The Challenging Relationship Between Fundamental Research and Action in Mathematics Education

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Abstract In this text, associated with my Felix Klein Medal awardee lecture at ICME-13, I develop a reflection on the relationships between fundamental research and action in mathematics education. This reflection is based on my experience as a teacher, teacher educator, and researcher and on what I learned from the responsibilities I had on the ICMI Executive Committee. Using as a filter the concept of didactical engineering, I address several issues: reproducibility, generalization, theoretical diversity, and values, that contribute to making these relationships especially challenging in mathematics education and point out promising evolutions in the field.

Introduction

The Felix Klein Medal awards lifetime achievement in mathematics education research, but what exactly is a lifetime achievement in this field? Different answers that express differences in personal visions can certainly be proposed. The theme I selected for my awardee lecture at ICME-13 expresses my personal vision that the field of mathematics education, even when seen as a field for fundamental research, as is the case in the French didactic culture I belong to, does not develop as a field of pure knowledge. Those who have for decades engaged in didactic research have done so with the desire that their research makes it possible to improve, ultimately, the teaching and learning of mathematics. They have held institutional positions that in fact forced them to combine research and action, and their engagement in action has nurtured their research achievements.

Felix Klein award.

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In this text for the proceedings, my intention is thus to contribute to the reflection on the relationship between fundamental research and action in our field. I do so not only by looking back at my personal experience as a researcher, teacher, and teacher educator, but also by relying on what I learned from the responsibilities that I have assumed in connection with ICMI. I use the concept of didactical engineering that I contributed to establishing (Artigue, 1989, 2014) as a specific filter, reflecting on both its possibilities and limitations in acting as a bridge between fundamental research and action.

A Vision of Relationships Between Research and Action Emerging from a Particular Culture

The field of mathematics education has not developed in the same way in all countries and cultures, which has an impact on how relationships between research and action are viewed. In a recent text regarding mathematics education research in Japan, for instance, Isoda (2015) showed the crucial role played in its emergence and development by the practice of lesson studies, a practice established more than one century ago. Even within Europe, the comparison of the didactic traditions of Germany, Italy, Netherlands, and France, which has been part of the thematic afternoon at ICME-13, has shown differences in that respect. In Germany, for instance, influential researchers such as Wittmann began early to promote a vision of mathematics education as a design science (Wittmann, 1998). In the Netherlands, the development of Realistic Mathematics Education from the seminal ideas of Freudenthal has also been tightly connected with design (Van den Heuvel-Panhuizen & Drijvers, 2014). In Italy, research emerged from a long tradition of pragmatic action-research collaboratively carried out by mathematicians interested in education and teachers, before consolidating within a paradigm of Research for Innovation leading to the development of specific theoretical frames and constructs (Arzarello & Bartolini Bussi, 1998). In France, there is also a long tradition of reflection on mathematics education issues, and famous mathematicians have contributed to it. However, when mathematics education emerged as a research field in the late 60s in the context of New Math, it was with the clear awareness that responsible action required much more knowledge of teaching and learning processes than existed at that time. The disillusions generated by the New Math reform quickly made it clear that "successful"[1] reforms need more than mathematical and pedagogical visions, even when these have solid epistemological foundations. Thus the conviction strongly expressed by the two fathers of the French didactics of mathematics, Brousseau (originally a primary mathematics

[1]I have used quotation marks because the idea of success is always relative to a set of values and aims; these can be influenced by research results but are situated out of range of scientific validation.

teacher) and Vergnaud (a psychologist having had Piaget as Ph.D. supervisor), that a genuine field of research had to develop with both fundamental and applied dimensions. While maintaining a strong connection with mathematics as a discipline and relying on the affordances of psychology, especially the Piagetian constructivist epistemology of that time, they felt that this field should develop its own problématiques, methodologies, and theoretical constructions. Priority had to be given to understanding the functioning of didactic systems, the classroom being the prototype of such systems, over action. Similar to other scientific fields, this field should by no means be a normative or prescriptive field.[2] As a researcher, there is no doubt that I have been influenced by this vision, as well as by the importance it attaches to the specificity of mathematics as a discipline and to epistemological issues. It has definitively influenced my vision and experience of relationships between fundamental research and action.

Giving priority to the understanding and building of the didactics of mathematics as a genuine scientific field means neither that research is free from values nor that it does not have a transformative aim. It simply acknowledges that action on educational systems, even based on estimable values, that is not based on appropriate knowledge, is at least risky. Brousseau, himself an elementary teacher, while interested in Piagetian epistemology, soon understood the limitation of it as a base for didactic action, and this was the source of the project he began in the 60s of developing an experimental epistemology of mathematics education, a project that would lead to the theory of didactical situations (TDS) (Brousseau, 1997).[3] Developing the kind of knowledge needed requires appropriate structures. Brousseau created the Centre d'Observation et de Recherche sur l'Enseignement des Mathématiques (COREM) in Bordeaux, a very innovative structure to which was attached an elementary school, the École Michelet, which for 25 years was a tool of inestimable value for researchers.

As I have explained elsewhere (Artigue, 2016), my first didactic research experience in the 1970s took place in the context of an experimental primary school attached to the recently created Institut de Recherche sur l'Enseignement des Mathématiques (IREM) in Paris, which shared some characteristics with the École Michelet, and this experience was very rewarding. It was in fact a mixture of research and local action. Two IREM colleagues and I had a great deal of freedom over several years to organize the teaching of mathematics with the teachers of the school, under the condition that pupils had covered at least the content of the official syllabus by the end of primary school. A lot of time was devoted to designing teaching situations jointly with teachers and observing and analyzing their implementation. The situations created by Brousseau for the École Michelet were a

[2]As shown by a recent inquiry launched by Gascón and Nicolas, this vision of the field, while in line with Weber's vision of science, is not universally shared among researchers in the field (Gascón & Nicolas, to appear).

[3]For a detailed explanation about how this occurred, one can read Brousseau, Brousseau, and Warfield (2014), Chapter 4, or the long interview of Brousseau realized for the ICME-13 thematic afternoon, accessible at http://www.cfem.asso.fr/cfem/ICME-13-didactique-francaise.

constant source of inspiration. It was in this context that I experienced the power of constructions such as those already offered by the TDS. This was a fascinating experience, and the young and enthusiastic scholar I was at that time was convinced that the knowledge gained through didactic research would change the face of mathematics education. I did not suspect that generalization of local achievements would be so problematic; I was unconscious of the networks of constraints conditioning the life of ordinary didactic systems. In the experimental school, we did not hesitate to free ourselves from these constraints, with the support of teachers and parents, thanks to the indisputable legitimacy IREM gave to our action.

The Fundamental Role of Didactical Engineering

As explained above, the vision of French didactics has been a systemic one since its origin. Research methodologies had to reflect this systemic view. This was the source of the concept of didactical engineering that I contributed to establishing. This concept emerged very early, and the name itself was introduced by Brousseau who had heard about the existence of didactic engineers in Québec. As explained by Chevallard in a seminal text written for the second Summer School of Didactics of Mathematics (Chevallard, 1982)—the first time didactical engineering was collectively discussed by the French didactic community—didactic research needed to create something comparable to the *clinique* in medicine, obliging researchers to access the intimacy of the systems they were studying and making it possible to produce and reproduce phenomena. As stressed by Chevallard, this vision of didactical engineering was in line with the vision of science as a phenomenotechnique developed by Bachelard.

Such a vision of didactical engineering is substantially different from the vision underlying the concept of design research, which has had increasing influence in mathematics education, despite evident similarities. The design work that accompanies the use of didactical engineering has been primarily put in service of understanding the economy and ecology of didactical systems, producing didactic phenomena, and establishing existence theorems. This is how Brousseau himself conceives of his long-term research on the extension of the number field to rational and decimal numbers, which has played an important role in the development of the theory of didactical situations and has been reproduced more than 25 times. In his retrospective book he writes:

The initial objective of the experiment was thus an attempt to establish an "existence theorem":

- Would it be possible to produce and discuss such a process (a constructivist process making minimal use of pieces of knowledge imported by the teacher for reasons invisible to the students)?
- Would the students—all of the students—be able to engage in it?

- Could the request of the process be, for each of the students, a state of knowledge at least equal to that obtained by current, standard methods? (Brousseau et al., 2014, p. 129)

Brousseau is well aware of the complexity of this construction and of the expertise its implementation required from teachers.

> This curriculum was not made to be used in other classes. The sole purpose of reproducibility was to consolidate the scientific observations that we needed in order to test certain hypotheses. The lessons had above all the property of making apparent the enormous complexity of the act of teaching. (ibid., p. 7)

Once again, this does not mean that French didacticians were not aware of their social responsibility. In the text mentioned above, for instance, Chevallard writes that didactics will be judged

> on its ability to realize the knowledge it produces, its ambition to move towards practical and workable answers to the concrete difficulties identified by the practitioners of didactical systems; among the forms of action that relate most directly to its object (its problematics and methodology), that of producing lessons and sequences of lessons practically workable obviously holds a central place. (Chevallard, 1982, p. 30, our translation)

He acknowledges also the difficulty of the enterprise:

> From the didactic realization, as it takes place within the research process, to the production of sequences of lessons, there is all the distance of a true decontextualization, acting in several registers (epistemological, human, institutional, etc.) …. This situation thus leads to the issue of the "user guide" for such productions, that is to say, the problem of conditions for a non-denaturing didactic recontextualization which must guarantee its successful integration from epistemological, human, and institutional points of view—cultural in a single word—to the didactical engineering, beyond the scientific value of the research findings which constitute its raw material. (ibid., pp. 31–32, our translation)

As I wrote at the beginning of this text, considering the field of mathematics education a genuine scientific field and stressing the importance to be given to fundamental research in this field does not mean that it is the pure desire for knowledge that motivates didactic research, even in its most fundamental aspects. This motivation comes, in fact, from the ultimate desire of improving mathematics education through the knowledge gained, with a diversity of possible views regarding what improvement means. This is clearly a leitmotiv in my research work, even if this aim is pursued through a diversity of forms of research. It partly explains my privileged use of didactical engineering as a research methodology. Beyond the identification of didactic phenomena and laws, I always have seen in this methodology a means:

- to explore forms of life of mathematical and didactical practices that could not be observed in ordinary classrooms, but seemed to me more satisfactory from an epistemological perspective;
- to study the conditions and constraints influencing their economy and ecology; and
- to understand what should be done in order to help them grow and expand.

An Example: Didactical Engineering for the Teaching of Differential Equations

For instance, when I began to work on the teaching of differential equations in the 80s, as a mathematician I was working in the area of dynamical systems and experienced a kind of schizophrenia between my activity as a researcher and as a university teacher. The standard course on differential equations I was giving to second-year students was not especially problematic, but I was convinced that this course focusing on the algebraic solving of classical equations (in finite terms or using Taylor series) gave them a wrong idea of the field and of the important questions in it, both those internal to it and those resulting from its connection with other fields. With colleagues from the University of Lille 1, we decided to explore the accessibility of a first-year university course respectful of the epistemology of the field, combining thus algebraic, numerical, and qualitative approaches and incorporating modelling activities. Our hypothesis was that the affordances of technology made such a course accessible. The design of the course was based on careful preliminary analyses, combining epistemological, institutional, and cognitive dimensions according to the standards of didactical engineering. The first experimentation globally confirmed the accessibility of the course, while showing an important gap between the students' ability to analyze phase portraits of differential equations provided by the software used, or even predict phase portraits, and their ability to prove their conjectures. To ensure the viability of the qualitative approach, from the second experimentation, we introduced and legitimated specific forms of reasoning and proof combining institutionalized graphical notions (such as the notion of fence) and arguments with analytical formulations. This move was effective and the ecological viability of this construction was again confirmed in the third year, when the experimental section concentrated the students entering math university courses with low grades in mathematics at the scientific *baccalauréat* (Artigue, 1992; Artigue & Rogalski, 1990).

This research was certainly motivated by the desire for improving the actual teaching of differential equations in French universities. However, I have to acknowledge that, while being very well received, it had limited influence in France beyond the University of Lille 1 where the course was implemented for more than 10 years. The use of this didactical engineering was generally limited to the first situations of the qualitative approach. As explained in Artigue (2016), using the conceptual tools provided by didactic research, I am currently able to better explain why. As mentioned above, in order to ensure the viability of the qualitative study with first-year students, which meant that they would be able to prove conjectures, we were obliged to legitimate theorems and proofs combining graphical and analytical formulations and arguments. Due to my research expertise in this mathematical area, I could attest that mathematicians used these reasoning modes, even if in published papers they adopted a more formal discourse. This was an important ingredient for legitimation. Moreover, at the University of Lille 1, in the experimental section, important work was systematically carried out at the beginning of each academic year on the graphical register of representation in order to make it operational and change its status. However, this institutional situation was exceptional, and the legitimation of such arguments violated the rules of the didactic contract governing university courses in Analysis at that time. The contrast between the interest raised by this didactical engineering and its very limited impact clearly showed that its ecological viability depended on conditions regulating the teaching of Analysis and, more globally, the status given to graphical representations in mathematics teaching. These conditions were situated at higher levels of the "hierarchy of didactic codetermination" than the didactical engineering itself, as can be expressed today using a construct of the anthropological theory of the didactic (ATD) that did not exist at that time (Chevallard, 2002).

This example well illustrates the fact that the extension of any didactical engineering, beyond the research and ecologically protected environment where it has generally been developed and tested and its conversion from a research to a development object, must seriously take into account these different levels of conditions and constraints. Even when research allows us to understand the complex system of conditions and constraints that condition the ecological viability of a didactic construction, which is in itself an important research outcome, acting on such conditions and constraints is hardly in the hands of researchers. Action requires the building of new partnerships and collaborations beyond those at play in the joint development of didactical engineering at a research level. This is a reality to which all those today engaged in design research in mathematics education are sensitive, even if they do not use the same words to express this sensitiveness (see for instance Swan, 2014).

Before moving to the next point, I would like to point out that this example also shows the role played in research by values that, quite often, remain implicit. As explained above, this research emerged from the desire to better align the teaching of differential equations with the current epistemology of the field from the first contact with it at university. The fact that such a move constitutes an educational improvement was a non-questioned starting point of an epistemological nature. Beyond that, my didactic culture has made me especially sensitive to the optimization of the mathematical responsibility of the students in the design of situations, to the precise choice of their didactic variables and to the organization of the "adidactic milieu" with the meaning given to these terms in the TDS. One can see here the clear influence of the vision of learning in this theory; it combines adaptation and acculturation processes, but adaptation processes are given a fundamental role. Adopting TDS as a theoretical reference means that such vision and associated principles are accepted. The research carried out and the expression of its results are thus conditioned by these visions and principles, even if a number of results, for instance, those regarding the didactic contract at stake in the teaching of analysis about graphical representations and more globally those issued from the epistemological and institutional analyses or those regarding students' cognitive difficulties with qualitative proofs, have and have been proved to have more general value. However, I have to confess that this question was never addressed in the publications associated with this research.

Issues of Reproducibility

In the French didactic community, from the early 80s, didactical engineering developed thus as a methodological tool primarily at the service of research and not as a development tool. Development, in fact, was not an object of scientific inquiry as it can be in design research. This did not prevent the designs produced by research to migrate in the educational system through different channels. Researchers were members of curricular commissions; many developed their research in close connection with the IREM network and contributed to the activities of resource development and in-service teacher education that this network had in charge; others worked with primary and secondary teachers at the Institut National de Recherche Pédagogique (INRP), now Institut Français de l'Éducation (IFÉ), in research-action groups, for instance the group on primary education producing the ERMEL collection of teacher books that has been very influential in teacher education (see, for instance, ERMEL, 2005); and a few also co-authored textbooks. These conditions favored the percolation of knowledge, but this percolation process was not taken as an object of study. However, it soon became evident that the dissemination of research engineering designs through such channels in many cases systematically resulted in their denaturation, and this observation attracted my attention to issues of reproducibility. I made the hypothesis that one of the sources of the observed denaturation could be the vision of the reproducibility of the didactical situations conveyed, more or less explicitly, by didactic texts and educational resources. Roughly speaking, didactical situations were proposed as objects to be reproduced, with the implicit idea that following the proposed trajectories would result in the expected learning effects being obtained.

To test this hypothesis, I built a stochastic mathematical model of this vision. Using direct computations and complementing these by computer simulations using Monte-Carlo methods, I invalidated the model, thus invalidating the vision of reproducibility conveyed by the literature. More precisely, I showed that if such reproducibility was observed, it could not generally result from the reasons and characteristics of the design invoked. Other forces were at play whose action and mechanisms remained tacit. Using data coming from a previous research on primary students' conceptions of the circle, I showed that the model allowed researchers to expect the appearance of regularities, but, as is generally the case in non-linear dynamic systems, these regularities would be situated at structural levels other than those usually expected (Artigue, 1986). This led me to articulate a kind of principle of incertitude between the internal reproducibility aimed a priori (conserving the meaning of actions and discourses despite possible variations in the trajectories) and external reproducibility (at the more superficial level of classroom trajectories). According to this principle, any effort made to ensure external reproducibility has a systematic cost in terms of internal reproducibility. This result showed, for instance, that the phenomenon of obsolescence at play in the reproduction of the COREM didactical engineering on rational numbers by the same teachers, year after year, that had been identified by Brousseau some years earlier

(Brousseau, 1981) was an instance of a more global didactic phenomenon. This work of mathematization of the didactic field itself was not developed further, but it strongly influenced my conception of resource development, a crucial point as far as action is considered.

For instance, in the resources associated with the research on differential equations mentioned above, I tried to overcome the trap of linear descriptions and to open the dynamics of situations, envisaging, for instance, possible bifurcations. I also tried to approach more explicitly the key issue of the sharing of mathematical responsibility between teacher and students than was usual in classical engineering design at that time and whose underestimation appeared as a major source of denaturation.

The spontaneous conception of reproducibility was thus proven to be an obstacle to the dissemination of didactical designs coming from research and their productive use for action. I would not say that this didactic obstacle has been overcome. Many current educational resources still implicitly convey the same notion of reproducibility by giving the impression to the reader that classroom and individual trajectories can be fixed by a succession of tasks and questions, without damage. However, this misunderstanding about what can and cannot be reproduced with what consequences is only one of the many difficulties met in the transition from research to action.

Issues of Generalization

Establishing productive relationships between fundamental research and action obliges one to address the difficult issue of generalization. I have already evoked one case in which generalization was out of range under the current institutional conditions and constraints with the research on differential equations. However, understanding difficulties of generalization in the field both requires "vertical" analyses of conditions and constraints as the one I have sketched above, and "horizontal" analyses, according to the distinction we introduced with Winslow in our meta-analysis of comparative studies in mathematics education (Artigue & Winslow, 2010). The main reason is that mathematics education is a field geographically and culturally situated. As stressed in (Artigue, 2016), we all know today, even when we belong to dominant cultures—mine is certainly one of them in the field of mathematics education—how our insufficient sensitivity to the diversity of social and cultural contexts has been the source of hegemonic visions and abusive generalizations and exportations (see, for instance, Nebres's, 2008 contribution at the Symposium organized for celebrating the centennial of ICMI). This does not mean that didactic research does not identify regularities, such as didactic phenomena that transcend cultural specificities, for instance, the necessary existence of a didactic contract in any didactical situation, which others might call socio-mathematical norms; the specific economy and ecology of taught knowledge that regulates the processes of didactic transposition that cannot be reduced to a process of elementarization of knowledge; and the existence of epistemological obstacles to overcome, for instance, in the transition from whole numbers to rational and decimal numbers or in the learning of more advanced concepts such as the concept of limit, to give just a few examples. This means that even when there are regularities, didactic phenomena that to some extent transcend cultural specificities, the way the knowledge of these can be put at the service of action is highly dependent on the conditions and constraints of each specific context.

Another important point is that we hardly know the exact field of validity of the regularities we identify. Quite often, we tend to over-generalize regularities inferred from local studies without enough evidence. Again, this does not mean that local studies cannot give access to rather general phenomena and didactic laws. The very powerful concept of didactic contract, for instance, emerged from one of Brousseau's studies known as the Gael's case (Brousseau & Warfield, 1999). This only became a fundamental concept of the TDS, however, because it proved its capacity to make a diversity of students' and teachers' behaviors and interactions beyond this one case understandable and because this understanding and the associated technological discourse in the sense of ATD was able to find its place in the global theory of didactical situations.

Issues of Theoretical Diversity

I would like to come now to another crucial issue when thinking about the relationship between research and action: theoretical diversity. I became especially sensitive to this issue when I entered the ICMI Executive Committee in 1998. As I have explained elsewhere:

> Many times, in recent years, due to my ICMI responsibilities I have been confronted with questions about existing knowledge on particular educational issues that might inform teaching practices, curricular decisions, or teacher education. Faced with such questions, most often I was unable to give a clear answer, and often even unable to orient my interlocutor towards a set of references that would help her (him) develop a coherent and synthetic vision. Of course, things are not so simple in education as in mathematics. We must accept that most of the certainties we acquire are, except for the most general ones, situated both in time and space, and that it is difficult to know their exact domain of validity. The question of how research knowledge may inform practice in particular contexts is a difficult question, still insufficiently addressed. Nevertheless, the theoretical explosion of the field, the diversity of approaches, constructions, discourses, and the lack of connection substantially increases the difficulties of capitalization and dissemination (Artigue, 2016, p. 262).

In the last decade, I have been involved in different projects developed at the European level in order to address this issue in the framework of what is often known today as the "Networking of theories." These projects have also revealed to what extent theoretical diversity deeply permeates our research practices or, in ATD terms, our research praxeologies (Artigue & Bosch, 2014), making connection efforts directly situated at the level of theoretical objects hopeless. We certainly underestimated this point until recently. Personally, in the last decade, I have learnt the price to pay in order to overcome the current state, the necessary effort of decentration, and the uncompromising questioning required to understand the actual use we make of theoretical frameworks beyond their mere ritual invocation. I have also learned the necessity of developing specific devices that can allow us to take our research practices as objects of study without distorting them and the importance of developing metalanguages to support joint work and communication. One example is the metalanguage of key concerns, initially created in the Technology Enhanced Learning in Mathematics (TELMA) European team (Artigue, 2009), then refined in the project ReMath (Lagrange & Kynigos, 2014), which I also used as a guide when, together with Blomhøj, I investigated what the major didactic approaches have to offer to the conceptualization of inquiry-based learning in mathematics education (Blomhøj & Artigue, 2013).

Seen from the outside, such a form of research may be perceived as just theoretical and without possible practical interest. I would like to reaffirm here my conviction that this is not at all the case. Limiting the current fragmentation of the field and inventing forms of discourse that improve the quality of communication and support capitalization of knowledge is an absolute necessity for us if we want to be able to determine exactly what we know and what we do not know, as is legitimately expected from a mature research field, and if we want to create solid grounds for productive relationships between research and action.

Issues of Values

I have already briefly evoked this issue in a final comment regarding the research on differential equations, but this fundamental issue of values certainly needs more than a small comment. Mathematics education, for better or for worse, is a field in which science and values strongly intertwine. Some years ago, I was asked by UNESCO to pilot the realization of a document on the challenges in basic mathematics education (UNESCO, 2011). The group of experts involved agreed that the main challenge to be addressed was that of "quality mathematics education for all." However, coming to an agreement on what was the exact meaning that should be given to this commonly used expression was another story. We had long discussions that reflected differences in perceptions and values. Of course, these also had an impact on the vision we each had of the types of actions to be promoted in order to progress towards this goal.

Even within my own culture, even for theories with close epistemology, such as the theory of didactical situations and the anthropological theory of the didactic, there is no doubt that the forms of didactical engineering research developed are different. TDS relies on a constructivist vision of learning, which is not the case for ATD. The vision of didactic engineering in ATD, which expresses in terms of finalized and non-finalized study and research paths (Chevallard, 2015) with the role given in these to the dialectics between media and milieu and the opening of trajectories within the global structure of a reference epistemological model, is substantially different from the traditional vision of didactical engineering supported by TDS, which is structured around the search for fundamental situations. Actions inspired by these two research works take rather different forms.

In the mathematics education field, I often have the feeling that values are not questioned enough, that communication is often based on fuzzy consensus, and that the fact that the very diverse epistemologies existing in the field have no reason to lead to compatible decisions in terms of action is not really addressed. Establishing adequate relationships between research and action certainly needs systematic efforts to improve the situation, making more explicit the values underlying research and how these have an impact on research results and the vision of action, while also questioning these values.

Moving Forward

Up to now, I have mainly listed and discussed difficulties and issues faced when trying to make research a source of inspiration for action, using the case of didactical engineering in particular to illustrate my reflection. I would like, however, to express my conviction that the evolution of the field of mathematics education research, both theoretically and empirically; the number of existing realizations at different scales and in diverse contexts; the reflexive work carried out on these; and the communities and institutions established have substantially and productively influenced our vision of the relationship between fundamental research and action. I would like also to insist on the fact that we can today rely on conceptual and methodological tools much more powerful than was the case a few decades ago to address these issues and can therefore move forward. In the next part, I will briefly evoke what I see as major advances in that direction, beyond those already mentioned.

Didactical Engineering and Design-Based Research

Staying within the perspective of didactical engineering, the current research work carried out on the transition from research to development of didactical engineering is one promising avenue. As already explained, didactical engineering has developed in France as a research methodology, despite the fact that initially the exact role that would be given to it was not so clear. In the seminal text by Chevallard mentioned above, for instance, Chevallard distinguishes a priori between engineering work for research, for action, or both of them. For a long time, as already explained, the migration of didactical engineering designs or pieces of them from the research sphere to the action sphere was not an object of study. It developed outside any form of theoretical control, and the negative consequences of this state of affairs have been pointed out. This is no longer the case. I give as an example the research developed by Perrin-Glorian and her colleagues around the idea of second generation didactical engineering (Perrin-Glorian, 2011), but other promising projects have been developed, for instance, within the structure of the Lieux d'Éducation Associés (LÉA), joining schools and research laboratories, recently created by the IFÉ. The Arithmétique et Compréhension à l'Ecole Elémentaire (ACE) project[4] piloted by Sensevy is a good example.

Obviously, beyond the sole concept of didactical engineering that emerged in the French didactic tradition, another important evolution is the increasing role given to design-based research in the field and its associated theoretical and empirical work, with the consideration of scaling up as a major issue requiring specific research and methodology. This evolution is evidenced, for instance, by Cobb's research, for which he was awarded the Hans Freudenthal ICMI medal in 2005,[5] or the research and development work carried out by Swann and Burkhardt, who have been jointly awarded the first Emma Castelnuovo ICMI medal in 2016.[6]

Beyond these two evolutions directly linked to design, more global evolutions of the field offer substantial help to move forward the relationships between research and action in the field. I focus here on three of them.

The Increased Importance Taken by Socio-cultural and Anthropological Perspectives

Socio-cultural and anthropological perspectives allow us to better take into consideration the complexity and diversity of institutional, societal, and cultural conditions and constraints to which didactical systems are submitted. They provide conceptual and methodological tools to identify these and their respective strengths, to understand how they interact and shape the dynamics of didactical systems, and to reflect on how they can be moved when it seems a condition necessary to effective action. Beyond that, they help enlarge our vision of design. I have no doubt that this is indeed the case, for instance, with the conception of didactical engineering recently developed in ATD, especially:

- with the concept of non-finalized study and research path already mentioned, which provides an interesting theoretical framework for the conception of teaching strategies based on project and interdisciplinary work and
- with the dialectics between media and milieu, a powerful tool to take into consideration the important changes in access to information and inquiry practices induced by the technological evolution, especially the internet.

[4]http://python.espe-bretagne.fr/ace/.

[5]http://www.mathunion.org/icmi/icmi/activities/awards/past-recipients/the-hans-freudenthal-medal-for-2005/.

[6]http://www.mathunion.org/icmi/activities/awards/emma-castelnuovo-award-for-2016/.

The Development of Research on Teachers' Practices

I have lived the shift of attention of research from the student to the teacher. From unquestioned actors in the didactic relationship, teachers, with their beliefs, knowledge, systems of practices, and professional development, have become major figures of interest for research. The body of knowledge that has been built in that area since at least the early 90s is of the highest importance to improve the links between fundamental research and action. It helps understand the strong limitations of the strategies traditionally used to disseminate research results in the profession. Within this area of research, which is very diverse, I personally find constructions and approaches that address teacher work or activity in a rather global and systemic way to be especially useful for the reflection on relationships between research and action. This has been the case, for instance, in the double approach (didactic and ergonomic) of teachers' practices developed by Robert and Rogalski (Robert & Rogalski, 2002), in which teachers' practices are approached through five interconnected dimensions, including personal, social, and institutional determinations, and in the structuring features of classroom practice framework developed by Ruthven to analyze how teachers integrate or fail to integrate new technologies (Ruthven, 2009), to mention just a few examples.

The shift of attention of research from the student to the teacher has helped understand the exact nature of teaching expertise and better acknowledge its specificity. In many contexts, this has had an impact on the vision of the relationships between researchers and teachers, as attested, for instance, by the development of the idea of "community of inquiry" (Jaworsky, 2008), and as a consequence on the vision of relationships between research and action. Teachers can no longer be considered implementers of resources prepared by others, researchers, or those in charge of the educational transposition of research ideas and constructions. Teachers are themselves authors; they should be considered as such and supported in their authorship activity. The recently published ICMI Study on task design makes this clear (Watson & Ohtani, 2015).

The Development of Instrumental Approaches

Having contributed to the emergence and development of instrumental approaches (Artigue, 2002), I have had many opportunities to think about their affordances from a research perspective, but also in terms of the relationship between research and action. The first affordance I have seen is the fact that these approaches have made visible and understandable the essential processes of instrumental geneses that had nearly escaped the attention of those researching or promoting the educational use of technology. Blind points have thus been revealed, and the detrimental effects of such blindness identified. Distinctions, such as the one between the epistemic and pragmatic valence of techniques and schemes, the fact that technology disrupts the balance between these valences, and the fact that restoring appropriate balance requires new types of tasks, that have been established through didactical engineering research, have shown the profound inadequacy of teacher education in that area. They also have set conditions for the elaboration of educational resources. Beyond that, another interesting point is that this instrumental perspective has been progressively incorporated into a diversity of established theoretical frameworks, such as ATD in my initial work with close colleagues, activity theory for others, and the theory of semiotic mediation for Italian colleagues. Each of these incorporations has led to variations of the instrumental approach, despite the shared reference to the seminal work by Rabardel (2002). These different incorporations influence the resulting propositions in terms of design, as has been shown, for instance, in the ReMath project already mentioned (Lagrange & Kynigos, 2014). Another important point for my purpose here is the shift of attention from the student to the teacher, which, once again, has led to the extension of the approach to teachers' instrumental geneses, both personal and professional, then incorporated into a wider notion of genesis of use (Abboud-Blanchard & Vandebrouck, 2012). It is also the extension of this approach to the documentary work of the teacher, a domain of study today very active (Gueudet, Pepin, & Trouche, 2012). There is no doubt that a better knowledge of this essential dimension of teacher work and how it is affected by the technological evolution is of the highest importance for the relationships between research and action.

The last positive evolution I would like to mention is the development of projects of different scales in a diversity of contexts that provide new empirical bases to the reflection on these difficult issues. In recent years, I have been involved in a variety of European projects[7] aiming at the large-scale dissemination of inquiry-based education in mathematics and science following the publication of the report known as Rocard's report (Rocard et al., 2007). I have seen the intense and creative reflection and work that has gone into these projects to develop a more adequate vision of dissemination processes. I have seen the importance of the empirical work carried out. I have also again experienced up to what point attempts at making research at the service of action are themselves the source of questions for fundamental research.

However …

However, we cannot deny that such accomplishments must come to grips with growing social and political pressures exerted on both research and educational systems by economic and competitive visions and values of education that are often

[7]The Fibonacci, Primas, and Mascil European projects (see their respective websites: www.fibonacci-project.eu, www.primas-project.eu and http://www.mascil-project.eu.)

at odds with ours. Tensions and inconsistencies result from this situation that are imposed upon all educational actors. I could make a long list of such inconsistencies. I will just mention some recently experienced in the frame of the European projects just mentioned. As I have explained, such projects aim to organize the dissemination of inquiry-based practices in mathematics and science education. However, in most countries, this goes along with institutional forms of assessment guided by another logic that are contradictory to the form of mathematical and sciences practices that inquiry-based education wants to promote. They put teachers in a double-bind situation.

Eight years ago, in our plenary lecture with Kilpatrick at ICME-11, we denounced the *diktat* of randomized controlled trials as the best if not the only acknowledged source of knowledge in the field. As Kilpatrick said:

> There are far too many research questions for which either randomized controlled trials would be impossible or an appropriate study would require so many controls as to make the interventions, whatever they are, unrealistic …. When narrow criteria are applied, what happens—in the cases I have seen—is that too much is left to untested opinion and individual experience. Not enough use is made of the professional community's judgment and experience. (Artigue & Kilpatrick, 2008, p. 10)

We could say the same today and this pressure did not at all vanish. As an international community, we must denounce these pressures and inconsistencies and try to counter their negative effects on the establishment of productive relationships between research and action. ICMI has here a fundamental role to play.

Conclusion

In this lecture, I have only addressed very partially the difficult issue of the relationship between fundamental research and action. I have used the filter of didactical engineering, that is to say, a "design" filter, as a guide for the reflection, but I perfectly know that action on didactic systems may take a diversity of forms and that this filter, as any filter, is reductive. The primary reason for my choice is not my personal investment in the development of didactical engineering. Rather, despite the fact that I am deeply convinced that it would be an error to reduce mathematics education to a design science, as has been proposed sometimes, I am convinced that design activities, whatever they are named and considered, have a fundamental role to play in the development of this field of scientific knowledge and in the way the knowledge gained can be put at the service of action. I have pointed out and discussed some of the major issues that arise when the relationship between research and action is looked at through this filter, such as reproducibility, generalization, and values, but part of the discussion has certainly more general value. In my opinion, up to now, these three issues have not found satisfactory answers and need to be addressed more seriously by the community. I have also tried to show that advances in design, more global evolutions of the field as a scientific field, and not least the growing number of projects trying to put research at the service of action in a controlled way make us today better equipped to move forward, but I have also stressed the counter-productive effects of politically related abusive pressures and inconsistencies that must be vigorously denounced.

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