Game Control Problem for Systems of Distributed Equations

Vyacheslav Maksimov[1,2(✉)]

[1] Moscow State University, Moscow, Russia
maksimov@imm.uran.ru
[2] Institute of Mathematics and Mechanics of UB RAS, Yekaterinburg, Russia

Abstract. We consider a game problem of guaranteed positional control for a distributed system described by the phase field equations under incomplete information on system's phase states. This problem is investigated from the viewpoint of the first player (the partner). For this player, a procedure for forming feedback controls is specified. This procedure is stable with respect to informational noises and computational errors and is based on the method of extremal shift and the method of stable sets from the theory of guaranteed positional control. It uses the idea of stable dynamical inversion of controlled systems.

Keywords: Guaranteed control · Phase field equations

1 Introduction

The control theory for distributed systems has been intensively developed in recent time as a part of mathematical control theory. At present, there exists a number of monographs devoted to control problems for distributed systems [1–3]. As a rule, the emphasis is on program control problems in the case when all system's parameters are precisely specified. Along with this, the investigation of control problems for systems with uncontrollable disturbances (the problems of game control) is also reasonable. Similar problems have been poorly investigated. In the early 70es, N.N. Krasovskii suggested an effective approach to solving game (guaranteed) control problems, which is based on the formalism of positional strategies. The detailed description of this approach for dynamical systems described by ordinary differential equations is given in [4]. The goal of the present work is to illustrate possibilities of this approach for investigating a game problem for systems described by the phase field equations.

We consider a system modeling the solidification process and governed by the phase field equations (introduced in [5])

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi + l\frac{\partial}{\partial t}\varphi &= k\Delta_L\psi + Bu - Cv \quad \text{in} \quad Q = \Omega \times (t_0, \vartheta], \\
\tau\frac{\partial}{\partial t}\varphi &= \xi^2\Delta_L\varphi + g(\varphi) + \psi, \quad \vartheta = \text{const} < +\infty,
\end{aligned}
\tag{1}
$$

with the boundary condition $\frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0$ on $\partial\Omega \times (t_0, \vartheta]$ and the initial condition $\psi(t_0) = \psi_0$, $\varphi(t_0) = \varphi_0$ in Ω. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$, Δ_L is the Laplace operator, $\partial/\partial n$ is the outward normal derivative, $(U, |\cdot|_U)$ and $(V, |\cdot|_V)$ are Banach spaces, $B \in \mathcal{L}(U; H)$ and $C \in \mathcal{L}(V; H)$ are linear continuous operators, $H = L_2(\Omega)$, and the function $g(z)$ is the derivative of a so-called potential $G(z)$. We assume that $g(z) = az + bz^2 - cz^3$.

Systems of form (1) have been investigated by many authors. In what follows, for the sake of simplicity, we assume that $k = \xi = \tau = c = 1$. Further, we assume that the following conditions are fulfilled: (A1) the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, has the boundary of C^2-class; (A2) the coefficients a and b are elements of the space $L_\infty(T \times \Omega)$, $T = [t_0, \vartheta]$, and vrai sup $c(t, \eta) > 0$ for $(t, \eta) \in [t_0, \vartheta] \times \Omega$; (A3) the initial functions ψ_0 and φ_0 are such that $\{\psi_0, \varphi_o\} \in \mathcal{R} = \{\psi, \varphi \in W^2_\infty(\Omega) : \frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\varphi = 0$ on $\partial\Omega\}$.

Introduce the notation: $W^{2,1}_p(Q) = \left\{u \mid u, \frac{\partial u}{\partial \eta_i}, \frac{\partial^2 u}{\partial \eta_i \partial \eta_j}, \frac{\partial u}{\partial t} \in L^p(Q)\right\}$ for $p \in [1, \infty)$ is the standard Sobolev space with the norm $\|u\|_{W^{2,1}_p(Q)}$; $(\cdot, \cdot)_H$ and $|\cdot|_H$ are the scalar product and the norm in H, respectively. Let some initial state $x_0 = \{\psi_0, \varphi_0\}$ and functions $u(\cdot) \in L_\infty(T; U)$ and $v(\cdot) \in L_\infty(T; V)$ be fixed. A solution of system (1), $x(\cdot; t_0, x_0, u(\cdot), v(\cdot)) = \{\psi(\cdot; t_0, \psi_0, u(\cdot), v(\cdot)), \varphi(\cdot; t_0, \varphi_0, u(\cdot), v(\cdot))\}$, is a unique function $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot), v(\cdot)) \in V^{(1)}_T = V_1 \times V_1$, $V_1 = W^{2,1}_2(Q)$ satisfying relations (1). By virtue of the corresponding embedding theorem, without loss of generality, one can assume that the space $V^{(1)}_T$ is embedded into the space $C(T; H \times H)$. Therefore, the element $x(t) = \{\psi(t), \varphi(t)\} \in H \times H$ is the phase state of system (1) at the time t. The following theorem takes place

Theorem 1. [6, p. 25, Assertion 5] *Let conditions* (A1)–(A3) *be fulfilled. Then for any* $u(\cdot) \in L_\infty(T; U)$ *and* $v(\cdot) \in L_\infty(T; V)$ *there exists a unique solution of system* (1).

The paper is devoted to the investigation of the game control problem. Let us give the informal formulation of this problem. Let a uniform net $\Delta = \{\tau_i\}^m_{i=0}$, $\tau_i = \tau_{i-1} + \delta$, $\tau_0 = t_0$, $\tau_m = \vartheta$ with a diameter $\delta = \tau_i - \tau_{i-1}$ be fixed on a given time interval T. Let a solution of system (1) be unknown. At the times $\tau_i \in \Delta$, a part of the phase states $x(\tau_i)$ (namely $\phi(\tau_i)$) is inaccurately measured. The results of measurements $\xi^h_i \in H$, $i \in [1 : m - 1]$, satisfy the inequalities

$$|\xi^h_i - \phi(\tau_i)|_H \leq h. \tag{2}$$

Here, $h \in (0, 1)$ is a level of informational noise. Let the following quality criterion be given: $I(x(\cdot), u(\cdot)) = \sigma(x(\vartheta)) + \int_{t_0}^{\vartheta} \chi(t, x(t), u(t))\, dt$, where $\sigma : H \times H \to \mathbb{R}$ and $\chi : T \times H \times H \times U \to \mathbb{R}$ are given functions satisfying the local Lipschitz conditions. Let also a prescribed value of the criterion, number I_*, be fixed. The control problem under consideration consists in the following. There are two players-antagonists controlling system (1) by means of u and v, respectively.

One of them is called a partner; another, an opponent. Let $P \subset U$ and $E \subset V$ be given convex bounded and closed sets. The problem undertaken by the partner is as follows. It is necessary to construct a law (a strategy) for forming the control u (with values from P) by the feedback principle (on the base of measurements of $\varphi(\tau_i)$) in order to provide the prescribed value of the quality criterion for any (unknown) realization $v = v(\cdot)$.

To form the control u providing the solution of the problem, along with the information on the "part" of coordinates of the solution of system (1) (namely, on the values ξ_i^h satisfying inequalities (2)), it is necessary to obtain some additional information on the coordinate $\psi(\cdot)$, which is missing. To get such a piece of information during the control process, it is reasonable, following the approach developed in [7–9], to introduce an auxiliary controlled system. This system is described by a parabolic equation (the form is specified below). The equation has an output $w_*(t)$, $t \in T$, and an input $p^h(t)$, $t \in T$. The input $p^h(\cdot)$ is some new auxiliary control; it should be formed by the feedback principle in such a way that $p^h(\cdot)$ "approximates" the unknown coordinate $\psi(\cdot)$ in the mean square metric. Thus, along with the block of forming the control in the real system (it is called an controller), we need to incorporate into the control contour one more block (it is called an identifier) allowing to reconstruct the missing coordinate $\psi(\cdot)$ in real time. Note that, in essence, the identifier block solves a dynamical inverse problem, namely, the problem of (approximate) reconstruction of the unknown coordinate $\psi(\cdot)$. In he recent time, the theory of inverse problems for distributed systems has been intensively developed. Among the latest investigations, it is possible to mark out the research [10].

2   Problem statement

Before passing to the problem formulation, we give some definitions. Furthermore, we denote by $u_{a,b}(\cdot)$ the function $u(t)$, $t \in [a, b]$, considered as a whole. The symbol $P_{a,b}(\cdot)$ stands for the restriction of the set $P_T(\cdot)$ onto the segment $[a, b] \subset T$. Any strongly measurable functions $u(\cdot) : T \to P$ and $v(\cdot) : T \to E$ are called program controls of the partner and opponent, respectively. The sets of all program controls of the partner and opponent are denoted by the symbols $P_T(\cdot)$ and $E_T(\cdot) : P_T(\cdot) = \{u(\cdot) \in L_2(T; U) : u(t) \in P$ a.e. $t \in T\}$, $E_T(\cdot) = \{v(\cdot) \in L_2(T; V) : v(t) \in E$ a.e. $t \in T\}$. Elements of the product $T \times \mathcal{H}$ are called positions, $\mathcal{H} = H \times H \times \mathbb{R} \times H \times H \times \mathbb{R}$. Any function (perhaps, multifunction) $\mathcal{U} : T \times \mathcal{H} \to P$ is said to be a positional strategy of the partner. The positional strategy corrects the controls at discrete times given by some partition of the interval T. Any function $\mathcal{V} : T \times H \times H \to H$ is said to be a strategy of reconstruction. The strategy $\mathcal{V}$ is formed in order to reconstruct the unknown component $\psi(\cdot)$.

Consider the following ordinary differential equation

$$\dot{q}(t) = \chi(t, x(t), u(t)), \quad q(t_0) = 0. \tag{3}$$

Introducing this new variable q, we reduce the control problem of Bolza type to a control problem with a terminal quality criterion of the form $I = \sigma(x(\vartheta)) + q(\vartheta)$.

In this case, the controlled system consists of phase field Eq. (1) and ordinary differential Eq. (3).

The scheme of an algorithm for solving the problem undertaken by the partner is as follows. In the beginning, auxiliary systems M_1 and M_2 (models) are introduced. The system M_1 has an input $u^*(\cdot)$ and an output $w(\cdot)$; the system M_2, an input $p^h(\cdot)$ and an output $w_*(\cdot)$, respectively. The model M_2 with its control law $\mathcal{V}$ forms the identifier, whereas the model M_1 and system (1) (with their control laws) form the controller. The process of synchronous feedback control of systems (1), (3), M_1, and M_2 is organized on the interval T. This process is decomposed into $m - 1$ identical steps. At the ith step carried out during the time interval $[\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time τ_i, according to some chosen rules $\mathcal{V}$ and $\mathcal{U}$, the elements $p_i^h \in \mathcal{V}(\tau_i, \xi_i^h, w_*(\tau_i))$, $u_i^h \in \mathcal{U}(\tau_i, \xi_i^h, p_i^h, \psi_i^h, w(\tau_i))$ are calculated. Here, ψ_i^h is the result of measuring $q(\tau_i)$. Then (till the moment τ_{i+1}), the control $p^h(t) = p_i^h$, $\tau_i \le t < \tau_{i+1}$, is fed onto the input of the system M_2; the control $u^h(t) = u_i^h$, $\tau_i \le t < \tau_{i+1}$, onto the input of system (1), (3). Under the action of these controls, as well as of the given control $u^*(t)$, $\tau_i \le t < \tau_{i+1}$, and the unknown control of the opponent $v(t)$, $\tau_i \le t < \tau_{i+1}$, the states $x(\tau_{i+1})$, $q(\tau_i)$, $w(\tau_{i+1})$, and $w_*(\tau_{i+1})$ are realized at the time τ_{i+1}. The procedure stops at the time ϑ.

Let models M_1 and M_2 with phase trajectories $w(\cdot)$ and $w_*(\cdot)$ be fixed. A solution $x(\cdot)$ of system (1) starting from an initial state (t_*, x_*) and corresponding to piecewise constant controls $u^h(\cdot)$ and $p^h(\cdot)$ (formed by the feedback principle) and to a control $v_{t_*,\vartheta}(\cdot) \in E_{t_*,\vartheta}(\cdot)$ is called an (h, Δ)-motion $x_{\Delta,w}^h(\cdot) = x_{\Delta,w}^h(\cdot; t_*, x_*, \mathcal{U}, \mathcal{V}, v_{t_*,\vartheta}(\cdot))$. This motion is generated by the positional strategies $\mathcal{U}$ and $\mathcal{V}$. Thus, the motions $x_{\Delta,w}^h(\cdot)$, $q_\Delta^h(\cdot), w(\cdot)$, and $w_*(\cdot)$ are formed simultaneously. So, for $t \in [\tau_i, \tau_{i+1})$, we define $x_{\Delta,w}^h(t) = x(t; \tau_i, x_{\Delta,w}^h(\tau_i), u_{\tau_i,\tau_{i+1}}^h(\cdot), v_{\tau_i,\tau_{i+1}}(\cdot))$, $q_\Delta^h(t) = q(t; \tau_i, q_\Delta^h(\tau_i), u_{\tau_i,\tau_{i+1}}^h(\cdot))$, $w(t) = w(t; \tau_i, w(\tau_i), u_{\tau_i,\tau_{i+1}}^*(\cdot)), w_*(t) = w_*(t; \tau_i, w_*(\tau_i), p_{\tau_i,\tau_{i+1}}^h(\cdot))$, where

$$u^h(t) = u_i^h \in \mathcal{U}(\tau_i, \xi_i^h, p_i^h, q^h(\tau_i), w(\tau_i)), \quad p^h(t) = p_i^h \in \mathcal{V}(\tau_i, \xi_i^h, w_*(\tau_i)) \qquad (4)$$

$$\text{for } t \in [\tau_i, \tau_{i+1}),$$

$$i \in [i(t_*) : m - 1], \quad |\xi_i^h - \phi_{\Delta,w}^h(\tau_i)|_H \le h, \quad |\psi_i^h - g_\Delta^h(\tau_i)| \le h, \qquad (5)$$

$$i(t_*) = \min\{i : \tau_i > t_*\}, \quad u^h(t) = u_*^h \in P, \quad p^h(t) = p_*^h \in H \text{ for } t \in [t_*, \tau_{i(t_*)}).$$

The set of all (h, Δ)-motions is denoted by $X_h(t_*, x_*, \mathcal{U}, \mathcal{V}, \Delta, w)$.

Problem. It is necessary to construct a positional strategy $\mathcal{U} : T \times \mathcal{H} \to P$ of the partner and a positional strategy $\mathcal{V} : T \times H \times H \to H$ of reconstruction with the following properties: whatever a value $\varepsilon > 0$ and a disturbance $v_T(\cdot) \in E_T(\cdot)$, one can find (explicitly) numbers $h_* > 0$ and $\delta_* > 0$ such that the inequalities

$$|I(x_{\Delta,w}^h(\cdot), u_T^h(\cdot)) - I_*| \le \varepsilon \quad \forall x_{\Delta,w}^h(\cdot) \in X_h(t_0, x_0, \mathcal{U}, \mathcal{V}, \Delta, w) \qquad (6)$$

are fulfilled uniformly with respect to all measurements ξ_i^h and ψ_i^h with properties (5), if $h \le h_*$ and $\delta = \delta(\Delta) \le \delta_*$.

3   Algorithm for Solving the Problem

To solve the Problem, we use ideas from [4], namely, the method of a priori
stable sets. In our case, this method consists in the following. Let a trajectory
of model M_1, $w(\cdot)$, possessing the property $\sigma(w_1(\vartheta)) + w_2(\vartheta) = I_*$ be known.
Then, a feedback strategy $\mathcal{U}$ providing tracking the prescribed trajectory of M_1
by the trajectory of real system (1) is constructed. This means that the (h, Δ)-
motion $x_\Delta^h(\cdot)$ formed by the feedback principle (see (4)) remains at a "small"
neighborhood of the trajectory $w(\cdot)$ during the whole interval T. This property
of the (h, Δ)-motion allows us to conclude that the chosen strategy solves the
considered control problem.

Let us pass to the realization of this scheme. Define $\Phi(t, x, u, v) = \{Bu - Cv, \chi(t, x, u)\} \in H \times \mathbb{R}$, $\Phi_u(t, x, v) = \bigcup_{u \in P} \Phi(t, x, u, v)$, $H_*(t; x) = \bigcap_{v \in E} \Phi_u(t, x, v)$,
$H_*(\cdot; x) = \{u(\cdot) \in L_2(T; H \times \mathbb{R}) : u(t) \in H_*(t; x) \; for \quad a.\, a.\; t \in T\}$. As a model
M_1, we take the system including two subsystems, i.e., the phase field equation

$$\frac{\partial}{\partial t} w^{(1)} + l \frac{\partial}{\partial t} w^{(2)} = \Delta_L w^{(1)} + u_1 \quad \text{in} \quad \Omega \times (t_0, \vartheta], \tag{7}$$

$$\frac{\partial}{\partial t} w^{(2)} = \Delta_L w^{(2)} + g(w^{(2)}) + w^{(1)}$$

with the boundary condition $\frac{\partial}{\partial n} w^{(1)} = \frac{\partial}{\partial n} w^{(2)} = 0$ on $\partial\Omega \times (t_0, \vartheta]$ and the
initial condition $w^{(1)}(t_0) = \psi_0$, $w^{(2)}(t_0) = \varphi_0$ in Ω, as well as the ordinary
differential equation

$$\dot{w}^{(3)}(t) = u_2(t), \quad w^{(3)} \in \mathbb{R}, \quad w^{(3)}(0) = 0. \tag{8}$$

By the symbol $w(\cdot)$, we denote the solution of system (7), (8). Then, the model
M_1 has the control $u(\cdot) = \{u_1(\cdot), u_2(\cdot)\}$. As a model M_2, we use the equation

$$\frac{\partial}{w_*} \partial t = \Delta_L w_* + p^h + g(w_*) \quad \text{in} \quad \Omega \times (t_0, \vartheta] \tag{9}$$

with the boundary condition $\frac{\partial w_*}{\partial n} = 0$ on $\partial\Omega \times (t_0, \vartheta]$ and the initial condition
$w_*(t_0) = \varphi_0$ in Ω.

Condition 1. There exists a control $u(\cdot) = u^*(\cdot) = \{u_1^*(\cdot), u_2^*(\cdot)\} \in H(t; w^{(1)}(t), w^{(2)}(t))$ for a.a. $t \in T$ such that $I_* = \sigma(w^{(1)}(\vartheta)) + w^{(2)}(\vartheta)$.

The strategies $\mathcal{U}$ and $\mathcal{V}$ (see (4)) are defined in such a way:

$$\mathcal{U}(t, \xi, p, \psi, w) = \arg\min\{L(u, y) + (\psi - w^{(3)})\chi(t, \xi, p, u) : u \in P\}, \tag{10}$$

$$\mathcal{V}(t, \xi, w_*) = \arg\min\{l(t, \alpha, u, s) : u \in U_d\}, \tag{11}$$

where $w = \{w^{(1)}, w^{(2)}, w^{(3)}\}$, $L(u, y) = (-y, Bu)_H$, $y = w^{(1)} - p + l(w^{(2)} - \xi)$,
$l(t, \alpha, u, s) = \exp(-2b_* t)(s, u)_H + \alpha|u|_H^2$, $s = w_* - \xi$, $b_* = |a + 1/3b|_{L_\infty(Q)}$,
$\alpha = \alpha(h)$, $U_d = \{p(\cdot) \in L_2(T; H) : |p(t)|_H \leq d$ for a.a. $t \in T\}$, $d \geq \sup_{t \in T}\{|\psi(t; t_0, x_0, u(\cdot), v(\cdot))|_H : u(\cdot) \in P_T(\cdot), v(\cdot) \in E_T(\cdot)\}$.

Condition 2. Let $h, \delta(h)$, and $\alpha(h)$ satisfy the conditions: $\alpha(h) \to 0, \delta(h) \to 0, (h + \delta(h))\alpha^{-1}(h) \to 0$ as $h \to 0$.

Let us pass to the description of the algorithm for solving the Problem. Namely, we describe the procedure of forming the (h, Δ)-motion $x^h_{\Delta,w}(\cdot) = \{\psi^h_{\Delta,w}(\cdot), \varphi^h_{\Delta,w}(\cdot)\}$ and trajectory $g^h_\Delta(\cdot)$ corresponding to some fixed partition Δ and the strategies $\mathcal{U}$ and $\mathcal{V}$, see (10) and (11). Before the algorithm starts, we fix a value $h \in (0,1)$, a function $\alpha = \alpha(h)$: $(0,1) \to (0,1)$, and a partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ with diameter $\delta = \delta(h) = \tau_{i+1} - \tau_i, \tau_i = \tau_{h,i}, \tau_{h,0} = t_0, \tau_{h,m_h} = \vartheta$. The work of the algorithm is decomposed into $m_h - 1$ identical steps. We assume that

$$u^h(t) = u^h_0 \in \mathcal{U}(t_0, \xi^h_0, p^h_0, 0, w(t_0)), \quad p^h(t) = p^h_0 \in \mathcal{V}(t_0, \xi^h_0, \varphi_0)$$

$(|\xi^h_0 - \varphi_0|_H \le h)$ on the interval $[t_0, \tau_1)$. Under the action of these piecewise-constant controls as well as of an unknown disturbance $v_{t_0,\tau_1}(\cdot)$, the (h, Δ)-motion $\{x^h_{\Delta,w}(\cdot)\}_{t_0,\tau_1} = \{\psi^h_{\Delta,w}(\cdot; t_0, \psi_0, u^h_{t_0,\tau_1}(\cdot), v_{t_0,\tau_1}(\cdot)), \varphi^h_{\Delta,w}(\cdot, t_0, \varphi_0, u^h_{t_0,\tau_1}(\cdot), v_{t_0,\tau_t}(\cdot))\}_{t_0,\tau_1}$ of system (1), the trajectory $\{q^h_\Delta(\cdot)\}_{t_0,\tau} = \{q(\cdot; t_0, q(t_0)), u^h_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of Eq. (3), the trajectory $\{w_*(\cdot)\}_{t_0,\tau_1} = \{w_*(\cdot; t_0, \phi_0, p^h_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{t_0,\tau_1} = \{w(\cdot; t_0, x_0, u^*_{t_0,\tau_1}(\cdot))\}_{t_0,\tau_1}$ of the model M_1 are realized. At the time $t = \tau_1$, we determine u^h_1 and p^h_1 from the conditions

$$u^h_1 \in \mathcal{U}(\tau_1, \xi^h_1, p^h_1, \psi^h_1, w(\tau_1)), \quad p^h_1 \in \mathcal{V}(\tau_1, \xi^h_1, w_*(\tau_1))$$

$(|\xi^h_1 - \varphi^h_{\Delta,w}(\tau_1)|_H \le h, |\psi^h_1 - g^h_\Delta(\tau_1)| \le h)$, i.e., we assume that $u^h(t) = u^h_1$ and $p^h(t) = p^h_1$ for $t \in [\tau_1, \tau_2)$. Then, we calculate the realization of the (h, Δ)-motion

$$\{x^h_{\Delta,w}(\cdot)\}_{\tau_1,\tau_2} = \{\psi^h_{\Delta,w}(\cdot; \tau_1, \psi^h_{\Delta,w}(\tau_1), u^h_{\tau_1,\tau_2}(\cdot), v_{\tau_1,\tau_2}(\cdot)),$$
$$\varphi^h_{\Delta,w}(\cdot; \tau_1, \varphi^h_{\Delta,w}(\tau_1), u^h_{\tau_1,\tau_2}(\cdot), v_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2},$$

the trajectory $\{q^h_\Delta(\cdot)\}_{\tau_1,\tau_2} = \{q(\cdot; \tau_1, q^h_\Delta(\tau_1), p^h_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2}$ of Eq. (3), the trajectory $\{w_*(\cdot)\}_{\tau_1,\tau_2} = \{w_*(\cdot; \tau_1, w_*(\tau_1), p^h_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{\tau_1,\tau_2} = \{w(\cdot; \tau_1, w(\tau_1), u^*_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2}$ of the model M_1.

Let the (h, Δ)-motion $x^h_{\Delta,w}(\cdot)$, the trajectory $q^h_\Delta(\cdot)$ of Eq. (3), the trajectory $w_*(\cdot)$ of the model M_2, and the trajectory $w(\cdot)$ of the model M_1 be defined on the interval $[t_0, \tau_i]$. At the time $t = \tau_i$, we assume that

$$u^h_i \in \mathcal{U}(\tau_i, \xi^h_i, p^h_i, \psi^h_i, w(\tau_i)), \quad p^h_i \in \mathcal{V}(\tau_i, \xi^h_i, w_*(\tau_i)) \tag{12}$$

$(|\xi^h_i - \varphi^h_{\Delta,w}(\tau_i)|_H \le h, |\psi^h_i - g^h_\Delta(\tau_i)| \le h)$, i.e., we set $u^h(t) = u^h_i$ and $p^h(t) = p^h_i$ for $t \in [\tau_i, \tau_{i+1})$. As a result of the action of these controls and an unknown disturbance $v_{\tau_i,\tau_{i+1}}(\cdot)$, the (h, Δ)-motion

$$\{x^h_{\Delta,w}(\cdot)\}_{\tau_i,\tau_{i+1}} = \{\psi^h_{\Delta,w}(\cdot; \tau_i, \psi^h_{\Delta,w}(\tau_i), u^h_{\tau_i,\tau_{i+1}}(\cdot), v_{\tau_i,\tau_{i+1}}(\cdot)),$$
$$\varphi(\cdot; \tau_i, \varphi^h_{\Delta,w}(\tau_i), u^h_{\tau_i,\tau_{i+1}}(\cdot), v_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}},$$

the trajectory $\{q_\Delta^h(\cdot)\}_{\tau_i, \tau_{i+1}} = \{q(\cdot; \tau_i, q_\Delta^h(\tau_i), p_{\tau_i, \tau_{i+1}}^h(\cdot))\}_{\tau_i, \tau_{i+1}}$ of Eq. (3), the trajectory $\{w_*(\cdot)\}_{\tau_i, \tau_{i+1}} = \{w_*(\cdot; \tau_i, w_*(\tau_i), p_{\tau_i, \tau_{1+1}}^h(\cdot))\}_{\tau_i, \tau_{i+1}}$ of the model M_2, and the trajectory $\{w(\cdot)\}_{\tau_i, \tau_{i+1}} = \{w(\cdot; \tau_i, w(\tau_i), u_{\tau_i, \tau_{i+1}}^*(\cdot))\}_{\tau_i, \tau_{i+1}}$ of the model M_1 are realized on the interval $[\tau_i, \tau_{i+1}]$. This procedure stops at the time ϑ.

Theorem 2. *Let Conditions 1 and 2 be fulfilled. Let also the models M_1 and M_2 be specified by relations (7), (8), and (9), respectively. Then, the strategies $\mathcal{U}$ and $\mathcal{V}$ of form (10) and (11) solve the Problem.*

Proof. To prove the theorem, we estimate the variation of the functional

$$\Lambda(t, x_{\Delta,w}^h(\cdot), q_\Delta^h(\cdot), w(\cdot)) = \Lambda^0(x_{\Delta,w}^h(t), q_\Delta^h(t), w(t))$$

$$+ 0.5 \int_0^t \left\{ \int_\Omega |\nabla \pi^h(\varrho, \eta)|^2 \, d\eta + l^2 \int_\Omega |\nabla \nu^h(\varrho, \eta)|^2 \, d\eta \right\} d\varrho,$$

where $x_{\Delta,w}^h(\cdot) = \{\psi_{\Delta,w}^h(\cdot), \phi_{\Delta,w}^h(\cdot)\}$, $w(\cdot) = \{w^{(1)}(\cdot), w^{(2)}(\cdot), w^{(3)}(\cdot)\}$, $\pi^h(t) = w^{(1)}(t) - \psi_{\Delta,w}^h(t)$, $\nu^h(t) = w^{(2)}(t) - \varphi_{\Delta,w}^h(t)$, $g^h(t) = \pi^h(t) + l\nu^h(t)$, $\Lambda_1(x_{\Delta,w}^h(t), w(t)) = 0.5|g^h(t)|_H^2 + 0.5l^2|\nu^h(t)|_H^2$, $\lambda(q_\Delta^h(t), w^{(3)}(t)) = 0.5|q_\Delta^h(t) - w^{(3)}(t)|^2$, $\Lambda^0(x_{\Delta,w}^h(t), q_\Delta^h(t), w(t)) = \Lambda_1(x_{\Delta,w}^h(t), w(t)) + \lambda(q_\Delta^h(t), w^{(3)}(t))$. It is easily seen that the functions $\pi^h(\cdot)$ and $\nu^h(\cdot)$ are solutions of the system

$$\frac{\partial \pi^h(t, \eta)}{\partial t} + l\frac{\partial \nu^h(t, \eta)}{\partial t} = \Delta_L \pi^h(t, \eta) + u_1^*(t, \eta) - (Bu^h)(t, \eta) + (Cv)(t, \eta), \quad (13)$$

$$\frac{\partial \nu^h(t, \eta)}{\partial t} = \Delta_L \nu^h(t, \eta) + R^h(t, \eta)\nu^h(t, \eta) + \pi^h(t, \eta) \quad \text{in } \Omega \times (t_0, \vartheta],$$

with the initial condition $\pi^h(t_0) = \nu^h(t_0) = 0$ in Ω and with the boundary condition $\frac{\partial \pi^h}{\partial n} = \frac{\partial \nu^h}{\partial n} = 0$ on $\partial\Omega \times (t_0, \vartheta]$. Here, $R^h(t, \eta) = a(t, \eta) + b(t, \eta)(w^{(1)}(t, \eta) + \varphi_{\Delta,w}^h(t, \eta)) - ((w^{(1)}(t, \eta))^2 + w^{(1)}(t, \eta)\varphi_{\Delta,w}^h(t, \eta) + (\varphi_{\Delta,w}^h)^2(t, \eta))$. Multiplying scalarly the first equation of (13) by $g^h(t)$, and the second one, by $\nu^h(t)$, we obtain

$$(g^h(t), g_t^h(t))_H + \int_\Omega \{|\nabla \pi^h(t, \eta)|^2 + l\nabla \pi^h(t, \eta)\nabla \nu^h(t, \eta)\} \, d\eta$$

$$= (g^h(t), u_1^*(t) - Bu^h(t) + Cv(t))_H, \quad (14)$$

$$(\nu^h(t), \nu_t^h(t))_H + \int_\Omega |\nabla \nu^h(t, \eta)|^2 \, d\eta \le (\pi^h(t), \nu^h(t))_H + b|\nu^h(t)|_H^2 \quad \text{for a.a. } t \in T.$$

Here, we use the inequality $\operatorname*{vrai\,max}_{(t,\eta) \in T \times \Omega} \{a(t, \eta) + b(t, \eta)(v_1 + v_2) - (v_1^2 + v_1 v_2 + v_2^2)\} \le b$, which is valid for any $v_1, v_2 \in \mathbb{R}$. It is evident that the inequality

$$\int_\Omega l(\nabla \pi^h(t, \eta), \nabla \nu^h(t, \eta)) \, d\eta \ge -0.5 \int_\Omega \{|\nabla \pi^h(t, \eta)|^2 + l^2|\nabla \nu^h(t, \eta)|^2\} \, d\eta \quad (15)$$

is fulfilled for a.a. $t \in T$. Let us multiply the first inequality of (14) by l^2 and add to the second one. Taking into account (15), we have for a.a. $t \in T$

$$(g^h(t), g^h_t(t))_H + l^2(\nu^h(t), \nu^h_t(t))_H + 0.5 \int_\Omega \{|\nabla \pi^h(t, \eta)|^2 + l^2 |\nabla \nu^h(t, \eta)|^2\} \, d\eta \quad (16)$$

$$\leq (g^h(t), u^*_1(t) - Bu^h(t) + Cv(t))_H + l^2(\pi^h(t), \nu^h(t))_H + bl^2|\nu^h(t)|^2_H.$$

Note that $\pi^h(t) = g^h(t) - l\nu^h(t)$. In this case, for a.a. $t \in T$

$$(\pi^h(t), \nu^h(t))_H + b|\nu^h(t)|^2_H = (g^h(t) - l\nu^h(t), \nu^h(t))_H + b|\nu^h(t)|^2_H \quad (17)$$

$$= (g^h(t), \nu^h(t))_H + (b - l)|\nu^h(t)|^2_H \leq 0.5(|g^h(t)|^2_H + (0.5 + |b - l|)|\nu^h(t)|^2_H.$$

Combining (16) and (17), we obtain

$$\frac{d}{dt} \Lambda^0(x^h_{\Delta,w}(t), q^h_\Delta(t), w(t)) + 0.5 \int_\Omega \{|\nabla \pi^h(t, \eta)|^2 + l^2|\nabla \nu^h(t, \eta)|^2\} \, d\eta \quad (18)$$

$$\leq 2l^2 \lambda^2 \Lambda^0(x^h_{\Delta,w}(t), q^h_\Delta(t), w(t)) + \gamma^{(1)}_t + \gamma^{(2)}_t,$$

where $\gamma^{(1)}_t = (q^h_\Delta(t) - w^{(3)}(t))(\chi(t, x^h_{\Delta,w}(t), u^h(t)) - u^*_2(t)), \gamma^{(2)}_t = (g^h(t), u^*_1(t) - Bu^h(t) + Cv(t))_H$ for a.a. $t \in T$. For a.a. $t \in \delta_i = [\tau_i, \tau_{i+1}]$, it is easily seen that

$$\gamma^{(1)}_t \leq (q^h_\Delta(\tau_i) - w^{(3)}(\tau_i))(\chi(t, x^h_{\Delta,w}(t), u^h(t) - u^*_2(t)) + k_0(t - \tau_i)^{1/2}, \quad (19)$$

$$|\chi(t, x^h_{\Delta,w}(t), u^h(t)) - \chi(\tau_i, \xi^h_i, p^h_i, u^h(t))| \leq k_1\{h + (t - \tau_i)^{1/2} + |p^h_i - \psi^h_{\Delta,w}(t)|^2_H\}. \quad (20)$$

From (19) and (20) we have for a.a. $t \in \delta_i$

$$\gamma^{(1)}_t \leq (q^h_\Delta(\tau_i) - w^{(3)}(\tau_i))(\chi(\tau_i, \xi^h_i, \psi^h_i, u^h(t)) - u^*_2(t)) \quad (21)$$

$$+ k_2|q^h_\Delta(\tau_i) - w^{(3)}(\tau_i)|\{h + (t - \tau_i)^{1/2} + |p^h_i - \psi^h_{\Delta,w}(t)|_H\}.$$

Estimate the last term in the right-hand side of inequality (18). For a.a. $t \in [\tau_i, \tau_{i+1}]$

$$|g^h(t) - y^h_i|_H = |\pi^h(t) + l\nu^h(t) - y^h_i|_H \leq \lambda_{1,i}(t) + \lambda_{2,i}(t), \quad (22)$$

where $y^h_i = w^{(1)}(\tau_i) - p^h_i - l(w^{(2)}(\tau_i) - \xi^h_i), \lambda_{1,i}(t) = |w^{(1)}(t) - \psi^h_{\Delta,w}(t) - w^{(1)}(\tau_i) + p^h_i|_H, \lambda_{2,i}(t) = l|w^{(2)}(t) - \varphi^h_{\Delta,w}(t) - w^{(2)}(\tau_i) + \xi^h_i|_H$. For a.a. $t \in \delta_i = [\tau_i, \tau_{i+1}]$, it is easily seen that $\lambda_{1,i}(t) \leq |p^h_i - \psi^h_{\Delta,w}(t)|_H + \int_{\tau_i}^t |\dot{w}^{(1)}(\tau)|_H \, d\tau, \lambda_{2,i}(t) \leq lh +$

$l \int_{\tau_i}^t \{|\dot{\varphi}^h_{\Delta,w}(\tau)|_H + |\dot{w}^{(2)}(\tau)|_H\} \, d\tau$. From this equation and (22), for a.a. $t \in \delta_i$, it follows that

$$|g^h(t) - y^h_i|_H \leq lh + \int_{\tau_i}^t \{l|\dot{\varphi}^h_{\Delta,w}(\tau)|_H + |\dot{w}^{(1)}(\tau)|_H + |\dot{w}^{(2)}(\tau)|_H\} \, d\tau + |p^h_i - \psi^h_{\Delta,w}(t)|_H.$$

$$(23)$$

By virtue of (11), taking into account results of [8,9], from estimate (23) we derive

$$\sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} M(t;\tau_i)\,dt \le k_3(h+\delta) + k_4 \int_{t_0}^{\vartheta} |p^h(\tau) - \psi_{\Delta,w}^h(\tau)|_H\,d\tau \le k_5 \nu^{1/2}(h), \quad (24)$$

where $M(t;\tau_i) = |g^h(t) - y_i^h|_H \{|Bu_i^h|_H + |Cv(t)|_H + |u_1^*(t)|_H\}$ for a.a. $t \in \delta_i$, $\nu(h) = (h + \delta(h) + \alpha(h))^{1/2} + (h + \delta(h))\alpha^{-1}(h)$. Note that, for a.a. $t \in \delta_i$,

$$\gamma_t^{(2)} \le (y_i^h, u_1^*(t) - Bu_i^h + Cv(t))_H + M(t;\tau_i). \quad (25)$$

Let the symbol $(\cdot,\cdot)_{H\times\mathbb{R}}$ denote the scalar product in the space $H \times \mathbb{R}$. Let us define elements v_i^e from the conditions

$$\inf_{u\in P}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v_i^e))_{H\times\mathbb{R}} \ge \sup_{v\in E}\inf_{u\in P}(s_i, \Phi(\tau_i, \xi_i^h, u, v))_{H\times\mathbb{R}} - h, \quad (26)$$

where $\Phi(\tau_i, p_i^h, \xi_i^h, u_i^h, v(t)) = \{Bu_i^h - Cv(t), \chi(\tau_i, \xi_i^h, p_i^h, u_i^h)\}$, $s_i = \{-y_i^h, \psi_i^h - w^{(3)}(\tau_i)\}$. It is obvious (see Condition 1) that $u_*(t) \in H(t, w^{(1)}(t), w^{(2)}(t)) \subset \bigcup_{u\in P} \Phi(t, w^{(1)}(t), w^{(2)}(t), u, v_i^e)$ for a. a. $t \in [\tau_i, \tau_{i+1})$. Then, for a.a. $t \in \delta_i$, there exists a control $u^{(1)}(t) \in P$ such that

$$\Phi(t, w^{(1)}(t), w^{(2)}(t), u^{(1)}(t), v_i^e) = u^*(t) \quad \text{for a. a.} \quad t \in [\tau_i, \tau_{i+1}]. \quad (27)$$

Using the rule of definition of the strategy $\mathcal{U}$, we deduce that

$$(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u_i^h, v(t)))_{H\times\mathbb{R}} \le \inf_{u\in P}\sup_{v\in E}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v))_{H\times\mathbb{R}} + h. \quad (28)$$

Here (see (12)), $u_i^h \in \mathcal{U}(\tau_i, \xi_i^h, p_i^h, \psi_i^h, w(\tau_i))$; $v_{\tau_i,\tau_{i+1}}(\cdot)$ is an unknown realization of the control of the opponent. In turn, from (26) we have

$$\sup_{v\in E}\inf_{u\in P}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v))_{H\times\mathbb{R}} \le \inf_{u\in P}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v_i^e))_{H\times\mathbb{R}} + h. \quad (29)$$

Moreover, it is evident that the equality

$$\inf_{u\in P}\sup_{v\in E}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v))_{H\times\mathbb{R}} = \sup_{v\in E}\inf_{u\in P}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v))_{H\times\mathbb{R}} \quad (30)$$

is valid. From (28)–(30) we have

$$(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u_i^h, v(t)))_{H\times\mathbb{R}} \le \inf_{u\in P}(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u, v_i^e))_{H\times\mathbb{R}} + 2h \quad (31)$$

$$\le (s_i, \Phi(t, p_i^h, \xi_i^h, u^{(1)}(t), v_i^e))_{H\times\mathbb{R}} + 2h + L|\psi_i^h - w^{(3)}(\tau_i)|(t - \tau_i).$$

Here, L is the Lipschitz constant of the function $\chi(\cdot)$ in the corresponding domain. In this case, for a.a. $t \in \delta_i$, it follows from (27), (31) that

$$(s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u_i^h, v(t)) - u^*(t))_{H\times\mathbb{R}} \quad (32)$$

$$\leq 2h + L|\psi_i^h - w^{(3)}(\tau_i)|\{t - \tau_i + |\xi_i^h - w^{(1)}(t)|_H + |p_i^h - w^{(2)}(t)|_H\}.$$

By virtue of (32) and (21), for the interval $[\tau_i, \tau_{i+1}]$, we derive

$$\gamma_t^{(1)} + \gamma_t^{(2)} \tag{33}$$

$$\leq \pi_i^*(t) + k_6\{h^2 + t - \tau_i + \Lambda^0(x_{\Delta,w}^h(t), q_\Delta^h(t), w(t)) + |p_i^h - \psi_{\Delta,w}^h(t)|_H^2\},$$

where

$$\pi_i^*(t) = (s_i, \Phi(\tau_i, p_i^h, \xi_i^h, u_i^h, v(t)) - u^*(t))_{H \times \mathbb{R}}.$$

We deduce from inequalities (18) and (33) that

$$\frac{\mathrm{d}\Lambda(t, x_{\Delta,w}^h(\cdot), q_\Delta^h(\cdot), w(\cdot))}{\mathrm{d}t} \leq M(t; \tau_i) \tag{34}$$

$$+k_6\Lambda(t, x_{\Delta,w}^h(\cdot), q_\Delta^h(\cdot), w(\cdot)) + k_7\{h^2 + t - \tau_i + |p_i^h - \psi_{\Delta,w}^h(t)|_H^2\} \quad \text{for a.a. } t \in \delta_i.$$

Using (34) and (24), by virtue of the Gronwall lemma, we obtain for $t \in T$

$$\Lambda(t, x_{\Delta,w}^h(\cdot), q_\Delta^h(\cdot), w(\cdot)) \leq k_8 \sum_{i=0}^{m-1} \left\{ \int_{\tau_i}^{\tau_{i+1}} M(t; \tau_i)\, \mathrm{d}t + \delta(h^2 + \delta) \right\} \leq k_9\nu^{1/2}(h).$$

The statement of the theorem follows from the last inequality.

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