Chapter 13
Connecting Whole Number Arithmetic Foundations to Other Parts of Mathematics: Structure and Structuring Activity

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13.1 Introduction

The focus of this chapter is on the use of structure and structuring activities as key means through which whole number arithmetic (WNA) can be connected to other areas of mathematics. As with the other chapters in this volume, several of the studies that we use to exemplify attention to structure and structuring in this chapter are drawn from contributions to ICMI Study 23: Primary Mathematics Study on Whole Numbers that was the key precursor to this volume. We begin this chapter with an

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introductory preface of contributions to the working group focused on the connections between WNA and other parts of mathematics, and use this overview to explain why attention to structure and structuring came to figure as a key overarching theme for looking across these connections. In this preface, we also note the ways in which our focus in this chapter connects with the focus of broader discussions in the other working groups and in the panel presentations at ICMI Study 23, which are also taken up across a range of the chapters in this volume.

13.1.1 What Was Presented at the Conference: Overview

Several of the contributions to the theme focused on connections between WNA and other areas were centrally concerned with the ways in which structuring activity and structure figured in supporting these connections. These studies attend to structuring and structure at the levels of learning, teaching and teacher education, and curricula and were drawn from research studies undertaken across the Americas, Europe, Africa, Asia and Australia, including input from ICMI CANP observer Estela Vallejo, from Peru.

At the conference, the presentations were organised into five sessions: two on whole number arithmetic and early algebra; two on whole number arithmetic and multiplicative reasoning, at the learner level and at the teacher level; and one session on whole number arithmetic competence as it relates to language ability and teacher development.

13.1.1.1 Whole Number Arithmetic and Early Algebra

Two papers concerned ways to visually depict and organise relationships. Mellone and Ramploud (2015) analysed the 'pictorial equation', which is used in Russian and Chinese primary schools to teach additive relationships. They discussed the cultural transposition that is involved in using this tool with Italian students. The authors found increased visibility of structural and algebraic approaches to additive relationships. Xin (2015) reported on substantial improvements in the mathematics performance of U.S. children with learning difficulties when using an approach that models the grammar of additive and multiplicative situations. The approach draws attention to the algebraic structure of such situations.

Two papers concerned tasks about patterns. Eraky and Guberman (2015) found that 5th and 6th grade Israeli learners working with numerical patterns were better able to generalise than those working with visual-pictorial patterns. These authors emphasised the need to push for more complex structural generalisations and multiple stages of generalisation. Ferrara and Ng (2015) reported on 3rd grade Italian students working with a figural pattern task. Working from a framework of assemblage, in which learning is an output of agency distributed between body and material, they explore arithmetic awareness within the development of algebraic thinking.

13.1.1.2 Whole Number Arithmetic and Multiplicative Reasoning

Looking at the learning level, Venenciano et al. (2015) reported on a study in Hawaii in which place value understandings were developed through ideas of measurement. They found that learners' initial attention to comparing quantities grew into an awareness of the need for intermediate regrouped units. Larsson and Pettersson (2015) investigated how Swedish learners solved mixed sets of additive and multiplicative covariation problems. They found that stronger performance was associated with inferring distance relationships from information about speed relationships, whereas weaker performance was associated with reliance on single procedures and attention to superficial contextual differences. Chen et al. (2015) investigated Chinese learners' performance on learning and assessment tasks about multiplication and division by rational numbers. The learning tasks were of three different types: computation, problem solving or problem posing. The findings point to performance on problem-posing tasks as important.

Moving to the teacher level, Beckmann et al. (2015) discussed their use of a quantitative definition of multiplication to help future middle grades teachers in the USA organise their thinking around topics including ratio and proportional relationships. Dole et al. (2015) reported on a curriculum analysis in Australia and the finding that teachers are often unaware of how many topics from the early grades through grade 9 offer opportunities for proportional reasoning. Venkat (2015) discussed research on a teacher education project in South Africa showing how representational approaches used with whole number scaling up can simultaneously support teachers' mathematical learning and their mathematics teaching.

13.1.1.3 Whole Number Arithmetic Competence: Language/Teacher Development

Zhang et al. (2015) presented the results of an investigation into how Chinese kindergarteners' language ability is related to their mathematical skills. They found that language ability was more strongly associated with informal mathematical skills (e.g. counting) than with more formal ones (e.g. addition and subtraction).

Baldin et al. (2015) shared data on an in-service teacher development model based on pedagogical content knowledge frameworks. The model was used in Brazil to strengthen teachers' knowledge and practice with whole number arithmetic.

13.1.2 The Discussion in the Working Group

The presentations and discussions in the working group sessions were lively, and the group developed a real sense of community and friendship while discussing the presentations and thinking together about next steps. A central focus of the discussions was on environments in which key 'glueing' ideas of mathematics related to WNA are featured. Across the papers and discussion sessions, these ideas encompassed multiplicative thinking and proportionality, measurement, generalising and mathematical models that attended to structure and generality. Central to some contributions, and implied or assumed in others, was the need to further develop teacher education in ways that promote understanding of connections and relations within and beyond WNA. The presentations and discussions emphasised the importance of creating representations such as actions, gestures, mental models and diagrams to construct mathematical relations.

In synthesising the presentations and discussions, the working group identified and developed crosscutting themes. A concept map was produced that organised the presentations into seven themes: justification; additive versus multiplicative thinking; structural relations; language; models, modelling and representations; general/specific; and teacher education. From these themes, the overarching theme of structure and structuring emerged as one that could organise and tie the presentations and discussions together into a coherent whole.

13.1.3 Connections to Other Working Groups, Panels and Plenary Presentations

The ubiquity of connections between mathematical topics and the overarching nature of mathematical processes makes it both difficult and unhelpful, in many ways, to compartmentalise discussions about mathematical thinking, learning and teaching into discrete categories. Thus, there are numerous connections between the ideas discussed in working group 5 and the other working groups, panels and plenary sessions at the conference. We note here a few that stand out as particularly pertinent to our focus.

On the surface, a tension is possible to discern relating to the timelines of introduction of attention to early algebra. In Ma's plenary presentation (this volume, Chap. 18), she expressed concern over pushing algebra down into the early grades of elementary school. In contrast, a number of presentations in the working group focused on attending to algebraic structure in the context of early WNA. Ma pointed to the theoretical core of school mathematics as (1) the basic concept of a unit and (2) two basic quantitative relations (adding and multiplying). However, as some of the papers that we discuss in the body of this chapter make clear, several of the 'early algebra' approaches discussed in the presentations seem also to be in the service of developing the concept of a unit and an understanding of the structure of arithmetic operations. Thus, the extent of the tension perhaps relates more to issues of what comes to be named as early algebra, rather than of its substance.

Taking a pattern-oriented route to focus on early algebra, Mulligan, in the panel on special needs (this volume, Chap. 16), reported on a long-term research project investigating the role of pattern and structure in mathematics learning. Findings included (1) an awareness of mathematical pattern and structure can be taught and (2) early school mathematics achievement is associated with children's level of this awareness. Mulligan concluded that mathematics curricula should promote structural thinking.

As with any issue concerning teaching and learning, connections between whole numbers and other topics will necessarily relate to teacher education. It is therefore no surprise that issues of teacher education arose repeatedly throughout the presentations and discussions of working group 5. The panel on teacher education (this volume, Chap. 17) connected to working group 5 in several ways. As one example, Kaur discussed the model method used in Singapore. This method relates directly to (1) the 'pictorial equation' approach discussed by Mellone and Ramploud (2015), (2) the measurement approach taken by Venenciano et al. (2015) and (3) one of the approaches to proportional relationships taken by Beckmann et al. (2015). Bass' plenary presentation (this volume, Chap. 19) also connected to this theme in highlighting the role of the number line and the idea that numbers can be viewed as an outcome of measurement activities.

We highlight one additional paper related to working group 5 because it indicates just how deep the connections are across mathematical ideas and how much everyone, even professional mathematicians, stand to learn about them. Cooper (2015), in his paper for working group 1, discussed how a mathematician – in his role as the instructor of a professional development course for teachers – gained a deeper understanding of division with remainder and its connection to topics beyond whole number arithmetic, including fractions and tests for divisibility.

13.1.4 The Structure of This Chapter

Connections between whole number arithmetic (WNA) and other parts of mathematics as a title suggest, at first glance, a focus on the need to, and ways in which to, build bridges between other content areas and WNA. This focus remains important in the face of ongoing evidence of frequently localised and highly fragmented and inflexible approaches to problem solving. This fragmentation has been described as an outcome of encounters with content in highly compartmentalised ways (Schoenfeld 1988). Given that WNA is, in mathematics curricula in most parts of the world that we have seen, the area in which induction into mathematics occurs, it is particularly important that this induction occurs in ways that allow for expansion into, and connections between, mathematical topics. In this chapter, we attend to this concern with a more general proposal about the ways in which the traditional contents of WNA instruction might be approached: attending, on the one hand, to mathematical structures that begin in the context of WNA but transcend these boundaries and, on the other hand, to providing openings to teachers and to students to engage in structuring activity as a key mathematical practice that again can begin in the context of, and also transcend, WNA. In this chapter, we begin by using literature to describe what we mean by attending to structure and structuring activity. In the body of the chapter, we present and discuss examples of ways in which attention to structure and engagement with structuring can support moves beyond WNA. These examples work across student mathematical working in classrooms, the teaching of mathematics and teacher education and curricula.

13.2 Mathematical 'Structure' and 'Structuring'

While there is broad agreement of the importance of 'structure' within mathematics, what structure refers to is frequently less clear. Sfard (1991) has contrasted 'structural' conceptions with 'operational' conceptions, with the former described in terms of processes that come to be solidified into encapsulating objects that have a 'static structure' (p. 20). Mason et al. (2009) describe mathematical structure in terms of:

> the identification of general properties which are instantiated in particular situations as relationships between elements. (p. 10)

Mathematical properties are important in this formulation in that, for these authors, recognising relationships between elements is not, in itself, a marker of structural thinking. Rather, it is when these relationships are recognised as 'instantiations of properties' that the onset of structural thinking is marked. Thus, while building attunement to pattern- and relation-recognition is critical within 'structuring' activity and viewed as a valuable precursor to attending to structure, instruction needs to provide openings for these relationships to be associated with fundamental properties. In the context of WNA, a range of fundamental properties are introduced. These include, for example, ideas related to equivalence, associativity and compensation and the nature of and distinction between additive and multiplicative structures. All of these properties are usually initially exemplified in natural number contexts, but can be extended beyond the boundaries of WNA. Rational number provides a key ground for studies focused on these extensions, with the expanded terrain providing grounds for looking at the ways in which the impacts of operational properties shift – e.g. multiplication no longer necessarily 'making bigger' and, conversely, division no longer necessarily 'making smaller'.

Given the centrality of linking specific relationships to more general properties, approaches focused on structure and structuring activity are commonly linked to algebraic thinking. Algebra topics and algebraic thinking are predictably, therefore, key foci for looking at connections between WNA and other areas.

Two broad positions on mathematical structure can be inferred within the literature base. These can be distinguished on the basis of structures that are presented 'ready-made' to support problem-solving activity or structures that emerge through structuring activity. In mathematics education, particular approaches have tended to align more within one, or other, of these camps. Bourbakian approaches, for example, have worked from the vantage point of emphasis on structure (Corry 1992), while Realistic Mathematics Education placed more emphasis on structuring activity as the means through which mathematical structures are reinvented (van den Heuvel-Panhuizen and Drijvers 2014). Working from the basis of definitions linked to properties therefore provides a key hallmark of working with structure. Working in more emergent ways for 'taken as shared' reinventions of structure provides, in contrast, a key hallmark of structuring activity. In either case, Mason et al. (2009) note that it is awareness of general properties, rather than awareness of internal relations within instances, that indicates, for them, possible presences and potential for structures to figure as thinking tools.

While both of these positions have advocates, there are also critiques that are important to be mindful of within operationalisation of the positions and their ensuing claims. A key issue that Freudenthal (1973) pointed to as problematic about traditional mathematics teaching was what he described as the 'anti-didactical' use of models in a 'top-down instructional design strategy in which static models are derived from crystallized expert mathematical knowledge' (Gravemeijer and Stephan 2011, p. 146). Artigue (2011) has more recently echoed this critique, noting that 'pupils do not know which needs are met by the mathematical topics introduced' and, concomitantly, that they therefore have 'little autonomy in their mathematical work' (p. 21). Presenting structures in a 'ready-made' format can be construed as incorporating some elements of this orientation. Venkat et al. (2014) have noted that within early primary years' teacher education in South Africa, attention to definitions of properties – key markers of structural relations – may not be sufficient without supporting attention to the example spaces in which the properties can be strategically applied. They point to data drawn from a small study in which early primary teacher educators, when asked to propose a set of examples for working on commutativity, included examples with the first number larger than the second number alongside examples where the second number was larger. One excerpt they point to includes the following explanation from a teacher educator with the offer of 9 + 3:

> 'We can say that this is the same as 3 + 9 using commutativity'.

In this response, there is clearly awareness of the commutativity property and what it means to apply the commutativity property to an addition example, but perhaps more limited attention to *when* it might be useful to apply this property. There was also no explanation of the distinctions between the totality of the space of all additive examples to which the former general definition applies as a structural property and the latter subspace in which useful application holds. Venkat et al. (2014) described these shortcomings in terms of a 'definitional' rather than a 'strategic' orientation to structural properties. These findings point to limitations of dealing with definitions as the sole source of structure and point to additional features that need to be part of the discussion if flexible and strategic working with mathematical properties is sought.

Conversely, the focus on individual reinvention activity has also been critiqued from several perspectives: these include arguments that the approach has always been more a function of political ideology than educational effectiveness and, therefore, dependent for its suitability on a broader political climate of autonomy (Tabulawa 2003), to reviews of key areas of education research arguing that there is more support for the efficacy of:

> direct, strong instructional guidance rather than constructivist-based minimal guidance during the instruction of novice to intermediate learners. […] Not only is unguided instruction normally less effective; there is also evidence that it may have negative results when students acquire misconceptions or incomplete or disorganized knowledge. (Kirschner et al. 2006, pp. 83–84)

In relation to structuring activities specifically, Schifter (2011) – while not arguing for the direct instruction position – does indicate that attention to structure is developed through experience with tasks that promote attention to structure and emphasises that this attention can begin in the context of WNA. She provides useful contrastive examples of two ways of dealing with the following task:

> Oscar had 90 stickers and decided to share some with his friends. He gave 40 stickers away. Becky also had 90 stickers. She gave away 35 stickers. Who has more stickers now? (p. 207)

In one class, no further discussion follows, and the children proceed to calculate Oscar and Becky's remaining stickers and then compare the two answers in order to answer the question. In the second class, after checking that the children are aware that the subtraction sentences 90 – 40 and 90 – 35 can be used to represent the two scenarios, the teacher explicitly tells her class that she wants them to consider who would be left with more stickers *without* calculating. She proceeds to orchestrate a discussion which is focused on comparing the effects of 'taking away more' and 'taking away less' from a quantity. Considering and articulating the properties of subtraction, rather than the operation of subtracting, are therefore at the fore here. Other writers have echoed this broadly 'cultural' position in that skill with structuring is seen as dependent upon, and an outcome of, participating in structuring activities (e.g. Wright et al. 2006). In parallel, though, there are also studies that have pointed to the need for teachers, at least, to have prior cultural familiarity with WNA structures – such as WNA representations based on the decimal number system structure – as a precursory support for being able to work with these structures constructively in mathematics classrooms if the mathematical ideality of these artefacts, inlaid into their material structure, is to be realised (Bakhurst 1991).

In this chapter, our focus is on studies that exemplify these two positions. The studies themselves encompass links between WNA and a range of other mathematical topic areas, including rational number and measurement, but our focus in this chapter is specifically on the position they take in relation to working with structures and structuring activity. The approaches used within these two positions, and differences specifically in the ways in which models are viewed and produced, are explored. We do this in order to explore overlaps and contrasts between the two positions in terms of the ways in which connections between WNA and algebraic thinking can be achieved via an emphasis on structures and structuring. We also attend to whether there is evidence that one or other of these approaches may be more appropriate when focusing on primary mathematics teacher education.

13.3 Investigating and Supporting Structuring Activity

Pattern and sequence tasks are commonly promoted in mathematics curricula across the world as contexts in which attention to structure and generalisation can be encouraged (Driscoll 1999). Hewitt (1992) has pointed to openings for linking spatially based orientations to pattern (rather than numerically based orientations) to openings for generalising. He also notes the latter ‘pattern-spotting’ orientation through translating spatial arrangements into tabular numerical summaries as limiting openings for attending to the various ways in which a particular spatial structure can be constructed and considering constructions that have structural similarities across instances. Spatial approaches to visual-pictorial patterns are seen in Eraky and Guberman’s (2015) inclusion of sequences presented in this format. While these authors note that primary students in Israel largely found it harder to make general statements in visual-pictorial pattern formats in comparison with numerical formats, their claim of the need to ‘go deeper into the rules of building a sequence’ (p. 548) at least partially reflects Hewitt’s argument that spatial pattern formats and attention to pattern construction and verbalisation of this construction provide better routes into attending to structure than the more common numerically oriented routes.

Of broader interest for us is the linking here between numerical, algebraic and spatial approaches to working with structuring. Ferrara and Ng (2015) provide a more distributed notion of the development of algebraic thinking in the context of visual pattern tasks, focusing on the ways in which two Grade 3 children’s identifications of mathematical structures develop in the assemblage of human and material resources. In looking at the emergence of structuring, rather than at children’s competence with identifying a correct overall generalised functional representation, these authors emphasise the ways in which specific spatial arrangements give rise to increasing emphasis on the numerosity of partial elements, or more holistic views of these arrangements and numerical relations between elements in children’s talk. Ferrara and Ng note also the ways in which the assemblage of material artefacts including the task and task conditions, and the children’s gestures and talk, all feed into supporting moves into WNA and functional thinking.

Warren and Cooper (2009) introduced two complementary representations, the balance scale and the number line, to model equivalence in a longitudinal study through Grades 2–6. From their study, they suggest a theory for a learning/teaching trajectory which supports generalised understanding of equivalence. Their conjectures propose that the act of translation between effective representations is one of the key points for constructing structured understanding of WNA that can be generalised beyond WNA. They offer the construction of superstructures, where multiple models are nested and integrated, as elucidation and conclude that 'the role of superstructures cannot be underestimated' (p. 92).

A small cluster of work involved studies working at the interface of structures and structuring. In these studies, tasks incorporating situations underscored by differences in their structure or tasks in which the emergent introduction of structure and relation produces substantial efficiencies in calculation are used to encourage children to focus on structural aspects. In the Measure Up programme in Hawaii, teaching begins with general ideas in a measurement context without using numbers, based on ideas from Davydov (Venenciano et al. 2015). Venenciano and her colleagues reported on first grade students' learning and conceptual understanding of place value by measurement of continuous quantities with different bases rather than focusing on special cases of the decimal structure and discrete numbers. They demonstrate students' ability to focus on the constant ratio between the units of adjacent places and emphasise that the unit of measure is a 'critical tool for both the conceptual and the physical development of partial units (e.g. thirds in base three or tenths in base ten)' (p. 581). By approaching place value in the measurement context through this approach, these authors noted that children were provided with opportunity to experience the notion of referent units as a general idea as well as instances of different bases as the measure unit. Tasks and tools underlain by a pedagogic awareness of the importance of structure are seen as central in this work to support children's structuring activity.

Paying attention to distinguishing multiplicative situations from additive situations, Larsson and Pettersson's (2015) paper presented details of a study in which Swedish sixth grade students were engaged in solving and comparing two covariation problems, one multiplicative and one additive, both set in the same context of children swimming lengths in a pool. They found that children who successfully solved both problems discerned the mathematically significant feature of the intensive quantity speed. These students further inferred from the speed what impact the speed had on the distance between the swimmers – indicating understanding of the properties that could be associated with this structural relation. These authors provide examples of this understanding in excerpts drawn from the talk of two students – Jonathan and Marcus (all names are pseudonyms):

> Jonathan: Because he swims faster [Jonathan moved two fingers simultaneously along the table with one finger moving faster]
>
> Marcus: If they are equally fast then of course she keeps that distance. [Marcus holds his hands on a fixed distance from each other and moves them forward at the same pace.] (p. 562)

Less successful students did not discern the speed as significant or, in spite of discerning it, did not make any inference from the speed about the distance between the swimmers. Matilda and Hanna expressed these differences when they compared the two problems, but that did not lead them to question their solutions, which were to treat both problems as if they were of an additive nature:

> Matilda: They start at the same time and they do not start at the same time.
>
> Hanna: And those two do not swim equally fast and those two swim equally fast. (p. 563)

Here, the ability to discern and distinguish the nature of structural relations between quantities in given situations is noted as central to successful mathematical problem solving.

The ability to distinguish between additive and multiplicative situations, but also to reason about the mathematical structure of a problem in terms of different additive or multiplicative situations, is discussed by Nunes et al. (2012) as what they denote to be mathematical reasoning. An example from their longitudinal study, involving 1680 children over a 5-year time period, is based on two similar problems where the distance between two persons is to be calculated, where one problem is solved by subtraction and the other by addition, since the persons travelled in the same or different directions along a road. They found the ability to recognise the mathematical structure to be a strong predictor for later achievement in mathematics, much stronger than arithmetic skills, logical thinking or working memory, hence recommending more emphasis on reasoning about the mathematical structure in mathematics instruction. This study's results coincide with findings from van Dooren et al. (2010), where students who categorised problems before solving similar problems were more successful than those who solved problems before categorising. The problems were similar to the problems in Larsson and Pettersson's (2015) study, i.e. additive and multiplicative covariation problems and also no variation problems formulated in the same format, such as if it takes 8 mins to boil 5 eggs, how much time do you need to boil 10 eggs concurrently. The amount of time it takes to boil the eggs does not vary here no matter how many eggs there are. The students who first categorised and later solved problems were not only better at distinguishing the mathematical structure and solved more problems correctly; they were also better at the categorisation task than their peers who solved problems before categorising.

This finding links with Ellis' (2007) distinction between arithmetic operations and quantitative operations. While arithmetic operations are driven towards evaluating quantities, quantitative operations are driven towards evaluating the structural relationships between quantities in a given situation. Working with quantitative operations, in these terms, is therefore at the fore of Larsson and Petterson's tasks rather than arithmetical operations, with this orientation reflected too in the emphasis on identifying structural similarity that is seen across task sequences in Askew's (2005) 'Big Book of Word Problems' series. Of further interest in relation to structuring activity more generally is Ellis' finding that different types of 'generalising actions' were prevalent in classrooms promoting one or other of these two approaches. Her notion of 'relating' as one key aspect of generalising activity – described in terms of the creation of relationships between:

> two or more problems, situations, ideas, or mathematical objects. Relating includes recalling a prior situation, inventing a new one, or focusing on similar properties or forms of present mathematical objects. (p. 454)

– was much more prevalent in classrooms promoting attention to quantitative rather than arithmetical operations.

Chen et al.'s (2015) study adds a further dimension to this work by adding in consideration of students' problem-posing competence, alongside attention to their calculation and problem-solving competence. Aligning with the earlier work of Dole et al. (2012), Chen et al. (2015) emphasise that variations in the emphases in their classroom learning experiences (across calculation, contextualised problem solving and problem posing), and the specific numbers and number relations in the problem sets, impact upon the ways in which children interpret structural relationships in problem situations. In the calculation activities, students were required to compute eight number sentences represented in different combinations of number types (i.e. combining a multiplier/divisor and multiplicand/dividend smaller and larger than 1): four decimal multiplications (i.e. 1.3 × 2.7, 2.4 × 0.9, 0.8 × 3.6 and 0.6 × 0.7) and four decimal divisions (i.e. 3.6 ÷ 1.2, 5.4 ÷ 0.9, 0.8 ÷ 1.6 and 0.6 ÷ 0.2). In the contextualised problem-solving activities, students had to solve eight word problems on decimal multiplication and division containing the number sentences from the calculation (e.g. A kilo of bananas costs 1.3 Yuan. I buy 2.7 kilos. How much do I pay?). In the contextualised problem-posing activities, students were required to pose problems according to the same eight number sentences as in the calculation. For example, students were required to pose problems according to the number sentence:

$$1.3 \times 2.7$$

Chen et al. (2015) found that students did well in interpreting the structural relationships in terms of multiplication/division operations in calculation and contextualised problem-solving activities but not in contextualised problem-posing activities. Alongside this, they also found that across the three different learning experiences, it was more difficult to interpret the structural relationships in terms of multiplication/division operations with a decimal multiplier/divisor smaller than 1 than those with a decimal multiplier/divisor larger than 1. Additionally, it was more difficult to interpret the structural relationships with a dividend smaller than the divisor than those with a dividend larger than the divisor. For example, quite a few students (7%) gave a wrong answer '2' for the calculation item '0.8 ÷ 1.6', and a substantial proportion of students (33%) provided a wrong answer '1.6 ÷ 0.8 = 2' for the problem-solving item:

> 1.6 kilos of carrots is 0.8 Yuan. How much are carrots per kilo?

Another sizeable proportion of students (15%) provided a wrong answer such as 'Xiao Hua bought 0.8 kg of bananas, and she spent 1.6 Yuan. How much are bananas per kilo?' in response to the problem-posing item '0.8 ÷ 1.6 = 0.5'.

Taken together, these findings suggest that there may be a useful distinction to be made within Ellis' (2007) quantitative operations category, between tasks geared more towards problem solving and those geared more towards problem posing. This suggests that Watson and Mason's (2005) emphasis on encouraging students to generate examples given specific constraints and/or relations as a way of encouraging attunement to structure and a move away from the calculation orientations that dominate more traditional mathematics instruction may be particularly important. While the examples presented in their work range across several mathematical topics and levels, there are a number of examples presented of tasks and approaches that encourage attention to structure in the context of WNA and to the ways in which properties and relations shift in the moves beyond WNA boundaries. Examples include tasks such as (adapted slightly for our purposes):

Write a pair of numbers that multiply to give 100.
And another pair…
And another pair…
Now write a pair of numbers that multiply to give 100, but one of your numbers has to be between 50 and 100.
Now write a pair of numbers that multiply to give 100, but one of your numbers has to be bigger than 100.

Attention to structural relations and equivalence are at the fore here, with an explicit focus on the boundaries of the example spaces that are usually constructed around multiplication. In this approach, attention to the ways in which properties shift and need reconstruction in order for more general conceptions to be created is opened up for students to work with.

Another example of a route into attention to structural and quantitative operations that stimulated students to find general relations was to prompt sixth grade students to evaluate suggested but erroneous strategies (Larsson 2015). Among the suggested strategies was the idea that 19 × 26 can be calculated as 20 × 25 instead, based on the reasoning that you can move one from 26 to 19, just as in addition. This calculation strategy came from earlier interviews in the same group of students who participated in the study. When the students investigated the strategy rather than the numerical example (and hence working quantitatively and not arithmetically), some succeeded not only in finding the strategy invalid, but also building explanations for why the strategy was invalid. These explanations included structural arguments as well as elaborations of the conditions under which the answer would get bigger than the original task. After an elaborated discussion with a peer, one student concluded thus:

> if you increase the smaller number and decrease the larger number, then it always gets bigger.

This student also stated that he had only investigated the strategy in WNA and that his statement was not yet tested with rational numbers. Students who solved the task by only checking the two answers by calculation (and hence construing the task as an arithmetic operation exercise) practised their calculation skills and could tell that the strategy was invalid but without any reasoning of why. They typically stated that it became another task when you moved one from one factor to the other. This example can be linked to what Smith and Thompson (2008) have distinguished as 'numerical/computational solutions' and 'quantitative/conceptual solutions' (p. 107). In their discussion of what early algebra should focus on, they argue that quantitative reasoning in WNA during the early school years prepares students in much broader and more flexible ways for algebra in the later school years. In their argumentation for more focus on quantitative reasoning, they describe the border between numerical and quantitative reasoning as indistinct and note that numerical reasoning can and should serve as a starting point to think about relations.

Reasoning in the context of numerical examples with a focus on structure is predominant in the 'Peter's method' activity, which Stephens (2004) employed in a study with elementary students. Peter's method was to avoid subtraction that involves renaming, or carrying over, by adding the number that can be followed by subtracting ten. It was presented to students with the example of subtraction of 5 from a two-digit number, shown as involving the adding of five and then subtracting of ten, as in 43 – 5 = 43 + 5 – 10. If the students demonstrated structural reasoning when this and similar examples were discussed, the task was elaborated to include other numbers, for example, subtraction by 6, with the challenge to find what number to add in order to then subtract ten to evaluate the answer, for example:

$$34 - 6 = 34 + ? - 10$$

The findings from his study indicated that students who could ignore the starting number (the minuend quantity) could also answer why and how Peter's method *always* worked, in contrast to students who first undertook the calculation of the left side of the equality sign. The desire to find a numerical answer rather than a general explanation appeared to be associated with hindering students to reason quantitatively. Nevertheless, it is an example of quantitative reasoning that originated in the context of a numerical operations task, with an adaptation that encouraged the structure to be brought into focus. In this sense, the approach overlaps with the approach discussed in the study about evaluation of erroneous calculation strategies that Larsson (2015) presented. Similar activities and classroom observations starting in investigations of WNA examples (but engaging young students to discern relational and structural properties) are described by Bastable and Schifter (2008) as a way to prepare for the transfer of operations beyond WNA into rational numbers.

13.4 Working with Presented Structures with Students

Attention to structure is seen within the presentation and discussion of generalised definitions, models and representations of relations between quantities, with similarities between the artefacts offered to learners and teachers but often differences in the emphases of the ensuing conversations around these artefacts. In this section, we discuss studies that have foregrounded structure at the student level and at teacher level.

Mellone and Ramploud (2015), using the working concept of *cultural transposition*, explored the processes involved in providing Italian children with a figural equation model representing an additive relation structure that has commonly been used in Russia and China. *Cultural transposition* is a process where 'the different cultural backgrounds generate possibilities of meaning and of mathematics education perspectives that, in turn, organise the contexts and school mathematics prac-

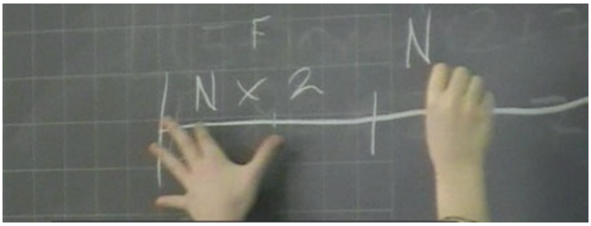

Fig. 13.1 The child's hand

tices in different ways' (Mellone and Ramploud 2015, p. 571). The authors argue that there are differences in emphases and role of the diagrammatic part-part-whole model in the Russian and Chinese contexts that link with cultural and linguistic differences in orientations to meaning, distinction and categorisation. In their analyses of the common practice in China of presenting sets of 'variation problems', the figural equation is explicitly presented as a unifier across variations and, thus, an explicit representation of a generalised property of additive relation situations. In contrast, they argue that Davydov's (1982) presentations of the part-part-whole model functions instead as a transition step between a graphic (non-quantified) situation and a symbolic algebra-based model. This 'intermediary' role for the figural equation has, though, been disputed within Davydov-linked approaches in the work of Dougherty and Slovin (2004), who argue the need for the simultaneous, rather than sequential, presentation of graphic, figural and symbolic models of structure in order to support student meaning-making. Both approaches though emphasise a push towards algebraic thinking from the outset of work in the context of WNA, rather than the more common deferral of algebraic work to a subsequent point.

Analysis of the ways in which a Grade 5 Italian primary school class solved the following problem provides empirical data related to both structural attention and the cultural transpositions involved in taking on approaches with origins in different contexts:

> Grandmother gifts 618 euros to her grandchildren, Franca, Nicola and Stefano. Franca receives twice Nicola's amount; Stefano receives 10 euros more than Nicola. How many euros will each grandchild receive?

Children solved this problem in activity groups. Given the focus in this chapter on working with structure, an important feature of the data presented by Mellone and Ramploud (2015) relates to the ways in which children developed and used figural equations to help themselves to find the solution. A key aspect, documented in the dataset (Electronic Supplementary Material: Ramploud et al. 2017), dataset, shows one group's sharing of their solution approach with the whole class. It is evident from the movement of the child's hand (Fig. 13.1) that the amount for Nicola is used as a measure to draw the figural equation. Of importance to us is that there is an abandonment of the focus on an arithmetical problem and that this is replaced by a focus on structures in ways that are related to informal algebra. The writing of the expression $N \times 2$ (Fig. 13.2) is an important part of this shift in orientation. In this example, we can see the Italian cultural transposition of the Russian tradition, which emphasises

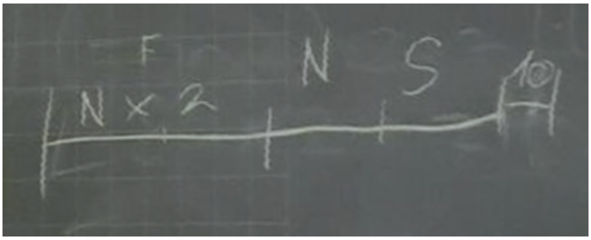

Fig. 13.2 The expression N × 2

continuous representations of quantity, and of the Chinese tradition in which the emphasis on numerical size is retained.

Mellone and Ramploud (ibid.) report positive results in their *cultural transposition* of the figural additive relations equation into an Italian Grade 5 classroom, noting that the structural rather than numerical emphasis of this model was associated with supporting pupils towards a more natural and flexible recourse to algebraic language in this context. This result supports and adds cultural nuance to earlier work pointing to the importance of encouraging attention to structure and generality in the context of WNA as a means of supporting both numerical calculation and later transitions into more formal algebra (Cai and Knuth 2011; Schifter 2011).

Building further on cross-cultural curriculum evaluation, Xin and colleagues developed the conceptual, model-based, problem-solving (COMPS) programme (Xin 2012) that is consistent with the theoretical framework of mathematical modelling and conceptual models (e.g. Blomhøj 2004; Lesh et al. 1983). The COMPS approach places emphasis on algebraic representation of generalised mathematical relations in equation models. For instance, 'Part + Part = Whole' is a conceptual model for additive word problems; 'Unit Rate × Number of Units = Product' (Xin 2012, p. 5) is a conceptual model for multiplicative equal-group (EG) problems. Giving the generalised mathematical models provided by COMPS, a range of arithmetic word problems involving the four basic operations can be represented and modelled in an algebraic equation, which can be used to help students solve problems.

To this end, Xin developed a set of word problem (*WP*) *story grammar* questions (see Figs. 13.3, 13.4, 13.5 and 13.6) to facilitate students' efforts in representing various word problems in the model equation. The algebraic equation then drives the solution process, that is, solve for the unknown quantity in the equation. During this process, the choice of operation for solving various arithmetic word problems is determined by the model equation (Part + Part = Whole; or Factor × Factor = Product). In the case of EG problem-solving (see Fig. 13.3), for instance, when the *number of units* is the unknown (e.g. *Dan has a total of $114 for buying gifts for his friends. If each gift costs $19, how many gifts can he buy?*), the model equation ($19 \times a = 114$) indicates that dividing the product (114) by the known factor (19) will solve for the unknown factor ($a = 6$). In the case of *multiplicative compare* (MC) problem solving (see Fig. 13.4), for instance, when the *referent* unit is the unknown (e.g. Pat has 204 marbles. Pat has 17 times as many marbles as Bob. How many marbles does Bob have?), the model equation ($a \times 17 = 204$) shows that dividing the *product*

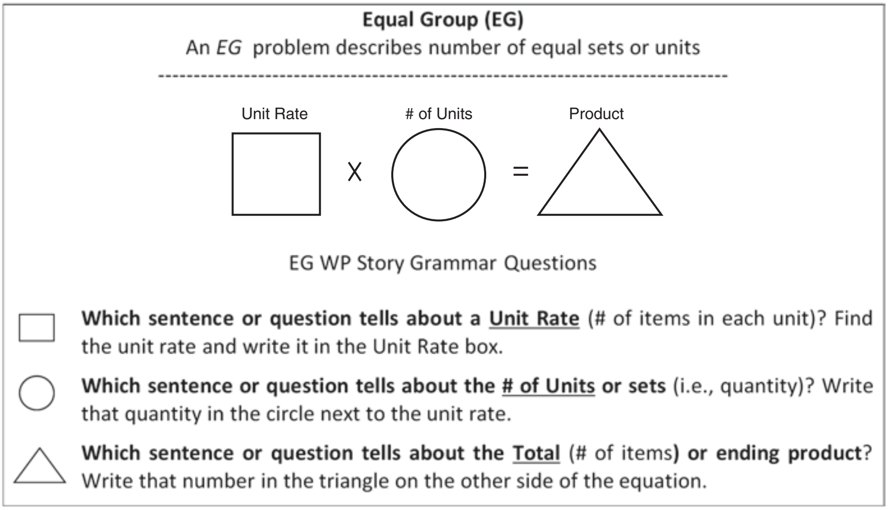

Fig. 13.3 Conceptual model of *equal groups* problems (Xin 2012, p. 105)

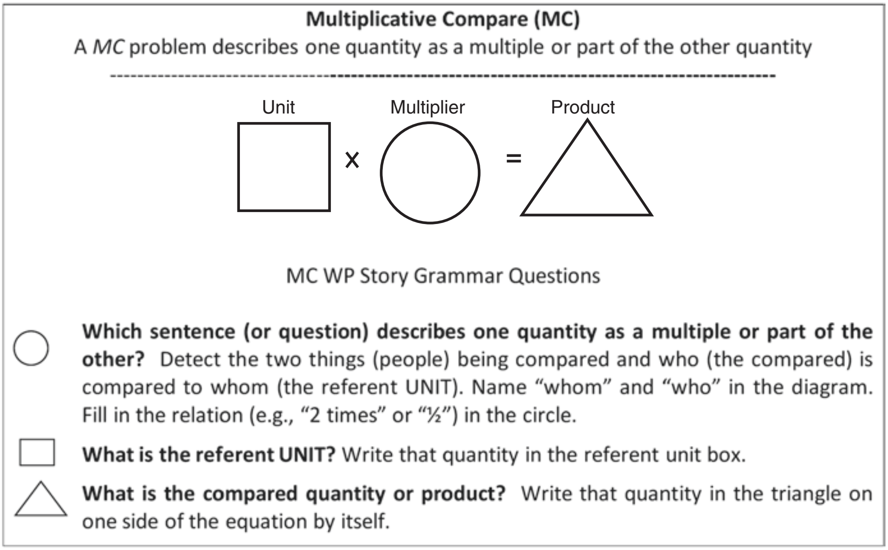

Fig. 13.4 Conceptual model of *multiplicative compare* problems (Xin 2012, p. 123)

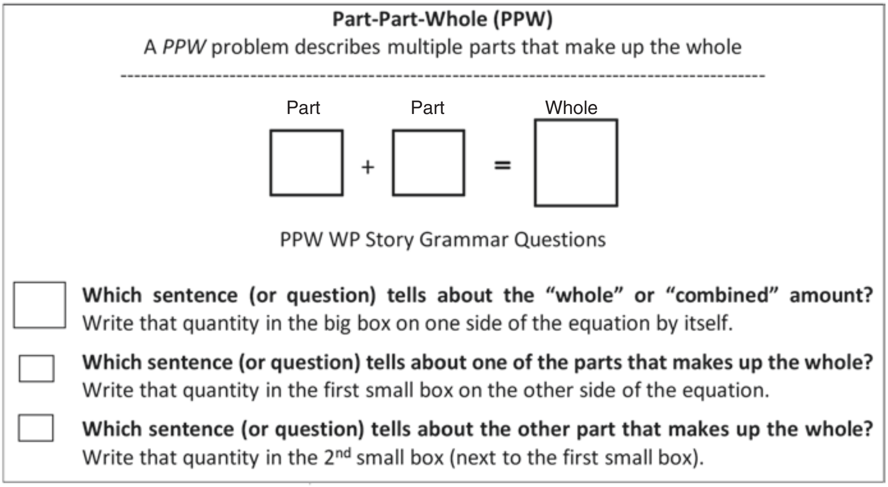

Fig. 13.5 Conceptual model of *part-part-whole* problems (Xin 2012, p. 47)

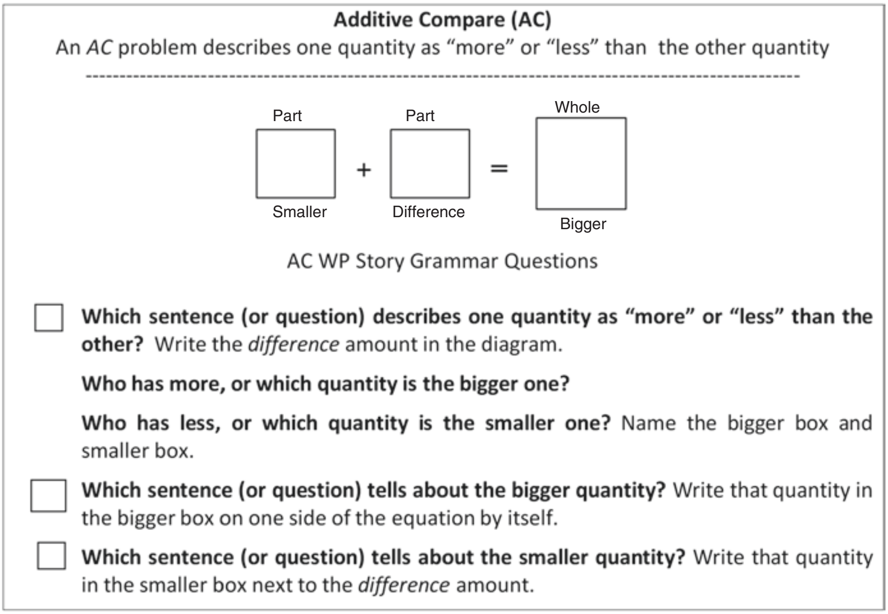

Fig. 13.6 Conceptual model of *additive compare* problems (Xin 2012, p. 67)

(204) by the *multiplier* (17) will solve for the unknown quantity ($a = 12$). Students no longer need to 'gamble' on the choice of operation or experience risk on the 'keyword' strategy.

As shown in Figs. 13.3, 13.4, 13.5 and 13.6, Xin (2015) presents structure in the form of algebraic representations of generalised conceptual models of additive and multiplicative situations and uses them as a heuristic model to support the problem-solving activities of elementary and middle school students with learning difficulties. Empirical data drawn from using this approach indicate that students who used the COMPS programme showed significant improvement in mathematics problem-solving performance (Xin et al. 2011).

13.5 Working with Presented Structures with Teachers

At the level of teacher education, a range of approaches driven by the need for teachers to make greater use of the power of structure in their pedagogy has been presented in recent studies. We discuss several studies that investigated how teachers reasoned about multiplicative and proportional relationships using presented structures. Quite predictably, there are also overlaps in this section with the chapter on teacher education in this publication (Chap. 17).

In their work with future middle grades teachers (Grades 4–8), Beckmann et al.[1] (2015) view multiplicative structure quantitatively in terms of multiplier, multiplicand and product and use the different roles taken by the multiplier and multiplicand as a route into thinking about proportional relationships in two different ways, as either 'multiple batches' or 'variable parts' (Beckmann and Izsák 2015). In their approach, multiplication is defined by an equation:

$$M \bullet N = P$$

where M, the multiplier, is a number of groups; N, the multiplicand, is the number of units in 1 group; and P, the product, is the number of units in M groups. The future teachers in the study adopted the definition as a class norm. This definition is quantitative, as opposed to purely numerical, because the multiplier, multiplicand and product have measurement units attached to them ('groups' and 'units') and therefore refer to quantities. Because the definition is quantitative, the multiplier and multiplicand play different roles and, using the definition, requires teachers to look for and identify structure in situations. The rationale for a presented structure is therefore to foster close examination of situations and attention to detail in constructing arguments.

For example, Figs. 13.7 and 13.8 show two ways that a future teacher in Beckmann et al.'s (2015) study reasoned to solve a proportion problem: A fertiliser

[1] Research was supported by the National Science Foundation under Grant No. DRL-1420307. The opinions expressed are those of the author and do not necessarily reflect the views of NSF.

is made by mixing nitrogen and phosphate in an 8:3 ratio and the question is how much phosphate to mix with 35 kg of nitrogen. The teacher's first solution (Fig. 13.7) takes a multiple-batches perspective. She views the fertiliser as some number of batches of an 8 kg nitrogen and 3 kg phosphate mixture, structures the 35 kg of nitrogen as a number of groups of 8 kg of nitrogen and structures the required amount of phosphate as that same number of groups of 3 kg of phosphate.

The same teacher's second solution to the same fertiliser problem (Fig. 13.8) takes a variable-parts perspective. This time the teacher views the fertiliser as eight parts nitrogen and three parts phosphate, where all parts are the same size. She determines the size of each part by structuring the eight parts nitrogen as eight groups that contain a total of 35 kg and structures the required amount of phosphate as three groups of that size.

Although the results are preliminary and part of a larger, ongoing project, in Beckmann et al.'s (2015) study, many pre-service teachers were similarly able to present well-constructed arguments for solving proportionality problems from two perspectives. We also note that a presented structure could be a useful organiser for the field. Although a multiple-batches perspective on proportional relationships has been well known in mathematics education research for many years, the existence of a separate variable-parts perspective was only recently discussed in mathematics education research (Beckmann and Izsák 2015). This is even though limitations of a multiple-batches perspective were recognised (Kaput and West 1994) and variable-parts solution methods were known. For example, the model method used in Singapore (see Kaur 2015) lends itself to a variable-parts perspective. By structuring proportional relationships through a quantitative definition of multiplication, we see the existence of two distinct quantitative ways of conceptualising proportional relationships.

Venkat (2015) similarly pointed to improvements in in-service teacher performance in South Africa on a ratio task following exposure to generalisable double number line models, introduced in the context of whole number situations but usable and used in decimal number (money) contexts as well. Structure, in her approach, was presented in the form of key representations of the structure of multiplicative situations – double number lines and ratio or 'T-tables' – that were introduced, discussed and used in their in-service teacher education programme. Evidence drawn from teacher assessment tasks in the course indicated that for some teachers, these 'structured' representations were taken up as tools that facilitated moves towards successful mathematical problem solving that allowed for the production of correct answers, while for other teachers, the same structured representations were taken up as pedagogic objects with associated explanations that could better support students to produce correct answers. Across both groups, there was evidence of greater elaboration of problem-solving processes, in ways that broader literature suggests are useful for teaching.

This finding is of interest in relation to the literature base on primary mathematics teacher knowledge where there is broad evidence that being able to do mathematics for oneself provides limited promise for competence with teaching mathematics to others. Essentially, the argument is that the latter competence

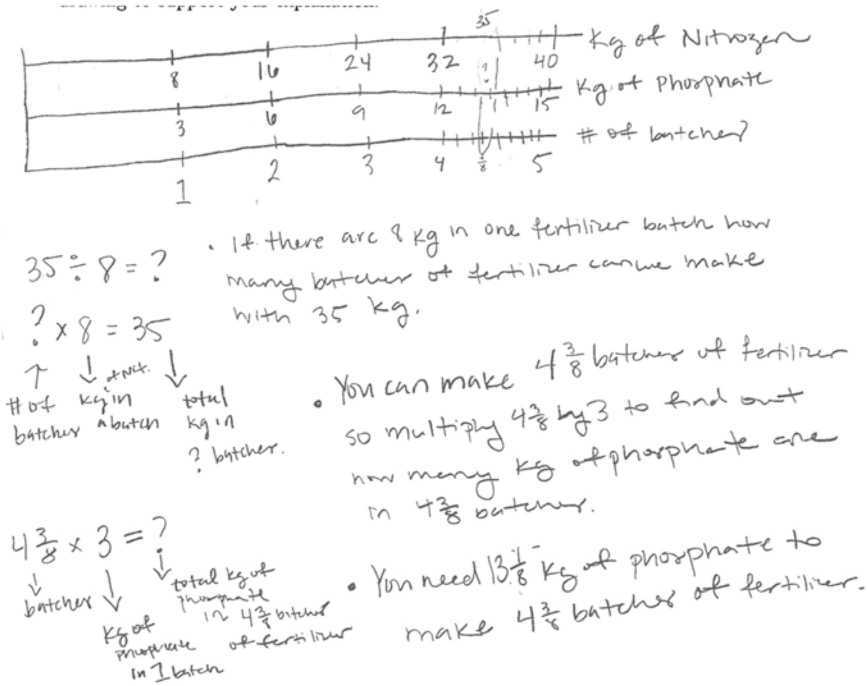

Fig. 13.7 Reasoning about proportional relationships from a multiple-batches perspective

requires an additional, 'specialised' knowledge base (Ball et al. 2008). As a group, we concur with this argument, but note from Venkat's (2015) work that a focus on key representations, introduced and discussed in the context of WNA, appeared to be an important component that worked simultaneously to support the development of competences related to teachers' working with mathematics and their teaching of mathematics. In a South African context marked by discursive gaps in pedagogy, this kind of approach, focused on generalised representations of structure, serves to address development in what a trajectory of work, of which Adler and Ronda's (2015) paper is the most recent, has described as teachers' 'mathematical discourse in instruction'.

Dole et al. (2015) attribute problems for students in recognising and working with multiplicative reasoning to 'the limited capacity of primary school curricula to promote multiplicative structures' (p. 535). In trying to address this shortcoming through an in-service teacher development project, Dole et al.'s team presented and discussed a range of proportional relation situations drawn from the Australian curriculum document, from other subject areas and from real life, encouraging focus on their structural similarity and their contrast with the structure of additive relation situations. Through this approach, they noted shifts in teachers' awareness of links between topic areas in the curriculum that had previously been viewed as separate

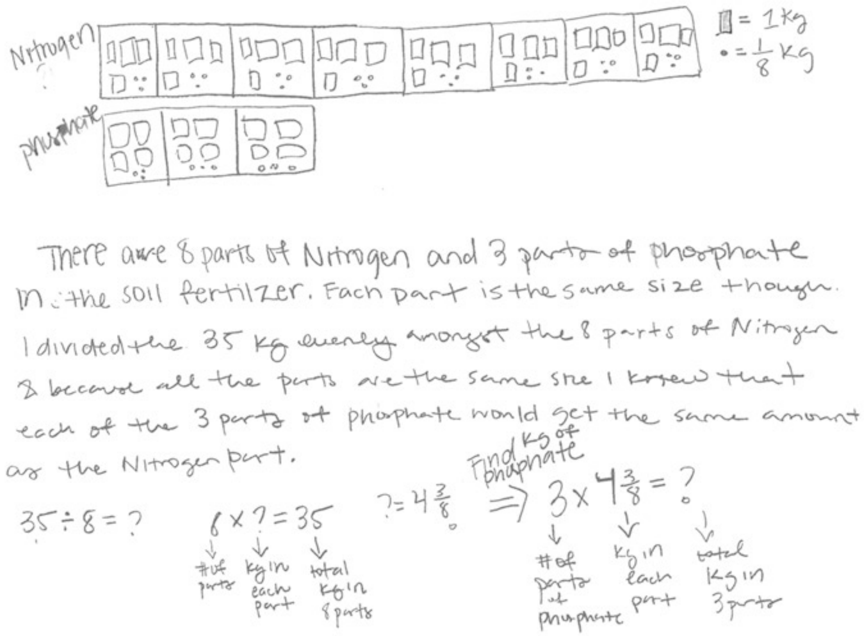

Fig. 13.8 Reasoning about proportional relationships from a variable-parts perspective

and competence with pointing to this similarity when working with situations underpinned by a multiplicative structure.

13.6 Conclusions, Implications and Future Directions

Across the papers in the two broad sections focused on 'structures' and 'structuring activity', there is general agreement that incorporating attention to developing awareness of structure should be an important component of work in WNA, in order to support early algebraic thinking. There are also several useful pointers towards approaches that appear to hold promise for the development of attention to structure in the WNA context in ways that have longevity beyond the boundaries of WNA. Given the evidence of students' cognitive difficulties in the transition from natural numbers to rational numbers (van Hoof et al. 2013), and the concomitant evidence of 'natural number bias' (Ni and Zhou 2005), the latter aspect is particularly important. We summarise these approaches here, noting emphases on particular features and phases within this evidence:

- There are indications that situations involving spatial awareness can provide useful springboards for WNA working in ways that relatively 'naturally' and usefully include attention to structural relations.

- Distinguishing between additive and multiplicative situations, as well as between different structures within additive and multiplicative situations, appears to be an important avenue into developing understanding of the different underlying structures of these situations. Problem posing in relation to given structures appears to be particularly complex and, therefore, openings for encouraging students to engaging with linking or constructing problems with given structural relations would seem to be an important area for further attention.
- For older children and for teachers, more 'top-down' presentations of structure in generalised word sentence or algebraic formats seem to have purchase in drawing attention to the nature of quantitative relations being worked with. This could well be related to, and acknowledging of, extensive prior encounters with additive and multiplicative situations. Parallel approaches for younger children appear to be better supported by the presentation of pictorial models of underlying structure that can be used in similar ways to develop more powerful discourses about the nature of quantitative relations in additive, multiplicative and other patterned situations involving some structural relations.

The importance of awareness of structural relations in a range of problem contexts has been widely acknowledged in mathematics education research. Our focus in this chapter has been on distinguishing between two key alternatives into developing this awareness. Whether working with offered structures or being invited to construct relations through structuring activity, the common centrepiece is inviting students and/or teachers to think more deeply about the mathematical structure of problems. Nunes et al. (2012) have noted that this kind of thinking can and should be developed in the context of WNA, with work in additive and multiplicative relation situations providing fertile ground from the earliest stages of mathematical learning for both learning about structure and learning to distinguish between structures. Both approaches provide the means for seeing numerical and spatial situations (and quite possibly mathematical situations more generally) as contexts that are open to structuring and to seeing in terms of structure. Mathematical activity in this orientation is viewed as fundamentally concerned with identifying structure and possible generality. At one level, this focus opens possibilities for seeing WNA as a ground with continuities into rational and real number arithmetic. A larger outcome, though, of the focus on structure and structuring is a breaking down of some of the high walls of arithmetic operations around WNA contexts, which so many children (and, importantly, many teachers) in so many parts of the world appear to have such difficulty in scaling and seeing beyond.

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