Real Options and Threshold Strategies

Vadim Arkin and Alexander Slastnikov(✉)

Central Economics and Mathematics Institute, Russian Academy of Sciences,
Nakhimovskii pr. 47, 117418 Moscow, Russia
slast@cemi.rssi.ru

Abstract. The paper deals an investment timing problem appearing in real options theory. The present values from an investment project are modeled by general diffusion process. We find necessary and sufficient conditions under which the optimal investment time is induced by a threshold strategy. We study also conditions for optimality of the threshold strategy (over all threshold strategies) and discuss the connection between the solutions to the investment timing problem and the free-boundary problem.

Keywords: Real options · Investment timing problem · Diffusion process · Optimal stopping · Threshold stopping time · Free-boundary problem

1 Introduction

One of the fundamental problems in real options theory concerns the determination of the optimal time for investment into a given project (see, e.g., the classical monograph [6]).

Let us think of an investment project, for example, a founding of a new firm in the real sector of economy. This project is characterized by a pair $(X_t, t \geq 0,\ I)$, where X_t is the present value of the firm founded at time t, and I is a cost of investment required to implement the project (for example, to found the firm). The input and the output production prices are assumed to be stochastic, so $X_t, t \geq 0$ is considered as a stochastic process, defined on a general filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$. This model assumes that:

– at any moment, a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information;
– investment are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes.

The investor's problem is to evaluate the project and determine an appropriate time for the investment (investment timing problem). In real options theory investment times are considered as stopping times (adapted to the flow of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$).

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In real options theory there are two different approaches to solving investment timing problem (see [6]).

The project value under the first approach is the maximum of the net present value from the implemented project over all stopping times (investment rules):

$$F = \max_{\tau} \mathbf{E}(X_\tau - I)e^{-\rho\tau}, \tag{1}$$

where ρ is a given discount rate. An optimal stopping time τ^* in (1) is viewed as the optimal investment time (investment rule).

Within the second approach an opportunity to invest is considered as an American call option – the right but not obligation to buy the asset on predetermined price. The exercise time is viewed as an investment time, and the option value is accepted as the investment project value. In this framework a project is spanned with some traded asset, whose price is completely correlated with the present value X_t of the project. In order to evaluate the (rational) value of this real option one can use methods of financial options pricing theory, especially, contingent claims analysis (see, e.g., [6]).

In this paper we follow the approach that the optimal investment timing decision can be mathematically determined as the solution of an optimal stopping problem (1). Such an approach originated in the well-known McDonald–Siegel model (see [6,11]), in which the underlying present value's dynamics is modeled by a geometric Brownian motion. The majority of results on this problem (optimal investment strategy) has a threshold structure: to invest when the present value of the project exceeds a certain level (threshold). At a heuristic level this is so for the cases of geometric Brownian motion, arithmetic Brownian motion, mean-reverting process and a few others (see [6]). However the following general question arises: For which underlying processes the optimal decision in the investment timing problem will have a threshold structure?

Another investment timing problem (with additional restrictions on stopping times) was considered by Alvarez [1], who established sufficient (but not necessary) conditions for optimality.

In this paper we focus our attention on finding of necessary and sufficient conditions for optimality of threshold strategies in the investment timing problem. Since this problem is a special case of the optimal stopping problem, a similar question may be addressed in the general optimal stopping problem: Under what conditions (on both process and payoff function) the optimal stopping time will have a threshold structure? Some results in this direction (in the form of necessary and sufficient conditions) were obtained in [2,3,5] under additional assumptions on underlying process and/or payoffs.

The paper is organized as follows. After a formal description of the investment timing problem and the assumptions on the underlying process (Sect. 2.1), we turn to study the threshold strategies in this problem. Since the investment timing problem by threshold strategies is reduced to one-dimensional maximization problem, then a related problem is to find the optimal threshold. In Sect. 2.2 we give necessary and sufficient conditions for the optimal threshold (over all thresholds). Solving a free-boundary problem (based on smooth-pasting principle) is the most commonly used method (but this is not the only method, see, e.g., [12]) that allows to find a solution to the optimal stopping problem. In Sect. 2.3 we discuss the connection between solutions to the investment timing problem and the free-boundary problem. Finally, in Sect. 2.4 we prove the main result on necessary and sufficient conditions under which the optimal investment time is generated by a threshold strategy.

2 Investment Timing Problem

Let I be the cost of investment required for implementing a project, and X_t the present value from the project started at time t. As usual the investment is supposed to be instantaneous and irreversible, and the project—infinitely-lived.

At any time a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until she/he obtains new information regarding its environment (prices of the product and resources, demand etc.). The goal of a decision-maker in this situation is to use the available information and find the optimal time for investing in the project (investment timing problem), i.e., find a time τ that maximizes the net present value from the project:

$$\mathbf{E}^x \left(X_\tau - I \right) e^{-\rho\tau} \mathbf{1}_{\{\tau<\infty\}} \to \max_{\tau\in\mathcal{M}} . \tag{2}$$

Here $\mathbf{E}^x$ is the expectation for the process X_t starting from the initial state x, $\mathbf{1}_A$ is indicator function of the set A, and the maximum is taken over all stopping times τ from a certain class $\mathcal{M}$ of stopping times[1].

We treat the interesting case $I < r$; otherwise the optimal time in (2) will be $+\infty$.

2.1 Mathematical Assumptions

Let $X_t, t > 0$ be a diffusion process with values in the interval $D \subseteq \mathbb{R}^1$ with boundary points l and r, where $-\infty \le l < r \le +\infty$, open or closed (i.e. it may be (l, r), $[l, r)$, $(l, r]$, or $[l, r]$), which is the solution to the stochastic differential equation:

$$dX_t = a(X_t)dt + \sigma(X_t)dw_t, \quad X_0 = x, \tag{3}$$

where w_t is a standard Wiener process, $a : D \mapsto \mathbb{R}^1$ and $\sigma : D \mapsto \mathbb{R}^1_+$ are the drift and the diffusion coefficients, respectively. Denote $\mathcal{I} = \mathrm{int}(D) = (l, r)$.

The process X_t is assumed to be regular; this means that, starting from an arbitrary point $x \in \mathcal{I}$, this process reaches any point $y \in \mathcal{I}$ in finite time with positive probability.

It is known that the following local integrability condition:

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |a(y)|}{\sigma^2(y)} dy < \infty \quad \text{for some } \varepsilon > 0, \tag{4}$$

[1] We consider stopping times which can take infinite values (with positive probability).

at any $x \in \mathcal{I}$ guarantees the existence of a weak solution of equation (3) and its regularity (see, e.g. [10]).

The process X_t is associated with the infinitesimal operator

$$\mathbb{L}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x). \tag{5}$$

Under the condition (4) there exist (unique up to constant positive multipliers) increasing and decreasing functions $\psi(x)$ and $\varphi(x)$ with absolutely continuous derivatives, which are the fundamental solutions to the ODE

$$\mathbb{L}f(p) = \rho f(p) \tag{6}$$

almost sure (with respect to Lebesque measure) on the interval $\mathcal{I}$ (see, e.g. [10, Chap. 5, Lemma 5.26]). Moreover, $0 < \psi(p),\ \varphi(p) < \infty$ for $p \in \mathcal{I}$. Note, if the functions $a(x),\ \sigma(x)$ are continuous, then $\psi,\ \varphi \in C^2(\mathcal{I})$.

2.2 Optimality of Threshold Strategies

Let us define $\tau_p = \tau_p(x) = \inf\{t \geq 0 : X_t \geq p\}$—the first time when the process X_t, starting from x, *exceeds* level p. The time τ_p is said to be a threshold stopping time generated by a threshold strategy—to stop when the process exceeds threshold p. Let $\mathcal{M}_{\text{th}} = \{\tau_p,\ p \in \mathcal{I}\}$ be a class of all such threshold stopping times.

For the class $\mathcal{M}_{\text{th}}$ the investment timing problem (2) can be written as follows:

$$(p - I)\,\mathbf{E}^x e^{-\rho\tau_p} \to \max_{p \in (l,r)}. \tag{7}$$

Such a problem appeared in [7] as the heuristic method for solving a general investment timing problem (2) over the class of all stopping times.

We say that the threshold p^* is optimal for the investment timing problem (7) if the threshold stopping time τ_{p^*} is optimal in (7). The following result gives necessary and sufficient conditions for the optimal threshold.

Theorem 1. *Threshold $p^* \in \mathcal{I}$ is optimal in the problem (7) for all $x \in \mathcal{I}$, if and only if the following conditions hold:*

$$\frac{p - I}{\psi(p)} \leq \frac{p^* - I}{\psi(p^*)} \quad \text{if } p < p^*; \tag{8}$$

$$\frac{p - I}{\psi(p)} \quad \text{does not increase for } p \geq p^*, \tag{9}$$

where $\psi(p)$ is an increasing solution to the ODE (6).

Proof. Let us denote the left-hand side in (7) by $V(p; x)$. Obviously, $V(p; x) = x - I$ for $x \geq p$.

Along with the above stopping time τ_p let us define the first hitting time to the threshold p: $T_p = \inf\{t \geq 0 : X_t = p\},\ p \in (l, r)$.

For $x < p$, clearly, $\tau_p = T_p$ and using known formula $\mathbf{E}^x e^{-\rho T_p} = \psi(x)/\psi(p)$ (see, e.g., [4,9]) we obtain:

$$V(p;x) = (p-I)\mathbf{E}^x e^{-\rho\tau_p}\mathbf{1}_{\{\tau_p<\infty\}} = (p-I)\mathbf{E}^x e^{-\rho T_p} = \frac{p-I}{\psi(p)}\psi(x). \tag{10}$$

Denote $h(x) = (x-I)/\psi(x)$.

(i) *Necessity.* Let $p^* \in \mathcal{I}$ be an optimal threshold in the problem (7) for all $x \in \mathcal{I}$. Then for $p < p^*$ we have

$$V(p;p) = p - I \le V(p^*;p) = \frac{p^*-I}{\psi(p^*)}\psi(p),$$

i.e. (8) holds. If $p^* \le p_1 < p_2$, then

$$V(p_2;p_1) = h(p_2)\psi(p_1) \le V(p^*;p_1) = p_1 - I = h(p_1)\psi(p_1),$$

and it follows that (9) is true.

(ii) *Sufficiency.* Now, suppose that conditions (8) and (9) hold.
Let $p < p^*$. If $x \ge p^*$, then $V(p;x) = x - I = V(p^*;x)$.
If $p \le x < p^*$, then, due to (8), $V_p(x) = x - I = h(x)\psi(x) \le h(p^*)\psi(x) = V(p^*;x)$.
Finally, if $x < p$, then, using (8) and (10), we obtain:
$V(p;x) = h(p)\psi(x) \le h(p^*)\psi(x) = V(p^*;x)$.
Consider the case $p > p^*$. If $x \ge p$, then $V(p;x) = x - I = V(p^*;x)$.
Whenever $p^* \le x < p$, then, due to (9), $V(p;x) = h(p)\psi(x) \le h(x)\psi(x) = x - I = V(p^*;x)$.
When $x < p^*$, then $V(p;x) = h(p)\psi(x) \le h(p^*)\psi(x) = V(p^*;x)$, since $h(p) \le h(p^*)$.
Theorem is completely proved.

Remark 1. The condition (9) is equivalent to the inequality

$$(p-I)\psi'(p) \ge \psi(p) \quad \text{for } p \ge p^*.$$

This relation implies, in particular, that the optimal threshold p^* must be strictly greater than the cost I (because the values $\psi(p^*)$, $\psi'(p^*)$ are positive).

Remark 2. Assume that $\log\psi(x)$ is a convex function, i.e. $\psi'(x)/\psi(x)$ increases. In this case there exists a unique point p^* which satisfies the equation

$$(p^*-I)\psi'(p^*) = \psi(p^*). \tag{11}$$

This value p^* constitutes the optimal threshold in the problem (7) for all $x \in \mathcal{I}$. Indeed, the sign of the derivative of the function $(p-I)/\psi(p)$ coincides with the sign of $\psi(p) - (p-I)\psi'(p)$. Therefore, in the considered case the conditions (8) and (9) in Theorem 1 are true automatically.

There are a number of cases of diffusion processes X_t which are more or less realistic for modeling the present values of a project. Some of them are listed below.

(1) *Geometric Brownian motion (GBM)*:

$$dX_t = X_t(\alpha dt + \sigma dw_t). \tag{12}$$

In this case $\psi(x) = x^\beta$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - \rho = 0$.

(2) *Arithmetic Brownian motion (ABM)*:

$$dX_t = x + \alpha dt + \sigma dw_t. \tag{13}$$

In this case $\psi(x) = e^{\beta x}$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta^2 + \alpha\beta - \rho = 0$.

(3) *Mean-reverting process (or geometric Ornstein–Uhlenbeck process)*:

$$dX_t = \alpha(\bar{x} - X_t)X_t dt + \sigma X_t\, dw_t. \tag{14}$$

In this case $\psi(x) = x^\beta {}_1F_1\left(\beta, 2\beta + \frac{2\alpha\bar{x}}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)$, where β is the positive root of equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\bar{x}\beta-\rho = 0$, and ${}_1F_1(p, q; x)$ is the confluent hypergeometric function satisfying Kummer's equation $xf''(x) + (q - x)f'(x) - pf(x) = 0$.

(4) *Square-root mean-reverting process (or Cox–Ingersoll–Ross process)*:

$$dX_t = \alpha(\bar{x} - X_t)dt + \sigma\sqrt{X_t}\, dw_t. \tag{15}$$

In this case $\psi(x) = {}_1F_1\left(\frac{\rho}{\alpha}, \frac{2\alpha\bar{x}}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)$.

The above processes are well studied in the literature (in connection with real options and optimal stopping problems see, e.g., [6,8]).

For the first two processes, (12) and (13), Theorem 1 gives explicit formulas for the optimal threshold in the investment timing problem:

$$p^* = \frac{\beta}{\beta - 1}I \text{ for the GBM,} \quad \text{and } p^* = I + \frac{1}{\beta} \text{ for the ABM.}$$

On the contrary, for mean-reverting processes (14) and (15) the function $\psi(x)$ is represented as an infinite series, and the optimal threshold can be find only numerically.

So, Theorem 1 states that optimal threshold p^* is a point of maximum for the function $h(x) = (x - I)/\psi(x)$. This implies the first-order optimality condition $h'(p^*) = 0$, i.e. the equality (11), and smooth-pasting principle:

$$V_x'(p^*; x)\big|_{x=p^*} = 1.$$

In the next section we discuss smooth-pasting principle and appropriate free-boundary problem more closely.

2.3 Threshold Strategies and Free-Boundary Problem

There is a common opinion (especially among engineers and economists) that the solution to a free-boundary problem always gives a solution to an optimal stopping problem.

A free-boundary problem in the case of threshold strategies in the investment timing problem can be formulated as follows: find the threshold $p^* \in (l, r)$ and a twice differentiable function $H(x)$, $l < x < p^*$, such that

$$\mathbb{L}H(x) = \rho H(x), \quad l < x < p^*; \tag{16}$$

$$H(p^* - 0) = p^* - I, \quad H'(p^* - 0) = 1. \tag{17}$$

If $\psi(x)$ is a twice differentiable function, then the solution to the problem (16) and (17) has the form

$$H(x) = \frac{p^* - I}{\psi(p^*)} \psi(x), \quad l < x < p^*. \tag{18}$$

Here $\psi(x)$ is an increasing solution to the ODE (6) and p^* satisfies the smooth-pasting condition (11). We call such p^* a solution to a free-boundary problem.

According to Theorem 1 the optimal threshold in problem (7) must be the point of maximum of the function $h(x) = (x - I)/\psi(x)$. However the smooth-pasting condition (11) provides only a stationary point for $h(x)$. Thus, we can apply standard second-order optimality conditions to derive relations between the solutions to the investment timing problem and the free-boundary problem.

Let p^* be a solution to the free-boundary problem (16) and (17). If p^* is also an optimal threshold in the investment timing problem (7), then, of course, $h''(p^*) \le 0$. This means that

$$\psi''(p^*) = -\frac{h''(p^*)\psi(p^*) + 2h'(p^*)\psi'(p^*)}{h(p^*)} = -\frac{h''(p^*)\psi(p^*)}{h(p^*)} \ge 0.$$

Thus, the inequality $\psi''(p^*) \ge 0$ may be viewed as a necessary condition for a solution of the free-boundary problem to be optimal in the investment timing problem. The inverse relation between solutions can be stated as follows.

Statement 1. *If p^* is the unique solution to the free-boundary problem* (16) *and* (17)*, and $\psi''(p^*) > 0$, then p^* is an optimal threshold in the problem* (7) *for all $x \in \mathcal{I}$.*

Proof. Since $h'(p^*) = 0$ and $\psi''(p^*) > 0$ then $h''(p^*) = -h(p^*)\psi''(p^*)/\psi(p^*) < 0$. Therefore, $h'(p)$ strictly decreases at some neighborhood of p^*.

Then, it is easy to see that $h'(p) > 0$ for $p < p^*$ and $h'(p) < 0$ for $p > p^*$. Otherwise $h'(q) = 0$ for some $q \ne p^*$, that contradicts the uniqueness of the solution to the free-boundary problem (16) and (17). Therefore, conditions (8) and (9) hold and Theorem 1 gives the optimality of threshold p^*.

The following result concerns the general case when the free-boundary problem has several solutions.

Statement 2. *Let p^* and $\tilde{p}$ be two solutions to the free-boundary problem* (16) *and* (17) *such that $\psi''(p^*) > 0$ and $(x-I)/\psi(x) \leq (p^*-I)/\psi(p^*)$ for $l < x < p^*$. If $\tilde{p} > p^*$ is such that $\psi^{(k)}(\tilde{p}) = 0, k = 2, \ldots, n-1$ and $\psi^{(n)}(\tilde{p}) > 0$ for some $n > 2$, then p^* is an optimal threshold in the problem* (7) *for all $x \in \mathcal{I}$.*

Proof. Let us prove that $h'(p) \leq 0$ for all $p > p^*$. The inequality $\psi''(p^*) > 0$ implies (as above) that $h''(p^*) < 0$, and, therefore, $h'(p) < 0$ for all $p^* < p < p_1$ with some p_1. If we suppose that $h'(p_2) > 0$ for some $p_2 > p^*$, then there exists $p_0 \in (p_1, p_2)$ such that $h'(p_0) = 0$ and $h'(p) > 0$ for all $p_0 < p < p_2$. Therefore, p_0 would be another solution to the free-boundary problem (16)–(17). The conditions of the Statement imply that $h^{(k)}(p_0) = 0,\ k = 2, \ldots, n-1$, $h^{(n)}(p_0) < 0$ for some $n > 2$, which contradicts the positivity of $h'(p)$ for $p_0 < p < p_2$.

Hence, $h'(p) \leq 0$ for all $p > p^*$ and conditions (8) and (9) hold. Thus, according to Theorem 1, p^* is an optimal threshold in the problem (7).

2.4 Optimal Strategies in the Investment Timing Problem

Now, let us return to the 'general' investment timing problem (2).

A specific version of the investment timing problem (2) over the class $\mathcal{M}_0$ of stopping times τ such that $\tau < \tau(0) = \inf\{t \geq 0 : X_t \leq 0\}$ was considered by Alvarez [1]. He derived sufficient conditions under which an optimal investment time in (2) over the class $\mathcal{M}_0$ will be a threshold stopping time. However these conditions are not necessary.

In this section we give necessary and sufficient conditions (criterion) for optimality of the threshold stopping time in the investment timing problem (2) over the class of *all stopping times*.

To reduce some technical difficulties we assume below that the drift $a(x)$ and the diffusion $\sigma(x)$ of the underlying process X_t are continuous functions.

Theorem 2. *The threshold stopping time τ_{p^*}, $p^* \in \mathcal{I}$, is optimal in the investment timing problem* (2) *for all $x \in \mathcal{I}$ if and only if the following conditions hold:*

$$(p-I)\psi(p^*) \leq (p^*-I)\psi(p) \quad \text{for } p < p^*; \tag{19}$$

$$\psi(p^*) = (p^*-I)\psi'(p^*); \tag{20}$$

$$a(p) \leq \rho(p-I) \quad \text{for } p > p^*. \tag{21}$$

Here $\psi(x)$ is an increasing solution to the ODE (6) *and $a(p)$ is the drift coefficient of the process X_t.*

Proof. Define the value function for the problem (2) over the class $\mathcal{M}$ of all stopping times as follows:

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}^x \left(X_\tau - I\right) e^{-\rho\tau} \mathbf{1}_{\{\tau < \infty\}}.$$

(i) *Sufficiency.* Let conditions (19)–(21) hold. Take the function

$$\Phi(x) = V(p^*; x) = \begin{cases} \dfrac{p^* - I}{\psi(p^*)}\psi(x), & \text{for } x < p^*, \\ x - I, & \text{for } x \ge p^*. \end{cases}$$

Obviously, $\Phi(x) > 0$ (due to condition (20)) and $V(x) \ge \Phi(x)$ for all $x \in \mathcal{I}$. On the other hand, condition (19) implies

$$\frac{p^* - I}{\psi(p^*)}\psi(x) \ge \frac{x - I}{\psi(x)}\psi(x) = x - I.$$

Therefore $\Phi(x) \ge x - I$ for all $x \in (l, r)$, i.e. $\Phi(x)$ is a majorant of the specific payoff function $x - I$.

For any stopping time $\tau \in \mathcal{M}$ and a real number $N > 0$ put $\tilde{\tau} = \tau \wedge N$. From Itô–Tanaka–Meyer formula (see, e.g. [10]) we have:

$$\begin{aligned} \mathbf{E}^x \Phi(X_{\tilde{\tau}}) e^{-\rho\tilde{\tau}} &= \Phi(x) + \mathbf{E}^x \int_0^{\tilde{\tau}} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt \\ &+ \frac{1}{2}\sigma^2(p^*)[\Phi'(p^*{+}0) - \Phi'(p^*{-}0)]\mathbf{E}^x \int_0^{\tilde{\tau}} e^{-\rho t} dL_t(p^*), \end{aligned} \tag{22}$$

where $L_t(p^*)$ is the local time of the process X_t at the point p^*.

By definition and in view of condition (20) we have

$$\Phi'(p^* + 0) - \Phi'(p^* - 0) = 1 - \frac{p^* - I}{\psi(p^*)}\psi'(p^*) = 0.$$

Denote $T_1 = \{t : 0 \le t \le \tilde{\tau}, X_t < p^*\}$, $T_2 = \{t : 0 \le t \le \tilde{\tau}, X_t > p^*\}$. We have:

$$\begin{aligned} \mathbb{L}\Phi(X_t) - \rho\Phi(X_t) &= \frac{p^* - I}{\psi(p^*)}\Big(\mathbb{L}\psi(X_t) - \rho\psi(X_t)\Big) = 0 \quad \text{for } t \in T_1, \\ \mathbb{L}\Phi(X_t) - \rho\Phi(X_t) &= a(X_t) - \rho(X_t - I) \le 0 \quad \text{for } t \in T_2. \end{aligned}$$

These relations follow from the definition of the function $\psi(x)$ and in view of condition (21). Then

$$\begin{aligned} \mathbf{E}^x \Phi(X_{\tilde{\tau}}) e^{-\rho\tilde{\tau}} &\le \Phi(x) + \mathbf{E}^x \left(\int_{T_1} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt + \int_{T_2} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt \right) \\ &\le \Phi(x). \end{aligned}$$

Since $\Phi(X_{\tilde{\tau}}) e^{-\rho\tilde{\tau}} \xrightarrow{\text{a.s.}} \Phi(X_\tau) e^{-\rho\tau} \mathbf{1}_{\{\tau<\infty\}}$ when $N \to \infty$, then due to Fatou's Lemma: $\mathbf{E}^x \Phi(X_\tau) e^{-\rho\tau} \mathbf{1}_{\{\tau<\infty\}} \le \Phi(x)$ for all $\tau \in \mathcal{M}$ and $x \in \mathcal{I}$. Therefore, $\Phi(x)$ is a ρ-excessive function, which majorates the payoff function $x - I$. Since, by Dynkin's characterization, the value function $V(x)$ is the least ρ-excessive majorant, then $V(x) \le \Phi(x)$.

Therefore, $V(x) = \Phi(x) = V(p^*; x)$, i.e. τ_{p^*} is the optimal stopping time in problem (2) for all x.

(ii) *Necessity.* Now, let τ_{p^*} be an optimal stopping time in the problem (2). Note, that τ_{p^*} will be also an optimal stopping time in the problem (7). Therefore, Theorem 1 implies conditions (19) and (20), since p^* is a point of maximum for the function $(x - I)/\psi(x)$.

Further, assume that inequality (21) is not true at some point $p_0 > p^*$, i.e. $a(p) > \rho(p - I)$ in some interval $J \subset (p^*, r)$ (by continuity). For some $\tilde{x} \in J$ define $\tau = \inf\{t \geq 0 : X_t \notin J\}$, where the process X_t starts from the point $\tilde{x}$. Then for any $N > 0$ from Dynkin's formula

$$\mathbf{E}^{\tilde{x}}(X_{\tau\wedge N} - I)e^{-\rho(\tau\wedge N)} = \tilde{x} - I + \mathbf{E}^{\tilde{x}} \int_0^{\tau\wedge N} [a(X_t) - \rho(X_t - I)]e^{-\rho t} dt > \tilde{x} - I.$$

Therefore, $V(\tilde{x}) > \tilde{x} - I$ which contradicts the relation $V(\tilde{x}) = V(p^*; \tilde{x}) = g(\tilde{x})$, since $\tilde{x} > p^*$.

Example 1. Let the process X_t be the geometric Brownian motion (12). Then Theorem 2 implies that the threshold stopping time τ_{p^*} will be optimal in the investment timing problem (2) over all investment times if and only if $p^* = I\beta/(\beta - 1)$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-\rho = 0$.

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