# The Hilbert Uniqueness Method for a Class of Integral Operators

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**Abstract.** The Hilbert Uniqueness Method introduced by J.-L. Lions in 1988 has great interest among scientists in the control theory, because it is a basic tool to get controllability results for evolutive systems. Our aim is to outline the Hilbert Uniqueness Method for first order coupled systems in the presence of memory terms in general Hilbert spaces. At the end of the paper we give some applications of our general results.

**Keywords:** Coupled systems · Convolution kernels · Reachability

## 1 Introduction

It is well known that heat equations with memory of the following type

$$y_t = \alpha \triangle y + \int_0^t K(t-s) \triangle y(s) ds, \tag{1}$$

with $\alpha > 0$, cannot be controlled to rest for large classes of memory kernels and controls, see e.g. [3,4]. The motivation for that kind of results is due to the smoothing effect of the solutions, because (1) is a parabolic equation when the constant $\alpha$ before the Laplacian is positive.

On the other hand the class of the partial integro-differential equations changes completely if in the Eq. (1) one takes $\alpha = 0$. The physical model relies on the Cattaneo's paper [1]. Indeed, in [1] to overcome the fact that the solutions of the heat equation propagate with infinite speed, Cattaneo proposed the following equation

$$y_t = \int_0^t K(t-s) \triangle y(s) ds, \tag{2}$$

with $K(t) = e^{-\gamma t}$, $\gamma$ being a positive constant. The interest for equations of the type (2) is in the property of the solutions to have finite propagation speed, the same property of the solutions of wave equations.


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From a mathematical point of view, a natural question is to study integro-differential equations of the type

$$u_t + \int_0^t M(t-s)\triangle^2 u(s)ds = 0\,,$$

where $M(t)$ is a suitable kernel, locally integrable on $(0,+\infty)$, and $\triangle^2$ denotes the biharmonic operator, that is in the $N$-dimensional case

$$\triangle^2 u = \sum_{i=1}^N \sum_{j=1}^N \partial_{ii}^2 \partial_{jj}^2 u\,.$$

The Hilbert Uniqueness Method has been introduced by Lions, see [7,8], to study control problems for partial differential systems. That method has been largely used in the literature, see e.g. [5].

Inspired by those problems, the goal of the present paper is to describe the Hilbert Uniqueness Method, for coupled hyperbolic equations of the first order with memory in a general Hilbert space, when the integral kernels involved are general functions $k_1, k_2 \in L^1(0,T)$ and integral terms also occur in the coupling:

$$\begin{cases} u_{1t} + \displaystyle\int_0^t k_1(t-s)\mathcal{A}u_1(s)ds + \mathcal{L}_1(1 * u_2) = 0 \\ & \text{in } (0,T)\,, \\ u_{2t} + \displaystyle\int_0^t k_2(t-s)\mathcal{A}^2 u_2(s)ds + \mathcal{L}_2(1 * u_1) = 0 \end{cases}$$

In another context, in [2] the authors study the exact controllability of the equation

$$y_t = \int_0^t K(t-s)\triangle y(s)ds + u\chi_\omega \qquad \text{in} \quad (0,T)\times\Omega, \tag{3}$$

where $\omega$ is a given nonempty open subset of $\Omega$. The hyperbolic nature of (3) allows to show its exact controllability under suitable conditions on the waiting time $T$ and the controller $\omega$, thanks to observability inequalities for the solutions of the dual system obtained by means of Carleman estimates.

For a different approach leading to solve control problems for hyperbolic systems, we refer to [6,11].

## 2 The Hilbert Uniqueness Method

Let $H$ be a real Hilbert space with scalar product $\langle\cdot\,,\,\cdot\rangle$ and norm $\|\cdot\|$.

We consider a linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ with domain $D(\mathcal{A})$, $k_1, k_2 \in L^1(0,T)$ and $\mathcal{L}_i$ $(i = 1,2)$ linear operators on $H$ with domain $D(\mathcal{L}_i) \supset D(\mathcal{A})$. We assume that $\mathcal{L}_2$ is self-adjoint and $\mathcal{L}_1$ is self-adjoint on a subset of its domain that will be precised later.

Moreover, let $H_1$ be another real Hilbert space with scalar product $\langle\cdot\,,\,\cdot\rangle_{H_1}$ and norm $\|\cdot\|_{H_1}$ and $\mathcal{B} \in L(H_0; H_1)$, where $H_0$ is a space such that $D(\mathcal{A}) \subset H_0 \subset H$. In the applications $\mathcal{B}$ could be, for example, a trace operator.

We take into consideration the following first order coupled system with memory

$$\begin{cases} u_{1t} + \displaystyle\int_0^t k_1(t-s)\mathcal{A}u_1(s)ds + \mathcal{L}_1(1 * u_2) = 0 \\ u_{2t} + \displaystyle\int_0^t k_2(t-s)\mathcal{A}^2u_2(s)ds + \mathcal{L}_2(1 * u_1) = 0 \end{cases} \quad \text{in } (0,T), \tag{4}$$

with null initial conditions

$$u_1(0) = u_2(0) = 0, \tag{5}$$

and satisfying

$$\mathcal{B}u_1(t) = g_1(t), \quad \mathcal{B}u_2(t) = 0, \quad \mathcal{B}\mathcal{A}u_2(t) = g_2(t), \qquad t \in (0,T). \tag{6}$$

For a reachability problem we mean the following.

**Definition 1.** *Given $T > 0$ and $u_{10}, u_{20} \in H$, a reachability problem consists in finding $g_i \in L^2(0,T;H_1)$, $i = 1,2$ such that the weak solution $u$ of problem* (4)–(6) verifies the final conditions

$$u_1(T) = u_{10}, \quad u_2(T) = u_{20}. \tag{7}$$

One can solve such reachability problems by means of the Hilbert Uniqueness Method. To show that, we proceed as follows.

To begin with, we assume the following conditions.

**Assumptions (H1)**

1. There exists a self-adjoint positive linear operator $A$ on $H$ with dense domain $D(A)$ satisfying

$$D(A) \subset D(\mathcal{A}), \qquad \mathcal{A}x = Ax \quad \forall x \in D(A), \qquad D(\sqrt{A}) = \mathrm{Ker}(\mathcal{B}).$$

2. $\mathcal{L}_2$ is self-adjoint and $\mathcal{L}_1$ is self-adjoint on $D(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$, that is

$$\langle \mathcal{L}_1\varphi, \xi\rangle = \langle \varphi, \mathcal{L}_1\xi\rangle, \quad \forall \varphi, \xi \in D(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B}). \tag{8}$$

3. There exists $D_\nu \in L(H_0; H_1)$ such that the following identity holds

$$\langle \mathcal{A}\varphi, \xi\rangle = \langle \varphi, A\xi\rangle - \langle \mathcal{B}\varphi, D_\nu\xi\rangle_{H_1}, \quad \forall \varphi \in D(\mathcal{A}), \xi \in D(A). \tag{9}$$

Now, we consider the *adjoint* system of (4), that is, the following coupled system

$$\begin{cases} z_{1t} - \displaystyle\int_t^T k_1(s-t)Az_1(s)ds - \int_t^T L_2z_2(s)ds = 0 \\ z_{2t} - \displaystyle\int_t^T k_2(s-t)A^2z_2(s)ds - \int_t^T L_1z_1(s)ds = 0 \end{cases} \quad \text{in } (0,T), \tag{10}$$

with given final data

$$z_1(T) = z_{1T}, \quad z_2(T) = z_{2T}. \tag{11}$$

We assume that for final data sufficiently regular an existence and regularity result for the solution of (10)–(11) holds. Precisely:

**Theorem 1.** *For any $z_{1T} \in D(A)$ and $z_{2T} \in D(A^2)$ there exists a unique solution $(z_1, z_2)$ of (10)–(11) such that $z_1 \in C^1([0,T],H) \cap C([0,T],D(A))$ and $z_2 \in C^1([0,T],H) \cap C([0,T],D(A^2))$.*

That type of result will be true in the applications, taking into account that backward problems are equivalent to forward problems by means of a change of the variable $t$ into $t - T$.

If Theorem 1 holds true, then the regularity of the solution $(z_1, z_2)$ of (10)–(11) and assumption (H1)-3 allow to obtain the following properties: the functions $D_\nu z_i$, $i = 1, 2$, belong to $C(0,T;H_1)$, because $D(A) \subset D(\mathcal{A}) \subset H_0$. So, we can consider the nonhomogeneous problem

$$\begin{cases} \phi_1'(t) + \displaystyle\int_0^t k_1(t-s)\mathcal{A}\phi_1(s)ds + \mathcal{L}_1(1 * \phi_2) = 0 \\ \phi_2'(t) + \displaystyle\int_0^t k_2(t-s)\mathcal{A}^2\phi_2(s)ds + \mathcal{L}_2(1 * \phi_1) = 0 & \text{in } (0,T) \\ \mathcal{B}\phi_1(t) = \displaystyle\int_t^T k_1(s-t)D_\nu z_1(s)ds, \\ \mathcal{B}\phi_2(t) = 0, \quad \mathcal{B}\mathcal{A}\phi_2(t) = \displaystyle\int_t^T k_2(s-t)D_\nu z_2(s)ds \\ \phi_1(0) = \phi_2(0) = 0. \end{cases} \tag{12}$$

If $(\phi_1, \phi_2)$ denotes the solution of problem (12), then we can introduce the following linear operator on $H \times H$:

$$\Psi(z_{1T}, z_{2T}) = (\phi_1(T), \phi_2(T)), \qquad (z_{1T}, z_{2T}) \in D(A) \times D(A^2).$$

We will prove the next result.

**Theorem 2.** *If $(\xi_1, \xi_2)$ is the solution of the system*

$$\begin{cases} \xi_1'(t) - \displaystyle\int_t^T k_1(s-t)A\xi_1(s)ds - \int_t^T \mathcal{L}_2\xi_2(s)ds = 0, \\ & \textit{in } (0,T) \\ \xi_2'(t) - \displaystyle\int_t^T k_2(s-t)A^2\xi_2(s)ds - \int_t^T \mathcal{L}_1\xi_1(s)ds = 0, \\ \xi_1(T) = \xi_{1T}, \quad \xi_2(T) = \xi_{2T}, \end{cases}$$

*where $(\xi_{1T}, \xi_{2T}) \in D(A) \times D(A^2)$, then the identity*

$$\begin{aligned} &\langle \Psi(z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle \\ &= \int_0^T \langle \mathcal{B}\phi_1(t), \int_t^T k_1(s-t)D_\nu\xi_1(s)\, ds\rangle_{H_1}\, dt \\ &\quad + \int_0^T \langle \mathcal{B}\mathcal{A}\phi_2(t), \int_t^T k_2(s-t)D_\nu\xi_2(s)\, ds\rangle_{H_1}\, dt, \end{aligned} \tag{13}$$

*holds true.*

*Proof.* We multiply the first equation in (12) by $\xi_1(t)$ and integrate on $[0, T]$, so we have

$$\int_0^T \langle \phi_1', \xi_1 \rangle \; dt \; + \int_0^T \langle \int_0^t \; k_1(t-s)\mathcal{A}\phi_1(s) \; ds, \xi_1 \rangle \; dt$$
$$+ \int_0^T \langle \mathcal{L}_1(1 * \phi_2), \xi_1 \rangle \; dt = 0. \tag{14}$$

In the second term of the above identity we change the order of integration and, since $\xi_1(t) \in D(A)$, we can use (9) to get

$$\begin{aligned}\int_0^T \langle \int_0^t k_1(t-s)\mathcal{A}\phi_1(s) \; ds, \xi_1(t) \rangle dt &= \int_0^T \; \int_s^T k_1(t-s) \langle \mathcal{A}\phi_1(s), \xi_1(t) \rangle \; dt \; ds \\ &= \int_0^T \; \langle \phi_1(s), \int_s^T \; k_1(t-s) A\xi_1(t) \; dt \rangle \; ds \\ &\quad - \int_0^T \langle \mathcal{B}\phi_1(s), \int_s^T \; k_1(t-s) D_\nu \xi_1(t) \; dt \rangle_{H_1} \; ds.\end{aligned}$$

Note that, in virtue of assumption (H1)-1, one has $D(A) \subset D(\mathcal{A}) \cap \mathrm{Ker}(\mathcal{B})$; so, changing again the order of integration and applying (8), we obtain

$$\int_0^T \langle \mathcal{L}_1(1 * \phi_2), \xi_1 \rangle \; dt = \int_0^T \; \langle \phi_2(s), \int_s^T \; \mathcal{L}_1 \xi_1(t) \; dt \rangle \; ds.$$

If we integrate by parts the first term in (14) and take into account the previous two identities, then, in view also of $\phi_1(0) = 0$, we get

$$\begin{aligned}&\langle \phi_1(T), \xi_1(T) \rangle - \int_0^T \langle \phi_1(t), \xi_1'(t) \rangle \; dt + \int_0^T \; \langle \phi_1(t), \int_t^T \; k_1(s-t) A\xi_1(s) \; ds \rangle \; dt \\ &- \int_0^T \; \langle \mathcal{B}\phi_1(t), \int_t^T \; k_1(s-t) D_\nu \xi_1(s) \; ds \rangle_{H_1} \; dt \\ &+ \int_0^T \; \langle \phi_2(t), \int_t^T \; \mathcal{L}_1 \xi_1(s) \; ds \rangle \; dt = 0.\end{aligned}$$

As a consequence of the former equation and

$$\xi_1'(t) - \int_t^T \; k_1(s-t) A\xi_1(s) ds = \int_t^T \; \mathcal{L}_2 \xi_2(s) ds,$$

we obtain

$$\begin{aligned}&\langle \phi_1(T), \xi_{1T} \rangle - \int_0^T \; \langle \mathcal{B}\phi_1(t), \int_t^T \; k_1(s-t) D_\nu \xi_1(s) \; ds \rangle_{H_1} \; dt \\ &+ \int_0^T \; \langle \phi_2(t), \int_t^T \; \mathcal{L}_1 \xi_1(s) \; ds \rangle \; dt - \int_0^T \langle \phi_1(t), \int_t^T \; \mathcal{L}_2 \xi_2(s) ds \rangle \; dt = 0.\end{aligned} \tag{15}$$

In a similar way, we multiply the second equation in (12) by $\xi_2(t)$ and integrate on $[0, T]$: if we integrate by parts the first term, take into account that

$\phi_2(0) = 0$ and change the order of integration in the other two terms, then we have

$$\begin{aligned}&\langle \phi_2(T), \xi_{2T}\rangle - \int_0^T \langle \phi_2(t), \xi_2'(t)\rangle \, dt \\ &+ \int_0^T \int_s^T k_2(t-s)\langle \, \mathcal{A}^2\phi_2(s), \xi_2(t)\rangle \, dt \, ds \\ &+ \int_0^T \int_s^T \langle \mathcal{L}_2\phi_1(s), \xi_2(t)\rangle \, dt \, ds = 0.\end{aligned} \tag{16}$$

Now, we observe that from (9) it follows for any $\varphi \in D(\mathcal{A}^2)$ and $\xi \in D(A^2)$

$$\langle \mathcal{A}^2\varphi, \xi\rangle = \langle \varphi, A^2\xi\rangle - \langle \mathcal{B}\varphi, D_\nu A\xi\rangle_{H_1} - \langle \mathcal{B}\mathcal{A}\varphi, D_\nu\xi\rangle_{H_1}.$$

Putting the above equation into (16) and taking into account that the operator $\mathcal{L}_2$ is self-adjoint yield

$$\begin{aligned}&\langle \phi_2(T), \xi_{2T}\rangle - \int_0^T \langle \phi_2(t), \xi_2'(t)\rangle \, dt \\ &+ \int_0^T \langle \phi_2(s), \int_s^T k_2(t-s)A^2\xi_2(t) \, dt\rangle \, ds \\ &- \int_0^T \langle \mathcal{B}\mathcal{A}\phi_2(s), \int_s^T k_2(t-s)D_\nu\xi_2(t) \, dt\rangle_{H_1} \, ds \\ &+ \int_0^T \langle \phi_1(s), \int_s^T \mathcal{L}_2\xi_2(t) \, dt\rangle \, ds = 0.\end{aligned}$$

In virtue of

$$\xi_2'(t) - \int_t^T k_2(s-t)A^2\xi_2(s)ds = \int_t^T \mathcal{L}_1\xi_1(s)ds,$$

we get

$$\begin{aligned}&\langle \phi_2(T), \xi_{2T}\rangle - \int_0^T \langle \mathcal{B}\mathcal{A}\phi_2(s), \int_s^T k_2(t-s)D_\nu\xi_2(t) \, dt\rangle_{H_1} \, ds \\ &+ \int_0^T \langle \phi_1(t), \int_t^T \mathcal{L}_2\xi_2(s)ds\rangle \, dt - \int_0^T \langle \phi_2(t), \int_t^T \mathcal{L}_1\xi_1(s)ds\rangle \, dt = 0.\end{aligned} \tag{17}$$

If we sum Eqs. (15) and (17), then we have

$$\begin{aligned}&\langle \Psi(z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T})\rangle = \langle \phi_1(T), \xi_{1T}\rangle + \langle \phi_2(T), \xi_{2T}\rangle \\ &= \int_0^T \langle \mathcal{B}\phi_1(t), \int_t^T k_1(s-t)D_\nu\xi_1(s) \, ds\rangle_{H_1} \, dt \\ &+ \int_0^T \langle \mathcal{B}\mathcal{A}\phi_2(t), \int_t^T k_2(s-t)D_\nu\xi_2(s) \, ds\rangle_{H_1} \, dt,\end{aligned} \tag{18}$$

that is, (13) holds true. □

If we take $(\xi_{1T}, \xi_{2T}) = (z_{1T}, z_{2T})$ in (13), then we have

$$\langle \Psi(z_{1T}, z_{2T}), (z_{1T}, z_{2T}) \rangle \\ = \int_0^T \left( \Big| \int_t^T k_1(s-t) D_\nu z_1(s)\, ds \Big|_{H_1}^2 + \Big| \int_t^T k_2(s-t) D_\nu z_2(s)\, ds \Big|_{H_1}^2 \right) dt.$$

Consequently, we can introduce a semi-norm on the space $D(A) \times D(A^2)$. Precisely, if we consider, for any $(z_{1T}, z_{2T}) \in D(A) \times D(A^2)$, the solution $(z_1, z_2)$ of the system (10)–(11), then we define

$$\|(z_{1T}, z_{2T})\|_F^2 := \\ \int_0^T \left( \Big| \int_t^T k_1(s-t) D_\nu z_1(s)\, ds \Big|_{H_1}^2 + \Big| \int_t^T k_2(s-t) D_\nu z_2(s)\, ds \Big|_{H_1}^2 \right) dt. \tag{19}$$

We observe that $\|\cdot\|_F$ is a norm if and only if the following uniqueness theorem holds.

**Theorem 3.** *If* $(z_1, z_2)$ *is the solution of problem* (10)–(11) *such that*

$$\int_t^T k_1(s-t) D_\nu z_1(s)\, ds = \int_t^T k_1(s-t) D_\nu z_2(s)\, ds = 0, \quad \text{on } [0,T],$$

*then*

$$z_1 = z_2 = 0 \quad \text{in } [0,T].$$

The validity of Theorem 3 is the starting point for the application of the Hilbert Uniqueness Method. Indeed, if we assume that Theorem 3 holds true, then we can define the Hilbert space $F$ as the completion of $D(A) \times D(A^2)$ for the norm $\|\cdot\|_F$. Thanks to (13) and (19) we have

$$\langle \Psi(z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle = \langle (z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle_F \\ \forall (z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \in D(A) \times D(A^2), \tag{20}$$

where $\langle \cdot, \cdot \rangle_F$ denotes the scalar product associated with the norm $\|\cdot\|_F$.

Consequently,

$$\big| \langle \Psi(z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle \big| \le \|(z_{1T}, z_{2T})\|_F \; \|(\xi_{1T}, \xi_{2T})\|_F \\ \forall (z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \in D(A) \times D(A^2).$$

Thanks to the above inequality, the operator $\Psi$ can be extended uniquely to a linear continuous operator, denoted again by $\Psi$, from $F$ into its dual space $F'$. By (20) it follows that

$$\langle \Psi(z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle = \langle (z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \rangle_F \\ \forall (z_{1T}, z_{2T}), (\xi_{1T}, \xi_{2T}) \in F,$$

and, as a consequence, we have that the operator $\Psi : F \to F'$ is an isomorphism.

Moreover, the key point to characterize the space $F$ is to establish observability estimates of the following type

$$\int_0^T \left( \Big| \int_t^T k_1(s-t) D_\nu z_1(s)\, ds \Big|_{H_1}^2 + \Big| \int_t^T k_2(s-t) D_\nu z_2(s)\, ds \Big|_{H_1}^2 \right) dt \asymp \|z_{1T}\|_{F_1}^2 + \|z_{2T}\|_{F_2}^2 \tag{21}$$

for suitable spaces $F_1$ , $F_2$. In that case, the uniqueness result stated by Theorem 3 holds true, so the operator $\Psi : F \to F'$ is an isomorphism, and in virtue of (19) and (21) we get

$$F = F_1 \times F_2$$

with the equivalence of the respective norms. Finally, we are able to solve the reachability problem (4)–(7) for $(u_{10}, u_{20}) \in F_1' \times F_2'$.

## 3 Applications

*Example 1.* Let $H = L^2(0,\pi)$ be endowed with the usual scalar product and norm. In [9] we take $\mathcal{A} = \frac{d^2}{dx^2}$ with null Dirichlet boundary conditions, $k_1(t) = \frac{\beta}{\eta} e^{-\eta t} + 1 - \frac{\beta}{\eta}$, $k_2 \equiv 1$. We examine the case in which $\mathcal{L}_i = a_i \mathrm{I}$, with $a_i \in \mathbb{R}$, $i = 1, 2$ and I the identity operator on $H$.

By writing the solutions as Fourier series, we are able to prove Theorems 1 and 3, thanks also to some properties of the solutions of integral equations. In particular, by showing suitable Ingham type estimates, we prove observability estimates of the type (21) where $F = H_0^1(0,\pi) \times H_0^1(0,\pi)$. Therefore, we can deduce reachability results by means of the Hilbert Uniqueness Method.

*Example 2.* We consider $H = L^2(0,\pi)$ endowed with the usual scalar product and norm. In [9] we take $\mathcal{A} = \frac{d^2}{dx^2}$ with null Dirichlet boundary conditions, $k_1(t) = \frac{\beta}{\eta} e^{-\eta t} + 1 - \frac{\beta}{\eta}$, $k_2 \equiv 1$, $\mathcal{L}_1 = a_1 \frac{d^2}{dx^2}$ and $\mathcal{L}_2 = a_2 \mathrm{I}$ with $a_i \in \mathbb{R}$, $i = 1, 2$.

*Example 3.* Let $H = L^2(\Omega)$ be endowed with the usual scalar product and norm. In [10] we take $\mathcal{A} = \triangle$ with null Dirichlet boundary conditions, $k_1(t) = \frac{\beta}{\eta} e^{-\eta t} + 1 - \frac{\beta}{\eta}$, $k_2 \equiv 1$, $\mathcal{L}_1 = a_1 \triangle$ and $\mathcal{L}_2 = a_2 \mathrm{I}$ with $a_i \in \mathbb{R}$, $i = 1, 2$.

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