strategies are significantly influenced by their previous experiences from their home, school and society (see, e.g. Pasch et al. 1995). Their previous experience can have a considerable impact on their ability to get insight into the cognitive processes of pupils, who meet new, and for them, often surprising concepts, properties and relations or obstacles. As noted by Even and Ball (2009):

> future elementary teachers in general use only weak mathematical conceptions, which often do not help them to realise their educational ambitions. On a general educational level, many of these students advocate discovery learning and collective problem solving, but when it comes down to the mathematical activities that have to be prepared, their experience of 'traditional' school mathematics is of little help. […] For both future teachers, elementary as well as secondary, building conceptions of mathematically rich and cognitively and socially stimulating school mathematical activities is at the heart of the process of their professional formation. (p. 35)

Teacher education in the Czech Republic is based on the idea that changes in mathematical education are substantially dependent on changes in teacher education. These changes must take into account the needs of practice. Resources from practice are collected in different ways, from official educational documents for schools through data collection organised by educational management to research organised at universities and research institutions such as the Czech Academy of Science. In this text we present information gathered from the research project *Kritická místa matematiky základní školy, analýza didaktických praktik učitelů* (*Critical places of basic school mathematics, analysis of teachers' didactical practices*). The project ran from 2011 to 2014 in the Czech Republic, and its results are being transferred into teacher education. The aim of the project was to discover which issues in compulsory school mathematics cause the main obstacles for further learning of Czech pupils.

In this text, we will restrict our focus to the primary school (the first 5 years of compulsory school attendance). The results from the project were published among others in two monographs: Rendl et al. (2013) and Vondrová (2015). While in Rendl et al. (2013), teachers' views and experiences are collected and analysed, in Vondrová (2015) these results are further elaborated and verified with pupils.

For data collection, in-depth interviews with individual respondents were used. The interviewer posed open questions and had the opportunity to add further specific questions in case of necessity.

The aim of the research with teachers (Rendl et al. 2013) was to find out which domains of school mathematics teachers evaluated as critical, how they dealt with them and what they saw as the reasons for pupils' difficulties. Textbooks that respondents used in his/her teaching were used as supporting material. From the domains related to WNA teachers evaluated rounding and estimating, arithmetical operations and word problems as the most difficult for primary pupils (Jirotková and Kloboučková 2013). The in-depth interviews were followed up with a questionnaire-based survey aimed at enriching some of the information gained in the interviews. In the research with pupils, during the in-depth interviews, pupils solved selected problems and described their mental processes (Vondrová 2015).

We present here only the results dealing with word problems as a part of the results linked directly with WNA (for more detailed information, see Havlíčková et al. 2015). The research showed that in the domain of word problems at the primary level, the most difficult aspects for pupils were:

- Understanding of text and grasping the problem.
- Recording the problem structure, pictures, schemes and models.
- Words in the function of an 'antisignal', i.e. that cues an operation different to the one required.
- States and their transformation in problems of comparison.
- Sets and their parts.
- Mathematical 'craft' – numerical errors and errors in algorithms.
- 'Chaining' numerical operations and pieces of information in the assignment.
- Fractions in a word problem.
- Conversion of units.

According to teachers, reading comprehension was a key issue. They pointed out pupils' inability to choose essential pieces of information in a text, to formulate the answer to the question in the assignment and several other problems they had encountered during their teaching practice.

In the questionnaire survey (Havlíčková et al. 2015), it was found that the textbook was the most important support for teachers in their teaching although they sometimes also used additional materials (other textbooks, collections of problems, own materials, etc.). The survey also provided other information directly linked with word problems in primary mathematics, concerning issues that teachers consider as important for their successful solving. For example, most participating teachers put emphasis on the need to automate pupils' calculations. For successful problem solving of a word problem, they considered working with 'problem types' and creating records of their differing structures to be very important. Nearly all teachers guided their pupils to choose words in the assignment that signalled the arithmetical operation. They believed that success in solving word problems was associated with pupils with higher cognitive dispositions, with weaker pupils facing difficulties when solving word problems. When discussing differences between successful and less successful solvers of word problems, teachers mostly mentioned differences in transforming a word problem text into a mathematical structure (i.e. with mathematisation) and differences in the ability to work systematically and to firstly determine a suitable solving procedure. Small differences were reported in the speed of performing arithmetical operations.

Although the findings from the survey do not speak directly about teacher education, they do provide useful pointers. Teachers need to be well prepared to work in an environment of diversity of pupils in all aspects. Preparing them for working for inclusion is one of the important tasks for teacher education, with teacher responses to word problems as demarcating between strong and weak students providing evidence of current beliefs standing against this view (Brousseau and Novotná 2008).

It is broadly accepted that during their pre-service, as well as in-service, education teachers' attention should be directed to structural relations; focus on keywords and operations is not sufficient (Hejný 2012).

In order to help their pupils to develop their knowledge of mathematics as well as positive attitudes towards mathematics, teachers will need a good command of mathematical and pedagogical knowledge. When entering the practice after graduating as primary teachers, they will not only work with their pupils but will also, importantly, cooperate and learn from (experienced) colleagues. Teacher education should prepare future teachers for all aspects of their teacher professional life; knowing the subject is not the only necessity. The results from the project are important for primary mathematics teacher education. Future teachers must undergo a suitable education for opening for them the possibility of helping their pupils to overcome critical obstacles. Lastly, teachers' approaches to overcoming their pupils' difficulties when learning mathematics are influenced not only by their mathematical and pedagogical content knowledge (which is a common part of all primary teacher education), but also by their personal characteristics.

17.4 Discussion

Mike Askew

In the presentations given at the conference, several common themes emerged that included, first, the recognition of the need for teaching WNA to have increased emphasis on problem solving (as opposed to simply focusing on teaching arithmetical fluency) and, second, the importance of the working with concrete and pictorial representations. Interested readers are likely not to be surprised to learn of these themes. Such issues have been on the mathematics education agenda for many years. Yet the evidence, both within the conference (see, e.g., the contribution by Mulligan and Woolcott 2015) and in the mathematics education literature, more generally (see, e.g., Cai 2003) suggests that we, the teacher education community, have had mixed success in achieving such changes.

For example, with regard to the role of representations, a discussion that emerged during a panel session at the conference illustrates how work still needs to be done on the balance and relationship between using ordinal (number line) and cardinal (base ten blocks) models as representation for WNA. A number of European participants argued strongly for the number line as a core model for WNA, but the more local speakers, while not dismissing using the number line altogether, clearly were less enamoured of it, being keener on cardinal images of number. Why might there be differences of opinion on this?

One argument put forward emphasising the use of number lines is that they provide learners with reinforcement of core whole number fluencies, such as making numbers up to the next multiple of ten or partitioning single digits. For instance, in adding 9 onto 24 on an empty number line, making a jump of 6 from 24 to 30 reinforces making 24 up to the next multiple of ten and simultaneously reinforces the bond of 9 = 6 + 3. There is also the argument that working with the number line encourages strategic thinking, through it being amenable to different approaches. Taking 25 + 9, jumping 10 on the number line to 35 can encourage thinking in terms of a compensation strategy by then subtracting 1 from 35.

Yet observing a demonstration lesson in Macao (see Chap. 11, this volume), the approach taken in the lesson was based on a cardinal model of 24 + 9 and yet still reinforced similar fluencies. In one method put forward by the learners, 9 was partitioned into 6 and 3 in anticipation of making 24 up to 30. And again, strategic thinking was encouraged – the learners produced at least three different solution methods. In contrast to the methods that emerge using the number line, all these methods were based on partitioning – in addition to partitioning 9 into 6 + 3, 24 was partitioned into 23 + 1 to create 23 + (1 + 9) or into 20 + 4 to create 20 + (4 + 9). This latter method, although recorded horizontally in this lesson, lends itself to being linked to the standard vertical algorithm, so it may be that the preference for a cardinal model is rooted in an eye to the mathematical horizon.

Thus, it could be argued that the Western preference of the number line model is based on a preference for encouraging flexible efficiency, developing and using methods that are related to the numbers in the calculation, thus using compensation to add, say, 9 or 19. This flexibility is often linked back to individual differences that the number line explicitly promotes as a model that allows for emergent approaches at a range of levels of sophistication. In contrast, while multiple methods are encouraged within cardinal models, there is much more reference in this work to strategies that relate to the underlying mathematical structure of the task situation than to individual differences.

The conference presentations thus point to the need for further research into how the two models of whole number – ordinal (number line) and cardinal (base ten blocks) – complement each other, rather than teacher educators across the globe ‘agreeing to disagree’ on which is the more favourable representation. But these bodies of research also suggest that it may be necessary to look ‘through’ such research to the basic aims and philosophy of education that guide the choice and use of models. And, importantly, could the mathematics teacher education community then come to some agreement on how best to work with pre-service teachers on both models, rather than privileging one over the other, or, as seems to be the case in many teacher education programmes, leaving it up to prospective teachers to choose for themselves which model they prefer to work with.

Such distinctions go to the heart of different perspectives of what it means to be a professional. For example, in many teacher education programmes in English-speaking nations, the approach is to introduce pre-service teachers to a range of pedagogical approaches and representations in the expectation that they, as professionals, can decide for themselves which they think may be most effective. In contrast, in places like Singapore and Shanghai, it is clear that there is more consensus on which approaches to use. In part, such consensus may be the result of structural and historical circumstances. Being a small nation, Singapore has only one teacher education institution and a small community of educators working with both preand in-service teachers. China has a long history of careful curriculum development and long-standing use of textbooks linked to the curriculum.

At the time of writing, there are initiatives in England to encourage teachers to adopt pedagogies based on those from Singapore and Shanghai, together with more use of textbooks. While teachers are welcoming these initiatives, there is less enthusiasm in some parts of the mathematics education community. Objections range from the argument that these initiatives are downplaying the good practices that already exist in England through to arguing that ‘training’ teachers to teach in particular ways is a threat to their professionalism.

Thus it would seem that difficulties in promoting, say, more problem solving and better use of representations may lie as much in the beliefs and practices of the mathematics education community as it does in the school and teaching community. While there is no shortage of research into teacher change, or the lack of it, are we, as mathematics educators and researchers, sufficiently self-reflective on whether or not our beliefs and practices need to change? Much of the research into teacher education addresses the question ‘What do teachers need to do differently?’, but do we pay sufficient attention to the parallel question ‘What could we (the teacher educators) do differently?’ We might begin to address this question by looking at some of our beliefs and, in particular, at our theory of knowledge.

Since Shulman’s seminal work on the distinction between content knowledge and pedagogic content knowledge, there has been a plethora of studies into knowledge for teaching, most of which theorise different models and try to tease apart exactly what the difference between content knowledge and pedagogic content knowledge might look like. In many higher education institutions, there is contestation over where the mathematics for teaching should be taught – should it be taught in education departments or in faculties of mathematics? And we know from research that the relationship between studying higher mathematics in contexts divorced from addressing issues of pedagogy is only weakly associated with later success as a teacher (see, e.g., Wilson et al. 2001) (which, of course, is not to say that knowledge of mathematics is not important, but that it is a particular sort of mathematical knowledge that matters). So there is a political dimension to the research in knowledge for teaching, but has that research now established a sound body of findings as to exactly what mathematics pre- and in-service teachers need to be taught? Does the culture of research in teacher education encourage the cumulative building of knowledge into effective teacher education, with each other’s work being built on and developed? Or is our culture one of individual reputations needing to be established? As Michael Billig argues, much writing in social sciences generally now has overtones of the culture of advertising, with different theories ‘positioning themselves’ in the field rather than complementing each other (Billig 2013).

Thus, a theory of knowledge that underpins much of the research in places like the USA and UK has at its heart the development of the individual (teacher and researcher). Knowledge is in the mind of the individual; it is the personal, individual knowledge and skills of the teacher that needs to be ‘developed’. Other traditions of development, for example lesson study in Japan (Lewis 2002), focus more, however, on teaching than on teachers. And, as the papers in Topic Group 4 suggest, within such cultures there may be less of a tradition of teachers choosing pedagogic approaches for themselves – the 'technology' of teaching is more in the hands of textbook and curriculum developers. The knowledge is not only in the heads of teachers, but also in the resources made available to them.

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