Chapter 1
Introduction to Fuzzy Sets

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Abstract The subject of this chapter is fuzzy sets and the basic issues related to them. The first section discusses concepts of sets: classic and fuzzy, and presents various ways of describing fuzzy sets. The second section is dedicated to t-norms, s-norms, and other terms associated with fuzzy sets. Subsequent sections describe the extension principle, fuzzy relations and their compositions, cylindrical extension and projection of a fuzzy set. The sixth section discusses fuzzy numbers and basic arithmetic operations on them. Finally, the last section summarizes the chapter.

1.1 Classic and Fuzzy Sets

The concept of a classic set is one of primitive notions, which do not have a definition. Most frequently a set is understood as a collection of objects (elements) having some features distinguishing them from other objects, such as the set of positive numbers less than 100 or the set of aquatic birds. Usually, sets are denoted in uppercase (e.g., a set A, B, ...), whereas objects are in lowercase (e.g., an object x, y, ...). Each set may be considered as a subset of an universe of discourse $\mathbb{X}$, which is a "super-set" containing all possible objects.

In the case of classic sets, a given object x may belong to a set A (be a member of a set A), or not belong to this set (not be a member of this set), and these two options are denoted by $x \in A$ or $x \notin A$. A classic set may be described by means of the characteristic function (χ_A) that takes two values: 1 (for the object belonging to a set A), and 0 (for the object not belonging to a set A) [19]

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$$\chi_A(x) = \begin{cases} 1, \ x \in A, \\ 0, \ x \notin A. \end{cases} \tag{1.1}$$

There are several operations defined on classic sets and the following are considered to be basic ones [19]:

- product (intersection, conjunction)

$$A \cap B = \{x \in \mathbb{X} | \ x \in A \text{ and } x \in B\}, \tag{1.2}$$

- sum (union, disjunction)

$$A \cup B = \{x \in \mathbb{X} | \ x \in A \text{ or } x \in B\}, \tag{1.3}$$

- negation (complement)

$$\overline{A} = \{x \in \mathbb{X} | \ x \notin A\}. \tag{1.4}$$

The above operations can also be defined on the basis of characteristic functions [19]:

$$\chi_{A \cap B} = \chi_A(x) \wedge \chi_B(x) = \min(\chi_A(x), \chi_B(x)), \tag{1.5}$$

$$\chi_{A \cup B} = \chi_A(x) \vee \chi_B(x) = \max(\chi_A(x), \chi_B(x)), \tag{1.6}$$

$$\overline{\chi_A(x)} = 1 - \chi_A(x). \tag{1.7}$$

Example 1.1 Let us consider the expression "The fetal heart rate (FHR) is about 120 bpm," which can be described by the classic set of FHR values in the interval, for example, [115, 125], defined in the universe $\mathbb{X} = [0, 240] \subset \mathbb{R}$. The characteristic function of this set is shown in Fig. 1.1a. According to the proposed interval, values 121 and 125 equally belong to this set, whereas values 125.1, 130, and 135 equally do not belong. However, the following observations can arise: a value 121 is closer to 120 than 125, thus it should belong "stronger", a value 125.1 should be the member of the set similarly as 125, and finally, a value 135 should belong "less" than 130. Fuzzy sets allow for taking into account these observations.

Fuzzy sets were introduced and described using membership functions by L.A. Zadeh in 1965 [24] and have many practical applications [10, 22]. As opposed to a classic set, in the case of a fuzzy set A an object x may belong to this set with varying membership degrees in the range [0, 1], where 0 and 1 denote, respectively, lack of membership and full membership.

One way of describing a fuzzy set A is to provide its membership function $\mu_A : \mathbb{X} \to [0, 1]$. There are various membership functions, three of them are presented below [5, 16].

- Gaussian membership function (where c and δ are parameters)

$$\mu_A(x; c, \delta) = \exp\left(-\frac{(x-c)^2}{2\delta^2}\right). \tag{1.8}$$

The parameter c specifies the center of a function; the parameter δ determines its dispersion.

- Trapezoidal membership function (where $p \leq q \leq r \leq s$ are parameters)

$$\mu_A(x; p, q, r, s) = \begin{cases} 0, & x \leq p, \\ \frac{x-p}{q-p}, & p < x \leq q, \\ 1, & q < x \leq r, \\ \frac{s-x}{s-r}, & r < x \leq s, \\ 0, & x > s. \end{cases} \tag{1.9}$$

A special case of a trapezoidal function (for $q = r$) is a triangular function.

- Singleton (where x_0 is a parameter)

$$\mu_A(x; x_0) = \delta_{x,x_0} = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases} \tag{1.10}$$

The parameter x_0 specifies the location of the singleton, that is, the single value of x which belongs to a set A (with a membership degree equal to 1).

An example of the Gaussian membership function is presented in Fig. 1.1b, and trapezoidal, triangular, and singleton functions are illustrated in Fig. 1.2.
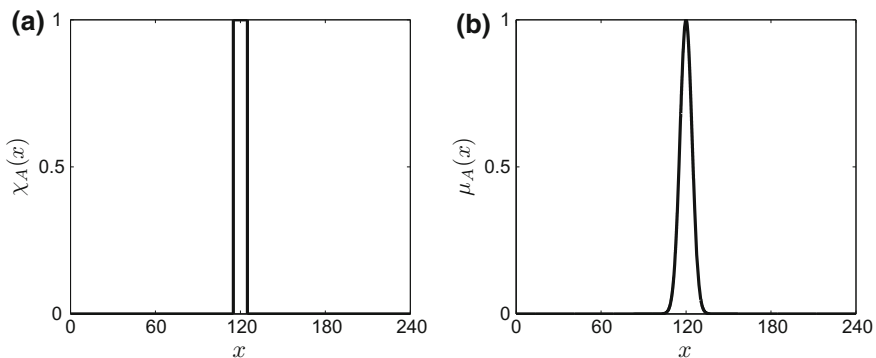
Fig. 1.1 "The fetal heart rate is about 120 bpm": **a** the characteristic function of the classic set, **b** the Gaussian membership function of the fuzzy set
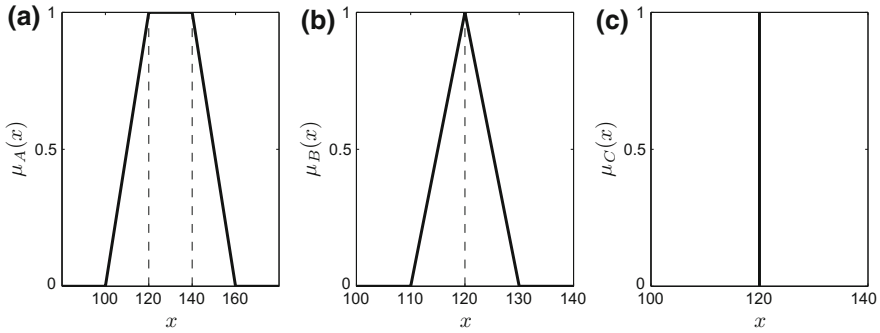
Fig. 1.2 Examples of membership functions: **a** trapezoidal ($p = 100$, $q = 120$, $r = 140$, $s = 160$), **b** triangular ($p = 110$, $q = r = 120$, $s = 130$) and **c** singleton ($x_0 = 120$)

Example 1.2 In the universe $\mathbb{X} = [0, 240] \subset \mathbb{R}$ let us define the fuzzy set A "The FHR is about 120 bpm." Such a set can be described by the Gaussian membership function (1.8) with $c = 120$ and $\delta = 4.25$, which is shown in Fig. 1.1b. In this case, in accordance with observations in Example 1.1, the FHR values discussed (121, 125, 125.1, 130, 135) belong to this set with different membership degrees: $\mu_A(121) = 0.973$, $\mu_A(125) = 0.500$, $\mu_A(125.1) = 0.486$, $\mu_A(130) = 0.062$ and $\mu_A(135) = 0.002$.

Example 1.3 One more example concerning fetal heart rate can be the expression "normal FHR," which means that the FHR value is in the physiological range. Based on FIGO guidelines, as the range of "normal FHR" values we can assume [110, 150] bpm and use the classic set of values in this range to describe the expression "normal FHR." However, this leads to the situation in which FHR value 151 bpm is not "normal," although it seems that it partially is. It suggests that it is better to use a fuzzy set to describe the expression "normal FHR."

Example 1.4 Another example can be the expression "new car." Assuming that a car is "new" when its age does not exceed three years, the expression "new car" can be described by the classic set of cars up to the age of three years. However, it results in a problem similar to the previous example: the car at the age of three years and one week is not "new," although it seems that it almost is. Also in this case it is better to use a fuzzy set.

The above examples suggest that fuzzy sets are a good tool for a formal description of vague and imprecise expressions such as "value about 120," "normal FHR," "new car," "medium height," "high salary," and so on. Examples of membership functions shown in Fig. 1.2 could be used to describe expressions such as: (a) "normal FHR," (b) the value of FHR is "about 120 bpm," and (c) FHR value is "exactly 120 bpm."

Another way of describing a fuzzy set is to list ordered pairs: an object x and its membership degree $\mu_A(x) \in [0, 1]$ in a set A [5]

$$A = \{(x, \mu_A(x)) \mid x \in \mathbb{X}\}. \tag{1.11}$$

To describe a fuzzy set, the notation proposed by Zadeh [25] can also be used:

- for discrete universe $\mathbb{X}$ (comprising ordered or nonordered objects)

$$A = \sum_{x \in \mathbb{X}} \mu_A(x) / x, \tag{1.12}$$

- for indiscrete universe $\mathbb{X}$

$$A = \int_{\mathbb{X}} \mu_A(x) / x. \tag{1.13}$$

In the above notation the symbol / is a separator, and symbols $\sum$ and $\int$ denote idempotent summation.

Example 1.5 In the discrete nonordered universe comprising selected fruits $\mathbb{X} = \{orange, pineapple, grape, apple, peach, banana, grapefruit\}$ let us define the fuzzy set A "Fruits, that the first author likes." Using the notation proposed by Zadeh we can write
$A = 1.0/orange + 0.6/pineapple + 0.2/grape + 1.0/apple + 0.8/peach$
$+0.6/banana + 0.5/grapefruit.$

Example 1.6 Let us consider the discrete ordered universe comprising values of possible temperatures to set in a car air-conditioning system $\mathbb{X} = \{low, 18, 19, \ldots, 23$ 24, $high\} \subset \mathbb{R}_+$, where "$low$" and "$high$" mean the lowest and the highest attainable temperatures. Using the notation of ordered pairs, the fuzzy set A "Adequate (according to the second author) temperature in the car" defined in the universe $\mathbb{X}$ can be described as follows
$A = \{(low, 0.1), (18, 0.4), (19, 0.5), (20, 0.8), (21, 0.9), (22, 1.0), (23, 0.8),$
$(24, 0.4), (high, 0.1)\}.$

Example 1.7 The set of FHR values from Example 1.2 using the notation proposed by Zadeh is described as

$$A = \int_{\mathbb{R}} \exp\left(-\frac{(x - 120)^2}{2 \cdot (4.25)^2}\right) / x.$$

Various extensions of fuzzy sets were proposed, for example, fuzzy sets of type-2 [26], interval-valued fuzzy sets [8, 11, 21, 26], probabilistic sets [9], rough sets [18], and intuitionistic fuzzy sets [2].

1.2 Fuzzy Sets—Basic Definitions

Similarly to classic sets, operations of product, sum, and complement are also established for fuzzy sets. Product and sum are defined by means of operators of t-norm and s-norm:

$$\forall_{x \in \mathbb{X}} \quad \mu_{A \cap B}(x) = \mu_A(x) \star_T \mu_B(x) = T(\mu_A(x), \mu_B(x)), \tag{1.14}$$

$$\forall_{x \in \mathbb{X}} \quad \mu_{A \cup B}(x) = \mu_A(x) \star_S \mu_B(x) = S(\mu_A(x), \mu_B(x)), \tag{1.15}$$

where $\star_T$ (T) and $\star_S$ (S) are operators of t-norm and s-norm. Both t-norm and s-norm (also called t-conorm) are mappings $[0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfy all necessary conditions [5, 16] presented in Table 1.1.

There are various t-norms and s-norms [5, 7, 13, 16, 23, 24, 27], three that are frequently used are presented below.

- Zadeh t-norm and s-norm:

$$x \star_T y = \min(x, y) = x \wedge y, \quad x \star_S y = \max(x, y) = x \vee y. \tag{1.16}$$

- Algebraic product and algebraic (also called probabilistic) sum:

$$x \star_T y = xy, \quad x \star_S y = x + y - xy. \tag{1.17}$$

- Lukasiewicz t-norm and s-norm:

$$x \star_T y = \max(x + y - 1, 0), \quad x \star_S y = \min(x + y, 1). \tag{1.18}$$

The complement of a fuzzy set A is defined as follows [5, 16]

$$\mu_{\overline{A}}(x) = n[\mu_A(x)], \tag{1.19}$$

where n denotes a negation function. Minimal assumptions about the function n are: n is a mapping $[0, 1] \rightarrow [0, 1]$, n satisfies conditions $n(0) = 1$, $n(1) = 0$, and n is

Table 1.1 Necessary conditions for t-norms and s-norms: A1 denotes boundary conditions, A2-commutativity, A3-monotonicity, and A4-associativity ($r, u, x, y, z \in [0, 1]$)

	t-norm	s-norm
A1	$T(x, 1) = x, \quad T(x, 0) = 0$	$S(x, 1) = 1, \quad S(x, 0) = x$
A2	$T(x, y) = T(y, x)$	$S(x, y) = S(y, x)$
A3	If $x \leq u$ then $T(x, y) \leq T(u, y)$	If $x \leq u$ then $S(x, y) \leq S(u, y)$
	If $y \leq r$ then $T(x, y) \leq T(x, r)$	If $y \leq r$ then $S(x, y) \leq S(x, r)$
A4	$T(x, T(y, z)) = T(T(x, y), z)$	$S(x, S(y, z)) = S(S(x, y), z)$
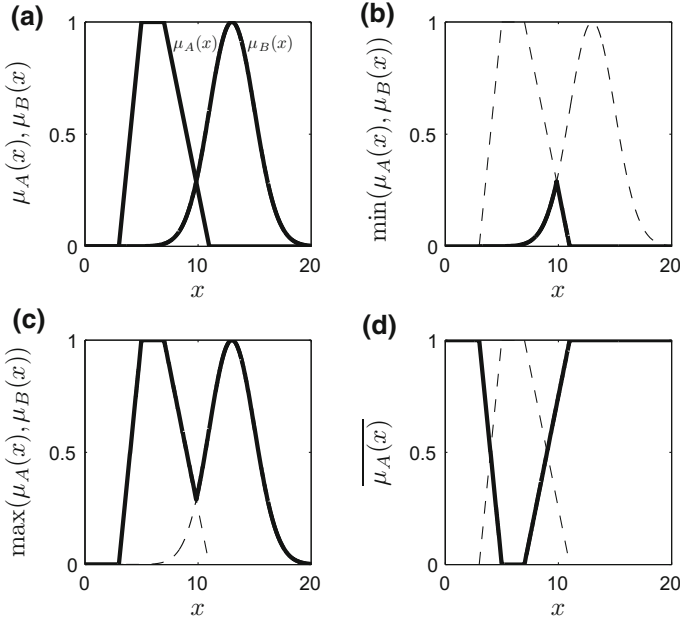
Fig. 1.3 Product, sum, and complement of fuzzy sets: **a** fuzzy sets A and B, **b** the product of A and B using Zadeh t-norm (minimum), **c** the sum of A and B using Zadeh s-norm (maximum), **d** complement of a set A using the standard negation

nonincreasing. Depending on specific conditions [5, 16], a negation function can be strict or strong. The strong negation in the form $N(x) = 1 - x$ (a standard negation) is most frequently used. Operations of product, sum, and complement are illustrated in Fig. 1.3.

In addition to definitions of t-norms and s-norms, other concepts that also characterize fuzzy sets are defined [3, 5–7, 16, 20, 23, 27]. Some of them are discussed below.

- An α-cut set (A_α) and a strong α-cut set ($A_{\underline{\alpha}}$)

$$A_\alpha = \{x \in \mathbb{X} | \mu_A(x) \geq \alpha\}, \qquad A_{\underline{\alpha}} = \{x \in \mathbb{X} | \mu_A(x) > \alpha\}. \qquad (1.20)$$

- Core of a fuzzy set (Core(A)), that is, the α-cut set with $\alpha = 1$.
- Support of a fuzzy set (Supp(A)), that is, the strong α-cut set with $\alpha = 0$.
- Width of a fuzzy set, Width(A) = $|x_2 - x_1|$, where x_1 and x_2 are crossover points of A defined below.
- Crossover points of a fuzzy set

$$\text{Crossover}(A) = \left\{ x \in \mathbb{X} \,\middle|\, \mu_A(x) = \frac{1}{2} \right\}. \qquad (1.21)$$

- A fuzzy set is "normal" if its core is not empty.
- A fuzzy set A is "convex" if and only if (iff)

$$\forall_{x_1, x_2 \in \mathbb{X}} \quad \forall_{\lambda \in [0,1]} \quad \mu_A \left[\lambda x_1 + (1 - \lambda) x_2 \right] \geq \min \left[\mu_A \left(x_1 \right), \mu_A \left(x_2 \right) \right]. \qquad (1.22)$$

- Two fuzzy sets A and B are equal iff

$$\forall_{x \in \mathbb{X}} \quad \mu_A \left(x \right) = \mu_B \left(x \right). \qquad (1.23)$$

- A fuzzy set A is a subset of a fuzzy set B iff

$$\forall_{x \in \mathbb{X}} \quad \mu_A \left(x \right) \leq \mu_B \left(x \right). \qquad (1.24)$$

Core(A), Supp(A), Width(A), and Crossover(A) are illustrated in Fig. 1.4. It is worth noting that Core(A) and Supp(A) specify classic sets ($\chi_{Core(A)}(x)$ and $\chi_{Supp(A)}(x)$ in Fig. 1.4 denote their characteristic functions).
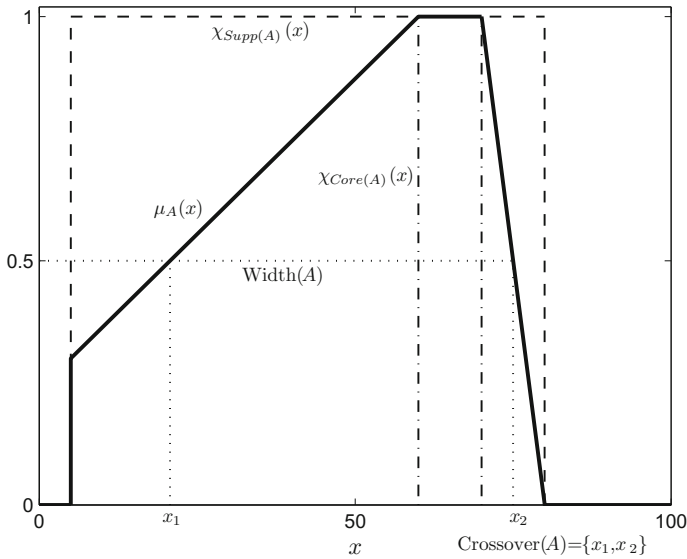
Fig. 1.4 The illustration of core, support, width, and crossover points of a fuzzy set

1.3 Extension Principle

The extension principle [26] allows for extension of the concept of mathematical functions (defined on classic sets) to fuzzy sets. In other words, mapping from a classic set to a classic set is extended to mapping from a fuzzy set to a fuzzy set.

Let $y = f(x)$ be a one-argument function, which is mapping from $\mathbb{X}$ to $\mathbb{Y}$, and A be a fuzzy set defined in a discrete universe $\mathbb{X} = \{x_1, x_2, \ldots, x_p\}$

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_p)/x_p. \tag{1.25}$$

As a result of mapping a set A by the function f we obtain the following fuzzy set B (defined in the universe $\mathbb{Y}$) [16]

$$B = f(A) = \mu_A(x_1)/f(x_1) + \mu_A(x_2)/f(x_2) + \cdots + \mu_A(x_p)/f(x_p), \tag{1.26}$$

where $+$ denotes a logical sum. For functions that are not injective (are not one-to-one mappings), a logical sum is performed using s-norm, and hence we can write [16]

$$\mu_B(y) = \begin{cases} \underset{\{x|f(x)=y\}}{\bigstar_S} \mu_A(x), & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset, \end{cases} \tag{1.27}$$

where $\bigstar_S$ stands for a multiargument s-norm, and $f^{-1}(y)$ denotes the domain of function $y = f(x)$.

Example 1.8 Suppose we have a fuzzy set A defined in the discrete universe $\mathbb{X} = \{-3, -2, \ldots, 3, 4\} \subset \mathbb{R}$
$A = \{(x_1 = -3, 0.8), (x_2 = -2, 0.5), (x_3 = -1, 0.3), (x_4 = 0, 0.5),$
$(x_5 = 1, 0.8), (x_6 = 2, 0.1), (x_7 = 3, 0.8), (x_8 = 4, 0.7)\},$
and the function $y = f(x) = 2x^4$, which is the mapping from $\mathbb{X}$ to $\mathbb{Y}$. Let us determine the mapping of A by the function f to the fuzzy set B.

Values of the function f arranged in ascending order are
$f(x_4) = 0, f(x_3) = f(x_5) = 2, f(x_2) = f(x_6) = 32, f(x_1) = f(x_7) = 162,$
$f(x_8) = 512.$

Using (1.27) and Zadeh s-norm (maximum), the fuzzy set B is determined in the following way
$B = \{(0, 0.5), (2, \max(0.3, 0.8)), (32, \max(0.5, 0.1)), (162, \max(0.8, 0.8)),$
$(512, 0.7)\} = \{(0, 0.5), (2, 0.8), (32, 0.5), (162, 0.8), (512, 0.7)\}.$

The above description concerned a one-argument function. Now let us consider a general case, a multiargument function $y = f(x_1, x_2, \ldots, x_n)$, which is a mapping from $\mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n$ to $\mathbb{Y}$. Suppose we have fuzzy sets $A_1, A_2, \ldots, A_n$, defined in universes $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n$, respectively. The mapping of these sets by the function f leads to the following fuzzy set B (defined in universe $\mathbb{Y}$) [16]

$$\mu_B(y) = \begin{cases} \displaystyle\bigstar_S \atop \{(x_1,\dots,x_n)|f(x_1,\dots,x_n)=y\} \left[\mu_{A_1}(x_1) \star_T \cdots \star_T \mu_{A_n}(x_n) \right], & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset. \end{cases}$$

(1.28)

1.4 Fuzzy Relations

A generalization of the concept of a fuzzy set is the idea of a fuzzy relation [26]. Let us consider a two-dimensional (binary) fuzzy relation R, which can be described by the set of ordered pairs: two objects x and y, and the membership degree $\mu_R(x, y)$. It can be written as [5, 16]

$$R = \{[(x, y), \mu_R(x, y)] | x \in \mathbb{X}, y \in \mathbb{Y}\}. \tag{1.29}$$

The membership degree $\mu_R(x, y)$ can be understood as the degree of relationship between objects x and y; the higher the value of $\mu_R(x, y)$, the greater is the degree of relationship. The membership degree $\mu_R(x, y)$ is the value of membership function $\mu_R : \mathbb{X} \times \mathbb{Y} \to [0, 1]$ of fuzzy relation R. The presented two-dimensional fuzzy relation is also a two-dimensional fuzzy set defined in an universe $\mathbb{X} \times \mathbb{Y}$.

Example 1.9 In universes $\mathbb{X} = \mathbb{Y} = [100, 160] \subset \mathbb{R}_+$ let us define the two-dimensional fuzzy relation R "Two FHR values (x and y) differ significantly." As the membership function of such a relation we can assume

$$\mu_R(x, y) = 1 - \exp\left(-\frac{(x - y)^2}{2\delta^2}\right),$$

which for parameter $\delta = 0.2$ is shown in Fig. 1.5.

Example 1.10 Let us consider two discrete universes: $\mathbb{X} = \{3500, 4000, 4500, 5000\} \subset \mathbb{R}_+$ and $\mathbb{Y} = \{2000, 3000, 3500, 5000, 5500\} \subset \mathbb{R}_+$, comprising possible salaries in companies A and B, respectively. In universe $\mathbb{X} \times \mathbb{Y}$ we can define the following fuzzy relation "The salary of an employee x in a company A is similar to the salary of an employee y in a company B."

A relation defined in discrete universes can be described by a relation matrix. In this case it is as follows

$$R = \begin{bmatrix} 0.32 & 0.88 & 1.00 & 0.32 & 0.14 \\ 0.14 & 0.61 & 0.88 & 0.61 & 0.32 \\ 0.04 & 0.32 & 0.61 & 0.88 & 0.61 \\ 0.01 & 0.14 & 0.32 & 1.00 & 0.88 \end{bmatrix},$$

Fig. 1.5 Example of the membership function of two-dimensional fuzzy relation R "Two FHR values (x and y) differ significantly"
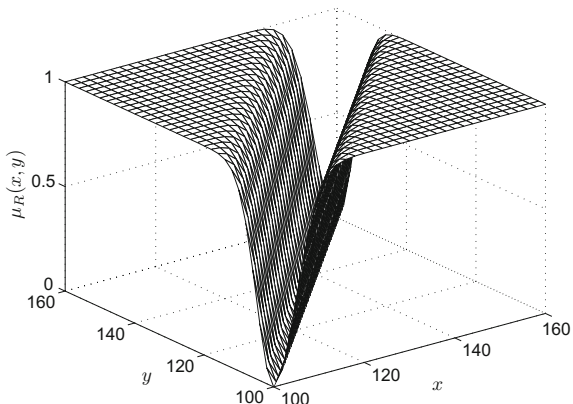
where rows correspond to elements of the universe $\mathbb{X}$, and columns to elements of the universe $\mathbb{Y}$. In other words, the element $R(i, j)$ determines the degree of relationship between the ith object in $\mathbb{X}$ and the jth object in $\mathbb{Y}$.

In general, a multidimensional fuzzy relation is defined as follows [5, 16]

$$R = \{[(x_1, x_2, \ldots, x_n), \mu_R(x_1, x_2, \ldots, x_n)] | x_1 \in \mathbb{X}_1, x_2 \in \mathbb{X}_2, \ldots, x_n \in \mathbb{X}_n\}, \quad (1.30)$$

where $\mu_R : \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \to [0, 1]$ is a membership function of an n-dimensional fuzzy relation R, that is, of an n-dimensional fuzzy set defined in universe $\mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n$.

Because fuzzy relations are fuzzy sets, they are subject to the same operations as fuzzy sets, for example, the product of sets or an α-cut set. Additionally, fuzzy relations may be composed. Let R_1 and R_2 be relations defined in universes $\mathbb{X} \times \mathbb{Y}$ and $\mathbb{Y} \times \mathbb{Z}$, respectively. Frequently used compositions are "supremum-t-norm" ($R_1 \circ R_2$) and "infimum-s-norm" ($R_1 \bullet R_2$), leading to the relation defined in an universe $\mathbb{X} \times \mathbb{Z}$ [16]:

$$\mu_{R_1 \circ R_2}(x, z) = \sup_{y \in \mathbb{Y}} \left[\mu_{R_1}(x, y) \star_T \mu_{R_2}(y, z) \right], \quad (1.31)$$

$$\mu_{R_1 \bullet R_2}(x, z) = \inf_{y \in \mathbb{Y}} \left[\mu_{R_1}(x, y) \star_S \mu_{R_2}(y, z) \right]. \quad (1.32)$$

For relations described by relation matrices, the above compositions can be achieved by multiplication of matrices with multiplication of elements replaced by t-norm (or s-norm), and the adding of elements replaced by maximum (or minimum).

Example 1.11 Suppose we have two fuzzy relations: R_1 and R_2 defined in discrete universes $\mathbb{X} = \{x_1, x_2, x_3\} \times \mathbb{Y} = \{y_1, y_2\}$ and $\mathbb{Y} = \{y_1, y_2\} \times \mathbb{Z} = \{z_1, z_2, z_3\}$, respectively, which are described by relational matrices:

$$R_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 1.0 \\ 0.6 & 0.5 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0.5 & 0.7 & 0.2 \\ 1.0 & 0.2 & 0.8 \end{bmatrix}.$$

The composition "maximum-t-norm" ($R_1 \circ R_2$) with Zadeh t-norm (minimum) leads to the following relation

$$R_1 \circ R_2 = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 1.0 & 0.2 & 0.8 \\ 0.5 & 0.6 & 0.5 \end{bmatrix},$$

where, for example, element (1, 1) was determined as

$\max(\min(0.5, 0.5), \min(0.2, 1.0)) = \max(0.5, 0.2) = 0.5$.

1.5   Cylindrical Extension and Projection of a Fuzzy Set

When analyzing fuzzy sets (fuzzy relations) defined in universes of different dimensionality, sometimes there is a need to increase or reduce dimensionality of one of the sets (one of the relations). To increase or reduce dimensionality, operations of cylindrical extension and projection were defined [26].

Cylindrical extension of a fuzzy set A leads to a fuzzy set (denoted by $Ce(A)$) of higher dimensionality. Let us assume we have a fuzzy set A defined in a one-dimensional universe $\mathbb{X}$. Its cylindrical extension in two-dimensional universe $\mathbb{X} \times \mathbb{Y}$ is defined as [16]

$$\mathop{\forall}_{x \in \mathbb{X}, y \in \mathbb{Y}} \quad \mu_{Ce(A)}(x, y) = \mu_A(x), \tag{1.33}$$

and is illustrated in Fig. 1.6a, which shows the cylindrical extension of the fuzzy set A from Example 1.2 in the universe $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{Y} = [0, 50] \subset \mathbb{R}$.

*Example 1.12* Let us consider the cylindrical extension of the fuzzy set A from Example 1.6; that is,

$A = \{(low, 0.1), (18, 0.4), (19, 0.5), (20, 0.8), (21, 0.9), (22, 1.0), (23, 0.8), (24, 0.4), (high, 0.1)\}$,

in the universe $\mathbb{X} \times \mathbb{Y}$, where $\mathbb{Y} = \{1, 2\} \subset \mathbb{R}_+$.

Using (1.33) the following fuzzy set is obtained

$Ce(A) = 0.1/(low, 1) + 0.4/(18, 1) + 0.5/(19, 1) + 0.8/(20, 1) + 0.9/(21, 1) +$
$+1.0/(22, 1) + 0.8/(23, 1) + 0.4/(24, 1) + 0.1/(high, 1) +$
$+0.1/(low, 2) + 0.4/(18, 2) + 0.5/(19, 2) + 0.8/(20, 2) + 0.9/(21, 2) +$
$+1.0/(22, 2) + 0.8/(23, 2) + 0.4/(24, 2) + 0.1/(high, 2)$.

In general, let us assume we have a fuzzy set A defined in an m-dimensional universe $\underline{\mathbb{X}} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_m$. The cylindrical extension of A in an $m + n$-dimensional universe $\underline{\mathbb{XY}} = \underline{\mathbb{X}} \times \underline{\mathbb{Y}}$, where $\underline{\mathbb{Y}} = \mathbb{Y}_1 \times \mathbb{Y}_2 \times \cdots \times \mathbb{Y}_n$, is defined as [16]
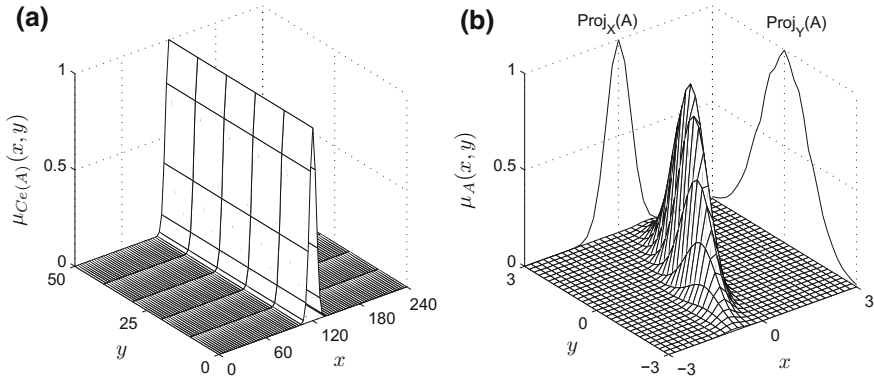
Fig. 1.6 Cylindrical extension (**a**) and projection of a fuzzy set (**b**)

$$\underset{\underline{x}_{XY} \in \underline{\mathbb{X}\mathbb{Y}}}{\forall} \; \mu_{Ce(A)} \left(\underline{x}_{XY} \right) = \mu_A \left(\underline{x}_X \right), \tag{1.34}$$

where $\underline{x}_{XY}$ and $\underline{x}_X$ denote objects from universes $\underline{\mathbb{X}\mathbb{Y}}$ and $\mathbb{X}$, respectively.

Projection of a fuzzy set [26] leads to fuzzy sets of lower dimensionality. For example, let us consider a fuzzy set A defined in a two-dimensional universe $\mathbb{X} \times \mathbb{Y}$ and described by the membership function presented in Fig. 1.6b. As a result of its projection in universes $\mathbb{X}$ and $\mathbb{Y}$ we can obtain two fuzzy sets [16]:

$$\underset{x \in \mathbb{X}}{\forall} \; \mu_{\text{Proj}_{\mathbb{X}}(A)} (x) = \underset{y \in \mathbb{Y}}{\sup} \mu_A (x, y), \tag{1.35}$$

and

$$\underset{y \in \mathbb{Y}}{\forall} \; \mu_{\text{Proj}_{\mathbb{Y}}(A)} (y) = \underset{x \in \mathbb{X}}{\sup} \mu_A (x, y), \tag{1.36}$$

which are also illustrated in Fig. 1.6b.

Example 1.13 In Example 1.11, the fuzzy relation R_1 defined in the universe $\mathbb{X} = \{x_1, x_2, x_3\} \times \mathbb{Y} = \{y_1, y_2\}$ was presented

$$R_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 1.0 \\ 0.6 & 0.5 \end{bmatrix}.$$

As a result of its projection in universes $\mathbb{X}$ and $\mathbb{Y}$, according to (1.35) and (1.36) the following fuzzy sets can be obtained.

$\text{Proj}_{\mathbb{X}} (R_1) = \max (0.5, 0.2) / x_1 + \max (0.1, 1.0) / x_2 + \max (0.6, 0.5) / x_3 =$
$= 0.5/x_1 + 1.0/x_2 + 0.6/x_3,$
$\text{Proj}_{\mathbb{Y}} (R_1) = \max (0.5, 0.1, 0.6) / y_1 + \max (0.2, 1.0, 0.5) / y_2 = 0.6/y_1 + 1.0/y_2.$

In general, let us assume we have a fuzzy set A defined in an $(m + n)$-dimensional universe $\underline{\mathbb{XY}} = \underline{\mathbb{X}} \times \underline{\mathbb{Y}}$. Its projection in an m-dimensional universe $\underline{\mathbb{X}}$ is defined as [16]

$$\underset{\underline{x}_X \in \underline{\mathbb{X}}}{\forall} \ \mu_{\mathrm{Proj}_{\underline{\mathbb{X}}}(A)}\left(\underline{x}_X\right) = \sup_{\underline{x}_Y \in \underline{\mathbb{Y}}} \mu_A\left(\underline{x}_{XY}\right), \tag{1.37}$$

where $\underline{x}_X$, $\underline{x}_Y$, and $\underline{x}_{XY}$ denote objects from universes $\underline{\mathbb{X}}$, $\underline{\mathbb{Y}}$, and $\underline{\mathbb{XY}}$, respectively.

1.6 Fuzzy Numbers

A separate class of fuzzy sets for describing imprecise expressions related to numbers (such as "about 5," "more or less 10," etc.) is distinguished [26]. Such sets are called fuzzy numbers and denoted by $\widetilde{A}, \widetilde{B}, \ldots$ [16]. Usually, fuzzy numbers are regarded as fuzzy sets that are defined over the real axis and fulfill given conditions; for example, they are normal, compactly supported, and in some sense convex [15].

Basic operations on fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ can be defined based on the extension principle in the following way [16]:

- addition

$$\mu_{\widetilde{A} \oplus \widetilde{B}}(z) = \sup_{\{(x,y)|x+y=z\}} \left[\mu_{\widetilde{A}}(x) \star_T \mu_{\widetilde{B}}(y) \right], \tag{1.38}$$

- subtraction

$$\mu_{\widetilde{A} \ominus \widetilde{B}}(z) = \sup_{\{(x,y)|x-y=z\}} \left[\mu_{\widetilde{A}}(x) \star_T \mu_{\widetilde{B}}(y) \right], \tag{1.39}$$

- multiplication

$$\mu_{\widetilde{A} \otimes \widetilde{B}}(z) = \sup_{\{(x,y)|xy=z\}} \left[\mu_{\widetilde{A}}(x) \star_T \mu_{\widetilde{B}}(y) \right], \tag{1.40}$$

- division

$$\mu_{\widetilde{A} \oslash \widetilde{B}}(z) = \sup_{\{(x,y)|x/y=z\}} \left[\mu_{\widetilde{A}}(x) \star_T \mu_{\widetilde{B}}(y) \right]. \tag{1.41}$$

Example 1.14 Let us calculate addition, subtraction, multiplication, and division of the following fuzzy numbers.

$$\widetilde{A} = 0.5/-2 + 1.0/-1 + 0.5/0,$$
$$\widetilde{B} = 0.8/4 + 1.0/5 + 0.8/6.$$

It can be noticed that the first number represents a value "about -1" and the second one "about 5," because membership degrees for -1 and 5 are equal to 1. Useful calculations are presented in Table 1.2.

Table 1.2 Arithmetic operations on fuzzy numbers defined based on the extension principle

x	y	$\mu_{\widetilde{A}}(x)$	$\mu_{\widetilde{B}}(y)$	$x+y$	$y-x$	xy	y/x
-2	4	0.5	0.8	2	6	-8	-2
-2	5	0.5	1.0	3	7	-10	-2.5
-2	6	0.5	0.8	4	8	-12	-3
-1	4	1.0	0.8	3	5	-4	-4
-1	5	1.0	1.0	4	6	-5	-5
-1	6	1.0	0.8	5	7	-6	-6
0	4	0.5	0.8	4	4	0	–
0	5	0.5	1.0	5	5	0	–
0	6	0.5	0.8	6	6	0	–

Using (1.38) and Zadeh t-norm (minimum), values of the membership function of the sum are calculated as follows:

$$\sup_{x+y=2}\,[\min(0.5, 0.8)], \quad \sup_{x+y=3}\,[\min(0.5, 1.0), \min(1.0, 0.8)],$$
$$\sup_{x+y=4}\,[\min(0.5, 0.8), \min(1.0, 1.0), \min(0.5, 0.8)],$$
$$\sup_{x+y=5}\,[\min(1.0, 0.8), \min(0.5, 1.0)], \quad \sup_{x+y=6}\,[\min(0.5, 0.8)].$$

Finally we get

$$\mu_{\widetilde{A}\oplus\widetilde{B}}(z) = 0.5/2 + 0.8/3 + 1.0/4 + 0.8/5 + 0.5/6.$$

Values of the membership function of the subtraction, multiplication, and division are calculated similarly applying (1.39)–(1.41); final results are given below:

$$\mu_{\widetilde{B}\ominus\widetilde{A}}(z) = 0.5/4 + 0.8/5 + 1.0/6 + 0.8/7 + 0.5/8,$$
$$\mu_{\widetilde{A}\otimes\widetilde{B}}(z) = 0.5/-12 + 0.5/-10 + 0.5/-8 + 0.8/-6 +$$
$$1.0/-5 + 0.8/-4 +$$
$$+0.5/0,$$
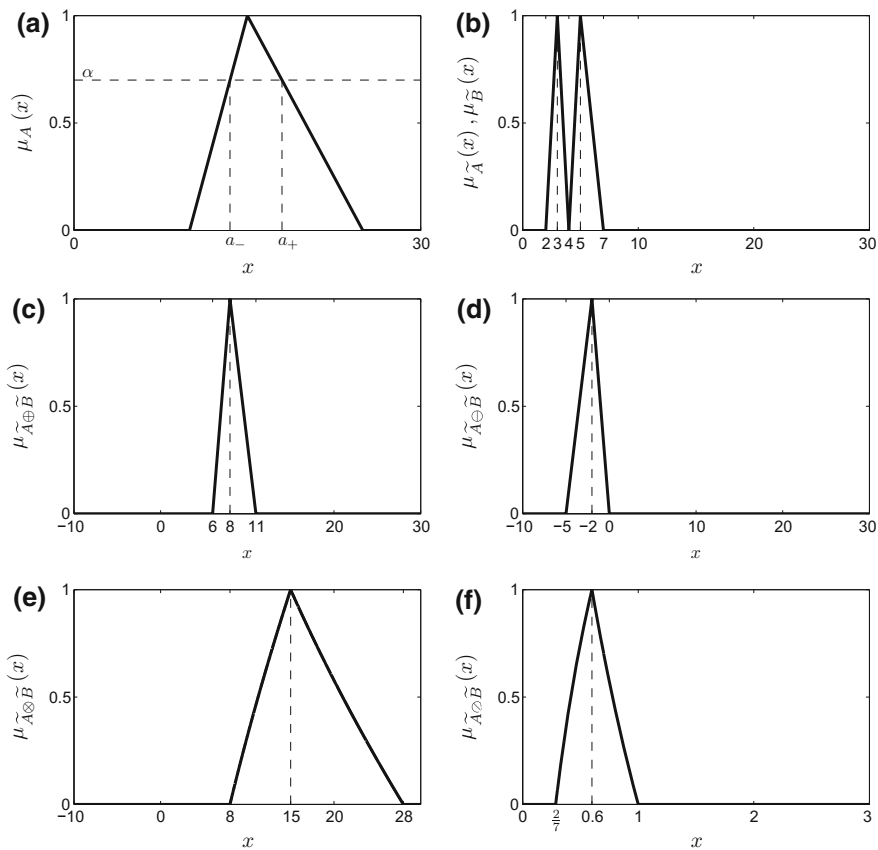$$\mu_{\widetilde{B}\oslash\widetilde{A}}(z) = 0.8/-6 + 1.0/-5 + 0.8/-4 + 0.5/-3 + 0.5/-2.5 + 0.5/-2.$$

It can be noted that the obtained results represent values: "about 4" (for the sum), "about 6" (subtraction), "about −5" (multiplication and division), which is consistent with classic arithmetic, for example, "about −1" + "about 5" = "about 4."

The considered arithmetic operations were defined based on the extension principle. Alternatively α-cuts of fuzzy numbers can be used. Figure 1.7a shows an α-cut of a fuzzy set A (see (1.20) in Sect. 1.2). According to the figure, as a result of an α-cut a classic set described by the interval $[a_-, a_+]$ is obtained. Arithmetic operations on fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ using α-cuts consist in application of interval arithmetic to intervals describing α-cuts of these numbers: $\widetilde{A}_\alpha = [\widetilde{a}_-, \widetilde{a}_+]$ and $\widetilde{B}_\alpha = [\widetilde{b}_-, \widetilde{b}_+]$. According to [1] arithmetic operations are defined as follows:

$$\left(\widetilde{A} \oplus \widetilde{B}\right)_\alpha = \left[\widetilde{a}_- + \widetilde{b}_-, \widetilde{a}_+ + \widetilde{b}_+\right], \tag{1.42}$$
$$\left(\widetilde{A} \ominus \widetilde{B}\right)_\alpha = \left[\widetilde{a}_- - \widetilde{b}_+, \widetilde{a}_+ - \widetilde{b}_-\right], \tag{1.43}$$
$$\left(\widetilde{A} \otimes \widetilde{B}\right)_\alpha = \left[\min\left(\widetilde{a}_-\widetilde{b}_-, \widetilde{a}_-\widetilde{b}_+, \widetilde{a}_+\widetilde{b}_-, \widetilde{a}_+\widetilde{b}_+\right),\right.$$

Fig. 1.7 Arithmetic operations on fuzzy numbers using α-cuts

$$\max\left(\widetilde{a}_-\widetilde{b}_-, \widetilde{a}_-\widetilde{b}_+, \widetilde{a}_+\widetilde{b}_-, \widetilde{a}_+\widetilde{b}_+\right)\right], \tag{1.44}$$

$$\left(\widetilde{A} \oslash \widetilde{B}\right)_\alpha = \left[\min\left(\widetilde{a}_-/\widetilde{b}_-, \widetilde{a}_-/\widetilde{b}_+, \widetilde{a}_+/\widetilde{b}_-, \widetilde{a}_+/\widetilde{b}_+\right),\right.$$
$$\left.\max\left(\widetilde{a}_-/\widetilde{b}_-, \widetilde{a}_-/\widetilde{b}_+, \widetilde{a}_+/\widetilde{b}_-, \widetilde{a}_+/\widetilde{b}_+\right)\right], \quad \text{if } 0 \notin \left[\widetilde{b}_-, \widetilde{b}_+\right]. \tag{1.45}$$

Example 1.15 Suppose we have two fuzzy numbers described by triangular membership functions:

$$\mu_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x; 2, 3, 4),$$
$$\mu_{\widetilde{B}}(x) = \mu_{\widetilde{B}}(x; 4, 5, 7),$$

which are presented in Fig. 1.7b. Let us calculate addition, subtraction, multiplication, and division of $\widetilde{A}$ and $\widetilde{B}$ using their α-cuts.

Based on equations of straight lines including sides of triangles we get intervals describing α-cuts of $\widetilde{A}$ and $\widetilde{B}$ (for any α in the range [0, 1]):

$\widetilde{A}_\alpha = [\alpha + 2, -\alpha + 4]$, $\widetilde{B}_\alpha = [\alpha + 4, -2\alpha + 7]$.

Applying (1.42) we get the interval

$$\left(\widetilde{A} \oplus \widetilde{B}\right)_\alpha = [2\alpha + 6, -3\alpha + 11],$$

where limits are functions describing locations of the beginning and the end of the interval describing an α-cut of the sum. Because functions are linear, replacing α with 0 and 1 provides parameters of the triangular membership function of the sum

$$\mu_{\widetilde{A} \oplus \widetilde{B}}(x) = \mu_{\widetilde{A} \oplus \widetilde{B}}(x; 6, 8, 11),$$

which is shown in Fig. 1.7c.

In a similar way, using (1.43) the results of the subtraction are obtained: the interval

$$\left(\widetilde{A} \ominus \widetilde{B}\right)_\alpha = [3\alpha - 5, -2\alpha]$$

and the membership function

$$\mu_{\widetilde{A} \ominus \widetilde{B}}(x) = \mu_{\widetilde{A} \ominus \widetilde{B}}(x; -5, -2, 0),$$

which is presented in Fig. 1.7d.

Determining the product requires more comments. Applying (1.44), the minimum and the maximum are searched among functions: $\alpha^2 + 6\alpha + 8$, $-2\alpha^2 + 3\alpha + 14$, $-\alpha^2 + 16$ and $2\alpha^2 - 15\alpha + 28$. From the analysis of values of these functions for α in the range [0, 1] the following interval is obtained

$$\left(\widetilde{A} \otimes \widetilde{B}\right)_\alpha = \left[\alpha^2 + 6\alpha + 8, 2\alpha^2 - 15\alpha + 28\right].$$

Limits of the above interval are not linear functions, thus replacing α with 0 and 1 provides only values of x, for which the membership function takes values 0 and 1; that is, $\mu_{\widetilde{A} \otimes \widetilde{B}}(8) = 0$, $\mu_{\widetilde{A} \otimes \widetilde{B}}(15) = 1$ and $\mu_{\widetilde{A} \otimes \widetilde{B}}(28) = 0$. To determine the membership function of the multiplication $\mu_{\widetilde{A} \otimes \widetilde{B}}(x)$, the following equations should be solved (with respect to α):

$$\alpha^2 + 6\alpha + 8 = x, 2\alpha^2 - 15\alpha + 28 = x.$$

The solutions are as follows: $\alpha_{1,2} = \left(-3 \pm \sqrt{1 + x}\right)$ for the first equation, and $\alpha_{1,2} = \left(15 \pm \sqrt{1 + 8x}\right)/4$ for the second. In the case of the first equation, the function $\left(-3 + \sqrt{1 + x}\right)$ is chosen as the membership function because it provides values in the range [0, 1] for $x \in [8, 15]$. Considering the second equation, for $x \in [15, 28]$ the values in the range [0, 1] are provided by the function $\left(15 - \sqrt{1 + 8x}\right)/4$. Finally, the product is described by the following membership function

$$\mu_{\widetilde{A} \otimes \widetilde{B}}(x) = \begin{cases} -3 + \sqrt{1 + x}, & 8 \le x \le 15, \\ \left(15 - \sqrt{1 + 8x}\right)/4, & 15 < x \le 28, \\ 0, & x < 8 \quad \text{or} \quad x > 28, \end{cases}$$

which is shown in Fig. 1.7e.

The result of the division is calculated similarly applying (1.45), however, there is no need to select solutions of equations since each of them has a single solution. Finally, we get the membership function

$$\mu_{\widetilde{A} \oslash \widetilde{B}}(x) = \begin{cases} \frac{7x-2}{2x+1}, & \frac{2}{7} \leq x \leq 0.6, \\ \frac{4(1-x)}{x+1}, & 0.6 < x \leq 1, \\ 0, & x < \frac{2}{7} \ \text{ or } \ x > 1, \end{cases}$$

which is presented in Fig. 1.7f.

Analyzing membership functions of the considered fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ it can be noted that they represent values "about 3" and "about 5," because membership degrees for 3 and 5 are equal to 1. The obtained results of arithmetic operations are correct; for example, the subtraction provided value "about −2."

As opposed to classic arithmetic, where two numbers are equal or are not equal, in fuzzy arithmetic a "partial equality" is possible. One of the methods of determining the degree of equality is based on the distance between compared fuzzy sets [16]. According to it, the equality index of sets A and B is defined as $Eq_1(A, B) = 1 - d_p(A, B)$, where $d_p(A, B)$ denotes Minkowski distance between sets described by membership functions $\mu_A(x)$ and $\mu_B(x)$

$$d_p(A, B) = \left(\int_{\mathbb{X}} |\mu_A(x) - \mu_B(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad p > 1. \tag{1.46}$$

Minkowski distance between sets is also the basis of one of the methods of ranking fuzzy numbers [16]. According to it, to compare fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the fuzzy number $\widetilde{C}$ such as $\widetilde{A} \leq \widetilde{C}$ and $\widetilde{B} \leq \widetilde{C}$ is established. The comparison of $\widetilde{A}$ and $\widetilde{B}$ consists in the analysis of their Minkowski distances from $\widetilde{C}$; it is stated that $\widetilde{A} \leq \widetilde{B}$ if $d_p(\widetilde{A}, \widetilde{C}) \geq d_p(\widetilde{B}, \widetilde{C})$. Most often $\widetilde{C} = \max(\widetilde{A}, \widetilde{B})$ is established based on the extension principle [16]

$$\mu_{\max(\widetilde{A}, \widetilde{B})}(z) = \sup_{\{(x, y)|\max(x, y)=z\}} \left[\mu_{\widetilde{A}}(x) \star_T \mu_{\widetilde{B}}(y) \right]. \tag{1.47}$$

Another way of ranking fuzzy numbers is to use their α-cuts [23].

The extension of the concept of fuzzy numbers are Ordered Fuzzy Numbers (OFNs) proposed in [14, 15]. The OFNs are ordered pairs of continuous real functions defined on the interval [0, 1] and their applications are the subject of research [4, 12, 17].

1.7 Summary

The chapter provides the review of basic issues concerning fuzzy sets, which – in contrast to classic sets – allow for partial membership of objects. As a result fuzzy sets are a good tool for representing vague and imprecise expressions of natural

language. Various ways of describing fuzzy sets and concepts related to them were shown. We discussed the extension principle, which allows for extension of traditional mathematical functions to fuzzy sets, as well as the idea of fuzzy relation, which makes possible a formal description of the relationship between two or more fuzzy sets. Operations of cylindrical extension and projection of a fuzzy set, which enable increasing and reducing its dimensionality, were also described. A separate section was dedicated to fuzzy numbers and basic arithmetic operations on them.

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