Strong Optimal Controls in a Steady-State Problem of Complex Heat Transfer

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Abstract. An optimal control problem of steady-state complex heat transfer with monotone objective functionals is under consideration. A coefficient function appearing in boundary conditions and reciprocally corresponding to the reflection index of the domain surface is considered as control. The concept of strong maximizing (resp. strong minimizing) optimal controls, i.e. controls that are optimal for all monotone objective functionals, is introduced. The existence of strong optimal controls is proven, and optimality conditions for such controls are derived. An iterative algorithm for computing strong optimal controls is proposed.

Keywords: Conductive-convective-radiative heat transfer · Diffusion approximation · Control problem · Strong optimal controls · Optimality condition

1 Introduction

The interest in studying problems of complex heat transfer (where the radiative, convective, and conductive contributions are simultaneously taken into account) is motivated by their importance for many engineering applications. The common feature of such processes is the radiative heat transfer dominating at high temperatures. The radiative heat transfer equation (RTE) is a first order integro-differential equation governing the radiation intensity. The radiation traveling along a path is attenuated as a result of absorption and scattering. The precise derivation and analysis of such models can be found in the monograph [1].

Solutions to the RTE can be represented in the form of the Neumann series whose terms are powers of an integral operator applied to a certain start function. The terms can be calculated using a Monte Carlo method, which may be

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interpreted as tracking the history of energy bundles from emission to adsorption on the boundary or within the participating medium. The method assumes that the bundles start from random points, propagate in random directions, and show the energy exchange due to random scattering (see e.g. [2]).

A way to avoid solving the integro-differential RTE is the use of expansions of the local intensity in terms of spherical harmonics, with truncation to N terms in the series, and substitution into the moments of the differential form of the RTE (see e.g. [1]). This approach leads to the P_N approximations, where N is the approximation order. Especially interesting is the P_1 (diffusion) approximation because it does not require high computational efforts. Using the diffusion model instead of the integro-differential RTE becomes popular, and this is substantiated in various applications (e.g., image reconstruction [3] and modeling the radiative transfer in biological tissues [4]). In this connection, the work [5] can also be mentioned: It is shown there that the diffusion approximation yields a good accuracy for temperatures up to 1200°C. Thus, the diffuse approximation can successfully be applied to various heat transfer problems where very high accuracy is not required.

Optimal control problems of complex heat transfer draw the interest of researchers working in applied fields, e.g. glass manufacturing [6–8], laser thermotherapy [9], the design of cooling systems [10,11], etc. A considerable number of works is devoted to control problems related to evolutionary systems describing radiative heat transfer (see e.g. [6–10,12–14]). In the above-mentioned works, the transfer of radiation is described by an integral-differential equation or by its approximations. The temperature field is simulated by the conventional evolutionary heat transfer equation with additional source terms accounting for the contribution of radiation.

As for steady-state problems of complex heat transfer, there are few results in this direction. It is worth to mention the work [11], where an optimal boundary multiplicative control problem for a steady-state complex heat transfer model is considered. The problem is formulated as the maximization of the energy outflow from the model domain by controlling reflection properties of the boundary. The solvability of this problem is proven based on new a priori estimates for solutions of the model equations. Moreover, an analogue of the bang-bang principle arising in control theory for ordinary differential equations is proven. Notice that the optimization of energy in/out flow, which improves heating/cooling systems, is a quite popular problem in many engineering applications. In [15–18], similar problems are considered in the context of shape optimization.

In the current work, a conductive-convective-radiative heat transfer control problem with monotone objective functionals is under consideration. The definition of monotone functionals looks as follows. Let θ and φ be the state variables of the model, and $J(\theta, \varphi)$ the objective functional. This functional is called monotone increasing (resp. decreasing) if the relations $\theta_1 \leq \theta_2$ and $\varphi_1 \leq \varphi_2$, a. e., imply the relation $J(\theta_1, \varphi_1) \leq J(\theta_2, \varphi_2)$ (resp. $J(\theta_1, \varphi_1) \geq J(\theta_2, \varphi_2)$). It should be noticed that such functionals appear very often in applications. For example, the objective functional in the problem of maximum energy outflow is a monotone one.

Moreover, the concept of strong minimizing (resp. maximizing) optimal controls, i.e. controls that yield minimal (resp. maximal) state functions is introduced. Sufficient optimality conditions that do not involve solutions of adjoint equations are derived.

An iterative algorithm for finding strong optimal controls is proposed, and its convergence is proven. This, in particular, proves the existence of strong optimal controls.

2 Problem Setting

The normalized diffusion model, P_1 approximation, of radiative, conductive, and convective heat transfer in a bounded domain $\Omega \subset \mathbb{R}^3$ looks as follow (see [1,19–21]):

$$- a\Delta\theta + \mathbf{v} \cdot \nabla\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = 0, \tag{1}$$

$$- \alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0. \tag{2}$$

Here, θ is the normalized temperature, φ the normalized intensity of radiation averaged over all direction, κ_a the absorbtion coefficient, and $\mathbf{v}$ a prescribed velocity field. The constants a, b, and α are defined by the formulas

$$a = \frac{k}{\rho c_p}, \quad b = \frac{4\sigma n^2 T_{max}^3}{\rho c_p}, \quad \alpha = \frac{1}{3\kappa - A\kappa_s},$$

where k is the thermal conductivity, c_p the specific heat capacity, ρ the density, σ the Stefan-Boltzmann constant, n the refractive index, T_{max} the maximum temperature in unnormalized model, $\kappa := \kappa_s + \kappa_a$ the extinction coefficient, κ_s the scattering coefficient. The coefficient $A \in [-1, 1]$ describes the anisotropy of scattering. The case $A = 0$ corresponds to isotropic scattering.

The following boundary conditions on $\Gamma := \partial\Omega$ are imposed:

$$a\partial_n\theta + \beta(\theta - \theta_b) = 0, \qquad \alpha\partial_n\varphi + u(\varphi - \theta_b^4) = 0. \tag{3}$$

Here, ∂_n denotes the derivative in the direction of the outward normal $\mathbf{n}$; $\theta_b = \theta_b(x)$, $x \in \Gamma$, and $\beta = \beta(x)$, $x \in \Gamma$, are given non-negative functions describing the normalized external temperature and the normalized overall heat transfer coefficient, respectively. The function $u = u(x)$, $x \in \Gamma$, describing the reflection properties of the boundary is considered as control input.

The minimum (resp. maximum) control problem is formulated as finding a control $\widehat{u} \in L^\infty(\Gamma)$, $\widehat{u}(x) \in [u_1(x), u_2(x)]$, a.e. on Γ, such that for any admissible control u the following relation holds: $y(\widehat{u}) \leq y(u)$ (resp. $y(\widehat{u}) \geq y(u)$, a.e. in Ω, where $y(\widehat{u})$ and $y(u)$ are solution pairs satisfying relations (1)–(3) with $\widehat{u}$ and u, respectively. Here, u_1, u_2 are given non-negative functions defining inequality constraints imposed on the control.

It is clear that the control $\widehat{u}$ is optimal in the sense of minimization (resp. maximization) of monotone functionals outlined in Sect. 1.

3 Problem Formalization

Assume that Ω be a bounded Lipschitz domain. Let L^p, $p \in [1, \infty]$, denotes the Lebesgue space, and $H^s(\Omega)$ the Sobolev space $W_2^s(\Omega)$. Moreover, let the following conditions be fulfilled:

(i) $\mathbf{v} \in H^1(\Omega)$, $\operatorname{div} \mathbf{v} = 0$;
(ii) $\beta, u_1, u_2, \theta_b \in L^\infty(\Gamma)$, $0 < \beta_0 \leq \beta$, $0 < u_0 \leq u_1 \leq u_2$, $\theta_b \geq 0$, β_0, u_0 are const;
(iii) $\beta + \mathbf{v} \cdot \mathbf{n} \geq 0$, if $\mathbf{v} \cdot \mathbf{n} < 0$.

Denote $H = L^2(\Omega)$, $V = H^1(\Omega)$, and define the norms, $\|\cdot\|$ and $\|\cdot\|_V$, in H and V, respectively, as follows:

$$\|f\|^2 = (f, f), \quad \|f\|_V^2 = \|f\|^2 + \|\nabla f\|^2, \quad (f, g) = \int_\Omega f(x)g(x)dx.$$

Definition 1. *A pair $\{\theta, \varphi\} \in V \times V$ is called a (weak) solution of the problem (1)–(3) if*

$$a(\nabla\theta, \nabla\eta) + (\mathbf{v} \cdot \nabla\theta + b\kappa_a(|\theta|\theta^3 - \varphi), \eta) + \int_\Gamma \beta(\theta - \theta_b)\eta d\Gamma = 0 \quad \forall \eta \in V, \quad (4)$$

$$\alpha(\nabla\varphi, \nabla\psi) + \kappa_a(\varphi - |\theta|\theta^3, \psi) + \int_\Gamma u(\varphi - \theta_b^4)\psi d\Gamma = 0 \quad \forall \psi \in V. \quad (5)$$

Theorem 1 (see [21]). *Let the conditions (i)–(iii) be true. Then the problem (1)–(3) is uniquely solvable, a weak solution $\{\theta, \varphi\}$ belongs to $\left(L^\infty(\Omega)\right)^2$ and satisfies the inequalities $0 \leq \theta \leq M$ and $0 \leq \varphi \leq M^4$, where $M = \|\theta_b\|_{L^\infty(\Gamma)}$, and the following estimate is true:*

$$\|\theta\|_V + \|\varphi\|_V \leq C, \quad (6)$$

where C depends only on Ω, M, $\|\beta\|_{L^\infty(\Gamma)}$, $\|u\|_{L^\infty(\Gamma)}$, $\|\mathbf{v}\|_V$, a, α, b, and κ_a.

Now, the conception of strong optimal controls will be introduced. Denote by $U_{ad} = \{u \in L^\infty(\Gamma) : u_1 \leq u \leq u_2\}$ the set of admissible controls.

Definition 2. *A function $\widehat{u} \in U_{ad}$ is called strong minimizing (resp. maximizing) optimal control if $\widehat{\theta} \leq \theta$ and $\widehat{\varphi} \leq \varphi$ (resp. $\widehat{\theta} \geq \theta$ and $\widehat{\varphi} \geq \varphi$), a.e. in Ω, for all $u \in U_{ad}$, where $\{\widehat{\theta}, \widehat{\varphi}\}$ and $\{\theta, \varphi\}$ are solution pairs corresponding to $\widehat{u}$ and u, respectively.*

Definition 3. *A functional $J : [V \cap L^\infty(\Omega)]^2 \to \mathbb{R}$ is called monotone if the relations $0 \leq \theta_1 \leq \theta_2$ and $0 \leq \varphi_1 \leq \varphi_2$, a.e. in Ω, imply the inequality $J(\theta_1, \varphi_1) \leq J(\theta_2, \varphi_2)$, where $\theta_1, \theta_2, \varphi_1$, and φ_2 are arbitrary functions from $V \cap L^\infty(\Omega)$.*

Consider examples of monotone functionals.

1. The sum of weighted norms of θ and φ:

$$J(\theta,\varphi)=\int_\Omega \left(r_1\theta^2+r_2\varphi^2\right)dx, \tag{7}$$

where r_1 and $r_2\in L^\infty(\Omega)$ are non-negative given functions.

2. Energy flow through a part of the boundary. Let $\Gamma_1\subset\Gamma$ be an outflow boundary part, i.e. $\mathbf{v}\cdot\mathbf{n}\geq 0$ on Γ_1. The density of the energy flow is defined by the formula

$$\mathbf{q}=-a\nabla\theta+\theta\mathbf{v}-\alpha b\nabla\varphi,$$

and therefore, the energy outflow through Γ_1 is given by the functional

$$J(\theta,\varphi)=\int_{\Gamma_1}\mathbf{q}\cdot\mathbf{n}\,d\Gamma=\int_{\Gamma_1}(\beta(\theta-\theta_b)+\theta\mathbf{v}\cdot\mathbf{n}+b\gamma(\varphi-\theta_b^4))d\Gamma. \tag{8}$$

Here, γ is a constant that replaces the function u in the boundary condition for φ on the opening Γ_1. If the Marshak boundary condition [22] is used, then $\gamma=0.5$.

Say that a triple $\{\theta,\varphi,u\}$ is admissible if $u\in U_{ad}$ and $\{\theta,\varphi\}\in V\times V$ is a solution of the problem (1)–(3) corresponding to the control u. Denote the set of all admissible triples by $\mathcal{U}$.

Let J be a monotone functional (see Definition 3). Consider the following optimization problems:

Problem 1:

$$J(\theta,\varphi)\to\min,\qquad \{\theta,\varphi,u\}\in\mathcal{U}.$$

Problem 2:

$$J(\theta,\varphi)\to\max,\qquad \{\theta,\varphi,u\}\in\mathcal{U}.$$

A solution $\{\theta,\varphi,u\}$ of Problem 1 or 2 will be called *optimal triple*, and its components $\{\theta,\varphi\}$ and u will be referred as *optimal state* and *optimal control*, respectively.

The following proposition is an obvious corollary of Definitions 2 and 3, accounting for that the objective functionals of Problems 1 and 2 are monotone.

Proposition 1. *A strong minimizing (resp. maximizing) optimal control solves Problem 1 (resp. Problem 2).*

The next section describes the derivation of sufficient optimality conditions characterizing strong optimal controls and discusses the question of uniqueness. These considerations give rise to an iterative numerical method that always converges to a strong optimal control, which, in particular, proves the existence of such controls.

4 Conditions of Optimality

Similar to [21], introduce nonlinear operators $F_1 : L^\infty(\Omega) \times U_{ad} \to L^\infty(\Omega) \cap H^1(\Omega)$ and $F_2 : L^\infty(\Omega) \to L^\infty(\Omega) \cap H^1(\Omega)$ as follows: $\varphi = F_1(\theta, u)$ if

$$\alpha(\nabla\varphi, \nabla v) + \int_\Gamma u\,(\varphi - \theta_b^4) v d\Gamma + \kappa_a(\varphi, v) = \kappa_a(|\theta|\theta^3, v) \quad \forall v \in V, \tag{9}$$

and $\theta = F_2(\varphi)$ if

$$a(\nabla\theta, \nabla v) + \int_\Gamma \beta(\theta - \theta_b) v d\Gamma + (\mathbf{v}\nabla\theta, v) + b\kappa_a(|\theta|\theta^3, v) = b\kappa_a(\varphi, v) \quad \forall v \in V. \tag{10}$$

Notice that $\{\widehat{\theta}, \widehat{\varphi}\}$ is a weak solution of the problem (1)–(3) with a control $\widehat{u}$ if and only if $\widehat{\theta} = F_2(F_1(\widehat{\theta}, \widehat{u}))$, $\widehat{\varphi} = F_1(F_2(\widehat{\varphi}), \widehat{u})$. The operators F_1 and F_2 have the following properties (see [21]):

1. If $u \in U_{ad}$, $M = \|\theta_b\|_{L^\infty(\Gamma)}$, $0 \le \theta \le M$, and $0 \le \varphi \le M^4$, a.e. in Ω, then $0 \le F_1(\theta, u) \le M^4$ and $0 \le F_2(\varphi) \le M$, a.e. in Ω.
2. If $u \in U_{ad}$, $\theta_1 \le \theta_2$, and $\varphi_1 \le \varphi_2$, a.e. in Ω, then $F_1(\theta_1, u) \le F_1(\theta_2, u)$ and $F_2(\varphi_1) \le F_2(\varphi_2)$, a.e. in Ω.

Define an operator $U\colon L^\infty(\Gamma) \to L^\infty(\Gamma)$ as follows:

$$U(\eta)(s) = \begin{cases} u_1(s), & \eta(s) - \theta_b^4(s) < 0, \\ u_2(s), & \eta(s) - \theta_b^4(s) > 0. \end{cases}$$

Lemma 1. *If $\widetilde{u} \in U_{ad}$, $\theta \le \widetilde{\theta}$ a.e. in Ω, $\varphi = F_1(\theta, u)$, $\widetilde{\varphi} = F_1(\widetilde{\theta}, \widetilde{u})$, where $u = U(\varphi)$ or $u = U(\widetilde{\varphi})$, then $\varphi \le \widetilde{\varphi}$ a.e. in Ω.*

Proof. Set $\overline{\theta} = \theta - \widetilde{\theta}$ and $\overline{\varphi} = \varphi - \widetilde{\varphi}$. Observe that Eq. (9) implies the equation

$$\alpha(\nabla\overline{\varphi}, \nabla v) + \int_\Gamma [u(\varphi - \theta_b^4) - \widetilde{u}(\widetilde{\varphi} - \theta_b^4)] v d\Gamma + \kappa_a(\overline{\varphi}, v) = \kappa_a(|\theta|\theta^3 - |\widetilde{\theta}|\widetilde{\theta}^3, v) \quad \forall v \in V. \tag{11}$$

Denote $\psi = \max(\overline{\varphi}, 0)$ and put $v = \psi$ into (11) to obtain the estimate

$$\alpha\|\nabla\psi\|^2 + \int_\Gamma [u(\varphi - \theta_b^4) - \widetilde{u}(\widetilde{\varphi} - \theta_b^4)]\psi d\Gamma + \kappa_a\|\psi\|^2 = \kappa_a(|\theta|\theta^3 - |\widetilde{\theta}|\widetilde{\theta}^3, \psi) \le 0.$$

Notice that the equality

$$u(\varphi - \theta_b^4) - \widetilde{u}(\widetilde{\varphi} - \theta_b^4) = \widetilde{u}\overline{\varphi} + (u - \widetilde{u})(\varphi - \theta_b^4) = u\overline{\varphi} + (u - \widetilde{u})(\widetilde{\varphi} - \theta_b^4)$$

implies the non-negativity of the boundary integral in the estimate provided that $u = U(\varphi)$ or $u = U(\widetilde{\varphi})$. Therefore, $\psi = 0$, i.e. $\varphi \le \widetilde{\varphi}$ a.e. in Ω.

Theorem 2. *In order for a function $u \in U_{ad}$ to be a strong minimizing optimal control, it is sufficient that $u = U(\varphi)$, where $\varphi = \varphi(u)$ is a solution of system (1)–(3) with the control u.*

Proof. Assume that u satisfies the conditions of Theorem 2. Let $\widetilde{u} \in U_{ad}$ be an arbitrary control, $\theta = \theta(u)$, $\varphi = \varphi(u)$, $\widetilde{\theta} = \theta(\widetilde{u})$, and $\widetilde{\varphi} = \varphi(\widetilde{u})$. Lemma 1 implies the inequality $\varphi = F_1(\theta, u) \leq F_1(\theta, \widetilde{u}) = \widetilde{\varphi}_1$. Let

$$\widetilde{\theta}_k = F_2(\widetilde{\varphi}_k), \;\; \widetilde{\varphi}_{k+1} = F_1(\widetilde{\theta}_k, \widetilde{u}), \;\; k = 1, 2, \ldots. \tag{12}$$

Since $\{\theta, \varphi\}$ is a solution pair, the equation $\theta = F_2(\varphi)$ holds, and therefore, due to above mentioned properties 1 and 2 of the operators F_1 and F_2, the following inequalities are true:

$$0 \leq \theta \leq \widetilde{\theta}_k \leq M, \;\; 0 \leq \varphi \leq \widetilde{\varphi}_k \leq M^4, \;\; k = 1, 2, \ldots.$$

Notice that the sequences $\{\widetilde{\theta}_k\}$ and $\{\widetilde{\varphi}_k\}$ are bounded in V. Therefore, there exist functions θ_*, $\varphi_* \in L^\infty(\Omega) \cap H^1(\Omega)$ such that

$$\widetilde{\theta}_k \to \theta_*, \;\; \widetilde{\varphi}_k \to \varphi_* \text{ a.e. in } \Omega, \;\; \text{weakly in } H^1(\Omega), \;\; \text{and strongly in } L^2(\Omega)$$

up to subsequences.

The above convergence allows us to pass to the limit in (12) as $k \to \infty$. Therefore, $\{\theta_*, \varphi_*\}$ is a weak solution of the problem (1)–(3) with the control $\widetilde{u}$. Moreover, $\theta \leq \theta_*$ and $\varphi \leq \varphi_*$. By the uniqueness of solutions of the problem (1)–(3), it holds that $\widetilde{\theta} = \theta_*$ and $\widetilde{\varphi} = \varphi_*$, and therefore $\theta \leq \widetilde{\theta}$ and $\varphi \leq \widetilde{\varphi}$, a.e. in Ω. This proves the theorem.

The proof of the next theorem is similar to that of Theorem 2.

Theorem 3. *In order for a function $u \in U_{ad}$ to be a strong maximizing optimal control, it is sufficient that*

$$u(s) = \begin{cases} u_1(s), & \text{if } \varphi(s) - \theta_b^4(s) > 0, \\ u_2(s), & \text{if } \varphi(s) - \theta_b^4(s) < 0, \end{cases}$$

where $\varphi = \varphi(u)$.

Theorem 4. *Let u and $\widetilde{u}$ be two strong optimal controls. Then $u = \widetilde{u}$ on Γ where $\varphi \neq \theta_b^4$.*

Proof. Let u and $\widetilde{u}$ be two strong optimal controls. From the definition of strong optimal controls, it follows that $\varphi(u) = \varphi(\widetilde{u}) = \varphi$ and $\theta(u) = \theta(\widetilde{u}) = \theta$, a.e. in Ω. Using Eq. (11) yields the relation

$$\int_\Gamma [u(\varphi - \theta_b^4) - \widetilde{u}(\varphi - \theta_b^4)] v d\Gamma = \int_\Gamma (u - \widetilde{u})(\varphi - \theta_b^4) v = 0 \;\; \forall v \in V,$$

which yields that $(u - \widetilde{u})(\varphi - \theta_b^4) = 0$ a.e. in Γ. Therefore, $u = \widetilde{u}$ at points of Γ where $\varphi \neq \theta_b^4$.

Corollary 1. *If a strong optimal control $\widehat{u}$ is arbitrarily changed at points of Γ where $\varphi(\widehat{u}) = \theta_b^4$, then it remains to be a strong optimal control.*

Proof. Let $\widehat{u}$ be a strong optimal control, and $\widehat{\theta}$ and $\widehat{\varphi}$ the corresponding solutions satisfying Eqs. (4) and (5). Assume that $\widehat{u}$ is changed on the set $\{s \in \Gamma : \widehat{\varphi} - \theta_b^4 = 0\}$ to obtain a new control $\widehat{u}_{\text{new}}$. Equations (4) and (5) imply that the pair $\{\widehat{\theta}, \widehat{\varphi}\}$ is also a weak solution corresponding to the modified control $\widehat{u}_{\text{new}}$ because the last integral of Eq. (5) remains unchanged. Thus, the control $\widehat{u}_{\text{new}}$ is a strong optimal control.

5 Iterative Algorithm for Finding the Optimal Control

By Theorem 2, a function $u \in U_{ad}$ such that $u = U(\varphi(u))$ is a strong minimizing optimal control. This gives rise to the idea to use an iterative procedure for finding strong minimizing optimal controls. Below, such a procedure will be proposed, and its convergence will be proven. This additionally proves the existence of strong minimizing optimal controls. The case of strong maximizing optimal controls is treated analogously.

Let $w \in U_{ad}$, $\varphi_0 = \varphi(w)$, $\theta_0 = \theta(w)$, i.e. $\varphi_0 = F_1(\theta_0, w)$, $\theta_0 = F_2(\varphi_0)$. Define the sequences

$$u_{k+1} = U(\varphi_k), \;\; \theta_{k+1} = F_2(\varphi_k), \;\; \varphi_{k+1} = F_1(\theta_{k+1}, u_{k+1}), \;\; k = 0, 1, 2, \ldots. \tag{13}$$

Using properties 1 and 2 of the operators F_1 and F_2 yields the following estimates:

$$0 \le \theta_k \le M, \;\; 0 \le \varphi_k \le M^4, \;\; k = 0, 1, 2, \ldots,$$

and, additionally, the sequences $\{\theta_k\}$ and $\{\varphi_k\}$ are bounded in V. Observe that $\theta_1 = F_2(\varphi_0) = \theta_0$, and hence $\varphi_1 = F_1(\theta_1, U(\varphi_0)) \le \varphi_0 = F_1(\theta_0, w)$ by Lemma 1. Then, the monotonicity of F_2 yields the inequality $\theta_2 \le \theta_1$.

Now, inductive arguments yield the following relations:

$$\varphi_k \le \varphi_{k-1}, \quad \theta_{k+1} \le \theta_k, \quad u_{k+1} \le u_k, \text{ a.e. pointwise, for all } k \ge 1.$$

Indeed, if $\varphi_k \le \varphi_{k-1}$ and $\theta_{k+1} \le \theta_k$ for some $k \ge 1$, then, by the Lemma 1,

$$\varphi_{k+1} = F_1(\theta_{k+1}, U(\varphi_k)) \le \varphi_k = F_1(\theta_k, u_k),$$

and therefore $\theta_{k+2} \le \theta_{k+1}$. Moreover, the monotonicity of the mapping U with respect to the a.e. pointwise order yields the inequality $u_{k+1} = U(\varphi_k) \le U(\varphi_{k-1}) = u_k$.

Similar to the arguments used in the proof of Theorem 2, the properties of boundedness and monotonicity of the sequences $\{u_k\}, \{\theta_k\}$, and $\{\varphi_k\}$ allow us to clime the existence of functions $\widehat{u} \in L^\infty(\Gamma)$ and $\widehat{\theta}$, $\widehat{\varphi} \in L^\infty(\Omega) \cap H^1(\Omega)$ such that

$$\begin{aligned} &u_k \to \widehat{u} \text{ a.e. in } \Gamma, \;\; \theta_k \to \widehat{\theta}, \;\; \varphi_k \to \widehat{\varphi} \text{ a.e. in } \Omega, \\ &\text{weakly in } H^1(\Omega), \;\; \text{and strongly in } L^2(\Omega). \end{aligned} \tag{14}$$

The convergence (14), taking into account the upper semi-continuity of the mapping U, allows us to pass to the limit in (13) to obtain the equations

$$\widehat{u} = U(\widehat{\varphi}), \;\; \widehat{\theta} = F_2(\widehat{\varphi}), \;\; \widehat{\varphi} = F_1(\widehat{\theta}, \widehat{u}).$$

Therefore, $\widehat{u} = U(\varphi(\widehat{u}))$, $\widehat{\theta} \leq \theta(w)$, and $\widehat{\varphi} \leq \varphi(w)$, i.e. $\widehat{u}$ is a strong minimizing optimal control.

Thus, the following statement is true:

Theorem 5. *There exists a strong minimizing (resp. maximizing) control u uniquely defined on the set $\Gamma \setminus \{\eta \in \Gamma : \varphi(u) = \theta_b^4\}$. The modification of such a control on the set $\{\eta \in \Gamma : \varphi(u) = \theta_b^4\}$ does not violate its strong optimality.*

6 Numerical Experiment

The following data are used in the numerical experiments. The region Ω being a channel of the following form (the units are centimeters):

$$\Omega = \{r = (x_1, x_2, x_3) : \ 0 < x_2 < 50, \ 0 < x_{1,3} < 10\}.$$

The boundary parts at $x_2 = 0$ and $x_2 = 50$ are inflow and outflow regions, respectively. The side faces, parallel to the x_2 axis, are solid walls of the channel. The velocity field is specified as $\mathbf{v} = (0, 9, 0)$ [cm/s]. The function θ_b is defined as follows:

$$\theta_b(x_1, 0, x_3) = 0.5, \quad \theta_b(x_1, 50, x_3) = 1,$$

$$\theta_b(0, x_2, x_3) = \theta_b(10, x_2, x_3) = \theta_b(x_1, x_2, 0) = \theta_b(x_1, x_2, 10) = 0.5 + 0.01x_2.$$

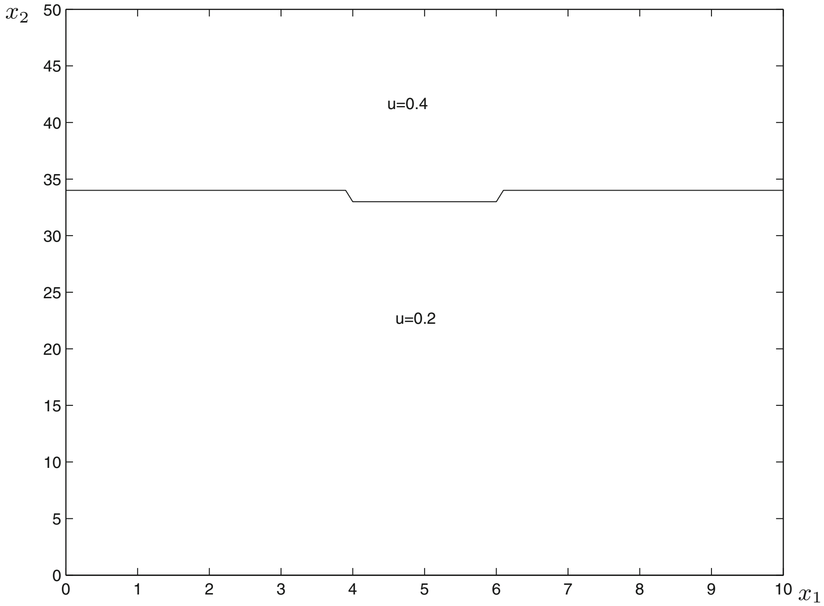

Fig. 1. Distribution of the control on the top face, $x_3 = 10$, of the channel.

The thermodynamical characteristics of the medium inside the channel correspond to air at the normal atmospheric pressure and the temperature of 400 °C. The maximum temperature in unnormalized model is chosen as $T_{max} = 500\,^{\circ}\mathrm{C}$. The extinction coefficient κ is equal to 0.1 [cm^{-1}], $\alpha = 3.3(3)$, the absorption coefficient κ_a equals 0.01 [cm^{-1}], the anisotropy coefficient A equals 0, and the coefficient γ equals 10.

It is assumed that the control u is variable on the upper face, $x_3 = 10$, and constrained by the inequalities $0.2 \leq u \leq 0.4$. On the other faces, the control assumes prescribed constant values as follows:

$$u(x_1, 0, x_3) = u(x_1, 50, x_3) = 0.5,$$

$$u(0, x_2, x_3) = u(10, x_2, x_3) = u(x_1, x_2, 0) = 0.3.$$

The iterative algorithm requires only two steps to deliver a strong maximizing optimal control. The distribution of this control on the upper face of the channel is shown in Fig. 1.

7 Conclusion

The current paper deals with a nonstandard problem of optimal control and proposes its complete solution. The notion of strong optimal controls seems to be a little bit unrealistic for common control problems. Nevertheless, the model considered in this work does have such solutions. They are unique in some sense and can be easily computed. Another surprising point is that the intuition fails when predicts that e.g. a strong maximizing optimal control should assume possibly maximal admissible values. In contrast to that, the example presented shows the opposite. Some analysis shows that a strong maximizing optimal control assumes minimal admissible values on a part of the surface where the absorption of thermal radiation occurs. Thus, the structure of strong optimal controls may be rather complicated, and therefore some practical heuristic solutions can be improved using the study presented. It would be also interesting to find other problems permitting strong optimal controls.

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