Chapter 5
Processing Direction with Ordered Fuzzy Numbers

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Abstract It was already mentioned in previous sections that the Ordered Fuzzy Number (OFN) model can represent a kind of tendency or direction. However, for a real practical use of this feature the tools for processing it are also needed. Of course some kind of quantitative processing is provided by the definitions of calculations, but there is much more potential for this feature apart from arithmetic operations. This part presents the idea of a property of processing data called **sensitivity to the direction**. The main focus here is placed on the proposition of a direction determinant parameter that can be understood as a kind of measure of a direction. This determinant is a basis for the definition of such elements as the compatibility between two OFNs and also for an inference operator for a rule where the OFNs were used. The propositions of such operations are the important part of these sections of the book.

5.1 Introduction

The Ordered Fuzzy Number (OFN) model introduces a new feature, the direction. It is the representation of order of the *up-part* and *down-part* of an OFN from Definition 4.1 in Chap. 4. It is used for defining those arithmetical calculations that do not have to produce more imprecise results. But there is another potential of this feature. In fact if we can use OFN to describe the situation, *"A vehicle speed is about 50 km/h and it is growing,"* it would be more efficient to have the potential to use it not only for calculations but also for more complex processing as, for example, in the rule, *"IF speed is 50 km/h and is growing, THEN safety of a city drive is 75% but it is lowering."* In general, it is similar to the idea of the gradual fuzzy system (see [8]), however, the source of the OFN concept is quite different. An interesting approach to trend modeling using the classical fuzzy numbers idea is also presented in [11], where the trend is understood as a gradual dependence between attributes. However,

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gradual fuzzy rules have a form, "*The more X is F, the more Y is G*," but here for the OFN model a more appropriate form is *IF X is in F which is growing/decreasing, THEN Y is in G which is growing/decreasing*. Moreover, the OFNs in a natural way represent a tendency unlike the classical fuzzy sets/numbers where modeling a trend requires additional actions.

It was already presented in the previous chapter (see Sect. 4.7) of this book that the reversing of the axes in the definition of OFNs compared to typical fuzzy conceptions reverses focus in the analysis of the problems. Functions that form an OFN have a target set that is a universe of real numbers. It seems proper if we want to model a quantitative problem. This reversal does not prevent the OFN model from being a tool for an imprecise data representation. Additionally, arithmetic operations are not the only form of processing the quantitative values. One of great advances of fuzzy set theory is easy and intuitive modeling of the linguistic formulas with the reference rules. If we want to retain this advantage also for the OFN model we need a basic tool for comparing two values that can be called a compatibility. For practical linguistic use the compatibility is a result of the sentence or statement type *A is B*, where *A* and *B* are imprecise values, the OFNs in this case.

Presenting the tools for the processing of OFNs other than direct calculations is the main goal of these sections of the book. The basic idea here is to preserve good intuitiveness of the general fuzzy approach and combine it with the tendency modeling potential of the OFN model. The methods presented in the next sections are *sensitive to the direction* (see [23, 25]).

Remark 1 As the **sensitivity to the direction** we understand the property of operation. This property means that the result can be different if we change the direction of the OFNs used in the operation.

It should be noted that the above remark is a general postulate, not a formal definition of the property. The problem is quite complex, thus more explanation is needed. We especially postulate that the result changes if only one of the components (OFN) of the operation will change the direction (see Definition 4.2 the *reversal of direction* operation from Chap. 4). In many cases where two data items change a direction our intuition suggests the result should not be changed. For example, let us look at the linguistically described rules that consider tendency:

- **IF** *speed* is *decreasing* **THEN** *safety* is *increasing*.
- **IF** *speed* is *increasing* **THEN** *safety* is *decreasing*.

Both of them express the same intuition, yet with opposite tendency, thus the change of direction for both values *speed* and *safety* should not really change the result. In addition, when analyzing sensitivity to the direction in the OFN methods, it is necessary to consider their specificity such as the improper OFNs (see previous chapter Sect. 4.4). Thus, a method that is generally sensitive to the direction may give the same result despite change where the *up-part* and *down-part* of the given OFN are equal. Apart from many improper OFNs such a situation will also arise in the singleton case. Thus the lack of change in the result for some specific situations does not negate the method as one that is sensitive to the direction. Therefore when

we postulate for a given method to be sensitive, the words "change of direction **can** (not must) change the result" are a clue.

It is worth noting that the basic arithmetic operations on the OFN model presented in Sect. 4.5 are generally sensitive to the direction. If the *up-part* and *down-part* of OFN A are not equal, then the *reversal of direction* operation (see Definition 4.2 from previous chapter) generates $A|^- \neq A$. Therefore the result of an arithmetic operation will be different after reversal of the single input value.

The purpose of this chapter is to propose a full set of methods and operations to define fuzzy systems based on OFNs that are sensitive to the direction feature. Therefore in the next sections, a general tool for processing a tendency of OFNs is presented. It is called the **direction determinant** (see also [24, 25]) as it is a kind of measure of direction for a given element of OFN support. Next the compatibility of OFNs as a result of statement *A is B* is proposed see [25] that uses the direction determinant. Finally a proposal of a technical inference method is presented that is meant to be a practical realization of the rule *IF X is A THEN Y is B*.

5.2 Direction Measurement Tool

The key element of the OFN model is the order between the *up-part* and *down-part*, which is independent of the real numbers. This can also be called the direction or orientation. It is taken into account in the definitions of arithmetic operations and their extensions, which make the calculations flexible and unified and more importantly, their properties and relationships are consistent with calculations on real numbers (see previous part of this book as well as [22]). Thus it seems natural that information processing methods based on OFNs also take into account the direction. Here the tool that allows meeting this assumption in defining methods is presented. However, it is helpful to start with a supporting structure that simplifies further description.

In general, the propositions presented in this section refer to the concept of the membership function for the OFN model presented in Sect. 4.3.3.
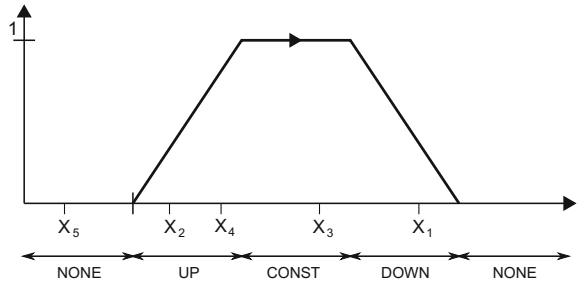
5.2.1 The PART Function

The *PART* function as the result presents information about the part of the OFN that contains the given argument [24].

Definition 1 For the OFN A defined on X the PART function $X \rightarrow Y$ is determined as follows.

$$
PART_A(x) = y = \begin{cases} CONST_A : x \in CONST_A, \\ UP_A : x \in UP_A, \\ DOWN_A : x \in DOWN_A, \\ NONE_A : x \in NONE_A. \end{cases} \tag{5.1}
$$

Fig. 5.1 Specific parts of the support of an OFN

where:

$x \in X$,

$Y = \{CONST_A, UP_A, DOWN_A, NONE_A\}$,

$CONST_A$ – A subset of X for which the membership function of A number is equal to 1.

UP_A – A subset of X for which the inverse of the up-part has values.

$DOWN_A$ – A subset of X for which the inverse of the down-part has values.

$NONE_A$ – A subset of X for which the membership function of A number is 0.

Figure 5.1 illustrates the effect of the $PART$ function. Example results presented there are as follows.

$$PART(x_1) = DOWN$$
$$PART(x_2) = UP$$
$$PART(x_3) = CONST \qquad (5.2)$$
$$PART(x_4) = UP$$
$$PART(x_5) = NONE$$

Fuzzy numbers are fuzzy sets defined over the space (or subspace) of real numbers. Thus the sets UP, $CONST$, and $DOWN$ can be treated as numerical intervals (see also Sect. 4.2 from previous chapter). We use the following denotations of their boundaries.

$$UP = (s, 1^-)$$
$$CONST = [1^-, 1^+] \qquad (5.3)$$
$$DOWN = (1^+, e)$$

5.2.2 The Direction Determinant

The direction of the OFN is an additional property in comparison with classical fuzzy numbers and its meaning is different from the degree of membership. Therefore, if we want to process the full information contained in the OFN, we need an additional parameter that will represent a new property.

Fig. 5.2 Proportional direction determinant calculations
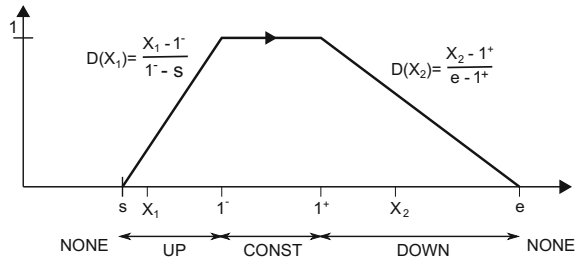
The proposition is **direction determinant** (see [23–25]). The purpose of this parameter is to represent a kind of direction "intensity" of the argument. The direction determinant is strictly connected with a particular OFN and is defined only for its support (see Sect. 4.3.2). The general idea is to measure a distance of argument from the core of the OFN. It is calculated from the ratio of the position in support of the considered argument in relation to the whole fuzzy boundary of the OFN, to which this argument belongs. It is well illustrated in Fig. 5.2.

Such an approach is connected with one of the useful interpretations of the OFN direction [13, 14]. The intuition behind the **direction determinant** is that the partial membership at the fuzzy boundaries can represent the imprecise concept of "now". If we treat this imprecision as symmetrical, then our fuzzy "now" in the context includes as much time forwards as backwards. Hence, UP and $DOWN$ in the scale of time (independently of the arguments) are equal. Thus there is a reason for calculating the determinant of the element situated on UP or $DOWN$ to the proportion of the respective intervals and not only to the value.

Definition 2 Let A denote the OFN, and x be an element of the support. **The proportional direction determinant** of x in relation to A marked dir_x^A is calculated as a result of directional function $D : supp_A \rightarrow (-1; 1)$ for the argument x in the following way.

$$dir_x^A = D_A(x) = \begin{cases} 0 & : \text{for PART(x)} = \text{CONST} \\ \frac{(x-1^-)}{(1^--s)} & : \text{for PART(x)} = \text{UP} \\ \frac{(x-1^+)}{(e-1^+)} & : \text{for PART(x)} = \text{DOWN} \end{cases} \qquad (5.4)$$

The above-mentioned determinant is called proportional because this is a certain simplification/approximation of the general idea. This facilitates practical implementation and still serves its purpose.

It is worth noting that, if the degree of membership is equal to zero, the direction determinant is undefined, because the argument is not part of function domain D (the value is outside OFN support). It should also be noted that for the arguments in the $CONST$ interval, we have the direction determinant that is equal to zero, which is justified, as these are the values about which we have no doubt: their membership is full (equal to 1). According to this intuition we should also expect (and this is taken

into account) that, the closer the arguments are to the kernel of the fuzzy number, their direction "intensity" (i.e., the direction determinant) is smaller. We should also note that the sign of the determinant clearly shows its membership to a selected part. If it is negative, it means that the argument belongs to UP, and if it is positive, then the argument is part of $DOWN$. Let's call this the **sign dependency**. Thus in certain situations we can simplify the analysis. When processing the data represented by the OFNs we wish to include only the information about which part we deal with ($up - part$ or $down - part$); the information about the sign of the determinant is sufficient without considering the exact value.

Based on the above analysis, the trivial variant of direction determinant can be proposed.

Definition 3 Let A denote the OFN, and x be an element of the support. **The trivial direction determinant** in relation to number A for x marked as dir_x is calculated with the use of the value of the directional function $D_A : supp_A \to (-1; 1)$ for the argument x in the following way.

$$dir_x^A = D_A(x) = \begin{cases} 0 & : \text{for PART(x)} = \text{CONST} \\ -1 & : \text{for PART(x)} = \text{UP} \\ 1 & : \text{for PART(x)} = \text{DOWN} \end{cases} \tag{5.5}$$

As can be noted, the trivial direction determinant simply remaps a set ($UP, CONST, DOWN$) into the set $(-1, 0, 1)$.

Having a basic tool, we can now propose the methods that are sensitive to the direction.

5.3 Compatibility Between OFNs

The fuzzy expression (or statement) "A is B" where A and B are fuzzy sets is a basis for the analysis where we want to apply the fuzzy sets and their imprecise mechanisms. The calculation result of this statement can be called a similarity or compatibility of A with B. The idea of compatibility and similarity between fuzzy sets was discussed in many publications (e.g., [5, 6, 12, 27]).

In this section the idea for calculating compatibility between two OFNs is presented [25]. We search methods sensitive to the direction, therefore a solution is to use the direction determinant in processing. Thus, as the result of fuzzy statement A is B a pair of values is proposed. First is a truth value in classical fuzzy meaning: the value from interval [0, 1], which indicates a degree of compatibility between two pieces of imprecise data represented by the OFNs. The second is the direction determinant, which retains information about direction.

Definition 4 For Ordered Fuzzy Numbers A and B the result of expression "A is B" called **directed fuzzy compatibility** (DFC) and labeled $COMP_{AB}$ is composed of two values: the truth value T_{AB} and direction determinant D_{AB} calculated as follows.
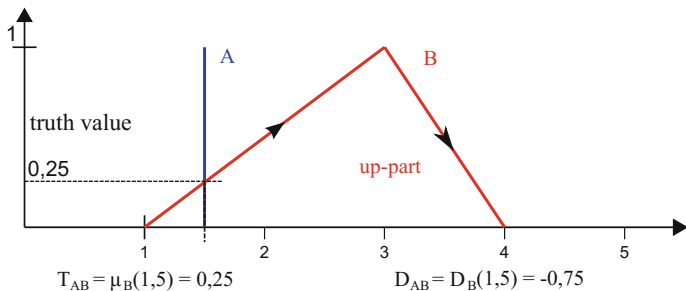
Fig. 5.3 The compatibility of a singleton with a general OFN

$$COMP_{AB} = (T_{AB}, D_{AB}) \tag{5.6}$$

$$T_{AB} = max(min(\mu_A(x), \mu_B(x))) : x \in \mathbf{X} \tag{5.7}$$

If T_{AB} is zero, then D_{AB} is unspecified, else

$$D_{AB} = D_B(x_0), \ x_0 = x : \mu_B(x) = T_{AB} \tag{5.8}$$

where $X \subset \mathbf{R}$ is a domain of given OFNs, $\mu_A(x)$, $\mu_B(x)$ are membership functions of A and B, and D_B is the direction determinant of B for given x.

Figure 5.3 shows the result of compatibility of A with B, when A is a singleton. For this example the truth value is $T_{AB} = 0, 25$ and the direction determinant is $D_{AB} = -0, 75$. The D_{AB} can be interpreted as an indication of shifts of A to B. The negative values mean the shift in the direction of the *up-part* of B, and the positive shift in the direction of the *down-part* of B. Such behavior can also be understood as a kind of directed relative dependence between values.

An Ordered Fuzzy Number can be understood as an extension of classical fuzzy numbers; the result of the fuzzy expression "*A is B*" should be an extension of the classical solution. It is important that the boundary dependencies for truth values are preserved in the new proposition. Especially when there is no shared part of the support between the numbers A and B, the truth value of the result is zero. On the other hand, when A is the same number as B, the truth value is equal to one regardless of the directions of the numbers. In addition to these results, we also achieve intuitive behavior of results with partial compatibility.

It is understandable that the expression "*A is B*" in a context of the truth value is symmetrical. However, if we want use direction-sensitive methods we need a tool that gives us different results in such contexts as presented above in Definition 4.

The examples in Figs. 5.4 and 5.5 present the results of DFC with different directions of the OFNs. For both cases we can observe that truth value results are the same. But the difference is specified just by the direction determinant.

However, for the opposite direction of OFNs (see Fig. 5.5) the direction determinants are the same. As we remember, the determinant part of the result indicates the
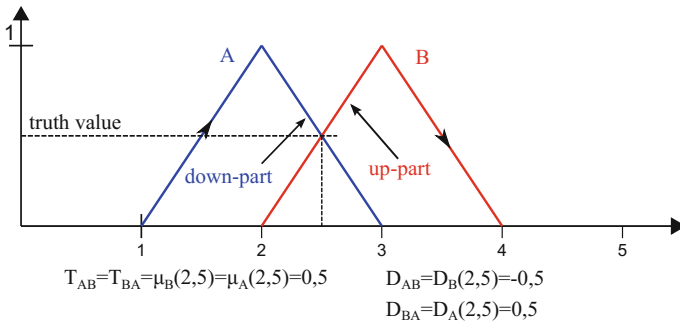
Fig. 5.4 The compatibility between two OFNs with the same direction
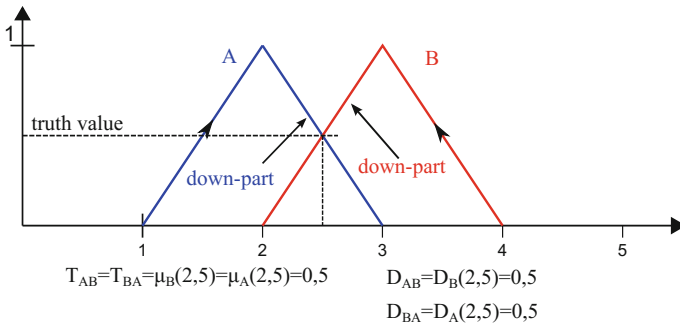
Fig. 5.5 The compatibility between the OFNs with the opposite directions

shift of A to B. For "A is B" A is shifted to B in the direction of the *down-part* of B, and for the "B is A" B is shifted to A in the direction of the *down-part* of A. Thus both shifts in the context of parts of OFNs have the same direction.

To preserve usefulness the above compatibility with classical fuzzy ideas is important to retain some clue behavior. That is, if the truth value of compatibility is equal to 1, this means we are not analyzing the direction determinants. Such a solution preserves the boundary dependencies: when the truth value is zero we have no compatibility, and with the truth value equal to one we have full compatibility regardless of the direction.

It must be emphasized that the direction determinant can be used as a tool for measurement and comparison of the directions of OFNs in different ways. Methods concerning direction should use that parameter as an element in their definition, but if we want to use OFNs in data processing only for their good arithmetic then we can ignore the direction determinant and use only the truth value for further processing.

5.4 Inference Sensitive to Direction

One of the main applications of fuzzy sets is a fuzzy system. Its core is a base of rules. Apart from the initial Chaps. 1 and 2 of this monograph, there are many publications that consist of the overviews of basic conceptions of fuzzy sets and modeling fuzzy systems [3, 16, 18, 19, 26]. The general advantage of fuzzy systems is the possibility to model the rules easily using linguistic description.

The basis for the processing of fuzzy rules is the operators of inference. They describe algorithms for transferring given fuzzy input into a fuzzy answer. Generally these methods are based on implications. However, there are also popular solutions including the MIN or PROD, which formally are not the implications, but their practical usefulness is proved. If we deal with quantitative imprecise data, we can use the OFNs instead of classical fuzzy numbers. We can ignore the direction and use the same methods. However, if we want to process additional information contained in the new model, we need methods sensitive to the direction.

When processing imprecise information using classical fuzzy methods, we often have fuzzy numbers at the input. However, during the process, in principle we ignore the quantitative nature of the data, focusing primarily on their qualitative aspect. Thus, even if input data are the fuzzy number, we rarely also get the fuzzy number at the output before defuzzification. In some cases it can be somehow inconsistent. For example, in the rule, *"IF temperature is about* 10 °C *THEN heating should be about 200 W,"* when processing data with classical fuzzy inference methods, in general, the output will not be a fuzzy number, although part of the rule, "Heating should be *about 200 W*," clearly suggests a quantitative output. It can be particularly difficult in the cases where the result of inference is to be used as fuzzy data without defuzzification later for the calculations in further processing of this information.

In the case of the OFN model and processing methods that can be called "arithmetic" (see [20, 21, 23, 24]), at each stage of the process we deal with the quantitative aspect of the data. Thus consistently we obtain fuzzy numbers at each step: the aggregation of premises, the inference, and the accumulation-aggregation of the rules answers.

5.4.1 Directed Inference Operation

An inference mechanism presented here is based on the generalized modus ponens (compare with the information in Chap. 2), where the main role is played by a rule of the type:

$$\textbf{IF } X \ is \ A \ \textbf{THEN } Y \ is \ B \qquad (5.9)$$

where A, B are fuzzy values that model a rule and X,Y, input and output variables. In the generalized modus ponens, where the data are represented by fuzzy numbers (or sets), the whole mechanism of inference is closed in the mathematical rule. This

rule describes an algorithm for calculation of the answer, Y value. Sometimes it is also called an inference operator (see [9, 17]).

The proposition presented here is dedicated for the OFN model, therefore in the rule (formula (5.9)) values are presented as such objects. The statement, "X is A," is calculated as compatibility between OFNs. The method was described in the previous section.

Definition 5 For the rule as in formula (5.9) let A and B be the OFNs. Let X be the input value also represented by an OFN. The result of "X is A" is calculated as directed fuzzy compatibility; $COMP_{XA} = (T_{XA}, D_{XA})$, where T_{XA} is the truth value and D_{XA} is the direction determinant part of $COMP_{XA}$.

The **directed inference by the multiplication with a shift** (DIMS) are the calculations of answer Y of the following rule: if $T_{XA} = 0$ there is no activation of the rule, therefore the answer is not calculated. In other cases,

$$Y = B + |D_{XA}| \cdot c$$

where

$$c = \begin{cases} s - B \ : \ D_{XA} < 0 \\ e - B \ : \ D_{XA} > 0 \end{cases} \qquad (5.10)$$

It is worth noting that this is not the classical logical inference. The truth value of the premise part of the rule is used to check whether the rule can be implemented at all. The specificity of the presented method is that the inference is made through arithmetic operations. We do processing of the quantitative data with calculations.

5.4.2 Examples

For better understanding of the proposed method, an example is useful. Let us assume that for the rule from formula (5.9) we have OFNs A as in Fig. 5.6a and B in Fig. 5.6b. In Fig. 5.6a we can also find the input value X.

According to the Definition 4 "X is A" is $COMP_{XA} = (T_{XA} = 0.66; D_{XA} = -0.33)$. Using the new inference we get the result shown in Fig. 5.6c. In Fig. 5.7a we have a situation where the X OFN value changes only a direction (but does not change the shape). This time the result of "X is A" is $COMP_{XA} = (T_{XA} = 0.66; D_{XA} = 0.33)$. As we can see in Fig. 5.7b the result of inference was changed. This is related to the change of direction determinant.

If we analyze the proposed method of inference in more detail, we can note that if the D_{XA} is closer to -1, the result of inference will be the narrow fuzzy number situated at the UP part side of support of OFN B. On the other hand, when the D_{XA} approaches 1, the result of inference is aimed at extreme values of support but on the $DOWN$ side. Finally, when $D_{XA} = 0$ and the $T_{XA} = 1$ it means that the X is fully compatible with A. Thus the result of inference is exactly the number B, the value from the conclusion.
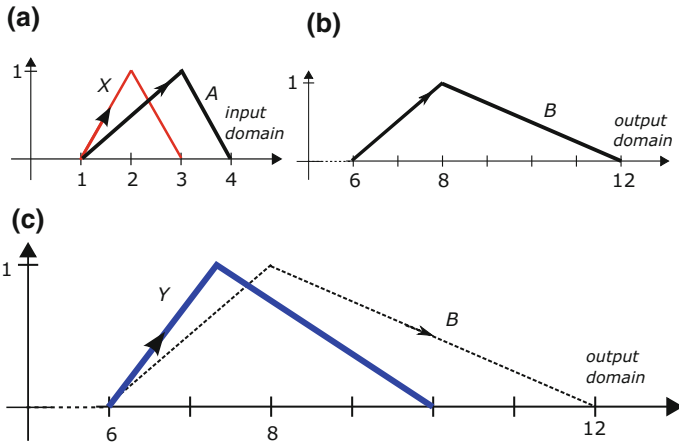
Fig. 5.6 **a** Example OFNs X and A; **b** OFN B from rule conclusion, **c** Y the result of inference operation
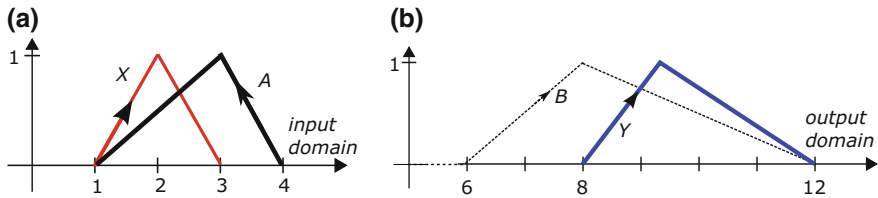
Fig. 5.7 **a** OFN A with opposite direction and X is the same as before; **b** the new Y result of inference operation

In practical applications (a fuzzy system, e.g.), a pair of values should be considered as a result of inference: the truth value of the premise part of a rule, and the OFN calculated in accordance with the Definition 5.

Comparability with conventional fuzzy inference operators is important to preserve in general similar usefulness in practical situations for the new conceptions. Therefore, behavior of the output of the inference in boundary cases is compatible with classical fuzzy solutions (see [3, 18, 19]). If there is no compatibility in the premise part "*X is A*", then the rule is not activated, and on other side if the activation is full, then the result is the exact value from a conclusion.

5.5 Aggregation of OFNs

A purpose of this section is to propose an aggregation operator that is generating intuitively good results as well as being consistent with the OFN model. The main basics of the proposition come from the paper [24]. The method presented here

maintains the expected properties of the aggregate functions [2, 4]. Additionally, it also takes into account the key idea of OFNs of the direction of the components.

5.5.1 The Aggregation's Basic Properties

Generally, an aggregation is an operation used in those situations when we need to find a single value representing the set of various numbers/data. There can be different application areas specified where an aggregation [2] is needed, for example, making decisions based on multiple criteria, or choosing from a variety of peer evaluations, one of which is treated as the result of them all. One important area of application is also the aggregation of the rule premise in a rule-based fuzzy system, where we have many input variables. The aggregation operation is a function that converts a number of input data into a single value. Transformation depends on the chosen method, but it is expected that in the process of determination of the result all of the input data were considered (in some way). Typically, aggregations where the number of input data is greater than one are used. Moreover, to call a function an aggregation, it should have two elementary properties (see [4]):

1. Boundary conditions. If all input data are minimal (or maximal), the result will also be the minimal (maximal) value. In the case of aggregation A for values from interval [0, 1] (the range of values of a fuzzy set), when all the arguments are equal to 1, the result of aggregation is also equal to 1 and similarly for zeros:

$$A(0, 0, ..., 0) = 0$$
$$A(1, 1, ..., 1) = 1 \qquad (5.11)$$

2. Nondecreasing. The function is nondecreasing against each input variable. This means that the growth of any of the input data cannot cause a decrease of the result of aggregation A.

$$\forall_{i=2..n} x_i \leq y_i \ \wedge \ (x_1, ..., x_n) \neq (y_1..., y_n) \Rightarrow A(x_1, ..., x_n) < A(y_1, ..., y_n) \qquad (5.12)$$

Apart from these two elementary properties a number of other important properties such as continuity, symmetry (anonymity), and idempotency are pointed out [2, 4, 10].

Continuity means that a small change in one input argument implies small change of the result. In the context of engineering applications, continuity corresponds to intuition, which is related to the fact that a small error in the entry cannot cause a large error in the output.

Symmetry means the independence of the result from the sequence of input data. This property is also called anonymity, because based on the output it is not possible to determine the sequence of input values.

Idempotency means that if each independent input has the same value, this particular value will be the result of aggregation. It may be noted that the boundary conditions are, in fact, idempotent for the maximal and minimal values.

There are also many different properties that can characterize an aggregation operator [2, 4, 10]. However, those mentioned above are the most essential and desirable in practical applications.

5.5.2 Arithmetic Mean Directed Aggregation

The basic, simple, and intuitive idea is to use an arithmetic mean idea in aggregation. As the arithmetic operations (thus the adding too) are *sensitive to the direction*, therefore the aggregation based on them also will be. The flexibility of the calculations grants a possibility for freely mixing the OFN objects with crisp numbers in mathematical formulas. Thus we can define the aggregation exactly like the arithmetic mean for the real numbers and it will preserve the *sensitivity to the direction*.

Definition 6 The result of **arithmetic mean directed aggregation** (**AMDA**) is OFN A calculated for L any set of OFNs such as:
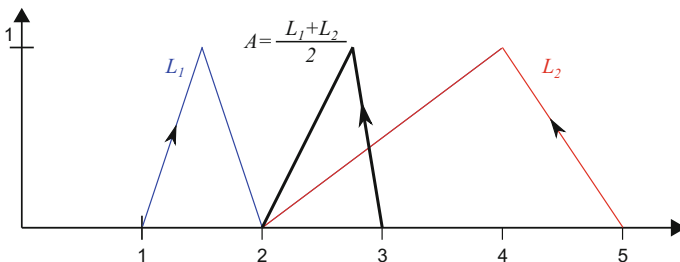
$$A = \Sigma_{i=1}^{n} \frac{L_i}{n},\tag{5.13}$$

where $L_i \in L$ is the ith OFN object from L, and n is the amount of elements in L.

Figure 5.8 presents the example of aggregation of two OFNs.

5.5.3 Aggregation for Premise Parts of Fuzzy Rules

Definition 6 from the previous section is simply the direct transfer of the idea of arithmetic mean into the $\mathbb{OFN}$ space of all OFNs. However, the popular application

Fig. 5.8 Result of AMDA operation for two OFNs L_1 and L_2

of the aggregations of fuzzy sets, and also fuzzy numbers, is a fuzzy rule with many input variables (see Chap. 2). Such rules have a premise part with a number of elementary fuzzy expressions of type *"X is L"*. For example,

$$\textbf{IF} \ \ X_1 \ is \ L_1 \ AND \ X_2 \ is \ L_2 \ AND \ ... \ AND \ X_n \ is \ L_n \ \textbf{THEN}... \quad (5.14)$$

where X_i are the fuzzy input data, L_i is the fuzzy set/number from a linguistic model, and $i = 1, ..., n$ is the number of input variables in the rule.

To use an OFN model in such a rule we need an aggregation consistent with the fuzzy expression's compatibility calculation presented in Sect. 5.3. Below is presented the proposition based directly on *AMDA* and designated specially for inference rules, and thus called **arithmetic mean directed inference aggregation (AMDIA)**. It uses the *direction determinant* idea. The main purpose of the proposal is to calculate the level of activation or firing strength for a rule.

Definition 7 Let's assume that the general pattern of the premise part of a rule R is specified in formula (5.14). The result of **arithmetic mean directed inference aggregation** A_R of fuzzy expressions from the premise part of the rule R is calculated as a DFC (*directed fuzzy compatibility* see Sect. 5.3), thus it is a pair: truth value T_R and direction determinant D_R.

$$A_R = (T_R, D_R) \quad (5.15)$$

The algorithm specifying A_R is presented as the following steps.

1. Calculation of set $A = \{A_1, A_2, ..., A_n\}$ containing elements that are the results of all fuzzy expressions from the premise part

$$A_i = COMP_{X_i L_i} = (T_{X_i L_i}, D_{X_i L_i}). \quad (5.16)$$

2.
$$\exists_{T_i=0} \Rightarrow T_R = 0, \ D_R \ is \ unspecified \quad (5.17)$$

   If there is at least one fuzzy expression with the truth value equal to 0, then the truth value of the aggregation result is also zero. Therefore this rule is inactivated, and the direction determinant is undefined.
3. Otherwise,
$$T_R = \Sigma_{i=1}^n \frac{T_i}{n},$$
$$D_R = \Sigma_{i=1}^n \frac{D_i}{n}. \quad (5.18)$$

The proposed aggregation operator for the OFN generates a result with two components. For the calculation of each of them the arithmetic mean is used. Because

the arithmetic mean is a function fulfilling the basic criteria of aggregation operators (see [2, 4, 10] and Sect. 5.5.1), the AMDIA also fulfills them.

It is worth noting that we are dealing with two different parameters: the truth value (degree of membership) and the direction determinant. However, they are not completely independent, therefore, it is worth having a look at some important dependencies between them. The direction determinant of the result equal to zero indicates that the activation is not moved from the $CONST$ interval in any direction. Note that this happens only in two cases:

1. When all truth values of the fuzzy expressions from the premise part are equal to one, then activation of the rule (truth value of aggregation result) will also be equal to one.
2. When the truth values of the fuzzy expressions on the UP side are precisely balanced with the resultant on the $DOWN$ side, then the truth value of the result will be greater than zero, and less than one.

Let's take a closer look at the first case. The level of activation may be only equal to 1 when the determinant is equal to zero. This means that in the case of complete compatibility of premises the given data do not represent any direction. This is especially important if we want to combine the concept of OFNs with the ideas for classical fuzzy sets. In such a way the fundamental meaning of full membership (also the full nonmembership) coincides in both solutions.

Finally, an alternative conception should be analyzed. It may be tempting to use the geometric mean instead of arithmetic in the aggregation. It seems good for truth values, due to the fact that if we have zero for at least one input, it is automatically zero for the truth of aggregation result and generally cancels the rule from further computations. Unfortunately, for the same reason it may not be used for calculating the direction determinant part of the result. The zero value of the direction determinant of elementary fuzzy expression means in most cases full compatibility (truth value equal to one). It is against intuition that only one full compatibility of one fuzzy expression will automatically grant no direction for the aggregation result, no matter how many other expressions have only partial compatibility.

5.6 Summary

All sections of this chapter can be treated as an introduction to tendency-sensitive data processing with the use of Ordered Fuzzy Numbers. The basic tool for linguistic modeling is the operation **directed fuzzy compatibility** used to calculate a result of the expression, "*X is A*". The inference operator **DIMS** is another important tool for the practical use of sensitivity to direction. Both propositions use an idea of the direction determinant, which can be treated as a general parameter for measuring direction. Together these propositions can also be used for practical defining and realization of the full fuzzy system based on rules type "*IF-THEN*", which is sensitive to the direction/tendency of information presented by OFNs. If a fuzzy system

needs rules that use more input values there is the proposal of the *arithmetic mean directed inference aggregation* method which is also based on the idea of direction determinant.

To generate one fuzzy answer from all rule outputs a simple calculation can be used. It is the idea of weighted mean where weights are the levels of activation of the rules (see [21]):

$$Y = \frac{\sum_{i=1}^{k} (a_i \cdot Y_{Ri})}{\sum_{i=1}^{k} a_i} \tag{5.19}$$

where k is the amount of rules, a_i is the value of activation for the i rule, and Y_{Ri} is the OFN output for the i rule. For such calculations the result will always be an OFN. A key observation for this solution is that rules that were not activated (activation equals 0) have no participation in the final result. Calculation of the fuzzy answer of all rules results is also a form of aggregation (see Sect. 5.5), sometimes also called an "accumulation."

When we have one OFN as the result of a system, we can defuzzify it. For this purpose, we use one of the classic fuzzy methods as the mean of maxima, or the *center of mean inclination* method mentioned in Definition 4.4 from previous chapter. It is based on the specific parameter of OFN, an *inclination*. As the aggregation of premises and inference operator are sensitive to the direction, the OFN-based fuzzy system will also be characterized by this property. Therefore, the accumulation and defuzzification methods proposed above do not need to fulfill the sensitivity postulate.

It should be underlined that defuzzification is a very important operation in terms of the practical usefulness of fuzzy concepts. There can be many applications where fuzzy elements are helpful but without rule/inference processing. This applies particularly to quantitative problems, when we need to calculate the result where data are fuzzy. Therefore developing the defuzzification methods independently of the fuzzy system application is an important issue. The next chapters in this part of the monograph (see also [1, 7, 15]) present other ideas and propositions to realize defuzzifications that consider specificity of the OFN model.

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