# Chapter 15
# Tradition in Whole Number Arithmetic

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## 15.1 General Introduction

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The Merriam-Webster Dictionary[1] defines tradition (Definition 1) as:

a: an inherited, established, or customary pattern of thought, action, or behavior (such as a religious practice or a social custom);
b: a belief or story or a body of beliefs or stories relating to the past that are commonly accepted as historical though not verifiable.

The dictionary also states that tradition concerns the 'handing down of information, beliefs, and customs by word of mouth or by example from one generation to

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[1] http://www.kamous.com/translator/merriam-webster.asp?book=Dictionary&va=tradition

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another without written instruction' (Definition 2) and represents 'cultural continuity in social attitudes, customs, and institutions' (Definition 3).

It is apparent from these definitions that the ways in which whole numbers are spoken, written, thought, taught and learnt sum up what we can address as a part of tradition. Hence, researchers and teachers need to consider these factors from the many different perspectives that make up the multifaceted cultural, epistemological, psychological and neurological nature of tradition.

Some of these components have a more or less strong 'local' connotation because they are linked to different cultures and traditions. However, other components are more general and seem to have universal traits. Hence, the concept of so-called near-universal conventional mathematics (NUC: Barton 2008, p. 10) may conflict with such local instances. This possible contrast can represent a significant problem for teachers because a reasonable learning trajectory for whole numbers cannot avoid discussing their traditional roots, while addressing the NUC as its main goal.

This general background shaped the panel discussions, which aimed to scientifically deepen the analysis of some of these different cultural roots, consider old and new findings from research and practice, and make explicit the main consequences of possible concrete didactical trajectories.

In the following sections, some general issues are considered before introducing the panellists' contributions.

## *15.1.1 Different Semiotic Representations of Numbers*

The historically and culturally different systems of whole number representation encompass a large variety of semiotic systems, including but not limited to language.

### 15.1.1.1 Numbers and Words

The manner in which numbers are articulated in different languages raises a complex issue that has been examined in a large body of research. From the pioneering book of Menninger (1969) to more recent works (Zaslavsky 1973; Ifrah 1985), all of these studies provide evidence of what Bishop has called the *mathematical enculturation* (Bishop 1991) of numbers (see also Ascher (1991), Selin and D'Ambrosio (2000) and Barwell et al. (2015)).

The manner in which whole numbers are articulated and written is a significant feature that can reveal various different cultural factors. This issue needs to be considered when teaching early arithmetic. Some well-known examples are summarised below (see also, this volume, Chap. 3).

In many languages, the numbers from 11 to 20 are spelled according to specific rules that differ from those for the following sequences, e.g. from 20 to 30. These rules may hide the mathematical structure of those numbers [12 vs 'twelve' (~two left); 14 vs 'quattordici' (~ four-ten); 17 vs 'diciassette' (~ ten-seven)]. Similarly, the French numbers from 60 to 99 are spelled according to an old base 20 root that is typical of some Celtic languages. For example, to say 97, a French girl/boy must learn to say ‘quatre-vingt-dix-sept’, that is, ‘four (times) – twenty – ten – seven’, whereas a German child must learn ‘Siebenundneunzig’ (seven and ninety) and an Italian child must say ‘novantasette’ (ninety-seven), and so on. In contrast, in Chinese the grammar of numbers is more regular, which may provide an advantage in learning numbers. An Italian teacher, Bruna Villa (2006), has developed an effective learning design for grade one children to teach them how to grasp the machinery of whole numbers (see this volume Sect. 15.3.3). She based her design on what, following Brissiaud, Clerc and Ouzoulias (2002), she called the method of the ‘small Chinese dragon’ (Villa 2006; Electronic Supplementary Material: Arzarello 2017), in which the children articulate numbers based on a uniform Chinese-like structure (e.g. 11 is ‘ten-one’ and not ‘undici’; 21 is two (times) ten-one and not ‘ventuno’) before passing to the Italian system. In this way, she has been able to shorten the time needed to master the whole numbers from one to 100 (in Italian words and standard arithmetic representation) and to use them to carry out arithmetic. In Sect. 15.3.3, this process is illustrated and discussed in more detail.

A further fascinating example, which shows strong differences between the way numbers are spelled in a language and their mathematical structure, is illustrated in Barton (2008), where he discusses the way numbers are articulated in Maori. Prior to European contact, numbers in Maori were similar to verbs in that they expressed actions, e.g. saying that ‘there were two persons’ was similar to saying that ‘those persons two-ed’. This difference was even more dramatic when negation was involved: ‘To negate a verb in Maori the word *kaore* is used. […] Unlike English, where negating both verbs and adjectives requires the word “not”, in Maori, to negate an adjective a different word is used, *ehara*’ (p. 4). Hence, when this verbal feature of Maori number words was ignored, in English translation the mathematics vocabulary process acted against the original ethos of the Maori language (this volume, Chap. 3).

Other researchers have indicated the ways in which the use of numbers in everyday language interferes with the mathematical meaning of numbers. In an excellent book, unfortunately available only in Italian, a researcher in linguistics, Carla Bazzanella (2011), points out that the expression of numbers in everyday language can convey an indeterminate and largely vague meaning rather than the canonical cardinal denotation (see other examples in this Sect. 15.2.2 and 15.4.2 and in Chaps. 3 and 4 of this volume).

#### 15.1.1.2 Non-verbal Representations of Numbers

Researchers have also discussed the ways in which numbers are represented in different nonlinguistic ways in different cultures (Joseph 2011), e.g. using parts of the body (typically digits, but not only; see Saxe 2014) or spatial arrangements in complex arithmetical calculations when number words are lacking (this volume, Chap. 4).

Many researchers have pointed out that there are some typical steps in the ways in which children progress in building up numbers by intertwining language and gesture, e.g. using their digits for counting and adding. For example, Vergnaud uses an adaptation of the Piagetian notion of the *schème* [which he defines as, 'the invariant organization of behaviour for a certain class of situations' (Vergnaud 1997, p. 12)]. He discusses how, when a child uses a counting scheme, a cognitive shift, related to gesture, may occur:

> Une autre caractéristique du schème concerne la marque énonciative de la cardinalisation: le dernier mot-nombre prononcé représente le cardinal de tout l'ensemble et non pas le dernier élément. Cette marque énonciative consiste soit dans la répétition (1, 2, 3, 4, 5,… 5), soit dans l'accentuation (1, 2 3, 4…5). On voit clairment avec ce premier example que l'activité langagière est étroitement associée au fonctionnement du schème, et qu'elle prend sa fonction dans un assemble de gestes perceptivomoteurs dont l'organisation dépend de la disposition des objects et de leur nature, et d'un problème à résoudre: associer un nombre invariant à une collection donnée. (Vergnaud 1991, p. 80)
>
> [Another characteristic of schemes concerns the way cardinalisation is marked in speech: the last number pronounced represents the cardinality of the whole collection and not just the last object. This marking with speech comprises not only the repetition (1, 2, 3, 4, 5,.. 5) but also the accentuation (1, 2, 3, 4…5). One can clearly see from this example that language is closely associated with the functioning of a scheme, and that it plays a role in producing perceptuo-motor gestures whose organisation depends on both the nature and arrangement of the objects, and the problem to solve; associating an invariant number with a given collection.]

Butterworth et al. (2011) describe a similar multi-step process for addition strategies that is based on a more neurological stance:

> Where two numbers or two disjoint sets, say 3 and 5, are to be added together, in the earliest stage the learner counts all members of the union of the two sets – that is, will count 1, 2, 3, and continue 4, 5, 6, 7, 8, keeping the number of the second set in mind. In a later stage, the learner will 'count-on' from the number of the first set, starting with 3 and counting just 4, 5, 6, 7, 8. At a still later stage, the child will count on from the larger of the two numbers, now starting at 5, and counting just 6, 7, 8. It is probably at this stage that addition facts are laid down in long term memory. (p. 631)

Recent studies in ethnomathematics and neurology have introduced a fresh and wider perspective on the issue of language and its role as a resource for arithmetic activities (for a survey from a neuroscientific perspective, see Dehaene and Brannon (2011)). An intriguing example is given in Butterworth et al. (2011), who point out that word counting strategies are not the only methods that people can use for developing arithmetic competencies:

> We tested speakers of Warlpiri and Anindilyakwa aged between 4 and 7 years old at two remote sites in the Northern Territory of Australia. These children used spatial strategies extensively, and were significantly more accurate when they did so. English-speaking children used spatial strategies very infrequently, but relied on an enumeration strategy supported by counting words to do the addition task. The main spatial strategy exploited the known visual memory strengths of Indigenous Australians, and involved matching the spatial pattern of the augend set and the addend. These findings suggest that counting words, far from being necessary for exact arithmetic, offer one strategy among others. They also

> suggest that spatial models for number do not need to be one-dimensional vectors, as in a mental number line, but can be at least two dimensional. (p. 630)

Further research in neurology has supported these claims in relation to the wider characteristics of mathematics. For example, Varley et al. (2002) show that:

> once these resources [mathematical ones] are in place, mathematics can be sustained without the grammatical and lexical resources of the language faculty. As in the case of the relation between grammar and performance on 'theory-of-mind' reasoning tasks (42), grammar may thus be seen as a co-opted system that can support the expression of mathematical reasoning, but the possession of grammar neither guarantees nor jeopardizes successful performance on calculation problems. (p. 470)

Monti et al. (2012) also point out that:

> Our findings indicate that processing the syntax of language elicits the known substrate of linguistic competence, whereas algebraic operations recruit bilateral parietal brain regions previously implicated in the representation of magnitude. This double dissociation argues against the view that language provides the structure of thought across all cognitive domains. (p. 914)

Finally, some studies have pointed out that the sense of numbers is not only based on discrete approaches that rely on the one-one correspondence between external symbols and numerical representations, but also on approximate number of systems (e.g. the estimation of the numbers of two sets when subitising is not possible) that are based on the ratio between their cardinality and not on their difference (see Gallistel and Gelman (2000)). According to these studies, this continuous, analogic system emerged during our evolution and became encoded in our brains prior to the discrete approach.

These findings have introduced a fresh perspective on the issues of tradition and language and their roles as resources for arithmetic activities.

In particular, some major questions for the panel are:

- How can teachers base their task designs for arithmetic on the linguistic and cultural roots of numbers?
- Does the embodied traditional approach to arithmetic need to be modified/extended by the findings of the neurological research on numbers?

#### 15.1.1.3 Representing Numbers in Artefacts

In the research on the semiotic representation of numbers, a specific stream of analysis concerns the calculation tools (typically, but not only, abaci) that incorporate both the specific representations of numbers and the corresponding practices for completing arithmetical operations (for a survey see Ifrah 2001). These tools are deeply intertwined with language and can be incorporated in the didactical designs used in primary school. Many teachers use the tools alongside modern technology to introduce concrete artefacts and their simulations in a virtual technological classroom environment. For example, Sinclair and Metzuyanim (2014) integrated such embodied and traditional representations using tablets based on the hypothesis that the touch-screen devices enable an intuitive, embodied interface for conducting arithmetic. The devices are also suitable for young learners because they allow them to use their fingers and gestures to explore mathematics ideas and express mathematical understandings. Furthermore, Soury-Lavergne and Maschietto (2015) use an old Pascal machine to approach arithmetic in concrete and virtual ways in primary school. These and further examples are discussed in Chap. 9 of this volume.

These types of research pose the following interesting questions for the panel:

- How are traditional instances embodied in the current technology?
- Does the possible integration of cultural roots within a technological environment allow the gap between the 'old-fashioned' tradition and the NUC to be bridged?

The panel consists of four scholars[2] who are representative of the different cultural traditions of teaching numbers, namely, Nadia Azrou (mathematics teacher at the University of Yahia Fares in Medea, Algeria, and a PhD student in math education), Maria G. Bartolini Bussi (full professor in mathematics education at the University of Modena and Reggio Emilia, Italy), Sarah Inés González de Lora Sued (full professor in mathematics education at Pontificia Universidad Católica Madre y Maestra, República Dominicana) and Xu Hua Sun (assistant professor in education at the University of Macau, China). Man Keung Siu (honorary fellow, The University of Hong Kong) acted as the discussant.

## 15.2 Spoken and Written Arithmetic in Different Languages: The Case of Algeria

Nadia Azrou

### *15.2.1 Post-colonial Countries: The Case of Algeria*

At the elementary level, numbers are learnt along with several technical concepts (e.g. the place value of digits, number line and decimal position system) that support learning or weaken it if not effectively acquired. Learning numbers and other basic arithmetic notions is also affected by culture and particularly by language. This is more visible in multicultural classes in schools that host migrants of different nationalities, but also in post-colonial countries such as Algeria, where history, cultural evolution and external and internal power influences have a direct influence on the school system.

[2] Sarah Inés González de Lora Sued was unable to take part in the panel for health reasons. However, she provided a text, which appears in this chapter.

How do we deal with such phenomena? The globalisation of school mathematics, which aims to unify the curricula in countries that have different cultural and linguistic backgrounds, and the assumption that learners have to submit to a country's dominant language, have been shown to have their limitations. From a different perspective, Usiskin (1992) claims that differences provide the best situation for curriculum development and implementation. Gorgorió and Planas (2001) point out that if language is the main carrier of a culture, then the 'language of the mathematics class' conveys the culture of the classroom as a social group doing mathematics, along with its norms and legitimate roles.

For teachers, this clearly represents a challenge, particularly for those who teach in a traditional, transmissible way. Most teachers presume that the 'normal' learning context is a monolingual classroom, that learners know the 'norms' of the school (which are usually shaped by the dominant culture) and that children already master the language of instruction. Given this situation, teachers should acknowledge the relevance of the issues related to cultural and linguistic diversity, understand how they influence the learning process and manage them to scaffold the children's learning in an effective way. In particular, teachers should be able to identify the possible difficulties that children experience when learning numbers in a language different from their mother tongue and to create opportunities to turn these difficulties into advantages. Moreover, I share the view of Gorgorió and Planas (2001) who believe that there is no classroom in which linguistic capital is equitably distributed. As a consequence, what may appear as an extremely 'different' setting not relevant for mainstream practice may be relevant for communication issues in all classrooms.

Further research is needed to clarify how mathematical language can be taught and to investigate the relationships among the 'language of the mathematics class', mathematical language and the process of construction of mathematical knowledge (Gorgorió and Planas, 2001). However, some elements and insights can already be provided to address the question of how teachers can concretely develop their task designs for basic notions of arithmetic by taking the linguistic characteristics and cultural roots of numbers into account. Some answers may be suggested by the analysis of the situation.

### *15.2.2 Number Naming, Place Value and Decimal Position System*

It is not unusual for a language to have irregularities in regard to number naming, and these irregularities are not the same in different languages. In Europe, for instance, every language possesses its own number naming system with its own set of irregularities. For example, similar to French, Spanish and Italian, the numbers between 13 and 19 in English have names that position the lowest place value digit first, in contradiction to the written form, which goes from left to right according to the decreasing place value of digits and the other spoken numbers. Moreover, although in English the word 'ten' is nearly present in thirteen (13) to nineteen (19) as in Italian, this is not the case in French where 'ten' ('dix') does not appear in the numbers 12 to 16 (e.g. quatorze 14, seize 16), although 17 is pronounced 'dix-sept'). In Arabic, the numbers from 11 to 99 are pronounced from the lowest place value digit to the highest and are in complete correspondence with the written form, which is presented from right to left. In Danish, seventy is named halvfjerds, which is a short for halvfjerd-sinds-tyve, meaning "fourth half times twenty", or "three score plus half of the fourth score" [3½ * 20]. Moreover, as in German, there is no correspondence between the written and spoken forms for the numbers from 13 to 99, which are pronounced from the lowest place value digit to the highest. In French, the numbers between 81 and 99 are expressed as 'four-twenty' plus a number between 1 and 19. Traces of the contamination between different languages and the historical roots in old number systems can be detected in these irregularities and differences. However, irregularities, in the same oral language or when shifting from one language to another, may be a source of difficulty for children. Research suggests that in some Asian countries, Asian speaking children perform better with place value, counting and decimal system tasks due to their regular number naming systems (Miura et al. 1994). Nonetheless, irregularities and differences also provide students (under the guidance of the teacher) with opportunities to notice important characteristics of the decimal position system of writing of numbers, such as the position value of digits and reflect on them. For instance, with reference to the above examples, the teacher may exploit the differences between the irregular forms of spoken numbers within the same language (in the case of most European languages) and between how numbers are spoken in one language and another.

The case of Algeria is interesting. About 10 years ago, a political decision was made to write formulas and symbols from left to right with the Latin alphabet (in the past they were written from right to left with the Arabic alphabet) when teaching mathematics at all levels, while maintaining comments and names in classical Arabic (from right to left). This change has subsequently influenced how children conceive, understand and learn arithmetic. Thus, teachers should use this as an opportunity to allow children to realise that mathematics is not separate from culture and language and to understand that its evolution is also affected by the historical and political dynamics.

### *15.2.3 Mathematics Register*

As defined by Halliday & Hasan (1985), the mathematical register records how everyday language is used in new ways to serve the meanings of mathematical words, even though words such as 'double, less, more' may have different meanings in ordinary language from those in mathematics. These differences may have resulted in some children failing to solve problems caused by misunderstanding the text. For example, in Arabic, the verb used to express the multiplication operation is to 'beat' (thus scaring the children), that is, 'we multiply 2 by 4' is 'we beat 2 to 4'. In English, expressions such as 'twice as much as' or 'twice as less as' may sound ambiguous. Accordingly, children need to learn the language patterns associated with these words, how to construct concepts in mathematics and the implicit logical relationships, because when children construct mathematics concepts in everyday language, the relationships they come up with are often technically incorrect. Learning mathematics and the language of mathematics, that is, the mathematics register, is a challenge for all children. Teachers can facilitate the learning process by using the mathematical register effectively and working to build language in deliberate ways, moving from everyday to technical linguistic expressions of mathematical knowledge and using spoken language, reading and writing. If learners have difficulties in verbalising a mathematical process, the teacher can promote mathematical thinking by using their mother language to tackle mathematical problems (Adler 1997). Thus, it is highly recommended that teachers have some knowledge about their learners' languages and consider the norms and contexts in which words are used. Teachers should aim to develop the mathematics register in the languages in which the children are instructed. However, until this is done, teachers need to face and overcome the difficulties in translating mathematics concepts into students' home languages (Schleppegrell 2007).

## 15.3 From the Number Line to the Productive Dialogue Between Different Cultural Traditions: Italy and China

Maria G. Bartolini Bussi

### *15.3.1 The Number Line*

The number line is a very popular teaching aid (Bartolini Bussi 2015; Electronic Supplementary Material: Bartolini Bussi 2017). Italian teachers can find specific references to the number line in the standards (MIUR 2012) for the mathematics curriculum. To begin with, the number line comprises whole numbers and it is then expanded to contain rational numbers. The following goals are stated at the end of the third grade (the first time the goals are explicitly listed):

> To read and write whole numbers in base ten, being aware of the place value; to compare and to order them, representing them on the number line. (p. 61)
>
> To read, write and compare decimal numbers, to represent them on the number line …. (p. 61)

The goals are summarised and reinforced at the end of primary school (fifth grade):

> To represent the known numbers on the line and to use graduated scales in contexts that are meaningful for science and technique. (p. 62)

The last goal hints at a possible use of the number line as a modelling tool. A similar use is stated in the history curriculum, where, at the end of primary school (fifth grade), the following goal is stated:

> To use the timeline to organize information, pieces of knowledge, periods and to detect sequences of events, concurrent events, duration of events. (p. 53)

The representation of numbers on a line is emphasised (at all levels) in the framework for the national assessment in mathematics (INVALSI 2012).

The general approach to the mathematics curriculum in Italy is stated in the programmes (MIUR 1985) for primary school:

> The development of the concept of whole numbers must be roused exploiting the previous experience of students, like counting and recognizing numerical symbols in play and in family and social life. It is advisable to consider that the idea of whole number is complex and requires a multifaceted approach (order, cardinality, measuring, …); it is acquired at higher and higher levels of internalisation and abstraction during primary school and beyond.

This idea is widely shared and has been confirmed in other curriculum documents (e.g. MIUR-UMI 2001), which have strongly influenced the elaboration of the more recent standards (e.g. MIUR 2012).

In the number line, order and measuring are in the foreground. However, the other properties of whole numbers (e.g. cardinality, place value representation) are not supported by the number line and must be taught independently. This choice is consistent with a multifaceted approach in which different routes are explored in parallel to develop a complex concept of whole numbers.

In contrast, as argued by Sun (2015), the Chinese tradition of whole number arithmetic appears to place less emphasis on the number line and to foreground other properties of whole numbers (e.g. the part-part-whole and associative law) in constructing a consistent teaching path in which these properties are pursued in a systematic way, step by step and without deflection.

### *15.3.2 The Dialogue Between Cultures: Towards Cultural Transposition*

The panel discussion on the number line is a paradigmatic example of the process that occurs when scholars from different cultural backgrounds engage in a true dialogue. The point is not to determine the best, 'universal' choice but to understand how and why the mathematics curriculum was developed in one's own context. Jullien (1996) stated that, 'every thought, when coming towards the other, questions itself about its own unthought' (p. iii). In this sense, noticing the different approaches used in Italy and China serves as a prompt to start a cultural analysis of the content (Boero and Guala 2008). Cultural artefacts, when carefully analysed, reveal a lot of things about the culture that has produced them. To implement activities by using cultural artefacts in a different culture, it is necessary to enter a process of *cultural transposition*, where:

> the different cultural backgrounds generate possibilities of meaning and of mathematics education perspectives, that, in turn, organize the contexts and school mathematics practices in different ways. (Mellone and Ramploud 2015, p. 578)

### *15.3.3 Examples of Cultural Transposition*

The first example concerns the use of counting rods for the development of place value. Some years ago, we analysed some Chinese textbooks for the first grade and noticed the use of counting rods and bundles of rods to link numbers and quantities. The book for the first semester of the first grade comprised 120 pages (from September 1 to the end of January). After having presented the numbers from 1 to 10 together with addition, subtraction and word problems, the numbers between 10 and 20 were introduced. On p. 85, the following activity was presented (Fig. 15.1). This was the first activity in which numbers with two digits (from 11) were introduced.

The teacher says: 'First count ten little rods and bind them to get one bundle. How can you go on counting?' The boy answers: 'To combine together one ten and one, it is ten-one'. The right classifier is always used: 个(*gè*) for rods and 捆(*kǔn*) for bundles; 个(*gè*) again for one and for ten, which is the origin of place value (this volume, Chap. 3).

The process is supposed to be very fast, as if the student is able to produce the right name without any help from the teacher. This natural process is possible in China, because the way of recognising numbers in Chinese is part of everyday experience and completely transparent in relation to the place value (this volume, Chap. 3). No specific teaching processes are needed in school. In contrast, a specific teaching process is needed to design school practices in other languages/cultures. For instance, the names of numbers in Italian are irregular and not transparent, and hence it is not possible for a student to name a number with one bundle and one rod ('undici' in Italian, 'eleven' in English). It is necessary to plan two parallel processes before linking the rod and bundle representations to the names and the symbols in which the bundles are bound to construct the concept of ten as a higher order unit, and the Italian names for the numbers are learnt. Only later is it possible to link these processes to one another. Thus, more instruction time is needed than in the Chinese classroom.

An Italian teacher, Bruna Villa, produced another example of cultural transposition in the same content (see this chapter Sect. 15.1.1.1; Electronic Supplementary Material: Arzarello 2017). She also introduced two parallel processes. At the beginning of the first grade, the teacher told a fairy tale of a small Chinese dragon who was visiting the classroom to teach the children how to say numbers. Hence, the students learnt to say the numbers in two ways:

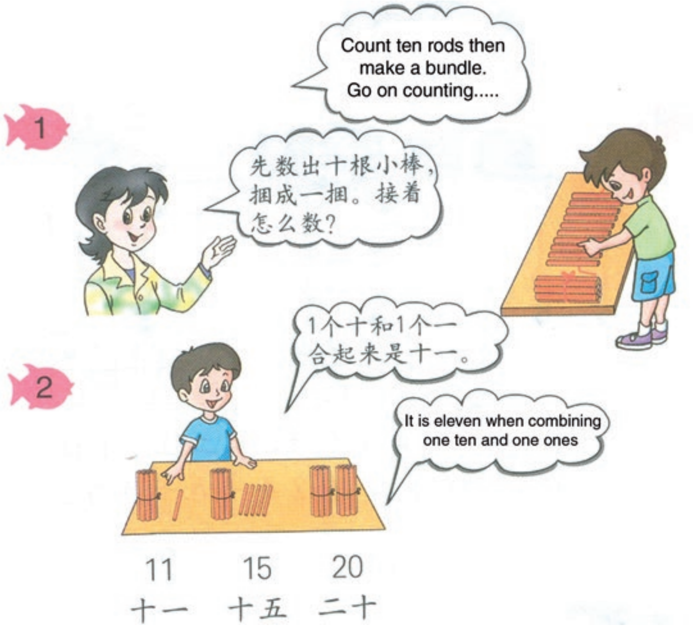


**Fig. 15.1** The introduction of tens in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005)

*The completely regular Chinese-like names: 11 is ten-one; 21 is two ten-one.*
*The Italian irregular names: 11 is 'undici', that is, 'eleven'; 21 is 'ventuno', that is, 'twentyone'.*

The teacher evoked the images of two imaginary characters, the 'small Chinese dragon' (with narratives, drawings and even hats to wear when acting as a dragon) and the 'mom', representing the (Italian) adult voice from the students' everyday experience, for some months to avoid ambiguity and help the students to make sense of the experience. In this way, the teacher succeeded in teaching place value in an exciting but robust way while introducing some ideas about different cultural contexts.

Another example concerns the planned transposition to Italian classrooms of the word problem of cakes observed in the first grade of the Hou Kong School (this volume, Chap. 11). In this case, the task draws on Chinese practice to consider a number as a system of part-part-wholes in different ways (a variation of the 'one problem, multiple solutions' (OPMS) approach, Sun (2011)). Italian students are not accustomed to these kinds of tasks, hence the cultural transposition to Italian classes requires additional tasks to be introduced in parallel to the existing ones.

There are other examples of variation problems (e.g. Bartolini Bussi et al. 2013). After being examined in pilot studies, some curriculum material has been developed in Italy (see the Italian project *PerContare*, this volume, Chap. 7) in which the cultural analysis has been made explicit for teachers. This is a possible answer to the last question posed to the panel, namely, where do the 'traditional'activities of different cultural traditions come from? We were able to exploit the cultural transposition because the Italian standards leave some freedom for teachers to conduct pilot experiments. My questions are now: Is it possible to conduct pilot experiments in China that exploit new activities? And how can cultural differences be incorporated into the established approach, which is mainly based on textbooks with fixed content for each lesson?

## 15.4 The Role of Using the Cultural Roots of Numbers and Artefacts for Children Learning Whole Number Arithmetic in Latin America

Sarah Inés González de Lora Sued

### *15.4.1 The Design of Learning Tasks for Arithmetic Based on the Linguistic and Cultural Roots of Numbers*

Most of the time, whole number arithmetic learning is not related to the reality in which children live, because the design of the learning tasks is divorced from the local cultural environment. Referring to the weak relationship between culture and mathematics in the classrooms, D'Ambrosio (2001) stated that:

> when teachers do acknowledge a connection between mathematics and culture often they engage their students in multicultural activities merely as a curiosity. Such activities usually refer to a culture's past and to cultures that are very remote from that of the children in the class. (p. 308)

And he also pointed out:

> As our students experience multicultural mathematical activities that reflect the knowledge and behaviors of people from diverse cultural environments, they not only may learn to value the mathematics but, just as important, may develop a greater respect for those who are different from themselves. (p. 308)

However, when mathematics, and whole number arithmetic in particular, is presented to students using examples that are a lively part of their culture (e.g. by introducing mathematical concepts and procedures through problematic situations drawn from their reality), the concepts become meaningful to them.

D'Ambrosio stresses that:

> We can help students realize their full mathematical potential by acknowledging the importance of culture to the identity of the child and how culture affects how children think and learn. We must teach children to value diversity in the mathematics classroom and to understand both the influence that culture has on mathematics and how this influence results in different ways in which mathematics is used and communicated. We gain such an understanding through the study of Ethnomathematics. (p. 308)

Moreover, if whole number arithmetic is taught in isolation from the culture of segregated populations, such as the many indigenous populations of Latin America, the situation becomes more complex. In the 2013 report 'Intercultural citizenship: Contributions from the political participation of indigenous peoples in Latin America', the United Nations Development Programme (UNDP) indicated that there are approximately 50 million indigenous people in Latin America, about 10% of the total population. However, in countries such as Peru and Guatemala, indigenous people account for almost half of population, while in Bolivia they comprise over 60% of the total population. These indigenous peoples speak their own languages and many are marginalised because they do not speak Spanish. These cultures also have their own ways of conceptualising whole numbers. Many studies have been conducted on the mathematics of indigenous Latin American cultures. To teach the children of these populations, teachers need to be able to understand these mathematical approaches to whole number arithmetic. In the case of Guatemala, by law, all children must be taught using the Mayan number system *and* the base 10 system.

### *15.4.2 Numbers and Words*

Table 15.1 lists the names of the numbers from 1 to 20 in Mayan, Quechua Collao and Spanish. The fourth column lists the Latin words from where the Spanish names proceed, according to the Real Academy of the Spanish Language. It is interesting to note the patterns in the table. For example, from 1 to 10, the names within each language do not have any relationships among them. For Mayan numbers, this is also the case for numbers 11 and 12. However, from 13 to 19, the names are composite words of the names 3 to 9 and the name of 10 (lahun). In the case of Quechua Collao, from 11 to 19, the names are composite words with the name of 10, 'Chunka', the names of the numbers from 1 to 9 and the word 'niyuk'. In this case, the word for 10 comes first. In the case of Spanish names, from 11 to 15, the relationships between the names and the numbers are not clear. However, when you see the Latin roots of the words, as described in the dictionary of the Real Academy of Spanish Language, it seems the meaning is 'one and ten' for 11, 'two and ten' for 12, and so on. The dictionary of the Real Academy of Spanish Language does not include the roots of numbers 16 to 19 but their meaning is clearer and easier for children to understand (ten and six, ten and seven, and so on).

Moreover, in Quechua Callao, 30 is kimsa chunka, 40 tawa chunka, 50 pichqa chunka and so on, which can be interpreted as having the multiplicative meanings of three times ten, four times ten and so on. In Spanish, 30 is treinta and its Latin root is triginta, 40 is cuarenta and its Latin root is quadraginta, 50 is cincuenta and its Latin root is quinquaginta and so on with the same meaning. It seems that it would be easier for Quechua Collao children to learn the names of numbers from the patterns of the names.

**Table 15.1** Some pre-Columbian names of numbers

| | Mayan names of numbers | Quechua Collao | Spanish | Latin root |
|---|---|---|---|---|
| *1* | *Hun* | *Huk* | *Uno* | *Unus* |
| *2* | *Caa* | *Iskay* | *Dos* | *Duo* |
| *3* | *Ox* | *Kimsa* | *Tres* | *Tres* |
| *4* | *Can* | *Tawa* | *Cuatro* | *Quattuor* |
| *5* | *Hoo* | *Pichqa* | *Cinco* | *Quinque* |
| *6* | *Uac* | *Suqta* | *Seis* | *Sex* |
| *7* | *Uuc* | *Qanchis* | *Siete* | *Septem* |
| *8* | *Uaxac* | *Pusaq* | *Ocho* | *Octo* |
| *9* | *Bolon* | *Isqun* | *Nueve* | *Novem* |
| *10* | *Lahun* | *Chunka* | *Diez* | *Decem* |
| *11* | *Buluc* | *Chunka hukniyuq* | *Once* | *Undecim* |
| *12* | *Lahca* | *Chunka iskayniyuk* | *Doce* | *Duodecim* |
| *13* | *Oxlahun* | *Chunka kimsaniyuk* | *Trece* | *Tredecim* |
| *14* | *Canlahun* | *Chunka tawaniyuk* | *Catorce* | *Quattuordecim* |
| *15* | *Hoolahun* | *Chunka pichqaniyuk* | *Quince* | *Quindecim* |
| *16* | *Uaclahun* | *Chunka suqtaniyuk* | *Dieciseis* | |
| *17* | *Uuclahun* | *Chunka qanchisniyuk* | *Diecisiete* | |
| *18* | *Uaxaclahun* | *Chunka pusaqniyuk* | *Diecioho* | |
| *19* | *Bolonlahun* | *Chunka isqunniyuk* | *Diecinueve* | |
| *20* | *Hun Kal* | *Iskay chunka* | *Veinte* | *Viginti* |

### 15.4.3 The Role of Semiotic Representation in Learning Whole Number Arithmetic

Radford (2014) discussed the influence of representation and artefacts on knowing and learning, indicating that:

> the problem of the epistemic and cognitive role of tools and signs continues to haunt us—perhaps now more than ever by virtue of the unprecedented technological dimensions of contemporary life. And so here we are still trying to make sense of how we think and learn with and through signs and artefacts. We assume that interaction with reality plays a crucial role in learning. Meanings constructed from one's experience lead to a deeper understanding of theoretical constructions. From this perspective, human perception and action and, more generally, interaction with artefacts, are of crucial importance for learning and doing mathematics. (p. 406)

Arzarello et al. (2005) pointed out:

> The embodied point of view of our 'acting is learning' stresses the importance of relating action and language to mental activity. Although such a claim is widely acknowledged from a theoretical point of view, our provocation consists in fostering its transposition in school practice. (p. 56)

In the Dominican Republic, for example, the use of concrete manipulatives, such as base-ten blocks and Cuisenaire rods, for learning whole numbers was only introduced in public schools around two decades ago (85% of schools in the country). This approach has been shown to motivate students and teachers to represent number concepts and operations and has produced higher levels of achievement when accompanied with teachers' professional development activities (González et al. 2015). In the case of Peruvian indigenous children, the use of the Yupana, an artefact used by the quipucamayos (accounting people in the Inca Empire), is more appropriate because it is grounded in their culture. There are several hypotheses with respect to how the Yupana was used by the Incas. Based on the hypothesis of Willliam Burns, in the 1980s Martha Villavicencio Ubillús (1990) developed a methodological sequence for the comprehensive learning of the decimal numeration system and the algorithms of the basic operations using the Yupana, which was first applied in the bilingual education experimental project of the Puno and is now used widely in bilingual schools in Perú. There are also technological applications for children to simulate arithmetic operations using the Yupana through computers, cellphones and tablets (Rojas-Gamarra and Stepanova 2015).

## 15.5 Chinese Arithmetic Tradition and Its Influence on the Current Curriculum[3]

Xu Hua Sun

### *15.5.1 Chinese Arithmetic Tradition*

The ancient Chinese called mathematics arithmetic (算術 an art of computation), which possibly reflects China's long history and tradition of arithmetic. It is interesting to note that the Chinese traditional approach to whole numbers was mainly cardinal rather than ordinal. Ordinal numbers are formed by adding 第 dì ('sequence') before the number. The Chinese use cardinal numbers in certain situations in which English and other Western languages use ordinals. For example, whereas a person lives on the fourth floor in a building in English, in Chinese, it is said the person lives on four floor, not the fourth floor. In Chinese, the 20th of July is expressed as 20 July.

Figure 15.2 shows the world's earliest decimal multiplication table. Made from bamboo slips, the table dates from 305 BCE, during the Warring States period in China. Place value is the most overarching principle used in Chinese numerals and calculation tools (counting rods and the Chinese abacus), which could provide advantages as a regular system (foundation) for whole arithmetic/algebra development.

[3]This study was supported by Research Committee, University of Macau, Macao, China (MYRG2015–00203-FED). The opinions expressed in the article are those of the author.

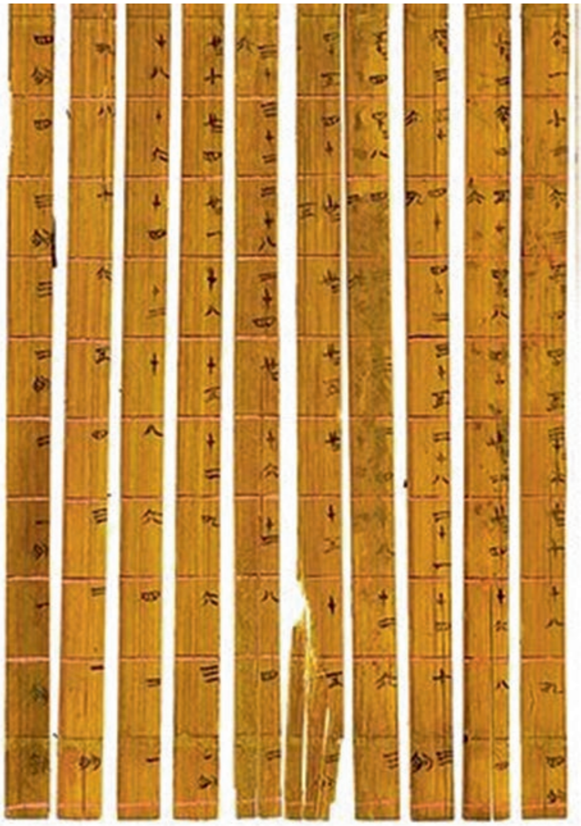

**Fig. 15.2** The world's earliest decimal multiplication table

The Chinese term for mathematics is *shuxue* (數學) or *suanxue*（算學）, which mean research on number or computation. Geometry was not included in the Chinese mathematics school curriculum until Euclid's *Elements* was introduced in the seventeenth century by Matteo Ricci (1552–1610) and Guangqi Xu (徐光啟)). The *Elements* mirrored the Chinese calculation tradition and was derived from the local world view. The ancient Chinese believed that the only way of knowing the world is through calculation, which is reflected in the *I Ching* (易經) in general. This is also expressed in the following quotation from the preface of *Sunzi Suanjing* (孫子算經):

> Calculation is the whole of heaven and earth, the origins of all life, the beginning and end of all laws, father and mother of yin-yang, the beginning of all stars, the inner and outer of three lights, the standards of five elements, the beginning of four seasons, the origins of ten thousands matters, and the general principles of six arts. (Lam and Ang 2004, p. 29)

Shushu Jiyi (數術記遺) (the first Chinese book – 220 CE – about calculation and its tools) systematically recorded 14 calculation approaches and 13 calculation tools. Among them, only bead calculation and base two (related to the computer) calculation have been handed down to today. All mathematics books are based on the decimal place number system, in which place value is the most overarching principle used in Chinese numerals and calculation tools (this volume, Chap. 3).

### 15.5.2 The Chinese Arithmetic Curriculum

Chinese mathematics education is famous for its stable basic education (Zhang 2006). The number line is rarely used in the Chinese curriculum, which may reflect some cultural trends in understanding number in China (Sun 2015).

The cardinal approach to whole numbers is embedded in the *suàn pán* (算盘, this volume, Chap. 9). The current Chinese curriculum standards mention *suàn pán* as a traditional tool for representing place value.[4] Bead calculation is not required, although some books for students are available. However, the spike abacus (which is similar to the *suàn pán*) is widely used in the current curriculum as a heritage item (Fig. 15.3).

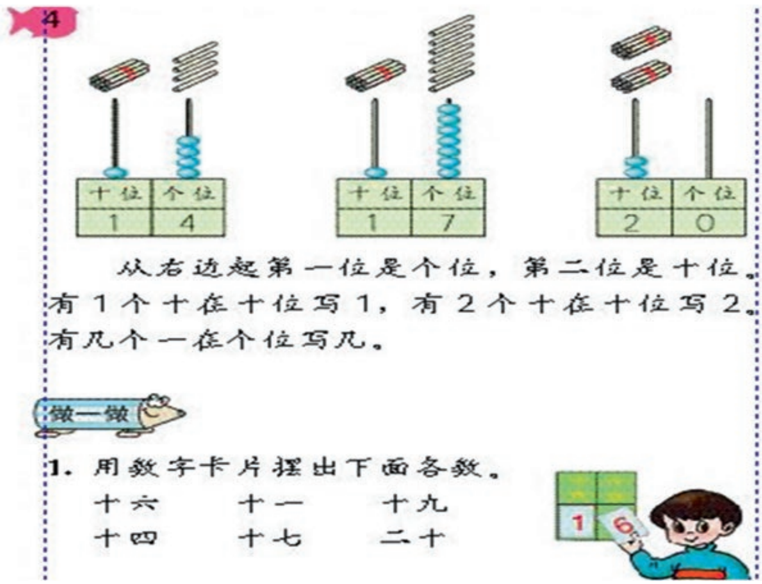


**Fig. 15.3** Spike abacus with rods in a Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005)

4 http://www.pep.com.cn/xxsx/jszx/xskcbj/201202/t20120224_1103348.htm

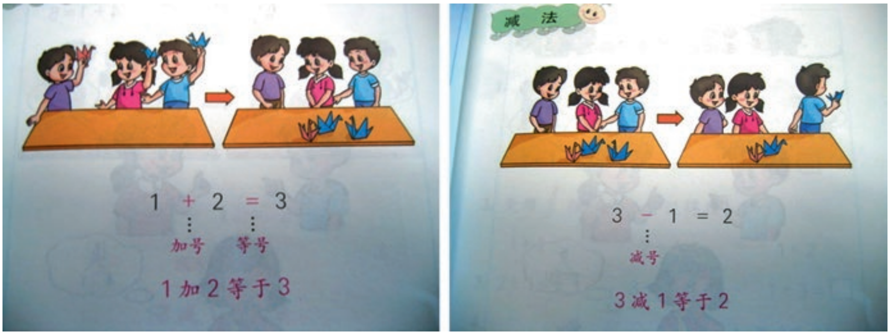


**Fig. 15.4** Addition and subtraction with concept connection in a Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, vol. 1)

It is worth noting that the Chinese curriculum aims to develop a system of numerative reasoning at an early stage. In addition to being closely related, implicit connections are made between the three concepts of addition, subtraction and number and the inverse relation between addition and subtraction ($a - b = c$ equals $a = b + c$). The authors of Chinese textbooks link the concept of subtraction to that of addition even in the first calculation lesson (Sun 2011). Figure 15.4 shows the paradigmatic example of $1 + 2 = 3$, $3 - 1 = 2$ (Mathematics Textbook Developer Group for Elementary School 2005, pp. 20–21). The aim of the problem is to help learners understand the relationship between addition and subtraction and the meaning of ‘equal’. The prototypical example from the Chinese textbook is an example of a *variation problem*. Further details of variation problems can be found in this volume (Chap. 11).

Chinese curriculum developers have connected the three core concepts of addition, subtraction and number in all of the chapters on addition and subtraction using the following explicit principles:

1. Adding one into a number obtains its adjacent number.
2. Subtracting one from the adjacent number gives the original number again.

This approach not only promotes rote counting and memorising but also reasoning. In contrast, in many Western curricula, the ideas of number, addition and subtraction are presented in separate chapters, isolated from one another.

Drawing on $1 + 2 = 3$; $3 - 1 = 2$ (Fig. 15.4), Chinese curriculum designers also aim to elicit a similar property for tens and thousands (Fig. 15.5).

*1ten + 2tens = 3tens; 3tens – 1ten = 2tens.*
*1thousand + 2thousands = 3thousands; 3thousands – 1thousand = 2thousands.*

This kind of inductive generalisation is widely used in the Chinese curriculum (Sun 2016a, b) and represents a further example of the variation problem from units to tens to thousands (Mathematics Textbook Developer Group for Elementary School 2005).

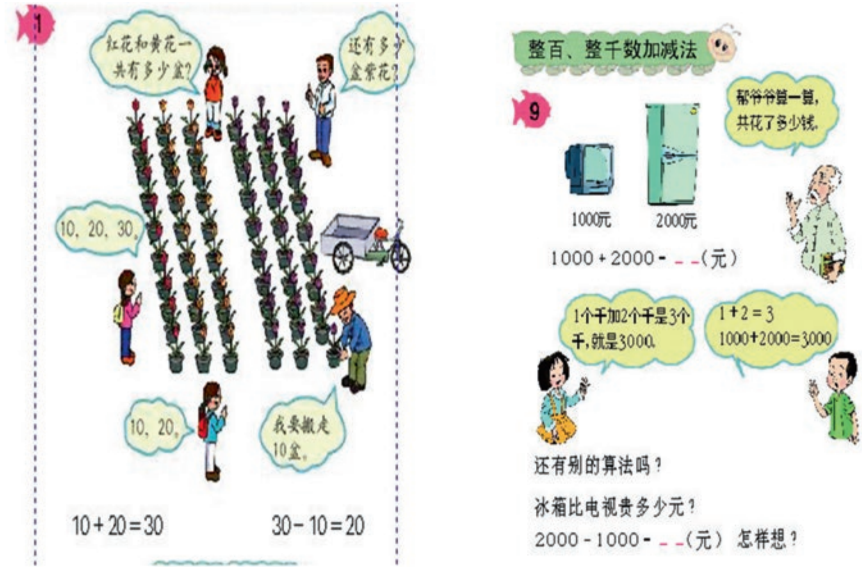


**Fig. 15.5** From units to tens (the *left*) and thousands (the *right*) in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005)

The Chinese curriculum for teaching number avoids counting as much as possible and is, therefore, different from the Western style of number teaching. The curriculum highlights the approach of composition/decomposition, which may have been inherited from ancient bead calculation. The approach is used seven times with 1–10 (decomposition of 4, 5, 6, 7, 8, 9, 10 in 7 lessons) in the Chinese curriculum (Sun 2013), which implicitly forms a core practice for learning number (Sun 2015). This approach aims to develop an implicit understanding of the associative and commutative laws, number properties and foundation of addition/subtraction operational flexibility. The partitions of ten are intensely studied, and numbers such as 12 are thought of from the beginning as one ten and two ones. The number names in Chinese are consistent with this method (for details see Chap. 3). The emphasis on partitioning and regrouping lends itself naturally to written algorithms.

The speed and accuracy of oral calculation are important requirements in the assessment of the Chinese curriculum standards. Great importance is traditionally attached to calculation because whole number is considered to be 'the first foundation of the whole subject' (Elementary Mathematics Department 2005, p. 1). This statement is consistent with the current Chinese curriculum standards, which state that:

> Oral calculation is the basis for learning mathematics. It should have very important influence on students' basic written calculation capacity. Its training can help students develop capability of observation, capability of comprehensive thinking, capability of creativity, and capability of reaction.

This heavy emphasis on calculation skills is reflected in the higher requirements for speed and accuracy in oral (mental) and paper-and-pencil calculations. For example, addition and subtraction between 1 and 20 should be completed at 8–10 operations per minute and from 20–100 at 3–4 per minute, whereas multiplication within 1–10 should be conducted as a rate of 3–4 per minute and two-digit multiplication at 1–2 per minute, all with 90% accuracy.

Chinese mathematics evolved in relation to the problems of land measurement, commercial trade, architecture, government records and taxes. The systematic word problems in the current curriculum include simple and complex problems called variation problems. These problems aim to deepen the concept connections and operations and increase flexibility (举一反三) (e.g. Bartolini Bussi et al. 2013; Sun 2011, 2016a, b). Further discussion of the variation problem can be found in Chap. 11.

### *15.5.3 Concluding Remarks*

This brief presentation has highlighted the significant differences between the Chinese and Western curricula (in the USA and Europe) inherited from tradition. Further details on this topic can be found in this volume (Chap. 3). The aim of this contribution is to start a dialogue between cultures. Bartolini Bussi, Sun and Ramploud (2013) have argued that the goal of a true dialogue between scholars with different cultural backgrounds is not to determine the best 'universal' choice but to understand the development of one's own mathematics curriculum, which is helpful for questioning the unthought characteristics of the educational context.

## 15.6 Discussion

Man Keung Siu

### *15.6.1 Introduction*

Most of the speakers on this panel have talked about the linguistic and cultural characteristics of the topic. As a mathematician, I will try to supplement the discussion by focusing more on the mathematical context. First, let me sketch a general framework of the teaching and learning of mathematics.

We begin with a world 'without mathematics'. This statement is to be taken with a grain of salt because mathematics is everywhere in our world and comes up frequently and unavoidably in our daily lives, perhaps even without our noticing it. You would know what I mean by that if you put yourself in the shoes of an infant who knows no 'formal mathematics'. Later, we come to see the world after learning some elementary mathematics and forming the ideas of mathematical objects, notions, theories and techniques. Then we come to understand more 'formal' mathematical concepts as we refine our ideas of these mathematical objects, notions, theories and techniques. This is what Bill Barton labels NUC (near-universal conventional) mathematics (see also Sect. 15.1.1.1). Finally, we try to apply the mathematics we have acquired to solve different kinds of problems. In some senses, NUC is culturally independent. However, when teaching mathematics, cultural transposition (as discussed in Sect. 15.3.2) is also a helpful concept.

Relatively few basic concepts are learnt in primary and secondary school, and these basic concepts come up time and again throughout a child's primary, secondary and even undergraduate education. Thus, I would like to extend the scope of whole number arithmetic and talk about other number systems. Here, my intervention is connected to the discussion of the Davydov approach in this volume (Chap. 19).

### *15.6.2 Numbers*

Let us begin with the notion of the number tree that often appears in school textbooks. Recalling our own learning experience, we can see that the concept of the number system is not presented in such a neatly packaged way in one stroke but is acquired in a vague and spiral fashion. Personally, I had been using the real numbers without much effort for many years in my secondary school days, but I did not know (and was not even aware that I did not know!) what real numbers were until I studied the subject as an undergraduate, and only gained a sufficient understanding when I came to teach mathematics as a young PhD student. The same process occurred in history as mankind went through the process of acquiring knowledge. Thus, it is likely that the number tree reveals itself in stages from kindergarten to primary school to secondary school to university as depicted below (Fig. 15.6).

The French mathematician Joseph-Louis Lagrange (1736–1813) referred to arithmetic and geometry as 'the wings of mathematics'. Another French mathematician, Henri Poincaré (1854–1912), commented on the construction of the real number system:

> If arithmetic had remained free from all intermixture with geometry, it would never have known anything but the whole number. It was in order to adapt itself to the requirements of geometry that it discovered something else. (Poincaré 2003, p. 135)

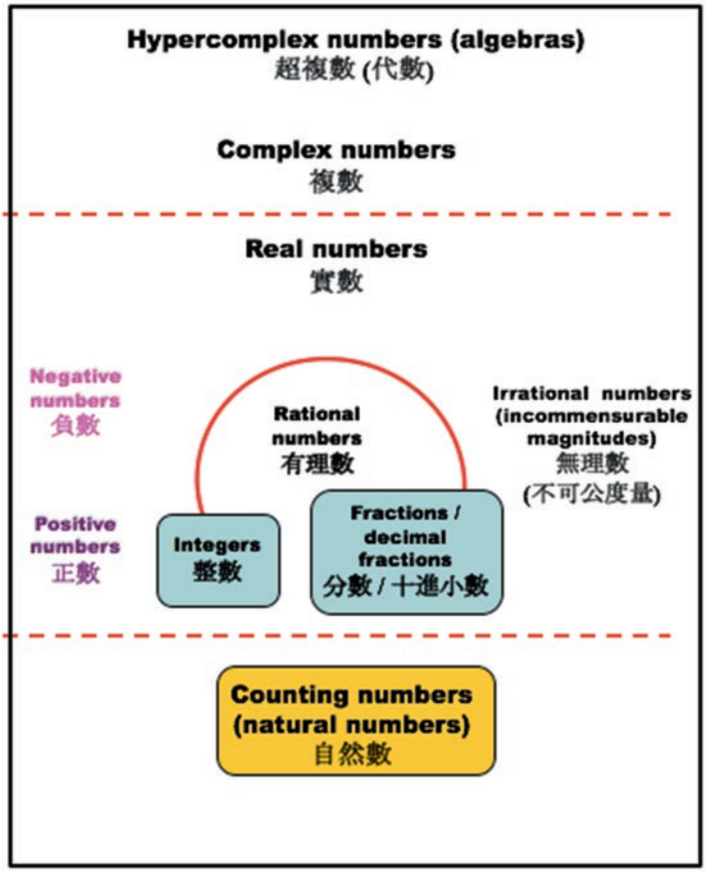


**Fig. 15.6** Different number systems

### *15.6.3 The Chinese Translation of Euclid's Elements*

Here, I will try to address the issues raised by the other panellists about arithmetic and geometry. The panellists mentioned the discrete/continuous and algebraic/geometric characteristics of the number system and asked why the number line does not feature as prominently in the Chinese classroom as in the Western classroom. What M. Bartolini Bussi (Sect. 15.3) has in mind is the discrete number line, but allow me to extend the discussion to the real number line.

When the Ming official scholar Xu Guang-qi (徐光啟1562–1633) collaborated with the Italian Jesuit Matteo Ricci (利瑪竇1552–1610) in translating Euclid's *Elements* into Chinese at the beginning of the seventeenth century, the title of the book was set as *Jihe Yuanben* (幾何原本literally 'source of quantities'). '*Jihe'* is now the modern translation in Chinese of the term 'geometry'. Some people assume that this translation arose as a transliteration of the Greek word '*geometria*'. There are ample reasons to refute this assumption. Indeed, the Chinese translation of Book V of *Elements* makes it clear that ‘*jihe’* is used as a translation of a technical term for magnitude, while its connotation for Xu Guang-qi is the term ‘how much or how many’ that appears frequently in ancient Chinese mathematical classics (Siu 2011). Thus, from the beginning, Xu Guang-qi noticed the significance of magnitude in Western mathematics and the metric nature of Euclidean geometry, so much so that he selected this term and incorporated it as part of the title of the translated text.

### *15.6.4* Jiuzhang Suanshu *(*九章算術 *Nine Chapters on the Mathematical Art)*

In the ancient Chinese tradition, geometry and algebra, or shapes and numbers, were integrated. Let us look at Problem 12 in Chap. 4 of the Chinese mathematical classic *Jiuzhang Suanshu* (九章算術Nine Chapters on the Mathematical Art) compiled between the first century BCE and first century CE (Chemla & Guo, 2004). The problem states, ‘Now given an area 55,225 [square] *bu*. Tell: what is the side of the square?’ The text offers and explains an algorithm for extracting the square root. The following picture may explain the method more clearly (Fig. 15.7).

The text goes on to explain: ‘If there is a remainder, [the number] is called unextractable, it should be defined as the side on which the square has the area of the *shi’*. It would be too much to claim that this indicates the awareness of an irrational number in this ancient epoch, but apparently it is the name of what is now called a surd. Thus, Chinese mathematicians in this ancient epoch knew about the estimate of the square root of an irrational number.

Let me cite another example in *Jiuzhang Suanshu* to show how algebra and geometry were integrated in the ancient Chinese mathematical tradition. Problem 20 in Chap. 9 states, ‘Now given a square city of unknown side, with gates opening in the middle. 20 *bu.* from the north gate there is a tree, which is visible when one goes 14 *bu.* from the south gate and then 1775 *bu.* westward. Tell: what is the length of each side?’ (Fig. 15.8). In modern-day mathematical language, we can solve this as a quadratic equation, $x^2 + 34x = 71{,}000$.

The method outlined in *Jiuzhang Suanshu* is an extension of the extraction of the square root, known as the extraction of the square root with an accompanying number (帶從開方法). Again, the following picture may explain the method more clearly (Fig. 15.9).

Even more interesting is the way the equation is set up in a geometric context (Fig. 15.10).

In offering these examples, my aim has been to let you see how algebra and geometry, shapes and numbers, come together in the ancient Chinese tradition. However, the number line does not seem to be a familiar representation in this tradition, perhaps as a result of its algorithmic nature.

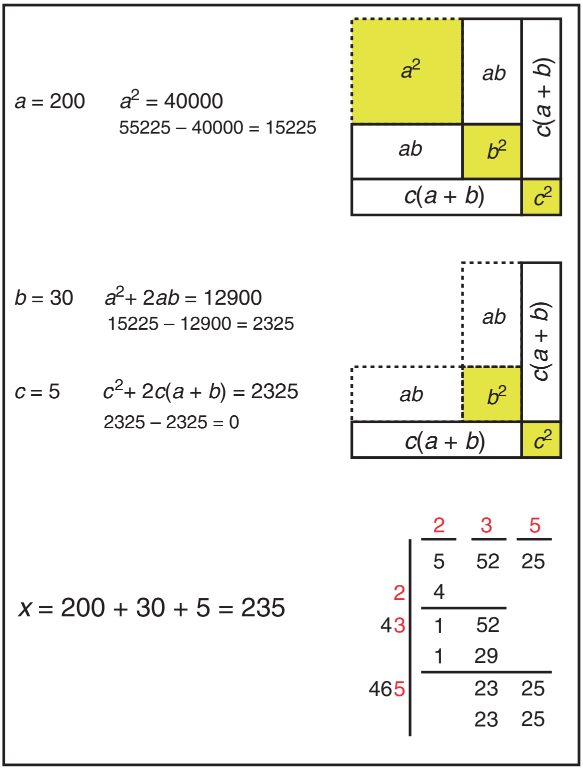


**Fig. 15.7** Extraction of the square root

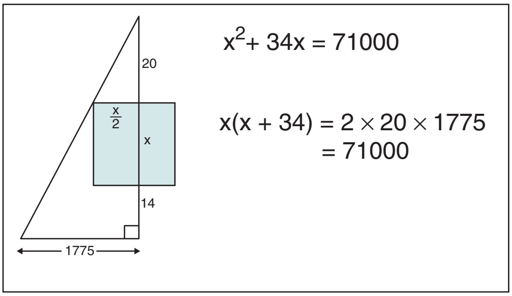


**Fig. 15.8** Problem of the square city and tree

## 15.6.5 *Tongwen Suanzhi* (同文算指)

Primary mathematics education is important and difficult. It is difficult because no academic subject is easy; in particular, no one subject is easier than another. However, there is an additional reason, in my humble opinion, why it is difficult. Most people are affected by this difficulty, not just the primary school teachers and their pupils but most people, particularly those who are parents of primary school pupils. However, most people think that they know what primary mathematics

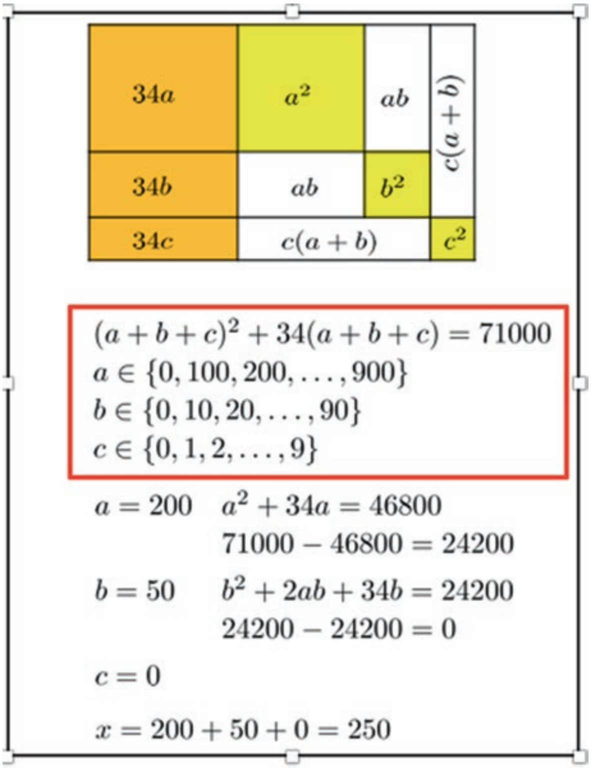


**Fig. 15.9** Extraction of the square root with accompanying number

education is and how it should be dealt with, because they were once primary school pupils. In other words, everybody considers themselves experts in this field! They do not think that they can learn from each other! As a semi-layman in this field, I am glad I have attended the ICMI Study 23. I have come to learn and, as the ancient Chinese classical text *Xueji* (學記) states, 'Teaching and learning help each other […] Teaching is the half of learning'. I am particularly glad to have joined the WG1 (Chap. 5) with culture as an emphasised component, and of course tradition, the subject of this panel.

My presentation in WG1 (Siu 2015a, b) concerns the book *Tongwen Suanzhi* (同文算指), compiled under the collaboration of the Italian Jesuit missionary Matteo Ricci and the Ming scholar official Li Zhi-zao (李之藻1565–1630). This book first transmitted into China the art of *bisuan* (筆算 written calculation) (Siu 2015a, b). The term *tongwen*, meaning literally 'common cultures', indicates a deep appreciation of the common cultural roots of mathematics despite the different mathematical traditions.

In the preface of *Tongwen Suanzhi*, Xu Guang-qi wrote that:

> The origin of numbers, could it not be at the beginning of human history? Starting with one, ending with ten, the ten fingers symbolise them and are bent to calculate them, [numbers] are of unsurpassed utility! Across the five directions and myriad countries, changes in cus-

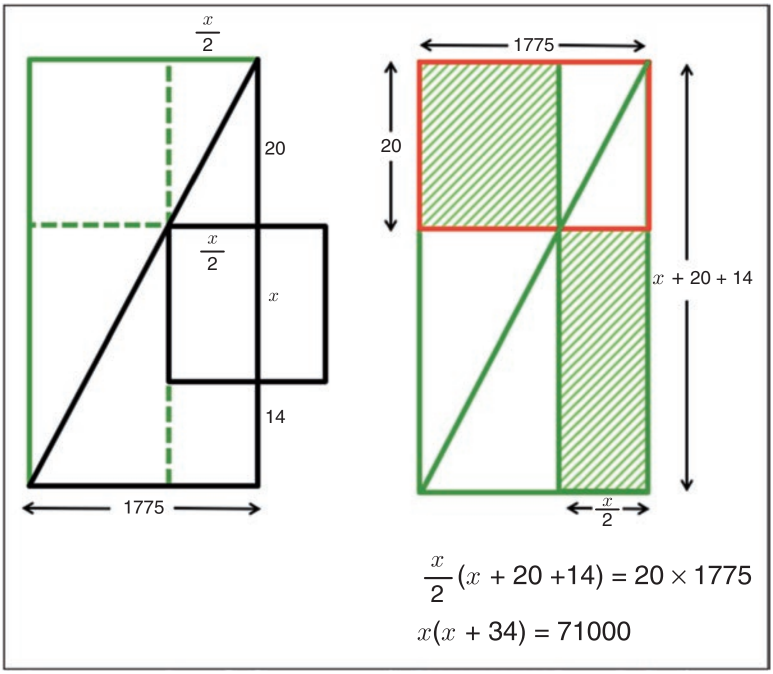


**Fig. 15.10** Setting up a quadratic equation in a geometric context

> toms are multitudinous. When it comes to calculating numbers, there are none that are not the same; that all possess ten fingers, there are none that are not the same.

In the preface to a reprinting of Matteo Ricci's *Tianzhu Shiyi* (天主實義*The True Meaning of the Lord of Heaven*), Li Zhi-zao wrote: 'Across the seas of the East and the West the mind and reasoning are the same [*tong*]. The difference lies only in the language and the writing'.

## 15.7 Conclusion

The quotation from Li Zhi-zao that concludes the previous section deeply illustrates the rationale and problems of teaching/learning whole numbers in primary school. In fact, language and writing incorporate different cultural meanings according to which numbers are processed and conceived in different cultures. This is further reflected in the ways in which representations are used and interpreted in different cultural environments.

Many basic concepts of mathematics and whole numbers show the dramatic duality between mathematics as a universal language and its specific features of enculturation. The contributions from this panel widely illustrate this point.

This situation poses a great didactic challenge, which has the flavour of a paradox. On the one hand, the abstract universal concepts of mathematics are the goals of teaching and learning, but on the other hand, this goal can only be achieved by dealing with the concrete ways in which the concepts have been shaped by specific cultural tools, from oral and written words, to a variety of forms of representation (drawings, bodily expressions, etc.). This challenge constitutes the fascinating main feature of our work as mathematics educators and makes conducting research on this problem worthwhile.

Acquiring new knowledge on this issue is crucial because of the great social and economic changes the world is now facing. In recent years, economic globalisation, universal technological development and the related needs for manpower skills have provided strong historical motivations for introducing unified standards for mathematics in school. However, only a multicultural perspective allows us to consider the existence of different epistemological and cultural positions concerning mathematics and its cultural relevance and to realise the distance of the proposed curricular reforms from the mathematical cultures of different countries. It is important to base any teaching programme on its relationships with the cultures of the students and the personal contributions that they bring to the classroom. This will help avoid alienating the students from their cultural environment and allow them to engage in learning in a productive way.

The contributions of this panel pinpoint the crucial issues that need to be addressed to avoid the dangers of both the cultural refusal of innovation and of cultural alienation and of losing the cultural richness that exists in the different regions of the world.

## References

Adler, J. (1997). A participatory-inquiry approach and the mediation of mathematical knowledge in a multilingual classroom. *Educational Studies in Mathematics, 33*(3), 235–258.

Arzarello, F., Robutti, O., & Bazzini, L. (2005). Acting is learning: Focus on the construction of mathematical concepts. *Cambridge Journal of Education, 35*(1), 55–67.

Ascher, M. (1991). *Ethnomathematics: A multicultural view of mathematical ideas*. Pacific Grove: Brooks/Cole Publishing.

Bartolini Bussi, M. G., Sun, X., & Ramploud, A. (2013). A dialogue between cultures about task design for primary school. In C. Margolinas (Ed.), *Proceedings of the International Commission on Mathematical Instruction Study 22: Task design in mathematics education* (pp. 549–558). Oxford. https://hal.archives-ouvertes.fr/hal-00834054. Accessed 20 Jan 2016.

Barton, B. (2008). *The language of mathematics: Telling mathematical tales*. New York: Springer.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati Phakeng, M.., Valero, P., & Villavicencio Ubillús, M. (Eds.). (2015). *Mathematics education and language diversity: The 21st ICMI study*. New York: Springer.

Bazzanella, C. (2011). *Numeri per parlare*. Bari: Laterza.

Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht: Kluwer Academic Publishers.

Boero, P., & Guala, E. (2008). Development of mathematical knowledge and beliefs of teachers: The role of cultural analysis of the content to be taught. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education* (vol. 1, pp. 223–244). Rotterdam: Sense Publishers.

Brissiaud, R., Clerc, P., & Ouzoulias, A. (2002). *J'apprends les maths – CP avec Tchou*. Paris: Retz.

Butterworth, B., Reeve, R., & Reynolds, F. (2011). Using mental representations of space when words are unavailable: Studies of enumeration and arithmetic in indigenous Australia. *Journal of Cross-Cultural Psychology, 42*(4), 630–638.

Chemla, K., & Guo, S.C.(郭書春). (2004). *Les Neuf Chapitres: Le Classique Mathématique de la chine Ancienne et Ses Commentaires*. Paris: Dunod.

D'Ambrosio, U. (2001). What is ethnomathematics, and how can it help children in schools? *Teaching Children Mathematics,7*(6), 308–310.

Dehaene, S., & Brannon, E. M. (Eds.). (2011). *Space, time, number in the brain: Searching for the foundations of mathematical thought*. London: Elsevier.

Elementary Mathematics Department. (2005). Mathematics teacher manual: Grade 1 (Vol. 1). Beijing: People Education Press. [in Chinese].

Gallistel, C. R., & Gelman, R. (2000). Non-verbal numerical cognition: From reals to integers. *Trends in Cognitive Sciences, 4*(2), 59–65.

González, S., Valverde, G., Roncagliolo, R., & Luna, E. (2015). *Reporte final del Programa de Escuelas Efectivas PUCMM – MINERD- USAID*. Santiago de los Caballeros: Pontificia Universidad Católica Madre y Maestra.

Gorgorió, N., & Planas, N. (2001). Teaching mathematics in multilingual classrooms. *Educational Studies in Mathematics, 47*(1), 7–33.

Halliday, M. A. K., & Hasan, R. (1985). *Language, context, and text: Aspects of language in a social-semiotic perspective*. Oxford: Oxford University Press.

Ifrah, G. (1985). *From one to zero. A universal history of numbers* (L. Bair, Trans.). New York: Viking Penguin Inc. (Original work published 1981.)

Ifrah, G. (2001). *The universal history of computing: From the abacus to the quantum computer*. New York: Wiley.

INVALSI. (2012). *Quadri di Riferimento. Primo ciclo di istruzione. Prova di Matematica*. http://www.invalsi.it/snv2012/documenti/QDR/QdR_Mat_I_ciclo.pdf. Accessed 20 Jan 2016.

Joseph, G.G. (2011). *The crest of the peacock: Non-European roots of mathematics* (3rd ed.). Princeton: Princeton University Press.

Jullien, F. (1996). *Si parler va sans dire: Du logos et d'autres ressources*. Paris: Seuil.

Lam, L., & Ang, T. (2004). *Fleeting footsteps*. Singapore: World Scientific Publishing.

Mathematics textbook developer group for elementary school. (2005). *Mathematics*. Beijing: People's Education Press. [In Chinese].

Menninger, K. (1969). *Number words and number symbols: A cultural history of numbers*. Cambridge, MA: The MIT Press. (Translated from the German edition of 1958).

MIUR. (1985). *Programmi della Scuola Elementare, D.P.R. 12 febbraio 1985, n 104*. http://www.edscuola.it/archivio/norme/programmi/elementare.html#MATEMATICA. Accessed 20 Jan 2016.

MIUR. (2012). *Indicazioni nazionali per il curricolo per la scuola dell'infanzia e il primo ciclo dell'istruzione*.http://www.indicazioninazionali.it/documenti_Indicazioni_nazionali/indicazioni_nazionali_infanzia_primo_ciclo.pdf. Accessed 20 Jan 2016.

MIUR-UMI. (2001). *Matematica 2001*. http://www.umi-ciim.it/wp-content/uploads/2013/10/mat2001.zip. Accessed 20 Jan 2016.

Miura, I. T., Okamoto, Y., Kim, C. C., Chang, C.-M., Steere, M., & Fayol, M. (1994). Comparisons of children's cognitive representation of number: China, France, Japan, Korea, Sweden, and the United States. *International Journal of Behavioral Development, 17*(4), 401–411.

Monti, M. M., Parsons, L. M., & Osherson, D. N. (2012). *Thought beyond language: Neural dissociation of algebra and natural language*. London: Psychological Science.

Poincaré, H. (2003). *Science and method*. (F. Maitland, Trans.). New York: Dover. (Original work published 1908.)

Radford, L. (2014). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*(3), 405–422.

Rojas-Gamarra, M., & Stepanova, M. (2015). Sistema de numeración Inka en la Yupana y el Khipu. *Revista Latinoamericana de Etnomatemática, 8*(3), 46–68.

Saxe, G. (2014). *Cultural development of mathematical ideas: Papua New Guinea studies*. Cambridge: Cambridge University Press.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly: Overcoming Learning Difficulties, 23*(2), 139–159.

Selin, H., & D'Ambrosio, U. (2000). *Mathematics across cultures: The history of non-western mathematics*. Dordrecht: Springer.

Sinclair, N., & Metzuyanim, E. (2014). Learning number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*(1), 81–99.

Siu, M. K.(蕭文強). (2011). 1607, a year of (some) significance: Translation of the first European text in mathematics — Elements — Into Chinese. In Barbin, E., Kronfellner, M., & Tzanakis, C. (Eds.). *History and epistemology in mathematics education* (pp. 573–589). Vienna: Verlag Holzhausen.

Siu, M. K. (蕭文強). (2015a). Tongwen Suanzhi (同文算指) and transmission of bisuan (筆算 written calculation) in China: From an HPM (History and Pedagogy of Mathematics) viewpoint. *Journal for History of Mathematics, 28*(6), 311–320.

Sun, X. (2011). Variation problems and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics, 76*(1), 65–85.

Sun, X. (2013). The structures, goals and pedagogies of "variation problems" in the topic of addition and subtraction of 0–9 in Chinese textbooks and reference books. *Proceedings of the Eighth Congress of European Research in Mathematics Education (CERME 8, WG 16)*, 2208–2218.

Sun, X. (2016a).《螺旋變式—中國內地數學課程與教學之邏輯》 新加坡八方文化創作室. *Spiral variation: A hidden theory to interpret the logic to design Chinese mathematics curriculum and instruction in mainland China*. Singapore: World Scientific Publishing.

Sun, X. (2016b). Uncovering Chinese Pedagogy: Spiral variation –the unspoken principle for algebra thinking to develop curriculum and instruction of "TWO BASICS". *Invited paper in 13th International Congress on Mathematical Education (ICME-13)*.

Usiskin, Z. (1992). Thoughts of an ICME regular. *For the Learning of Mathematics, 12*(3), 19–20.

Varley, R. A., Klessinger, N. J. C., Romanowsky, C. A. J., & Siegal, M. (2002). Agrammatic but numerate. *Nature Reviews. Neuroscience, 3*, 462–471.

Vergnaud, G. (1991). Langage et pensée dans l'apprentissage des mathématiques. *Revue Française de Pédagogie, 96*, 79–86.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 5–28). Hove: Psychology Press.

Villa, B. (2006). *Numeri cinesi & oltre*. http://gold.indire.it/nuovo/gen/cerca-s.php?parola=Numeri+cinesi+%26+oltre&submit=Cerca

Villavicencio Ubillús, M. (1990). *La matemática en la educación bilingüe: el caso de Puno*. Lima: Programa de Educación Bilingüe-Puno.

Zaslavsky, C. (1973). *Africa counts: Number and pattern in African culture*. Chicago: Lawrence Hill Books.

Zhang, D. (2006). *The "two basics": Mathematics teaching in mainland China*. [in Chinese]. Shanghai: Shanghai Education Press.

## Cited papers from Sun, X., Kaur, B., & Novotna, J. (Eds.). (2015). Conference proceedings of the ICMI study 23: Primary mathematics study on whole numbers. Retrieved February 10, 2016, from www.umac.mo/fed/ICMI23/doc/Proceedings_ICMI_STUDY_23_final.pdf

Bartolini Bussi, M. G. (2015). The number line: A ‘Western’ teaching aid (pp. 298–306).
Mellone, M., & Ramploud, A. (2015). Additive structure: An educational experience of cultural transposition (pp. 567–574).
Siu, M. K. (2015b). Pedagogical lessons from *Tongwen Suanzhi* (同文算指): Transmission of *bisuan* (筆算 written calculation) in China (pp. 132–9).
Soury-Lavergne, S., & Maschietto, M. (2015). Number system and computation with a duo of artifacts: The pascaline and the e-pascaline (pp. 371–378).
Sun, X. (2015). Chinese core tradition to whole number arithmetic (pp. 140–148).