Chapter 10
Artefacts and Tasks in the Mathematical Preparation of Teachers of Elementary Arithmetic from a Mathematician's Perspective: A Commentary on Chapter 9

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10.1 Introduction

The main focus of this paper is the mathematical preparation of primary school teachers in relation to the teaching of elementary – whole number – arithmetic. My comments are based mostly on my experience and reflections as a mathematician involved in the education of pre-service teachers, but with occasional inspiration from development activities for in-service teachers with whom I collaborated.

As a consequence, the prime intent of the paper is not to examine what may be happening about the learning of arithmetic by actual pupils in classrooms but rather to concentrate on the kind of 'adult experiences in mathematics' that, in my opinion, prospective teachers ought to encounter in order to better prepare for their role as guides accompanying their pupils in the acquisition of concepts and skills related to whole number arithmetic.

The context that induced the department of mathematics at my university to create two mathematics courses specifically for future primary school teachers – one of them being devoted to arithmetic – is briefly described in Hodgson and Lajoie (2015). Suffice it here to recall that this involvement has been ongoing now for more than four decades and that the responsibility of preparing prospective teachers for their mathematical duties is shared, and it goes without saying, with the Faculty of Education, where student teachers take three mathematics education courses (*didactique des mathématiques*, in French). The classroom reality and the pupils' needs are of course more significantly integrated in this didactical environment. An underlying theme of Hodgson and Lajoie (2015) is to stress the complementary roles played by mathematicians and mathematics educators (*didacticiens*) in this endeavour. It should however be noted, as indicated in the survey by Bednarz (2012, Tables

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M.G. Bartolini Bussi, X.H. Sun (eds.), *Building the Foundation: Whole Numbers in the Primary Grades*, New ICMI Study Series,
https://doi.org/10.1007/978-3-319-63555-2_10

1 and 2), that the model adopted at my university in that connection is rather unusual in the Canadian context (including in the province of Québec).

In order to provide an insight into the arithmetic course (entitled *Arithmétique pour l'enseignement au préscolaire/primaire*) that we have developed for primary school teachers and crystallise its main intents, I wish in this paper to examine both its spirit and some of its components. This will bring me to consider the two central themes at the heart of Chap. 9 (*Aspects that Affect Whole Number Learning: Cultural Artefacts and Mathematical Tasks*): the importance and variety of artefacts (often of a cultural and historical nature) that can be used to support the learning of whole number arithmetic, as well as the role played by mathematical tasks proposed in order to foster the 'mathematical message' that may be conveyed through the artefacts. I will discuss here concrete examples taken from our arithmetic course, intending by so doing to illustrate a crucial observation strongly emphasised in Chap. 9, namely that artefacts and tasks form an inseparable pair. However, I will first present some general observations about the mathematical preparation of primary school teachers with regard to whole number arithmetic.

10.2 Preparing Mathematically for the Teaching of Arithmetic

> One should not […] delay too late the moment when abstraction shall become the form and the condition of the whole teaching: finding for each pupil and for each study the right moment when it is advisable to move from the intuitive form to the abstract form is the great art of a true educator.[1] (Buisson 1911)

The philosophy underlying our arithmetic course is based on the conviction that in order to adequately fulfil their role as guides and become efficient communicators, primary school teachers should have developed such a level of mathematical competency that they see themselves as being in full possession of the mathematical tool with which they will be working, in other words that they feel *autonomous* with respect to their mathematical judgements concerning primary school arithmetic. We thus offer to the student teachers an opportunity for a personal reconstruction of elementary arithmetic through a mathematical pathway intended to allow them to clarify and develop basic notions underlying the teaching and learning of arithmetic at the primary level. This enterprise can be interpreted as aiming to both demystify and demythicise mathematics in general, but especially arithmetic, for prospective teachers.

[1] Original French text: 'Il ne faut (…) pas reculer trop tard le moment où l'on fera de l'abstraction la forme et la condition de tout l'enseignement: trouver pour chaque élève et pour chaque étude le moment précis où il convient de passer de la forme intuitive à la forme abstraite est le grand art d'un véritable éducateur'.

Hopefully, such an experience will allow them to gain confidence in their own expertise – of a very specific nature – about mathematics as seen in relation to the education of primary school pupils.

Part of what we aim at achieving on the subject of arithmetic with our student teachers is well captured by a quotation from the illustrious mathematician Leonhard Euler, the author, it should be reminded, of many influential textbooks intended for students. In the preface of an arithmetic textbook for Russian schoolchildren published in 1738 under the auspices of the St Petersburg Academy of Sciences, *Einleitung zur Rechenkunst* (*The Art of Reckoning*), Euler wrote:

> The learning of the art of reckoning without some basic principles is sufficient neither for solving all cases that may occur nor to sharpen the mind, as should be our specific aim. […] Thus when one not only grasps the rules [of reckoning], but also clearly understands their causes and origins, then one will to some extent be enabled to invent new rules of one's own, and to use these to solve such problems, for which the usual rules would not be sufficient. One should not fear that the learning of arithmetic might thus become more difficult and require more time than when the raw rules are presented without any explanation. Because any individual understands and retains much more easily those matters, whose causes and origins he clearly comprehends.[2] (Euler 1738, pp. 3–4)

It is far from me to suggest that the aims or means of our arithmetic course are in any way new or revolutionary. For a very long while, a number of authors have been reflecting on the need to improve the mathematical preparation of pre-service school teachers and proposing varied and at times highly innovative approaches. One outstanding example is given by Felix Klein, the very first President of ICMI (1908–1920), who presented in the early twentieth century a famous series of lectures intended for teachers (Klein 1932). Although Klein was then mainly addressing secondary school teachers of mathematics, parts of his comments, especially in the very first chapter devoted to 'calculating with natural numbers', can be seen as pertaining directly to elementary arithmetic, and the needed mathematical background and vision with which primary school teachers, in his opinion, should be familiar. Another example, having as a setting my own university but at the time of the 'New Math' era, is given by Wittenberg, Sister Sainte Jeanne de France and Lemay (Wittenberg et al. 1963) – the three authors were then all connected to the mathematics department. Resisting the 'bourbakised' vision of mathematics teaching (p. 91) then quite fashionable, the authors reflect on the numerous reform movements of those days which, they claim:

[2] Original German text: 'Die Erlernung der Rechenkunst ohne einigen Grund weder hinreichend ist, alle vorkommenden Fälle aufzulösen, noch den Verstand schärfet, als dahin die Absicht insonderheit gehen sollte. […] Dann wann man auf diese Art nicht nur die Regeln begreift, sondern auch den Grund und Ursprung derselben deutlich einsieht, so wird man einigermassen in Stand gesetzt, selbsten neue Regeln zu erfinden und vermittelst derselben solche Aufgaben aufzulösen, zu welchen die sonst gewöhnlichen Regeln nicht hinreichend sind. Man hat auch im geringsten nicht zu befürchten, dass die Erlernung der Arithmetik auf diese Art schwerer fallen und mehr Zeit erfordern werde, als wann man nur die blossen Regeln ohne einigen Grund vorträgt. Dann ein jeder Mensch begreift und behält dasjenige im Gedächtnis viel leichter, wovon er den Grund und Ursprung deutlich einsieht'.

> at times seem to lock themselves into a surprising and naïve conviction that it is enough, in that domain, to meditate on the thinking of a single man (multiplicatively reincarnated, it is true), and that: *whoever has read Bourbaki, has read everything*.[3] (Wittenberg et al., p. 11)

The authors propose a 'genetic approach' as a way to allow practising and prospective teachers to see elementary mathematics with new eyes – so to say, to see it as their pupils will – and to reflect on its internal structure (p. 13).

We do not aim, in our arithmetic course for teachers, at presenting a fully fledged 'genesis' of the basic concepts related to whole numbers. Still we wish to adhere to a vision that remains as primitive as possible. For that purpose our arithmetical journey relies, to a large extent, on a rudimentary yet very fruitful numerical artefact for tackling numbers: sequences of tallies (Sect. 10.4). This allows us to gradually build a body of knowledge about the (set of) whole numbers, a patent emphasis being put on establishing, with a certain level of rigour, the basic properties at stake. It is thus a structural perspective on elementary arithmetic that we propose to our student teachers, which brings into the discussion a level of abstraction possibly rather new, maybe even strange, to some of them. But as expressed by the renowned French educator Ferdinand Buisson at the turn of the twentieth century (see the quotation as epigraph to this section), such an abstract perspective is at the heart of the teaching and learning process.

Our mathematical approach to 'arithmetic for teachers' has clear links to several current or recent research works, such as that of Grossman, Wilson and Shulman (1989), stressing the importance of a teachers' sound knowledge of mathematical content. It also has connections with the famous study of Ma (1999) concerning a 'profound understanding of mathematics', as well as with the work of Ball and Bass (2003) about 'mathematical knowledge for teaching'. In that context, I think it is of interest to comment, from a mathematician's perspective, on some of the mathematical artefacts and tasks used in our teaching of arithmetic.

10.3 Accesses to the Concept of Number

'How do numbers emerge?' Such is the question raised by Hans Freudenthal, the eighth President of ICMI (1967–1970), at the very outset of a chapter entitled 'The Number Concept: Objective Accesses' from his monumental *Mathematics as an Educational Task* (1973, p. 170). He proposes a fourfold distinction, discussing successively – from a mixture of mathematical and didactical vantage points – the emergence of the number concept under the disguises of *counting* number, *numerosity* number, *measuring* number and *reckoning* number.

[3] Original French text: 'alors que prolifèrent des entreprises de réforme [qui] semblent parfois s'enfermer dans une surprenante et naïve conviction que c'est assez, dans ce domaine, de méditer la pensée d'un seul homme (multiplicativement réincarné, il est vrai), et que: *qui a lu Bourbaki, a tout lu*'.

(Another way of reacting to the question put forward by Freudenthal is found under the heading 'The Logical Foundations of Operations with Integers' of Klein (1932), notably pp. 11–13. While Klein sees issues of psychology and epistemology being at stake, he chooses, as indicated by the title of that section, to propose a mathematical reflection based on a spectrum of perspectives and arguments from logic, including a purely formal theory of numbers.)

Space prevents me here from entering into all the nuances of Freudenthal's very rich discussion of the various accesses he discusses in relation to the concept of number. But I would like to use his framework as a basis for my reflections and for establishing connections to some of the core ingredients of our arithmetic course for primary school teachers.

10.3.1 Counting Number

Freudenthal characterises the notion of counting number as being connected to 'the reeling off in time of the sequence of natural numbers' (1973, p. 170). He observes that grasping 'the whole unlimitedly continuing sequence' of numbers is for children 'a conceptual seizure that has no analogue in learning the names of colours and letters' (1973, pp. 170–171). At stake here is the notion of *successor*, at the heart of the axiomatic approach to arithmetic proposed by Giuseppe Peano, jointly with the principle of mathematical induction. As a consequence of the successor is de facto an order implicitly introduced: we are thus in presence of the general notion of ordinal number.

One possible implementation of the idea of counting numbers is via an artefact that actually plays a fundamental role in our course, sequences of tallies. The idea of successor is then readily perceived as the simple adjunction of a new tally to a given sequence, and this can 'obviously' be repeated indefinitely, at least in principle. I shall return to this particular artefact in Sect. 10.4.

One very early encounter of children with counting numbers is when learning a counting list, the successive terms being either written as numerals or expressed as words in their mother tongue. The memorising of the usual oral counting list is often supported by nursery rhymes. A possible task that may be proposed to teachers in that connection is to examine if an actual poem or a song could really be used for counting. What are the qualities that a good 'counting song' (or list) should possess? How far could one go with a particular song like *Au clair de la lune* (see https://en.wikipedia.org/wiki/Au_clair_de_la_lune) – for instance, in order to count the number of pupils in the classroom? (Other aspects of songs and poems being used in arithmetic are discussed in Sect. 9.2.2.5.)

One particular task I like to give to my student teachers on the very first day of the course is to build a written counting list using the symbols from some given alphabet. (It is understood here that the alphabet comes with a specific order among the symbols it contains.) Without having yet discussed the notion of positional-value numeration system, I propose to restrict the available digits to, say, 0, 1, 2 and

3 (in that order), and invite them to build a counting list, in the same spirit as our usual list of numerals, but using only these symbols. While some may come back with rather original lists, most of them would have built a list 'in base four', analogous to the usual base ten list. But given the alphabet comprising the symbols A, B, C and D, the typical answer

A, B, C, D, AA, AB, AC, AD, BA, BB, …, DD, AAA, AAB, …

clearly illustrates that the usual counting list they know very well (base ten) brings into play one symbol having a rather special behaviour: 0.

The learning of the usual oral counting list may introduce some linguistic peculiarities, usually specific to a given language. For instance, an interesting cultural task (in French) is to observe the distinction, when counting by tens, between the 'regular'

cinquante, soixante, septante, octante, nonante

and the more usual (but depending on countries)

cinquante, soixante, soixante-dix, quatre-vingts, quatre-vingt-dix

the latter being (partly) a remnant of vigesimal numeration (see this volume, Sect. 3.2.2).[4]

10.3.2 Numerosity Number

Using even animals as *cas de figure* for small numerosities, Freudenthal notes that 'perhaps the numerosity number is genetically earlier than the counting number' (1973, p. 171). His comments point to the fact that recognising at a glance – without counting – the number corresponding, say, to four dots (even if placed randomly), is an easy task (this capacity, called *subitising*, is described in this volume, Sect. 7.2.1): one can instantly 'see' the four dots. But the same is possibly not true for most people when looking quickly at the dots in Fig. 10.1a. However, the pattern used in Fig. 10.1b is such that the numerosity of the dots is immediate. (See also Sect. 9.3.4.2 for comments about strategies related to artefacts with a structural feature such as that of Fig. 10.1b.)

Numerosity rests on the possibility of identifying the 'number' corresponding to a certain situation without numbering the objects one by one. The idea is to associate the given situation with another one, the question at stake then being not 'how many?' but rather 'is it as many as?'. While the notion of equipotency (or one-to-one correspondence) on which rests this approach to whole numbers is a very natural one (and it will play a central role when discussing sequences of tallies in Sect. 10.4),

[4] One may be reminded here, for instance, of Molière's *L'Avare* (1668), when one of Harpagon's servants, fawning over him about his longevity, says: 'Par ma foi, je disais cent ans, mais vous passerez les six-vingts' (Act II, Scene 5).

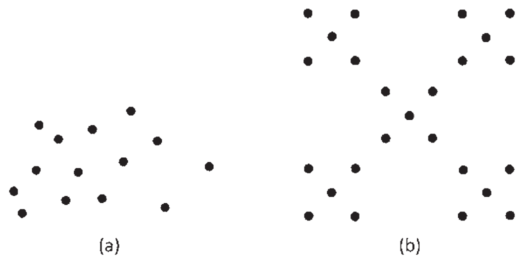

Fig. 10.1 Numerosity of dots aggregates (**a**): Randomly displayed dots (**b**): A pattern of dots

its formalisation via the general notion of cardinal number *à la Cantor* is definitely questionable, according to Freudenthal. (It may be noted here that Klein (1932, p. 12) speaks on the contrary in enthusiastic terms of this 'modern' approach due to Cantor.) The criticism made by Freudenthal (1973, p. 181) is as strong as can be:

(1) The opinion that the numerosity number, that is the potency, suffices as a foundation of natural numbers is mathematically wrong.
(2) The numerosity aspect of natural numbers is irrelevant if compared with the counting aspect.
(3) The numerosity aspect is insufficient for the didactics of natural numbers.

Freudenthal then spends more than fifteen pages expounding his objections.

Hodgson and Lajoie (2015, p. 309) mention that the first versions of our arithmetic course, in the 1970s, used a set-theoretic context to introduce natural numbers as cardinalities of finite sets, operations on numbers being defined via set-theoretic operations. This was in accordance with the spirit of the times, as can be seen through a quotation from a report of an ICMI-supported workshop organised by UNESCO in 1971, in a chapter on 'Primary Mathematics':

> All modern reformed programs have introduced the study of sets into mathematical instruction. This topic is perhaps the most visible trait of an actual change in primary mathematics teaching. […] There is a universal trend to use sets to develop the concept of cardinal or natural numbers, and the four rational operations on natural numbers. (UNESCO 1973, pp. 5–6)

It was eventually decided to approach whole numbers in our course not on the basis of a possible prior idea of sets, but more intrinsically as 'primitive' objects of their own, as well as the operations defined on them: sequences of tallies thus entered the picture.

10.3.3 Measuring Number

Freudenthal discusses the notion of measuring number in a general context, comparison with a given unit leading sometimes to exhaustion of the magnitude to be measured and sometimes to incomplete exhaustion. The latter case can be seen as giving rise either to division with remainder (most appropriately called Euclidean division) or, when the unit is divided, to fractions. Eventually issues of commensurability and incommensurability, in an ancient Greek spirit, may come into play.

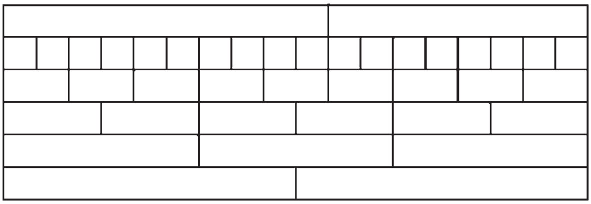

Fig. 10.2 The factors of 18, *à la Cuisenaire*

Central notions of elementary theory of numbers (the αριθμητική *arithmētikē* of the Greece of Antiquity) can be brought to the fore via the notion of measure: this is precisely how divisibility is introduced in Euclid's *Elements*. Two most fruitful artefacts to be used here are Cuisenaire rods (this volume, Sects. 8.2.1 and 9.3.1.1) and the number line (Sect. 9.2.2.4).

Although strongly attached, in the collective memory, to the 'New Math' era, the Cuisenaire rods have a strong merit of their own, and it may be seen as still pertinent for future teachers to be familiar with this artefact. It is thus a pity that many of our student teachers have never encountered them earlier in their own schooling (as if legions of boxes of Cuisenaire rods were left sleeping in school basements). The simple fact of observing with the rods how a 'train' of length, say, 18 (one orange rod and one brown rod), can be divided into series of identical 'wagons' in five different ways (eighteen white, nine red, six light green, three dark green, two blue) is without any doubt a most inspiring way of addressing the notion of factors (see Fig. 10.2).

A most relevant distinction can be made here, considering, for instance, the factorisation $18 = 2 \times 9$, between the 'unit' 2 being repeated nine times and the 'unit' 9 repeated twice.

Transfer, in more generality, to the number line is then immediate. As stressed in Sect. 9.2.2.4, line segments are precisely how Euclid 'saw' numbers, including whole numbers. And the issue of a number 'measuring' or not another one can readily be addressed via segments on the number line.

I wish to stress here another fertile artefact from ancient Greece for approaching elementary theory of numbers: the use of figurate numbers, that is, of certain geometrical arrangements of collections of dots. This vision of whole numbers does not emphasise an idea of measure, as I just discussed. Going back to the Pythagorean school, it provides a dynamic context revealing rich relations between given family of numbers. For instance, passing from a given square of side n to a square of side $n + 1$ involves the adjunction of a 'gnomon' (γνώμων) made of twice the length n and a unit. Fig. 10.3a presents the situation, for the case $n = 4$, in a figurate number style, while Fig. 10.3b uses a more traditional image based on area. It may be pointed out here that the latter artefact has as well a very long history, being present in essentially all the ancient mathematical traditions. Both artefacts of Fig. 10.3 can serve as supports for *visual proofs* – in the present case, of the fact that $(n + 1)^2 = n^2 + (2n + 1)$.

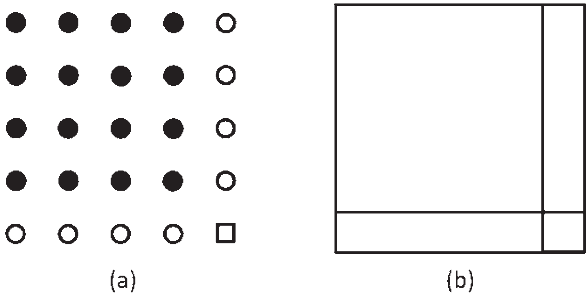

Fig. 10.3 Passing from the square of a whole number to the next

$$\begin{aligned}
23 \times 15 &= (2 \times 10 + 3) \times (1 \times 10 + 5) && (1)\\
&= ((2 \times 10 + 3) \times (1 \times 10)) + ((2 \times 10 + 3) \times 5) && (2)\\
&= ((2 \times 10) \times (1 \times 10) + 3 \times (1 \times 10)) + && (3)\\
&\qquad ((2 \times 10) \times 5 + 3 \times 5) && \\
&= ((2 \times 10) \times 10 + 3 \times 10) + ((2 \times 10) \times 5 + 3 \times 5) && (4)\\
&= ((2 \times 10) \times 10 + 3 \times 10) + (5 \times (2 \times 10) + 3 \times 5) && (5)\\
&= (2 \times 10 \times 10 + 3 \times 10) + (5 \times (2 \times 10) + 3 \times 5) && (6)\\
&= (2 \times 10 \times 10 + 3 \times 10) + ((5 \times 2) \times 10 + 3 \times 5) && (7)\\
&= (2 \times 10 \times 10 + 3 \times 10) + (10 \times 10 + 15) && (8)\\
&= (2 \times 10^2 + 3 \times 10) + (10^2 + 15) && (9)\\
&= (2 \times 10^2 + 3 \times 10) + (1 \times 10^2 + 15) && (10)
\end{aligned}$$

Fig. 10.4 Fragment of the calculation of 23 x 15 (detailed style)

10.3.4 Reckoning Number

Freudenthal (1973, p. 171) uses the expression reckoning number in order to highlight the algorithmic aspect attached to whole number arithmetic. This is the *logistikē* (λογιστική) of ancient Greece, that is, actual calculations involving operations of elementary arithmetic and, in particular, of course, the 'four operations'. We are now in the same ballpark as with Euler's *Rechenkunst* mentioned above.

Reckoning is considered from two different perspectives in our arithmetic course for teachers. One encounter with reckoning happens after about a month of the course, once the basic laws of arithmetic have been introduced and that positional-value numeration (base ten) has been fully reviewed, so that it is then possible to discuss and justify in a thorough manner the functioning of algorithms. A specific task proposed to teachers, concerning standard algorithms, is to identify minutely the way the arithmetic laws enter into action in a given algorithm. For example, concerning the multiplication 23 x 15, we provide them with a truly detailed computation (see Fig. 10.4) where the first ten lines from a series of twenty one are reproduced. A single mathematical basic event is enacted on each of these lines, and the students' task is to identify it.

Of course, we do not ask our students to construct by themselves calculations of such a type, as this is not really a task on which we would want them to devote energy and time. Nor do we have in mind them later using such a level of scrutiny with their own pupils. But for a teacher, seeing precisely (at least once in one's lifetime) the way the basic arithmetic laws intervene in standard algorithms is, we believe, a most valuable experience.

Another task part of the same discussion is to have our students look for, look at and understand non-standard algorithms for the four operations. We also present them other artefacts of a historical flavour, such as, for multiplication, the gelosia method or Napier's bones (Sect. 9.2.2.3) or the Egyptian algorithm.[5]

The other perspective we introduce in our course concerning reckoning happens on the very first day of the course: we give them as a task (to be done before the next class) to use their knowledge of algorithms for the four operations – these algorithms then play the role of artefacts – so to be able to compute in bases other than ten. We launch the task with them, reminding very briefly the idea of base ten numeration and asking then what if we were using, say, base eight. We make sure that they will go back home on that day having a reasonable intuition of what it means for a numeral to be expressed in a non-usual base such as eight, so that 'all that remains' for them is to do the calculations, using the knowledge they already have of basic arithmetic algorithms. Along the way we make them aware that in order to play the game fully in base eight (and not to 'cheat' via base ten), they will need to have access to information about additions and multiplications of one-digit numbers. An extra implicit task is thus for them to construct by themselves the artefacts for calculations in this new environment, namely, the Pythagorean tables for addition and multiplication in base eight. (See Sect. 9.2.2.3 for comments on Pythagoras tables from the perspective of an artefact.) The next class rests on the work they will have done in the meanwhile.

We eventually tell them explicitly that an aim behind this task is to destabilise them to a certain extent with regard to basic arithmetic skills that may seem, at first sight, trivial. This way a context is created enticing them to get into a deep reflective mode about algorithms that they already know how to perform but are probably not in a position to explain or justify. Many students later testify to this 'non-base-ten calculation' task as a genuine thought-provoking moment for them.

10.4 Defining and Representing Whole Numbers

In passages about basic arithmetic laws in the literature for teachers, one may find comments linking, say, the commutativity of addition to expressions such as

$$345 + 67 = 67 + 345.$$

While this equality may be seen as a fine illustration of the property, the use of such a vision introduces a confusion between many aspects related to numbers, in particular between the nature or essence of natural numbers (or of the operations

[5] https://en.wikipedia.org/wiki/Ancient_Egyptian_multiplication

defined on them) and the representation of these numbers via a numeration system – however important the latter may be in practice.

It is for us of crucial importance in our course for teachers to introduce the natural numbers 'in themselves', without any reference to a system for representing them. In the early days of the course, they were introduced as cardinalities of (finite) sets. A major shift occurred in our approach to basic arithmetic when it was decided to restrict sets to a role of 'linguistic' tools for communication, instead of primitive concepts on which the whole arithmetical building should be based (see Sect. 10.3.2 above) and to use the (historically primitive) notion of a *tally* (see this volume, Sect. 9.2.2.1) to introduce whole numbers.

10.4.1 Tallies: A Fruitful Artefact for Whole Numbers

> All writing may be put down, and nothing used but the score and the tally. Shakespeare (1594), *Henry the Sixth*, Part 2, Act IV, Scene vii (OED 2016)

Natural numbers, it was mentioned earlier, can be captured in a robust way by thinking of them as *counting* numbers (see Freudenthal's comments in Sect. 10.3). In a written form, this vision can be rendered concretely through a basic artefact, the notion of *tally,*[6] as well as *sequences of tallies*.

A natural number is 'naturally' defined as a sequence of tallies – a *finite* sequence, of course. Accepting such a sequence to be eventually empty is not a major issue (especially with adults) and allows the introduction of 0, of fundamental importance when addition enters into the picture. The set of natural numbers is thus constituted of all the finite sequences of tallies, and this can be accepted as a working definition with prospective teachers.

Leaving aside for the moment the empty sequence of tallies (a special symbol, such as an inverted triangle, could be introduced for that purpose), the (unlimited) sequence of counting numbers thus begins

$$| \qquad || \qquad ||| \qquad |||| \qquad ||||| \qquad |||||| \qquad \cdots$$

This artefact explicitly allows the notion of successor to be fully seen in action.

In order to gain a necessary level of generality, a notation such as

$$\overline{||\cdots||}^{\,n}$$

may be introduced in order to represent a sequence of tallies of arbitrary length, *n*.

[6] Other words traditionally used in a same sense include *notch, score* or *stroke*. A tally is to be considered simply as a mark, typically a short line segment. In our course, we speak of the '*bâton*' (in French), i.e. the *stick*.

Once we have such concrete models for natural numbers, a fundamental notion to discuss is the equality of two given natural numbers, which is to be captured through the verification that the corresponding sequences of tallies are identical. The natural way to render this idea is via the establishment of a bijective link between the two sequences. In the present context, this scheme of one-to-one correspondence between sequences of tallies appears as a most natural artefact, requiring no sophisticated set-theoretic support. It also leads to the definition of order among natural numbers, when one sequence happens to be exhausted before the other in a search for a one-to-one correspondence.

Such a setting in turn allows for operations on natural numbers to be introduced through operations on sequences of tallies which, in that context, can be accepted as natural and primitive. For instance, the addition of two given numbers *n* and *m* is defined as the juxtaposition of the corresponding sequences of tallies:

$$\overbrace{||\cdots||}^{n+m} \quad = \quad \overbrace{||\cdots||}^{n}\;\overbrace{||\cdots||}^{m}$$

(with equality here being per definition). The sum $n + m$ is readily seen to be a natural number. In a similar vein, the multiplication of *n* and *m* can be defined as the result of replacing each tally in the sequence for *n* by a replica of the sequence for *m*. For matters of convenience, the natural number *n* x *m* obtained as the product sequence can be displayed as a rectangular array (or matrix) of tallies:

$$n \left| \begin{array}{c} \overbrace{||\cdots||}^{m} \\ ||\cdots|| \\ \vdots \\ ||\cdots|| \end{array} \right.$$

From these definitions (using the notion of the empty sequence of tallies) follow immediately, for instance, two basic arithmetical facts: when a sum $n + m$ is 0, then both terms are 0; and when a product *n* x *m* is 0, then at least one of the factors is 0.

It may be noted that a similar artefact for counting numbers is the model of boxes of aligned dots used by Courant and Robbins (1947, p. 2 et seq.) in their study of the laws governing the arithmetic of whole numbers. Addition then corresponds to 'placing the corresponding boxes end to end and removing the partition' (p. 3):

while the multiplication *n* x *m* is defined via a box with *n* rows and *m* columns of dots (eventually reorganised as a box of aligned dots).

10.4.2 Establishing the Basic Laws of Arithmetic

The tally artefact has as a bonus the feature that the basic properties of arithmetic can be actually *proved* and not merely stated or illustrated. For instance, the property of commutativity of addition, mentioned at the beginning of this section, then amounts to the following: given two arbitrary sequences of n and m tallies, one shows that the order of juxtaposition does not matter by an appeal to the obvious one-to-one correspondence – the rightmost tally of m in the sequence $n + m$ is linked to the leftmost tally of m in $m + n$ and so forth. Both sequences $n + m$ and $m + n$ will then be exhausted simultaneously.

In turn all other fundamental properties of addition and multiplication can be proved similarly, thus leading to the establishment of 'the fundamental laws of reckoning' (associativity and commutativity of + and x, identity elements, compatibility of = with + and x, simplification for + and x, distributivity x/+, laws concerning order) (see, e.g. Klein (1932), p. 8, where such rules are simply stated). It is important, with teachers, to stress that these properties speak of the behaviour per se of numbers and certain operations defined on them and, as such, are totally independent of numeration, i.e. of ways of 'writing down' the numbers in a given system.

It may be noted that the artefact of tallies, as a concrete model for whole numbers, comes with practical limitations: for instance, given two sequences made, respectively, of, say, one thousand and one thousand and one tallies, it would be quite cumbersome to compare their size by searching for a possible one-to-one correspondence. But such is not the main point behind this artefact: the really crucial aspect is that we have a precise agreement about what any given natural number 'is'. Such a vision, independent of any numeration system, is a fundamental awareness for a teacher.

The interested reader will have noticed that we are now very close to the principle of mathematical induction and the full set of Peano axioms. While mentioning this en passant, we do not see it as appropriate to insist on such an abstract vision in our course for primary school teachers.

10.4.3 Representing Whole Numbers

Of course, the time comes, in our arithmetic course for teachers, when the concept of number emerging from the preceding tally-wise definition should be related to standard arithmetic practice, and in particular to our usual positional-value numeration system in base ten. The many devices and variants for representing numbers developed throughout history turn out to be very informative artefacts in order to distinguish between two fundamental issues at stake when writing or recording numbers: establishing a numeration scheme for memory or communication purposes, and establishing one for reckoning purposes.

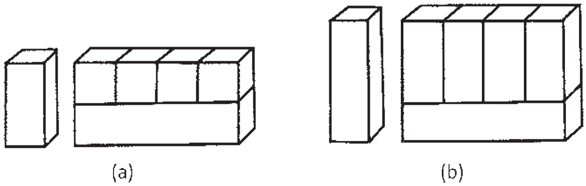

Fig. 10.5 The 'cargo method' of F. Lemay

Many interesting artefacts are available, often of a strongly cultural nature, as regards memory or communication devices for numbers: tallies, Egyptian numeration, Roman numeration, quipus, etc. When emphasis is put on devices facilitating calculations, a different set of artefacts may come to mind – in particular abaci – where the notion of position is part of the artefact itself. (See Sect. 9.2.2 for a rich catalogue of artefacts for whole number arithmetic used throughout history.)

The power of the positional nature of our usual numeration system should not be underestimated when it comes to consider it from a reckoning perspective.[7] Devices emphasising position thus play a crucial role in that connection. Many such artefacts are discussed in Chap. 9, in particular in Sects. 9.3.1.2 (multibase arithmetic blocks) and 9.3.1.3 (spike abaci). In the case of the blocks, the position is transmitted via the size of blocks, while for the abaci it is the place of a given spike on the physical device that indicates position.

Mention is made in Sect. 9.3.1.3 of the possibility of using beads of different colours being piled on the various spikes of an abacus in order to make more evident the difference in the roles played by the beads. The authors then observe that such a use of colours 'seems not advisable, as attention is focused on colours and exchange conventions rather than on order and position'.

While agreeing fully with the importance of stressing order and position when learning about positional value, I would suggest that an artefact integrating colours, and even emphasising exchanges through a colour code, may be of interest on its own, at least when working with teachers.

I wish to briefly describe here an artefact, due to Lemay (1975), based on the Cuisenaire rods and using both the lengths and the colours of the rods in order to develop the core for a positional-value system. The so-called 'cargo method' of Lemay (*méthode des cargaisons*, in French) leads to a mechanical way for exchanging, without counting, a certain sets of rods for a rod of the 'next type'.

We first agree on a certain rod serving as a basis for numeration – the word basis being taken here as well in a truly physical sense. In the example of Fig. 10.5, we use the pink rod (of length four) as the basis, lain horizontally. The exchange process rests on the following rule: if a number of rods of the same colour can be placed upright side by side so as to cover it completely, it can be exchanged for a rod whose length matches the total height of the freight (still standing vertically on the basis).

[7] One may be reminded here of a famous engraving from Gregor Reisch's *Margarita philosophica* (1503), where an allegorical figure of Arithmetic appears to express her preference in a kind of calculation contest between the 'Ancient' and the 'Modern' (see Swetz and Katz 2011).

Figure 10.5a shows a cargo of white rods being exchanged for one red rod, and Fig. 10.5b, a cargo of red rods replaced by a light green one.

This method clearly leads to a fully fledged positional-value numeration system (base four in the example), the length of the successive rods corresponding to the position of digits in a given numeral.

Such an artefact could be related to one involving strictly a colour code, the Cuisenaire rods being simply replaced by colour tokens. The exchange rule then requires grouping the tokens of a given colour (e.g., by counting) so as to form a heap to be exchanged for one token of the next colour. Although possibly considered more abstract, as no physical indication for the 'weight' of a digit (i.e. its position) is conveyed by the colour itself, such a system appears of unquestionable importance. It corresponds, for instance, to two important historical artefacts, the (additive) Egyptian and Roman numeration systems. The fact, in the latter case, that the symbol L, say, corresponds to fifty has no (immediate) physical connection – although the history of the Roman characters may be of interest on its own, as plainly shown by Ifrah (2000, pp. 187–200).

It is thus of importance to convey the idea that the value of a given element may depend strictly on the agreement made about it and not on its physical size. While clearly of fundamental interest for positional numeration, multibase arithmetic blocks, for that matter, do not convey the whole story of numeration. Non-physical (or abstract, if one wishes) exchange codes are also present in concrete artefacts (and daily situations), such as monetary systems. Among the Canadian coins, for instance, the 10¢ coin is smaller than the 5¢ coin, but children have no problem agreeing to this arbitrary value. (The fact that the 'unit', namely, the 1¢ coin, has recently disappeared from physical monetary transactions in Canada raises other interesting numerational issues, as the 5¢ does not really correspond to a heap of five 1¢ but rather takes its value from an abstract agreement.)

10.4.4 Historical, Logical and Didactical Background to the Tallies

Hodgson and Lajoie (2015) offer comments on the long and diverse history of the use of tallies as an artefact for counting numbers, from 'early stone age' to more recent centuries. Using passages from Ifrah (2000), they recall how present this approach to numbers is among many cultures.

Sequences of tallies can also be seen as a practical artefact of ancient times, but one still largely in use even today when counting a not-too-large population, often implemented by means of groupings by five:

In a modern context, this ‘unary’ vision of natural numbers is often encountered in works pertaining to logic, be it in an epistemological context, as ‘a primitive form of notation for natural numbers’ (Steen 1972, p. 4), or in relation to the notion of computability as defined via Turing machines (Kleene 1952, p. 359) (Davis 1958, p. 9). It may be of interest to note in the latter case that in order to have access to a simple notation for 0, a sequence of $n + 1$ tallies is used to represent the natural number n as a ‘tape expression’ on the Turing machine. Steen emphasises the generative aspect of sequences of tallies, starting with the sequence made of a single tally and accepting as a construction rule the adjunction of a tally to a given sequence (see also Lorenzen (1955, p. 121 et seq.), under a chapter entitled ‘Concrete mathematics’[8]).

As an example of a recent didactical application, this ‘*constructive* (or operative) *foundation* of natural number’ presented by Lorenzen, and in particular the ‘calculus of counting actions’ just described, is acknowledged by Wittmann (1975, p. 60) as the basis of the reflections he proposes about the teaching and learning of natural numbers.

10.5 A Miscellany of Artefacts and Tasks for Elementary Arithmetic

I now mention briefly a few other artefacts and tasks pertinent to elementary arithmetic and used in our course.

10.5.1 An Artefact for Focusing on Remainders: Clock Arithmetic

Today is Tuesday. What day of the week will it be in 18 days from today? Or: *It is now 15:30. What time will it be in 1000 hours from now?* Such questions, connected to daily basic arithmetic, emphasise the fact that in many contexts related to division, the remainder may be seen as more relevant that the quotient. Being familiar with the 12-hour (and the 24-hour) clock is an important basic learning. And this in turn may be the starting point supporting the main ideas of modular arithmetic in general, using as an artefact so-called clock arithmetic.

[8] It may be noted that another artefact for natural numbers found in logic, of a more advanced nature, is provided by the so-called von Neumann ordinals (von Neumann 1923). They amount to have each ordinal be defined as the set of ordinals that precede it (the empty set being taken as the starting point, ordinal 0). In such a context, we have, for instance, $3 = \{0,1,2\}$, that is, 3 is a specific set with three elements. A nice feature of this definition of ordinal numbers is that it allows easily the transfer to transfinite ordinals. But we are then a bit beyond primary school arithmetic.

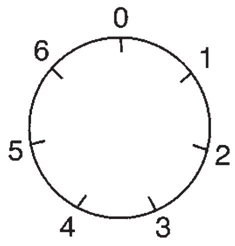

Fig. 10.6 The 7-hour clock

Fig. 10.7 The six-column Eratosthenes sieve (for primes up to 50)

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49

Figure 10.6 shows a slightly special clock, namely, a '7-hour' clock (with 0 used instead of 7, in distinction from a standard clock). It is easy to define basic arithmetical 'clock operations' in this environment, such as addition: 2 + 18 = 6 (which may be read as today being Tuesday, it will be Saturday in 18 days). Subtraction and multiplication are as well easily implemented: we are thus in the vicinity of the ring ***Z****/n****Z***.

10.5.2 An Artefact for Finding Prime Numbers: The Six-Column Sieve

The sieve of Eratosthenes is a well-known device for finding prime numbers. Experience shows that students will very frequently list the natural numbers in ten columns in order to do the sieving. However, using as an artefact a six-column sieve (Fig. 10.7) immediately reveals a nice phenomenon, once the proper multiples of the first two primes have been eliminated: *with the exception of* 2 *and* 3*, all prime numbers are of the form* $6k + 1$ *or* $6k - 1$. This is a fine example of a visual proof: the artefact in itself 'is' the proof of this result. The sieving process is then made considerably easier due to the format of the sieve. See Hodgson (2004, pp. 334–335) for further details.

A crucial related task here is of course to ask: why six columns? A bit of elementary theory of numbers – either using 6-hour clock arithmetic or examining a prime p jointly with its two neighbours, $p - 1$ and $p + 1$ – will be helpful in bringing out the fact that 'we knew' already that any prime, besides 2 and 3, is the neighbour of a multiple of 6. However, the artefact in itself is still of interest in primary education.

10.5.3 An Artefact for Observing Divisors: A Brick in the Wall

I have mentioned above (see Sect. 10.3.3) how Cuisenaire rods may be used for showing in action, so to say, the divisors of a given number. The same artefact may be used for finding concretely common divisors of two numbers, and this way ‘see’ the GCD. A similar remark applies to common multiples and LCM of numbers.

10.5.4 An Artefact for Applying Divisors: The Long Hotel

A well-known problem in the literature for primary school teachers is the ‘Long Hotel’ problem, using the parlance of Cassidy and Hodgson (1982). The underlying artefact is set as follows: there are n rooms along a long corridor, and the n guests consecutively apply an ‘open/close’ process on the doors, Guest #k changing the position of every k-th door (starting with door #k). The question of determining which doors are left open and which ones are closed at the end of the process boils down to identifying the divisors of a given door number, and more precisely the parity of the number of divisors: the perfect squares here stand out as of special interest.

10.5.5 An Artefact for Applying GCD and LCM: The Circle Hotel

Cassidy and Hodgson (1982) introduce a variant to the preceding problem, using the ‘what-if-not’ strategy in problem posing and transferring the process to a circular corridor. The ‘Circle Hotel’ problem provides a nice context to see clock arithmetic in action. It turns out in this case that a single door remains open at the end of the process, the precise nature of the door number depending on the parity of n.

Various additional questions can be asked in this context, related, for instance, to the number of times Guest #k will go round the corridor before stopping (i.e. before touching the same doors again) or the number of doors Guest #k will have touched during the process. The answers to these questions have to do with the GCD and the LCM of n and k and are connected to a famous artefact from a few decades ago, the so-called Spirograph. Figure 10.8a shows the figure generated with the Spirograph by rolling a small wheel of 30 teeth inside a big wheel of 105 teeth, which corresponds to the action of Guest #30, when the Circle Hotel has 105 rooms.

Considered as an artefact, the Spirograph is of special interest on its own, as the elegance and beauty of the figures it can generate may play the role of ‘attention-catcher’ and invite to indulge in further investigation, for instance, about the family of star polygons $\{n/d\}$ (Fig. 10.8b). These are (generalised) polygons obtained by connecting with line segments every d-th point on a circle where n equally spaced points are marked. Such aspects are discussed in Hodgson (2004, pp. 324–328).

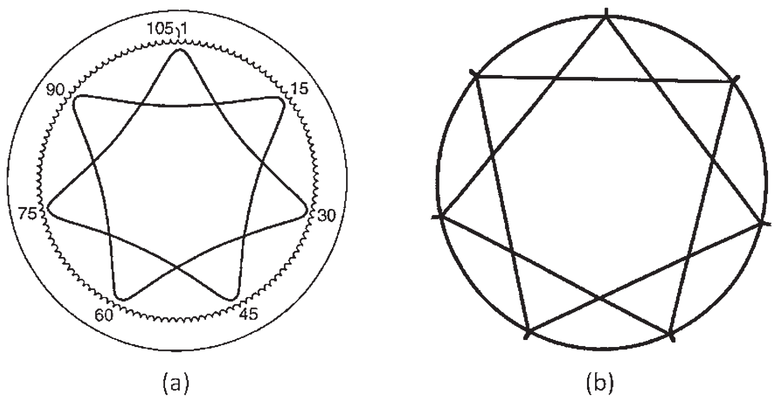

Fig. 10.8 The Spirograph curve 105/30 and the star polygon {7/2}

10.5.6 An Algorithmic Artefact: The Euclidean Algorithm

Understanding and applying the Euclidean algorithm for finding the GCD of two numbers can be seen as a task appropriate for prospective primary school teachers. It has a very positive impact, as this algorithm is totally new to most student teachers. This algorithm provides the occasion for a nice learning experience, as prospective teachers are then in the same situation as their future pupils with respect to encountering a certain algorithm for the first time.

This algorithm, via Bézout's identity, then becomes a fine artefact for solving 'popular' problems, such as whether it is possible or not to obtain a certain quantity of water using two pails whose capacity is known.

10.5.7 Visual Artefacts: Proofs Without Words

The use of a figure for supporting the proof of – or even for 'proving' by itself – a given arithmetic identity has a long history. Already in Euclid's *Elements,* for instance, are results with a substantial visual component (but such was not of course the spirit of Euclid's approach). A classical example is about the area of the square of side $a + b$ (proposition II.4, accompanied by a figure similar to Fig. 10.3b above). Another result of Euclid (proposition II.1) concerns the area of a large rectangle that has been divided into smaller rectangles (Fig. 10.9a). In modern terms, the situation can be interpreted as corresponding to the distributivity of multiplication over addition.

In a similar spirit, Klein (1932, p. 26) proposes Fig. 10.9b as a support for the formula:

$$(a-b)(c-d) = ac - ad - bc + bd.$$

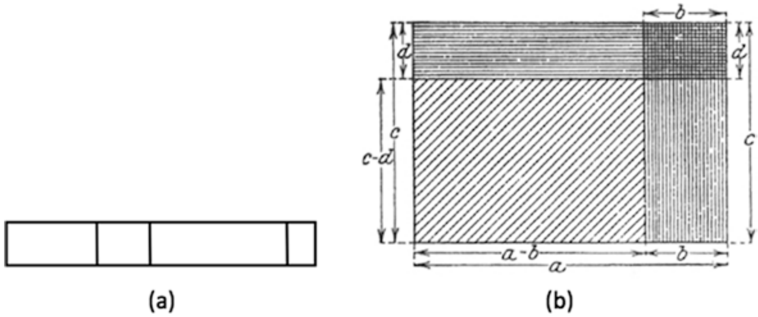

Fig. 10.9 Visual proofs of basic rules of arithmetic

As an additional example, the reader may wish to consider how a 'loaf' of balls packed in the shape of a rectangular prism of edges *a*, *b* and *c* may be seen as proving the associativity of multiplication: $a(bc) = (ab)c$. (Hint: One can imagine slices being cut vertically on the one hand and horizontally on the other.)

10.5.8 Tasks Supporting Positional Numeration

I end this brief survey of additional artefacts and tasks by considering a few tasks intended to foster the understanding of positional-value numeration systems (in particular ones in base ten).

10.5.8.1 Trading Bases

A natural question, when considering a numeration system in a base other than ten, is to consider how to transfer a given (base ten) numeral to the new base, and vice versa. Eventually such a question could be addressed considering arbitrary bases *a* and *b*, going directly from one system to the other without transiting via base ten. Necessary artefacts would then be the Pythagorean tables for the bases at stake. Experience shows that if student teachers are left on their own facing such a task, the following three methods will eventually appear:

- Dividing the given number by the grouping order of the target base (computations then take place in the source base).
- 'Exhausting' the number by multiples of powers of the target base (computations also take place in the source base).
- Evaluating the number in the target base (computations now take place in the target base).

Understanding the principles behind each of these methods sheds light on important aspects of numeration.

10.5.8.2 Paginating a Book

Here are problem-solving tasks providing nice insights into numeration:

- How many digits (i.e. printed characters) will one use in order to paginate a book of, say, 789 pages?
- Reciprocally, if so many characters have been used to paginate a book, how many pages does it have?
- In a similar spirit, how many times does one use the digit 7 when writing down all the numbers from 1 to 99,999?

10.5.8.3 Factorials and Fractions

Two problems emphasising the role played by the prime factors of the base are:

- By how many 0s does 77! end?
- Moving to rational numbers, one may ask when does a fraction a/b (given in its lowest terms) correspond to a terminating decimal expansion?

10.5.8.4 Casting Out Nines

A nice artefact with a long history is the ‘casting out nines’ procedure for testing the validity of a given reckoning, for instance, the computation of a product. Examining the functioning of this algorithm on the basis of clock arithmetic provides a rich context for entering into the functioning of our numeration system. A nice aspect that can be raised about this test is the issue of ‘false positives’. And what about ‘casting out threes’ or ‘casting out elevens’?

More generally, understanding divisibility criteria is a task fostering the understanding of positional numeration.

10.6 Conclusion

A central aim of the arithmetic course we propose to prospective primary school teachers is to help them develop a solid ‘conceptual understanding’ allowing to perceive mathematics not as a mere bunch of facts to be memorised, but rather as a coordinated system of ideas. We hope this way to contribute to the growth of their autonomy and critical analysis skills.

This paper has concentrated on the competency of teachers in mathematics (and especially in basic arithmetic), a crucial aspect of their preparation – but, it goes without saying, not the whole story (see Sect. 9.3.4.2). The interested reader will find in Hodgson and Lajoie (2015) brief comments about how the approach to whole number arithmetic described here can serve as a basis for other numerical contexts (***Z*** and ***Q***), as well as a discussion on how the didactical component of the preparation of primary school teachers can make use of artefacts such as sequences of tallies.

A crucial point raised at different places in Chap. 9 is the fact that artefacts and tasks are intimately linked together. While 'artefacts have the potential to foster students' construction of mathematical concepts in whole number arithmetic' (beginning of Sect. 9.4), they do not exist by themselves, pedagogically speaking. They must be related to some mathematical tasks. And reciprocally, as shown repeatedly in this paper, a given mathematical task is typically based on a certain artefact, be it a physical tool, an algorithm, or a device such as a sequence of tallies, coming both with a facet of concrete implementation and one of abstract conceptual object.

In spite of a comment made in the introduction, namely, that the learning of arithmetic by actual pupils is not an immediate aim of the work we do with our student teachers, I would maintain that many of the artefacts and tasks discussed in our arithmetic course (and in this chapter) can be transferred to pupils, but of course with a necessary adaptation, as our target audience comprises adults with already a substantial, even if at times frail, mathematical background and not young children new to such notions.

Acknowledgements I wish to thank Caroline Lajoie and Frédéric Gourdeau for their most inspiring discussions about the topic of this paper. I also wish to express my gratitude to Linda Lessard, sessional lecturer at Université Laval, with whom I have closely collaborated over a period of more than 35 years in the teaching of arithmetic and geometry to prospective primary school teachers.

This paper is dedicated to the memory of William S. Hatcher (1935–2005), my mentor in the field of mathematical logic and former colleague, who influenced in a significant way the basic vision of arithmetic developed in our courses supporting primary education.

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