Chapter 5
Symmetry in quantum mechanics

Roughly speaking, a ***symmetry*** of some mathematical object is an invertible transformation that leaves all relevant structure as it is. Thus a symmetry of a set is just a bijection (as sets have no further structure, whence invertibility is the only demand on a symmetry), a symmetry of a topological space is a homeomorphism, a symmetry of a Banach space is a linear isometric isomorphism, and, crucially important for this chapter, a symmetry of a Hilbert space H is a ***unitary operator***, i.e., a linear map $u : H \to H$ satisfying one and hence all of the following equivalent conditions:

- $uu^* = u^*u = 1_H$;
- u is invertible with $u^{-1} = u^*$;
- u is a surjective isometry (or, if $\dim(H) < \infty$, just an isometry);
- u is invertible and preserves the inner product, i.e., $\langle u\varphi, u\psi\rangle = \langle \varphi, \psi\rangle$ $(\varphi, \psi \in H)$.

The discussion of symmetries in quantum physics is based on the above idea, but the mathematically obvious choices need not be the physically relevant ones. Even in elementary quantum mechanics, where $A = B(H)$, i.e., the C*-algebra of all bounded operators on some Hilbert space H, the concept of a symmetry is already diverse. The main structures whose symmetries we shall study in this chapter are:

1. The ***normal pure state space*** $\mathscr{P}_1(H)$, i.e., the set of one-dimensional projections on H, with transition probability $\tau : \mathscr{P}_1(H) \times \mathscr{P}_1(H) \to [0,1]$ defined by (2.44).
2. The ***normal state space*** $\mathscr{D}(H)$, i.e. the convex set of density operators ρ on H.
3. The ***self-adjoint operators*** $B(H)_{\mathrm{sa}}$ on H, seen as a Jordan algebra (see below).
4. The ***effects*** $\mathscr{E}(H) = [0,1]_{B(H)}$, seen as a convex partially ordered set (poset).
5. The ***projections*** $\mathscr{P}(H)$ on H, seen as an orthocomplemented lattice.
6. The ***unital commutative C*-subalgebras*** $\mathscr{C}(B(H))$ ***of*** $B(H)$, seen as a poset.

Each of these structures comes with its own notion of a symmetry, but the main point of this chapter will be to show these notions are equivalent, corresponding in all cases to either unitary or—surprisingly—*anti-unitary* operators, both merely defined up to a phase. The latter subtlety will open the world of *projective* unitary group representation to quantum mechanics (without which the existence of spin-$\frac{1}{2}$ particles such as electrons, and therewith also of ourselves, would be impossible).

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5.1 Six basic mathematical structures of quantum mechanics

We first recall the objects just described in a bit more detail. We have:

$$\mathscr{P}_1(H) = \{e \in B(H) \mid e^2 = e^* = e, \mathrm{Tr}\,(e) = \dim(eH) = 1\}; \tag{5.1}$$
$$\mathscr{D}(H) = \{\rho \in B(H) \mid \rho \geq 0, \mathrm{Tr}\,(\rho) = 1\}; \tag{5.2}$$
$$B(H)_{\mathrm{sa}} = \{a \in B(H) \mid a^* = a\}; \tag{5.3}$$
$$\mathscr{E}(H) = \{a \in B(H) \mid 0 \leq a \leq 1_H\}; \tag{5.4}$$
$$\mathscr{P}(H) = \{e \in B(H) \mid e^2 = e^* = e\}; \tag{5.5}$$
$$\mathscr{C}(B(H)) = \{C \subset B(H) \mid C \text{ commutative C*-algebra}, 1_H \in C\}. \tag{5.6}$$

The point is that each of these sets has some additional structure that defines what it means to be a symmetry of it, as we now spell out in detail.

Definition 5.1. *Let H be a Hilbert space (not necessarily finite-dimensional).*

1. A **Wigner symmetry** *(of H) is a bijection*

$$\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H) \tag{5.7}$$

that satisfies

$$\mathrm{Tr}\,(\mathsf{W}(e)\mathsf{W}(f)) = \mathrm{Tr}\,(ef),\ e, f \in \mathscr{P}_1(H). \tag{5.8}$$

2. A **Kadison symmetry** *is an* **affine** *bijection*

$$\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H), \tag{5.9}$$

i.e. a bijection K that preserves convex sums: for $t \in (0,1)$ and $\rho_1, \rho_2 \in \mathscr{D}(H)$,

$$\mathsf{K}(t\rho_1 + (1-t)\rho_2) = t\mathsf{K}\rho_1 + (1-t)\mathsf{K}\rho_2. \tag{5.10}$$

3. a. A **Jordan symmetry** *is an invertible* **Jordan map**

$$\mathsf{J} : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}, \tag{5.11}$$

i.e., an $\mathbb{R}$-linear bijection that satisfies the equivalent conditions

$$\mathsf{J}(a \circ b) = \mathsf{J}(a) \circ \mathsf{J}(b); \tag{5.12}$$
$$\mathsf{J}(a^2) = \mathsf{J}(a)^2. \tag{5.13}$$

Here

$$a \circ b = \tfrac{1}{2}(ab + ba) \tag{5.14}$$

is the **Jordan product** *on $B(H)_{\mathrm{sa}}$, which turns the (real) vector space $B(H)_{\mathrm{sa}}$ into a* **Jordan algebra***, cf. §C.25.*

b. A **weak Jordan symmetry** *is an invertible* **weak Jordan map***, i.e., a bijection (5.11) of which the restriction $\mathsf{J}_{|C_{\mathrm{sa}}}$ is a Jordan map for each $C \in \mathscr{C}(B(H))$.*

4. A **Ludwig symmetry** *is an affine order isomorphism*

$$\mathsf{L} : \mathscr{E}(H) \to \mathscr{E}(H). \tag{5.15}$$

5. A **von Neumann symmetry** *is an order isomorphism*

$$\mathsf{N} : \mathscr{P}(H) \to \mathscr{P}(H) \tag{5.16}$$

preserving orthocomplementation, i.e. $\mathsf{N}(1-e) = 1 - \mathsf{N}(e)$ *for each* $e \in \mathscr{P}(H)$.

6. A **Bohr symmetry** *is an order isomorphism*

$$\mathsf{B} : \mathscr{C}(B(H)) \to \mathscr{C}(B(H)). \tag{5.17}$$

In nos. 3 and 5–6, an ***order isomorphism*** O of the given poset is a bijection that preserves the partial order $\leq$ (i.e., if $x \leq y$, then $\mathsf{O}(x) \leq \mathsf{O}(y)$) and whose inverse O^{-1} does so, too; cf. §D.1. The names in question have been chosen for historical reasons and (except perhaps for the first and third) are not standard.

Let us note that any Jordan map has a unique extension to a $\mathbb{C}$-linear map

$$\mathsf{J}_{\mathbb{C}} : B(H) \to B(H); \tag{5.18}$$

$$\mathsf{J}_{\mathbb{C}}(a^*) = \mathsf{J}_{\mathbb{C}}(a)^*, \tag{5.19}$$

which satisfies (5.12) for all a, b, as well as

$$\mathsf{J}_{\mathbb{C}}(a + ib) = \mathsf{J}(a) + i\mathsf{J}(b), \tag{5.20}$$

with notation as in Proposition 2.6. Conversely, such a Jordan map (5.18) defines a real Jordan map (5.11) by $\mathsf{J} = \mathsf{J}_{|B(H)_{\mathrm{sa}}}$. Similarly, a weak Jordan symmetry is equivalent to a map (5.18) that satisfies (5.19), preserves squares as in (5.13), and is linear on each subspace C of $B(H)$, with $C \in \mathscr{C}(B(H))$. In other words (in the spirit of Bohrification), $\mathsf{J}_{\mathbb{C}}$ is a homomorphism of C*-algebras on each commutative unital C*-subalgebra $C \subset B(H)$. Therefore, either way J and $\mathsf{J}_{\mathbb{C}}$ are essentially the same thing, and if no confusion may arise we call it J. Note that a weak Jordan map J *a priori* satisfies (5.12) only for *commuting* self-adjoint a and b. It follows that weak (and hence ordinary) Jordan symmetries are unital: since

$$\mathsf{J}(b) = \mathsf{J}(1_H \circ b) = \mathsf{J}(1_H) \circ \mathsf{J}(b) \tag{5.21}$$

for any b, we may pick $b = \mathsf{J}^{-1}(1_H)$ to find, reading (5.21) from right to left,

$$\mathsf{J}(1_H) = \mathsf{J}(1_H) \circ 1_H = 1_H. \tag{5.22}$$

The special role of unitary operators u now emerges: each such operator defines the relevant symmetry in the obvious way, namely, in order of appearance:

$$\mathsf{W}(e) = ueu^*; \tag{5.23}$$
$$\mathsf{K}(\rho) = u\rho u^*; \tag{5.24}$$
$$\mathsf{L}(a) = uau^*; \tag{5.25}$$
$$\mathsf{J}(a) = uau^*; \tag{5.26}$$
$$\mathsf{N}(e) = ueu^*; \tag{5.27}$$
$$\mathsf{B}(C) = uCu^*, \tag{5.28}$$

where $a^* = a$ in (5.26). If not, this formula remains valid also for the map $J_{\mathbb{C}}$. Furthermore, in (5.28) the notation uCu^* is shorthand for the set $\{uau^* \mid a \in C\}$, which is easily seen to be a member of $\mathscr{C}(B(H))$. Here, as well as in the other three cases, it is easy to verify that the right-hand side belongs to the required set, that is,

$$ueu^* \in \mathscr{P}_1(H),\ u\rho u^* \in \mathscr{D}(H),\ u\rho u^* \in \mathscr{E}(H), \tag{5.29}$$
$$uau^* \in B(H)_{\mathrm{sa}},\ u\rho u^* \in \mathscr{P}(H),\ uCu^* \in \mathscr{C}(B(H)), \tag{5.30}$$

respectively, provided, of course, that

$$e \in \mathscr{P}_1(H),\ \rho \in \mathscr{D}(H),\ a \in \mathscr{E}(H)\ a \in B(H)_{\mathrm{sa}},\ e \in \mathscr{P}(H),\ C \in \mathscr{C}(B(H)).$$

Indeed, if, in (5.23), $e = e_\psi = |\psi\rangle\langle\psi|$ for some unit vector $\psi \in H$, then

$$ue_\psi u^* = e_{u\psi}. \tag{5.31}$$

If $\rho \geq 0$ in that $\langle\psi, \rho\psi\rangle \geq 0$ for each $\psi \in H$, then clearly also $u\rho u^* \geq 0$, and if $\mathrm{Tr}\,(\rho) = 1$, then also $\mathrm{Tr}\,(u\rho u^*) = 1$. If $a^* = a$, then

$$(uau^*)^* = u^{**}a^*u^* = uau^*. \tag{5.32}$$

However, one may also choose u in these formulae to be *anti-unitary*, as follows:

Definition 5.2. *1. A real-linear operator $u : H \to H$ is* **anti-linear** *if*

$$u(z\psi) = \overline{z}\psi\ \ (z \in \mathbb{C}). \tag{5.33}$$

2. An anti-linear operator $u : H \to H$ is **anti-unitary** *if it is invertible, and*

$$\langle u\varphi, u\psi\rangle = \overline{\langle\varphi, \psi\rangle}\ \ (\varphi, \psi \in H). \tag{5.34}$$

The adjoint u^* of a (bounded) anti-linear operator u is defined by the property

$$\langle u^*\varphi, \psi\rangle = \overline{\langle\varphi, u\psi\rangle}\ \ (\varphi, \psi \in H), \tag{5.35}$$

in which case u^* is anti-linear, too. Hence we may equally well say that an anti-linear operator is anti-unitary if $uu^* = u^*u = 1_H$. The simplest example is the map

$$J : \mathbb{C}^n \to \mathbb{C}^n;$$
$$Jz = \bar{z}, \tag{5.36}$$

i.e., if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then $(Jz)_i = \bar{z}_i$. Similarly, one may define

$$J : \ell^2 \to \ell^2;$$
$$J\psi = \overline{\psi}, \tag{5.37}$$

and likewise on L^2, where complex conjugation is defined pointwise, that is,

$$(J\psi)(x) = \overline{\psi(x)}. \tag{5.38}$$

For any Hilbert space one may pick a basis (υ_i) and define J relative to this basis by

$$J\left(\sum_i c_i \upsilon_i\right) = \sum_i \bar{c}_i \upsilon_i. \tag{5.39}$$

For future use, we state two obvious facts.

Proposition 5.3. *1. The product of two anti-unitary operators is unitary.*
2. Any anti-unitary operator $u : H \to H$ takes the form $u = Jv$, where v is unitary and J is an anti-unitary operator on H of the kind constructed above.

It is an easy verification that (5.23) - (5.28) still define symmetries if u is anti-unitary. Note that in terms of the complexification $\mathsf{J}_{\mathbb{C}}$, eq. (5.26) should read

$$\mathsf{J}_{\mathbb{C}}(a) = ua^* u^*. \tag{5.40}$$

The goal of the following sections is to show that these are the only possibilities:

Theorem 5.4. *Let H be a Hilbert space, with* $\dim(H) > 1$.

1. *Each Wigner symmetry takes the form* (5.23)*;*
2. *Each Kadison symmetry takes the form* (5.24)*;*
3. *Each Ludwig symmetry takes the form* (5.25)*;*
4. *a. Each Jordan symmetry takes the form* (5.26)*;*
   b. If $\dim(H) > 2$*, also each weak Jordan symmetry takes this form;*
5. *If* $\dim(H) > 2$*, each von Neumann symmetry takes the form* (5.27)*;*
6. *Again if* $\dim(H) > 2$*, each Bohr symmetry takes the form* (5.28)*,*

where in all cases the operator u is either unitary or anti-unitary, and is uniquely determined by the symmetry in question up to a phase (that is, u and u' implement the same symmetry by conjugation iff $u' = zu$, where $z \in \mathbb{T}$).

As we shall see, the reason why the case $H = \mathbb{C}^2$ is exceptional with regard to weak Jordan symmetries, von Neumann symmetries, and Bohr symmetries is that in those cases the proof relies on Gleason's Theorem, which fails for $H = \mathbb{C}^2$.

To see this more explicitly, and also to prove the positive cases (i.e., nos. 1–4a) in a simple situation without invoking higher principles, before proving Theorem 5.4 in general it is instructive to first illustrate it in the two-dimensional case $H = \mathbb{C}^2$.

5.2 The case $H = \mathbb{C}^2$

We start with some background. Any complex 2×2 matrix a can be written as

$$a = a(x_0, x_1, x_2, x_3) = \tfrac{1}{2} \sum_{\mu=0}^{3} x_\mu \sigma_\mu \;\; (x_\mu \in \mathbb{C}); \tag{5.41}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.42}$$

i.e., the ***Pauli matrices***. Furthermore, if we equip the vector space $M_2(\mathbb{C})$ of complex 2×2 matrices with the canonical inner product (2.34), then the rescaled matrices $\sigma'_\mu = \sigma_\mu / \sqrt{2}$ form a basis ($\equiv$ orthonormal basis) of the ensuing Hilbert space.

Writing $\mathbf{x} = (x_1, x_2, x_3)$, some interesting special cases are:

- $x_0 \in \mathbb{R}$, $\mathbf{x} = i\mathbf{v}$ with $\mathbf{v} \in \mathbb{R}^3$ and $x_0^2 + v_1^2 + v_2^2 + v_3^2 = 1$, which holds iff $a \in SU(2)$;
- $x_\mu \in \mathbb{R}$ for each $\mu = 0, 1, 2, 3$, which is the case iff $a^* = a$.
- $x_0 = 1$, $\mathbf{x} \in \mathbb{R}^3$, and $\|\mathbf{x}\| = 1$, which holds iff a is a one-dimensional projection.

The first case follows because $SU(2)$ consist of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},\; \alpha, \beta \in \mathbb{C},\; |\alpha|^2 + |\beta|^2 = 1. \tag{5.43}$$

The second case is obvious, and the third follows from Proposition 2.9.

Assume the third case, so that $a = e$ with $e^2 = e^* = e$ and $\mathrm{Tr}\,(e) = 1$. If a linear map $u : \mathbb{C}^2 \to \mathbb{C}^2$ is unitary, then simple computations show that $e' = ueu^*$ is a one-dimensional projection, too, given by $e' = \frac{1}{2}\sum_{\mu=0}^{3} x'_\mu \sigma_\mu$ with $x'_0 = 1$, $\mathbf{x}' \in \mathbb{R}^3$, and $\|\mathbf{x}'\| = 1$. Writing $\mathbf{x}' = R\mathbf{x}$ for some map $R : S^2 \to S^2$, we have

$$u(\mathbf{x} \cdot \sigma)u^* = (R\mathbf{x}) \cdot \sigma, \tag{5.44}$$

where $\mathbf{x} \cdot \sigma = \sum_{j=1}^{3} x_j \sigma_j$. This also shows that R extends to a linear isometry $R : \mathbb{R}^3 \to \mathbb{R}^3$. Using the formula $\mathrm{Tr}\,(\sigma_i \sigma_j) = 2\delta_{ij}$, the matrix-form of R follows as

$$R_{ij} = \tfrac{1}{2}\mathrm{Tr}\,(u\sigma_i u^* \sigma_j). \tag{5.45}$$

Define $U(2)$ as the (connected) group of all unitary 2×2 matrices (whose connected subgroup $SU(2)$ of elements with unit determinant has just been mentioned). Also, recall that $O(3)$ is the group of all real orthogonal 3×3 matrices M, a condition that may be expressed in (at least) four equivalent ways (like unitarity):

- $MM^T = M^M M = 1_3$;
- M invertible and $M^T = M^{-1}$;
- M is an isometry (and hence it is injective and therefore invertible);
- M preserves the inner product: $\langle M\mathbf{x}, M\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

This implies $\det(M) = \pm 1$ (as can be seen by diagonalizing M; being a real linear isometry, its eigenvalues can only be ± 1, and $\det(M)$ is their product). Thus $O(3)$ breaks up into two parts $O_{\pm}(3) = \{R \in O(3) \mid \det(R) = \pm 1\}$, of which $O_+ \equiv SO(3)$ consists of rotations. Using an explicit parametrization of $SO(3)$, e.g., through Euler angles, or, using surjectivity of the exponential map (from the Lie algebra of $SO(3)$, which consist of anti-symmetric real matrices), it follows that $O_{\pm}(3)$ are precisely the two connected components of $O(3)$, the identity of course lying in $O_+(3)$.

Proposition 5.5. *The map $u \mapsto R$ defined by* (5.44) *is a homomorphism from $U(2)$ onto $SO(3)$. In terms of $SU(2) \subset U(2)$, this map restricts to a two-fold covering*

$$\tilde{\pi} : SU(2) \to SO(3), \tag{5.46}$$

with discrete kernel

$$\ker(\tilde{\pi}) = \{1_2, -1_2\}. \tag{5.47}$$

Proof. As a finite-dimensional linear isometry, R is automatically invertible (this also follows from unitarity and hence invertibility of u), hence $R \in O(3)$. It is obvious from (5.44) that $u \mapsto R$ is a continuous homomorphism (of groups). Since $U(2)$ is connected and $u \mapsto R$ is continuous, R must lie in the connected component of $O(3)$ containing the identity, whence $R \in SO(3)$. To show surjectivity of $\tilde{\pi}$, take some unit vector $\mathbf{u} \in \mathbb{R}^3$ and define $u = \cos(\frac{1}{2}\theta) + i\sin(\frac{1}{2}\theta)\mathbf{u} \cdot \sigma$. The corresponding rotation $R_\theta(\mathbf{u})$ is the one around $\mathbf{u}$ by an angle θ, and such rotations generate $SO(3)$.

Finally, it follows from (5.44) that $u \in \ker(\tilde{\pi})$ iff u commutes with each σ_i and hence, by (5.41), with all matrices. Therefore, $u = z \cdot 1_2$ for some $z \in \mathbb{C}$, upon which the the condition $\det(u) = 1$ (in that $u \in SU(2)$) enforces $z = \pm 1$. □

Note that the covering (5.46) is topologically nontrivial (i.e., $SU(2) \neq SO(3) \times \mathbb{Z}_2$), since $SU(2) \cong S^3$ is simply connected, whereas $SO(3)$ is doubly connected: a closed path $t \mapsto R_{2\pi t}(\mathbf{u})$, $t \in [0,1]$ in $SO(3)$ (starting and ending at 1_3) lifts to a path

$$t \mapsto \cos(\pi t) + i\sin(\pi t)\mathbf{u} \cdot \sigma$$

in $SU(2)$ that starts at the unit matrix 1_2 and ends at -1_2.

To incorporate $O_-(3)$, let $U_a(2)$ be the set of all anti-unitary 2×2 matrices. These do not form a group, as the product of two anti-unitaries is unitary, but the union $U(2) \cup U_a(2)$ is a disconnected Lie group with identity component $U(2)$.

Proposition 5.6. *The map $u \mapsto R$ defined by* (5.44) *is a surjective homomorphism*

$$\tilde{\pi}' : U(2) \cup U_a(2) \to O(3), \tag{5.48}$$

with kernel $U(1)$, seen as the diagonal matrices $z \cdot 1_2$, $z \in \mathbb{T}$. Moreover, $\tilde{\pi}'$ maps $U(2)$ onto $SO(3)$ and maps $U_a(2)$ onto $O_-(3)$.

Proof. The map $u \mapsto R$ in (5.44) sends the anti-unitary operator $u = J$ on $\mathbb{C}^2$ to $R = \mathrm{diag}(1, -1, 1) \in O_-(3)$. Since $U_a(2) = J \cdot U(2)$ and similarly $O_-(3) = R \cdot SO(3)$, the last claim follows. The computation of the kernel may now be restricted to $U(2)$, and then follows as in the last step op the proof of the previous proposition. □

We now return to Theorem 5.4 and go through its special cases one by one.

Part 1 of Theorem 5.4 is ***Wigner's Theorem***, which in the case at hands reads:

Theorem 5.7. *Each bijection* $\mathsf{W} : \mathscr{P}_1(\mathbb{C}^2) \to \mathscr{P}_1(\mathbb{C}^2)$ *that satisfies*

$$\mathrm{Tr}\,(\mathsf{W}(e)\mathsf{W}(f)) = \mathrm{Tr}\,(ef) \tag{5.49}$$

for each $e, f \in \mathscr{P}_1(\mathbb{C}^2)$ *takes the form* $\mathsf{W}(e) = ueu^*$*, where* u *is either unitary or anti-unitary, and is uniquely determined by* W *up to a phase.*

To prove, this we transfer the whole situation to the two-sphere, where it is easy:

Proposition 5.8. *The pure state space* $\mathscr{P}_1(\mathbb{C}^2)$ *corresponds bijectively to the sphere*

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

in that each one-dimensional projection $e \in \mathscr{P}_1(\mathbb{C}^2)$ *may be expressed uniquely as*

$$e(x,y,z) = \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \tag{5.50}$$

where $(x,y,z) \in \mathbb{R}^3$ *and* $x^2 + y^2 + z^2 = 1$*. Under the ensuing bijection*

$$\mathscr{P}_1(\mathbb{C}^2) \cong S^2, \tag{5.51}$$

Wigner symmetries W *of* $\mathbb{C}^2$ *turn into orthogonal maps* $R \in O(3)$*, restricted to* S^2.

Proof. The first claim restates Proposition 2.9. If ψ and ψ' are unit vectors in $\mathbb{C}^2$ with corresponding one-dimensional projections $e_\psi(x,y,z)$ and $e_{\psi'}(x',y',z')$ then, as one easily verifies, the corresponding transition probability takes the form

$$\mathrm{Tr}\,(e_\psi e_{\psi'}) = \tfrac{1}{2}(1 + \langle \mathbf{x}, \mathbf{x}' \rangle) = \cos^2(\tfrac{1}{2}\theta(\mathbf{x},\mathbf{y})), \tag{5.52}$$

where $\theta(\mathbf{x},\mathbf{y})$ is the arc (i.e., geodesic) distance between $\mathbf{x}$ and $\mathbf{y}$. Consequently, if $\mathsf{W} : \mathscr{P}_1(\mathbb{C}^2) \to \mathscr{P}_1(\mathbb{C}^2)$ satisfies (5.8), then the corresponding map $R : S^2 \to S^2$ (defined through the above identification $\mathscr{P}_1(\mathbb{C}^2) \cong S^2$) satisfies

$$\langle R(\mathbf{x}), R(\mathbf{x}') \rangle = \langle \mathbf{x}, \mathbf{x}' \rangle \ \ (\mathbf{x}, \mathbf{x}' \in S^2). \tag{5.53}$$

Lemma 5.9. *If some bijection* $R : S^2 \to S^2$ *satisfies* (5.53)*, then* R *extends (uniquely) to an orthogonal linear map (for simplicity also called)* $R : \mathbb{R}^3 \to \mathbb{R}^3$.

Proof. With $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ the standard basis of $\mathbb{R}^3$, define a 3×3 matrix by

$$R_{kl} = \langle \mathbf{u}_k, R(\mathbf{u}_l) \rangle. \tag{5.54}$$

It follows from (5.53) that $R^{-1}(\mathbf{u}_j)_k = R_{jk}$, which implies $\langle R^{-1}(\mathbf{u}_j), \mathbf{x} \rangle = \sum_k R_{jk} x_k$, or, once again using (5.53), $R(\mathbf{x})_j = \sum_k R_{jk} x_k$. Hence the map $\mathbf{x} \mapsto \sum_{j,k} R_{jk} x_k \mathbf{u}_j$, i.e., the usual linear map defined by the matrix (5.54), extends the given bijection R. Orthogonality of this linear map is, of course, equivalent to (5.53). □

Wigner's Theorem then follows by combining Propositions 5.6 and 5.8: given the linear map R just constructed, read (5.44) from right to left, where u exists by surjectivity of the map (5.48), and the precise lack of uniqueness of u as claimed in Theorem 5.4 is just a restatement of the fact that (5.48) has $U(1)$ as its kernel. □

Kadison's Theorem is part 2 of Theorem 5.4. Explicitly, for $H = \mathbb{C}^2$ we have:

Theorem 5.10. *Each affine bijection $\mathsf{K} : \mathscr{D}(\mathbb{C}^2) \to \mathscr{D}(\mathbb{C}^2)$ is given as $\mathsf{K}(\rho) = u\rho u^*$, where u is unitary or anti-unitary, and is uniquely determined by K up to a phase.*

Proof. We once again invoke Proposition 2.9, implying that any density matrix ρ on $\mathbb{C}^2$ takes the form

$$\rho = \tfrac{1}{2}\left(1_2 + \sum_{\mu=1}^{3} x_\mu \sigma_\mu\right), \tag{5.55}$$

with $\|\mathbf{x}\| \leq 1$. Moreover, the ensuing bijection $\mathscr{D}(\mathbb{C}^2) \cong B^3$, $\rho \mapsto \mathbf{x}$, is clearly affine, in that a convex sums $t\rho + (1-t)\rho'$ of density matrices correspond to convex sums $t\mathbf{x} + (1-t)\mathbf{x}'$ of the corresponding vectors in $\mathbb{R}^3$.

Lemma 5.11. *Any affine bijection K of the unit ball B^3 in $\mathbb{R}^3$ is given by an orthogonal linear map $R \in O(3)$.*

Proof. First, K must map the boundary $\partial_e B^3 = S^2$ to itself (necessarily bijectively): if $\mathbf{x} \in S^2$ and $\mathsf{K}(\mathbf{x}) = t\mathbf{x}' + (1-t)\mathbf{x}''$, then $\mathbf{x} = tK^{-1}(\mathbf{x}') + (1-t)K^{-1}(\mathbf{x}'')$, whence

$$K^{-1}(\mathbf{x}') = K^{-1}(\mathbf{x}''), \tag{5.56}$$

since $\mathbf{x}$ is pure, whence $\mathbf{x}' = \mathbf{x}''$, so that also $\mathsf{K}(\mathbf{x})$ is pure.

Second, the basis of all further steps is the property

$$\mathsf{K}(\mathbf{0}) = \mathbf{0}. \tag{5.57}$$

This is because $\mathbf{0}$ is intrinsic to the convex structure of B^3: it is the unique point with the property that for any $\mathbf{x} \in S^2$ there exists a unique $\mathbf{x}'$ such that $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}' = \mathbf{0}$, namely $\mathbf{x}' = -\mathbf{x}$. Thus $\mathbf{0}$ must be preserved under affine bijections. For a formal proof (by contradiction), suppose $\mathsf{K}(\mathbf{0}) \neq \mathbf{0}$, and define $\mathbf{y} = \mathsf{K}(\mathbf{0})/\|\mathsf{K}(\mathbf{0})\| \in S^2$. Then $\mathsf{K}(\mathbf{0})$ has an extremal decomposition $\mathsf{K}(\mathbf{0}) = t\mathbf{y} + (1-t)\mathbf{y}'$, with $\mathbf{y}' = -\mathbf{y}$ and $t = \frac{1}{2}(1 + \|\mathsf{K}(\mathbf{0})\|)$. Applying the affine map K^{-1} then gives

$$\|\mathsf{K}^{-1}(\mathbf{y}')\| = \|\mathsf{K}^{-1}(\mathbf{y})\| \cdot \frac{1 + \|\mathsf{K}(\mathbf{0})\|}{1 - \|\mathsf{K}(\mathbf{0})\|}.$$

Now $\mathbf{y} \in S^2$ and hence $\mathsf{K}^{-1}(\mathbf{y}) \in S^2$ by part one of this proof (applied to K^{-1}), so that $\|\mathsf{K}^{-1}(\mathbf{y})\| = 1$. But this implies $\|\mathsf{K}^{-1}(\mathbf{y}')\| > 1$, which is impossible because $\mathbf{y}' \in S^2$ and hence $\|\mathsf{K}^{-1}(\mathbf{y}')\| = 1$.

Third, for $\mathbf{x} \in B^3$ and $t \in [0,1]$ the preceding point implies that

$$\mathsf{K}(t\mathbf{x}) = \mathsf{K}(t\mathbf{x} + (1-t)\mathbf{0}) = t\mathsf{K}(\mathbf{x}) + (1-t)\mathsf{K}(\mathbf{0}) = t\mathsf{K}(\mathbf{x}). \tag{5.58}$$

The same then holds for $\mathbf{x} \in B^3$ and all $t \geq 0$ as long as $t\mathbf{x} \in B^3$: for take $t > 1$, so that $t^{-1} \in (0,1)$, and use the previous step with $\mathbf{x} \rightsquigarrow t\mathbf{x}$ and $t \rightsquigarrow t^{-1}$ to compute

$$\mathsf{K}(t\mathbf{x}) = tt^{-1}\mathsf{K}(t\mathbf{x}) = t\mathsf{K}(t^{-1}t\mathbf{x}) = t\mathsf{K}(\mathbf{x}).$$

Also, (5.58) and affinity imply that for any $\mathbf{x},\mathbf{y} \in B^3$ for which $\mathbf{x}+\mathbf{y} \in B^3$, we have

$$\mathsf{K}(\mathbf{x}+\mathbf{y}) = 2\mathsf{K}(\tfrac{1}{2}\mathbf{x}+\tfrac{1}{2}\mathbf{y}) = 2\cdot(\tfrac{1}{2}\mathsf{K}(\mathbf{x})+\tfrac{1}{2}\mathsf{K}(\mathbf{y})) = \mathsf{K}(\mathbf{x})+\mathsf{K}(\mathbf{y}). \tag{5.59}$$

With our earlier result (5.57), this also gives

$$\mathsf{K}(-\mathbf{x}) = -\mathsf{K}(\mathbf{x}). \tag{5.60}$$

For some nonzero $\mathbf{x} \in \mathbb{R}^3$, take $s \geq \|\mathbf{x}\|$ and $t \geq \|\mathbf{x}\|$. Then by (5.58) we have

$$s\mathsf{K}(\mathbf{x}/s) = s\mathsf{K}\left(\frac{t}{s}\frac{\mathbf{x}}{t}\right) = t\mathsf{K}(\mathbf{x}/t).$$

We may therefore define a map $R : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R(\mathbf{0}) = \mathbf{0}; \tag{5.61}$$

$$R(\mathbf{x}) = s\cdot\mathsf{K}(\mathbf{x}/s) \quad (\mathbf{x}\neq\mathbf{0}), \tag{5.62}$$

for any choice of $s \geq \|\mathbf{x}\|$. For $\mathbf{x} \in B^3$ we may take $s = 1$, so that R extends K.

To prove that R is linear, for $\mathbf{x} \in \mathbb{R}^3$ and $t \geq 0$ pick some $s \geq t\|\mathbf{x}\|$ and compute

$$R(t\mathbf{x}) = s\mathsf{K}\left(\frac{t}{s}\mathbf{x}\right) = s\mathsf{K}\left(\|\mathbf{x}\|\frac{t}{s}\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = s\cdot\|\mathbf{x}\|\frac{t}{s}\mathsf{K}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = tR(\mathbf{x}). \tag{5.63}$$

For $t < 0$, we first show from (5.60) and (5.62) that

$$R(-\mathbf{x}) = -R(\mathbf{x}), \tag{5.64}$$

upon which (5.63) gives

$$R(t\mathbf{x}) = R(|t|\cdot(-\mathbf{x})) = |t|R(-\mathbf{x}) = -|t|R(\mathbf{x}) = -tR(\mathbf{x}). \tag{5.65}$$

Furthermore, for given $\mathbf{x},\mathbf{y} \in B^3$, pick $s' > 0$ such that $s' \geq \|\mathbf{x}\|$ and $s' \geq \|\mathbf{y}\|$, so that $s = 2s' \geq \|\mathbf{x}+\mathbf{y}\|$ by the triangle inequality, and use (5.59) to compute

$$\begin{aligned} R(\mathbf{x}+\mathbf{y}) &= s\mathsf{K}\left(\frac{\mathbf{x}+\mathbf{y}}{s}\right) = s\mathsf{K}\left(\frac{\mathbf{x}}{s}+\frac{\mathbf{y}}{s}\right) = s\mathsf{K}(\mathbf{x}/s)+s\mathsf{K}(\mathbf{y}/s) \\ &= R(\mathbf{x})+R(\mathbf{y}). \end{aligned} \tag{5.66}$$

Finally, R is an isometry by (5.62) and step one of the proof. Being also linear and invertible, R must therefore be an orthogonal transformation. □

Given step one, an alternative proof derives this lemma from Proposition 5.18 below, which shows that the transition probabilities (5.52) on S^2 are determined by the convex structure of B^3, so that affine bijections must preserve them. In other words, the boundary map $S^2 \to S^2$ defined by K preserves transition probabilities and hence satisfies the conditions of Lemma 5.9. This reasoning effectively reduces Kadison's Theorem to Wigner's Theorem, a move we will later examine in general.

In any case, Theorem 5.10 now follows from Lemma 5.11 is exactly the same way as Theorem 5.7 followed from the corresponding Lemma 5.9. □

We have given this proof in some detail, because step 3 will recur on other occasions where a given affine bijection is to be extended to some linear map.

Ludwig's Theorem is part 3 of Theorem 5.4. For $H = \mathbb{C}^2$, we have:

Theorem 5.12. *Each affine order isomorphism* $\mathsf{L} : \mathscr{E}(\mathbb{C}^2) \to \mathscr{E}(\mathbb{C}^2)$ *reads* $\mathsf{L}(a) = uau^*$*, where u is unitary or anti-unitary, and is uniquely fixed by* L *up to a phase.*

Proof. Using the parametrization (5.41), we have $a(x_0,x_1,x_2,x_3) \in \mathscr{E}(\mathbb{C}^2)$ iff each x_μ is real and $0 \leq x_0 \pm \|\mathbf{x}\| \leq 2$. In particular, we have $0 \leq x_0 \leq 2$. This easily follows from (2.38), noting that $a \in \mathscr{E}(\mathbb{C}^2)$ just means that $a^* = a$ and that both eigenvalues of a lie in $[0,1]$. Thus $\mathscr{E}(\mathbb{C}^2)$ is isomorphic as a convex set to a convex subset C of $\mathbb{R}^4$ that is fibered over the x_0-interval $[0,2]$, where the fiber C_{x_0} of C over x_0 is the three-ball $B^3_{x_0}$ with radius $\|\mathbf{x}\| = x_0$ as long as $0 \leq x_0 \leq 1$, whereas for $1 \leq x_0 \leq 2$ the fiber is $B^3_{2-x_0}$, so at $x_0 = 1$ the fiber is $C_1 = B^3 \equiv B^3_1$ (in one dimension less, this convex body is easily visualizable as a double cone in $\mathbb{R}^3$, where the fibers are disks). The partial order on C induced from the one on $\mathscr{E}(\mathbb{C}^2)$ is given by

$$(x_0, \mathbf{x}) \leq (x'_0, \mathbf{x}') \text{ iff } x'_0 - x_0 \geq \|\mathbf{x}' - \mathbf{x}\|, \tag{5.67}$$

which follows from (5.41) and (2.38), noting that for matrices one has $a \leq a'$ iff $a' - a$ has positive eigenvalues. A similar argument to the one proving (5.57) then shows that any affine bijection L of C must map the base space $[0,2]$ to itself (as an affine bijection), and hence either $x_0 \mapsto x_0$ or $x_0 \mapsto 2 - x_0$. The latter fails to preserve order, so L must fix x_0. Similarly, L maps each three-ball C_{x_0} to itself by an affine bijection, which, by the same proof as for Kadison's Theorem above, must be induced by some element R_{x_0} of $O(3)$. Finally, the order-preserving condition $x'_0 - x_0 \geq \|\mathbf{x}' - \mathbf{x}\| \Rightarrow x'_0 - x_0 \geq \|R_{x'_0}\mathbf{x}' - R_{x_0}\mathbf{x}\|$ obtained from (5.67) and the property $\mathsf{L}(x_0) = x_0$ just found can only be met if R_{x_0} is independent of x_0. □

Part 3 of Theorem 5.4 does not carry an official name; it may be attributed to Kadison, too, but the hard part of the proof was given earlier by Jacobson and Rickart. Rather than a contrived (though historically justified) name like "Jacobson–Rickart–Kadison Theorem", we will simply speak of ***Jordan's Theorem*** (for $H = \mathbb{C}^2$):

Theorem 5.13. *Each linear bijection* $\mathsf{J} : M_2(\mathbb{C})_{\mathrm{sa}} \to M_2(\mathbb{C})_{\mathrm{sa}}$ *that satisfies* (5.13) *and hence* (5.12) *takes the form* $\mathsf{J}(a) = uau^*$*, where u is either unitary or anti-unitary, and is uniquely determined by* J *up to a phase.*

Proof. First, any Jordan map (and hence *a fortiori* any Jordan automorphism) trivially maps projections into projections, as it preserves the defining conditions $e^2 = e^* = e$. Second, any Jordan automorphism J maps *one-dimensional* projections into *one-dimensional* projections: if $e \in \mathscr{P}_1(H)$, then $\mathsf{J}(e) \neq 0$ and $\mathsf{J}(e) \neq 1_2$, both because J is injective in combination with $\mathsf{J}(0) = 0$ and $\mathsf{J}(1_2) = 1_2$, respectively. Hence $\mathsf{J}(e) \in \mathscr{P}_1(H)$, since this is the only remaining possibility (a more sophisticated argument shows that this is even true for any Hilbert space H). From (5.41) and subsequent text, as in (5.44), by linearity of J we therefore have

$$\mathsf{J}\left(\sum_{j=1}^{3} x_j \sigma_j\right) = \sum_{j=1}^{3} (R\mathbf{x})_j \sigma_j, \tag{5.68}$$

from some map $R : S^2 \to S^2$, which is bijective because J is. Linearity of J then allows us to extend R to a linear map $\mathbb{R}^3 \to \mathbb{R}^3$, with matrix

$$R_{jk} = \tfrac{1}{2} \sum_{j=1}^{3} \mathrm{Tr}\,(\sigma_k \mathsf{J}(\sigma_j)), \tag{5.69}$$

cf. (5.45). By (5.69), this linear map restricts to the given bijection $R : S^2 \to S^2$, which also shows that it is isometric. Thus we have a linear isometry on $\mathbb{R}^3$, which therefore lies in $O(3)$. The proof may then be completed as in Theorem 5.7. □

The case $H = \mathbb{C}^2$ was already exceptional in the context of Gleason's Theorem, and it remains so as far as weak Jordan symmetries and Bohr symmetries are concerned.

Proposition 5.14. *The poset $\mathscr{C}(M_2(\mathbb{C}))$ is isomorphic to $\{\perp\} \cup \mathbb{RP}^2$, where the real projective plane $\mathbb{RP}^2$ is the quotient $S^2/\sim$ under the equivalence relation $\mathbf{x} \sim -\mathbf{x}$, and the only nontrivial ordering is $\perp \leq p$ for any $p \in \mathbb{RP}^2$.*

Proof. It is elementary that $M_2(\mathbb{C})$ has a single one-dimensional unital *-subalgebra, namely $\mathbb{C} \cdot 1$, the multiples of the unit; this gives the singleton $\perp$ in $\mathscr{C}(M_2(\mathbb{C}))$.

Furthermore, any two-dimensional unital *-subalgebra C of $M_2(\mathbb{C})$ is generated by a one-dimensional projection e, in that C is the linear span of e and 1_2. Hence C is also the linear span of (the projection) $1_2 - e$ and 1_2. In our parametrization of all one-dimensional projections e on $\mathbb{C}^2$ by S^2 (cf. Proposition 2.9), if e corresponds to $\mathbf{x}$, then $1 - e$ corresponds to $-\mathbf{x}$. This yields the remainder $\mathbb{RP}^2$ of $\mathscr{C}(M_2(\mathbb{C}))$.

Finally, commutative unital *-subalgebras D of $M_2(\mathbb{C})$ of dimension > 2 do not exist. For any such algebra D would contain some two-dimensional C just defined, but a simple computation (for example, in a basis were C consists of all diagonal matrices) shows that the only matrices that commute with all elements of C already lie in C (i.e., are diagonal). Hence no commutative extension of C exists. □

Bohr symmetries B for $\mathbb{C}^2$ therefore correspond to bijections of $\mathbb{RP}^2$. Similarly, weak Jordan symmetries J for $\mathbb{C}^2$ corresponds to bijections of S^2 (the difference with Bohr symmetries lies in the fact that J may also map $C = \mathrm{span}(e, 1_2)$ to itself nontrivially, i.e., by sending e to $1_2 - e$, which for B would yield the identity map). In both cases, few of these bijections are (anti-) unitarily implemented.

5.3 Equivalence between the six symmetry theorems

If $\dim(H) > 1$, the first three claims of Theorem 5.4 are equivalent; if $\dim(H) > 2$, all claims are. We will show this in some detail, if only because the proofs of the various equivalences relate the six symmetry concepts stated in Definition 5.1 in an instructive way. We will do this in the sequence Wigner $\leftrightarrow$ Kadison $\leftrightarrow$ Jordan, and subsequently Jordan $\leftrightarrow$ Ludwig, Jordan $\leftrightarrow$ von Neumann, and Jordan $\leftrightarrow$ Bohr. Consequently, in principle only one part of Theorem 5.4 requires a proof. Although redundant, we will, in fact, prove both Wigner's Theorem and Jordan's (indeed, no independent proof of the other parts of Theorem 5.4 seems to be known!). The most transparent way to state the various equivalences is to note that in each case the set of symmetries of some given kind (i.e., Wigner, ...) forms a group. In all cases, the nontrivial part of the proof is the establishment of a "natural" bijection, from which the group homomorphism property is trivial (and hence will not be proved).

Proposition 5.15. *There is an isomorphism of groups between:*

- *The group of affine bijections* $\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H)$*;*
- *The group of bijections* $\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H)$ *that satisfy* (5.8)*, viz.*

$$\mathsf{W} = \mathsf{K}_{|\mathscr{P}_1(H)}; \tag{5.70}$$

$$\mathsf{K}\left(\sum_i \lambda_i e_{\upsilon_i}\right) = \sum_i \lambda_i \mathsf{W}(\upsilon_{\upsilon_i}), \tag{5.71}$$

where $\rho = \sum_i \lambda_i e_{\upsilon_i}$ *is some (not necessarily unique) expansion of* $\rho \in \mathscr{D}(H)$ *in terms of a basis of eigenvector* υ_i *with eigenvalues* λ_i*, where* $\lambda_i \geq 0$ *and* $\sum_i \lambda_i = 1$*. In particular,* (5.70) *and* (5.71) *are well defined.*

Proof. It is conceptually important to distinguish between $B(H)_{\rm sa}$ as a Banach space in the usual operator norm $\|\cdot\|$, and $B_1(H)_{\rm sa}$, the Banach space of trace-class operators in its intrinsic norm $\|\cdot\|_1$. Of course, if $\dim(H) < \infty$, then $B(H)_{\rm sa} = B_1(H)_{\rm sa}$ as vector spaces, but even in that case the two norms do not coincide (although they are equivalent). The proof below has the additional advantage of immediately generalizing to the infinite-dimensional case. We start with (5.70).

1. Since $\mathscr{P}_1(H) = \partial_e \mathscr{D}(H)$, by the same argument as in the proof of Lemma 5.11, any affine bijection of the convex set $\mathscr{D}(H)$ must preserve its boundary, so that K maps $\mathscr{P}_1(H)$ into itself, necessarily bijectively. The goal of the next two steps is to prove that (5.70) satisfies (5.8), i.e., preserves transition probabilities.
2. An affine bijection $\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H)$ extends to an isometric isomorphism $\mathsf{K}_1 : B_1(H)_{\rm sa} \to B_1(H)_{\rm sa}$ with respect to the trace-norm $\|\cdot\|_1$, as follows:

   a. Put $\mathsf{K}_1(0) = 0$ and for $b \geq 0$, $b \in B_1(H)$, i.e. $b \in B_1(H)_+$, and $b \neq 0$, define

$$\mathsf{K}_1(b) = \|b\|_1 \mathsf{K}(b/\|b\|_1). \tag{5.72}$$

By construction, K_1 is isometric and preserves positivity. For $b \in B_1(H)_+$ we have $\mathrm{Tr}\,(b) = \|b\|_1$, hence $b/\|b\|_1 \in \mathscr{D}(H)$, on which K is defined.
Linearity of K_1 with positive coefficients (as a consequence of the affine property of K) is verified as in the proof of Lemma 5.11; this time, use

$$a+b = (\|a\|_1 + \|b\|_1)\cdot\left(t\frac{a}{\|a\|_1} + (1-t)\frac{b}{\|b\|_1}\right), \tag{5.73}$$

with $t = \|a\|_1/(\|a\|_1+\|b\|_1)$. Note that if $a,b \in B_1(H)_+$, then $a+b \in B_1(H)_+$.

b. For $b \in B_1(H)_{\mathrm{sa}}$, decompose $b = b_+ - b_-$, where $b_\pm \geq 0$; see Proposition A.24 (this remains valid in general Hilbert spaces). We then define

$$\mathsf{K}_1(b) = \mathsf{K}_1(b_+) - \mathsf{K}_1(b_-). \tag{5.74}$$

To show that this makes K_1 linear on all of $B_1(H)_{\mathrm{sa}}$, suppose $b = b'_+ - b'_-$ with $b'_\pm \geq 0$. Then $b'_+ + b_- = b_+ + b'_-$, and since each term is positive,

$$\mathsf{K}_1(b'_+ + b_-) = \mathsf{K}_1(b'_+) + \mathsf{K}_1(b_-) = \mathsf{K}(b_+ + b'_-) = \mathsf{K}_1(b_+) + \mathsf{K}_1(b'_-),$$

by the previous step. Hence $\mathsf{K}_1(b'_+) - \mathsf{K}_1(b'_-) = \mathsf{K}_1(b_+) - \mathsf{K}_1(b_-)$, so that (5.74) is actually independent of the choice of the decomposition of b as long as the operators are positive. Hence for $a,b \in B_1(H)_{\mathrm{sa}}$ we may compute

$$\begin{aligned}\mathsf{K}_1(a+b) &= \mathsf{K}_1(a_+ + b_+ - (a_- + b_-)) = \mathsf{K}_1(a_+ + b_+) - \mathsf{K}_1(a_- + b_-)\\ &= \mathsf{K}_1(a_+) + \mathsf{K}_1(b_+) - \mathsf{K}_1(a_-) - \mathsf{K}_1(b_-) = \mathsf{K}_1(a) + \mathsf{K}_1((b),\end{aligned}$$

since $a_+ + b_+$ and $a_- + b_-$ are both positive.

The key point in verifying isometry of K_1 is the property $|b| = b_+ + b_-$, which follows from (A.76) or Theorem B.94. Using this property, we have

$$\begin{aligned}\|\mathsf{K}_1(b)\|_1 &= \mathrm{Tr}\,(|\mathsf{K}_1 b|) = \mathrm{Tr}\,(|\mathsf{K}_1(b_+) - \mathsf{K}_1(b_-)|) = \mathrm{Tr}\,(\mathsf{K}_1(b_+) + \mathsf{K}_1(b_-))\\ &= \mathrm{Tr}\,(b_+ + b_-) = \mathrm{Tr}\,(|b_+ - b_-|) = \mathrm{Tr}\,(|b|) = \|b\|_1.\end{aligned}$$

3. For any two unit vectors ψ, φ in H we have the formula

$$\|e_\psi - e_\varphi\|_1 = 2\sqrt{1 - \mathrm{Tr}\,(e_\psi e_\varphi)}, \tag{5.75}$$

which can easily be proved by a calculation with 2×2 matrices (since everything takes place is the two-dimensional subspace spanned by ψ and φ, expect when $\varphi = z\psi$, $z \in \mathbb{T}$, in which case (5.75) reads $0 = 0$ and hence is true also). Since K_1 is linear as well as isometric with respect to the trace-norm, we have

$$\|\mathsf{K}_1(e_\psi) - \mathsf{K}_1(e_\varphi)\|_1 = \|\mathsf{K}_1(e_\psi - e_\varphi)\|_1 = \|e_\psi - e_\varphi\|_1,$$

and hence, by (5.75), $\mathrm{Tr}\,(\mathsf{K}_1(e_\psi)\mathsf{K}_1(e_\varphi)) = \mathrm{Tr}\,(e_\psi e_\varphi)$. Eq. (5.70) then gives (5.8).

We move on to (5.71). The main concern is that this expression be well defined, since in case some eigenvalue $\lambda > 0$ of ρ is degenerate (necessarily with finite multiplicity, even in infinite dimension, since ρ is compact), the basis of the eigenspace H_λ that takes part in the sum $\sum_i \lambda_i e_{\upsilon_i}$ is far from unique. This is settled as follows:

Lemma 5.16. *Let* $\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H)$ *be a bijection that satisfies* (5.8)*, let* $L \subset H$ *be a (finite-dimensional) subspace, and let* (υ_j) *and* (υ'_i) *be bases of* L*. Then*

$$\sum_j \mathsf{W}(e_{\upsilon_j}) = \sum_i \mathsf{W}(e_{\upsilon'_i}). \tag{5.76}$$

Proof. As usual, for projections e and f on H we write $e \leq f$ iff $eH \subseteq fH$. From (B.212) and (B.214) we have $\sum_j |\langle \upsilon_j, \psi \rangle|^2 \leq 1$ for any unit vector $\psi \in H$, with equality iff $\psi \in L$. In other words, $e_\psi \leq e_L$ iff $\sum_j \mathrm{Tr}\,(e_{\upsilon_j} e_\psi) = 1$. Furthermore, by (5.8) the images $\mathsf{W}(e_{\upsilon_j})$ remain orthogonal; hence $\sum_j \mathsf{W}(e_{\upsilon_j})$ is a projection, and $e \leq \sum_j \mathsf{W}(e_{\upsilon_j})$ iff $\sum_j \mathrm{Tr}\,(\mathsf{W}(e_{\upsilon_j})e) = 1$. By (5.8), this condition is satisfied for $e = \mathsf{W}(e_{\upsilon_i})$, so that $\mathsf{W}(e_{\upsilon'_i}) \leq \sum_j \mathsf{W}(e_{\upsilon_j})$ for each j. Since also the projections $\mathsf{W}(e'_{\upsilon_i})$ are orthogonal, this gives $\sum_i \mathsf{W}(e'_{\upsilon_i}) \leq \sum_j \mathsf{W}(e_{\upsilon_j})$. Interchanging the roles of the two bases gives the converse, yielding (5.76). □

Finally, to prove bijectivity of the correspondence $\mathsf{K} \leftrightarrow \mathsf{W}$, we need the property

$$\mathsf{K}\left(\sum_i \lambda_i e_{\upsilon_i}\right) = \sum_i \lambda_i \mathsf{K}(e_{\upsilon_i}), \tag{5.77}$$

since this implies that K is determined by its action on $\mathscr{P}_1(H) \subset \mathscr{D}(H)$. In finite dimension this follows from convexity of K, and we are done. In infinite dimension, we in addition need continuity of K, as well as convergence of the sum $\sum_i \lambda_i e_{\upsilon_i}$ not only in the operator norm (as follows from the spectral theorem for self-adjoint compact operators), but also in the trace norm: for finite n, m,

$$\|\sum_{i=n}^{m} \lambda_i e_{\upsilon_i}\|_1 \leq \sum_{i=n}^{m} |\lambda_i| \|e_{\upsilon_i}\|_1 = \sum_{i=n}^{m} \lambda_i,$$

since $\|e_{\upsilon_i}\|_1 = 1$. Because $\sum_i \lambda_i = 1$, the above expression vanishes as $n, m \to \infty$, whence $\rho_n = \sum_{i=1}^n \lambda_i e_{\upsilon_i}$ is a Cauchy sequence in $B_1(H)$, which by completeness of the latter converges (to an element of $\mathscr{D}(H)$, as one easily verifies).

The proof of continuity is completed by noting that K is continuous with respect to the trace norm, for it is isometric and hence bounded (see step 2 above). □

It is enlightening to give a rather more conceptual proof that $\mathsf{K}_{|\mathscr{P}_1(H)}$ satisfies (5.8), which is based on a result to be used more often in the future. In what follows, for any convex set C, the notation $A_b(K)$ stands for the real vector space of *bounded* affine functions $f : C \to \mathbb{R}$, that is, bounded functions satisfying

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \; x, y \in C, t \in (0,1). \tag{5.78}$$

It is easily checked that $A_b(K)$ with the supremum-norm is a real Banach space.

Proposition 5.17. *For any Hilbert space H we have an isometric isomorphism*

$$A_b(\mathscr{D}(H)) \cong B(H)_{\mathrm{sa}}, \tag{5.79}$$
$$f \leftrightarrow a; \tag{5.80}$$
$$f(\rho) = \mathrm{Tr}\,(\rho a), \tag{5.81}$$

which preserves the unit (i.e., $1_{\mathscr{D}(H)} \leftrightarrow 1_H$) as well as the order (i.e, $f \geq 0$ iff $a \geq 0$).

Note that under the identification $\mathscr{D}(H) \cong S_n(B(H))$ (where in finite dimension the normal state space $S_n(B(H))$ simply coincides with the state space $S(B(H))$), where $\rho \leftrightarrow \omega$ as in (2.33), i.e., $\omega(a) = \mathrm{Tr}\,(\rho a)$, the above isomorphism simply reads

$$A_b(S_n(B(H))) \cong B(H)_{\mathrm{sa}}, \tag{5.82}$$
$$\hat{a} \leftrightarrow a; \tag{5.83}$$
$$\hat{a}(\omega) = \omega(a). \tag{5.84}$$

Proof. It is clear that for each $a \in B(H)_{\mathrm{sa}}$ the function $f : \rho \mapsto \mathrm{Tr}\,(\rho a)$ (or, equivalently, $\hat{a} : \omega \mapsto \omega(a)$) is affine as well as real-valued, and is bounded by (A.100) (supplemented, if $\dim(H) = \infty$, by Lemma B.142), noting that $\|\rho\|_1 = 1$ for $\rho \in \mathscr{D}(H)$, and in fact (B.483) yields the equality $\|f\|_\infty = \|a\|$ (or $\|\hat{a}\|_\infty = \|a\|$).

Conversely, $f \in A_b(\mathscr{D}(H))$ defines a function $Q : H \to \mathbb{R}$ by

$$Q(0) = 0; \tag{5.85}$$
$$Q(\psi) = \|\psi\|^2 f(e_{\psi/\|\psi\|}) \; (\psi \neq 0). \tag{5.86}$$

This function is clearly bounded on the unit ball of H, as in

$$|Q(\psi)| \leq \|f\|_\infty \|\psi\|^2. \tag{5.87}$$

To check that Q in fact defines a quadratic form on H, we verify the properties (A.8) - (A.9). The first is trivial. The second follows from the easily verified identity

$$te_{\frac{v+w}{\|v+w\|}} + (1-t)e_{\frac{v-w}{\|v-w\|}} = se_{\frac{v}{\|v\|}} + (1-s)e_{\frac{w}{\|w\|}}, \tag{5.88}$$

where $v, w \neq 0$, $v \neq w$, and the coefficients s, t are given by

$$t = \frac{\|v+w\|^2}{2(\|v\|^2 + \|w\|^2)}; \tag{5.89}$$
$$s = \frac{\|v\|^2}{\|v\|^2 + \|w\|^2}. \tag{5.90}$$

The affine property (5.78) then immediately yields (A.9). According to Proposition B.79, we obtain a unique operator $a \in B(H)_{\mathrm{sa}}$ such that $Q(\psi) = \langle \psi, a\psi \rangle$, i.e.,

$$\langle \psi, a\psi \rangle = f(e_\psi), \; \psi \in H, \|\psi\| = 1. \tag{5.91}$$

Since also $\langle\psi, a\psi\rangle = \mathrm{Tr}\,(e_\psi a)$, we have established (5.81) for each $\rho = e_\psi$, where $\psi \in H, \|\psi\| = 1$. To extend this result to general density operators $\rho = \sum_i \lambda_i e_{\upsilon_i}$, we use (A.100) as well as convergence of the above sum in the trace norm $\|\cdot\|_1$, cf. the proof of Lemma 5.16; the details are analogous to the proof of Theorem B.146. □

Proposition 5.18. *For any unit vectors $\psi, \varphi \in H$ we have*

$$\mathrm{Tr}\,(e_\psi e_\varphi) = \inf\{f(e_\psi) \mid f \in A_b(\mathscr{D}(H)), 0 \leq f \leq 1, f(e_\varphi) = 1\}. \tag{5.92}$$

The virtue of this formula is that the expression on the left-hand side, which defines the transition probabilities on $\partial_e\mathscr{D}(H) = \mathscr{P}_1(H)$, is intrinsically given by the convex structure of $\mathscr{D}(H)$. Consequently, any affine bijection of this convex set (which already preserves the boundary) must preserve these probabilities.

Proof. By the previous proposition, eq. (5.92) is equivalent to

$$\mathrm{Tr}\,(e_\psi e_\varphi) = \inf\{\langle\psi, a\psi\rangle \mid a \in B(H)_{\mathrm{sa}}, 0 \leq a \leq 1, \langle\varphi, a\varphi\rangle = 1\}. \tag{5.93}$$

Since $\mathrm{Tr}\,(e_\psi e_\varphi) = \langle\psi, e_\varphi\psi\rangle$, we are ready if we can show that the infimum is reached at $a = e_\varphi$. Therefore, we prove that for any a as specified we must have $\langle\psi, a\psi\rangle \geq \mathrm{Tr}\,(e_\psi e_\varphi) = |\langle\varphi, \psi\rangle|^2$. To do so, we are going to find a contradiction if

$$\langle\psi, a\psi\rangle < \mathrm{Tr}\,(e_\psi e_\varphi), \tag{5.94}$$

for some such a. Indeed, $\langle\varphi, a\varphi\rangle = 1$ with $\|a\| \leq 1$ (which follows from $0 \leq a \leq 1$) and $\|\varphi\| = 1$ imply, by Cauchy–Schwarz, that $a\varphi = \varphi$. Since $a^* = a$ (by positivity of a), we also have $a : (\mathbb{C}\cdot\varphi)^\perp \to (\mathbb{C}\cdot\varphi)^\perp$, so we may write $a = e_\varphi + a'$, with $a'\varphi = 0$ and a' mapping $(\mathbb{C}\cdot\varphi)^\perp$ to itself. Then $a \geq 0$ implies $a' \geq 0$. If (5.94) holds, then $\langle\psi, a'\psi\rangle < 0$, which contradicts positivity of a' (and hence of a). □

We now turn to the equivalence between Jordan's Theorem and Kadison's Theorem.

Proposition 5.19. *There is an isomorphism of groups between:*

- *The group of affine bijections $\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H)$;*
- *The group of Jordan automorphisms $\mathsf{J} : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$,*

such that for any $a \in B(H)_{\mathrm{sa}}$ one has

$$\mathrm{Tr}\,(\mathsf{K}(\rho)a) = \mathrm{Tr}\,(\rho\,\mathsf{J}(a)) \;\; (\rho \in \mathscr{D}(H)). \tag{5.95}$$

This immediately follows from the following lemma (of independent interest):

Lemma 5.20. *1. There is a bijective correspondence between:*

- *affine bijections $\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H)$;*
- *unital positive (i.e. order-preserving) linear bijections $\alpha : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$,*

such that for any $a \in B(H)_{\mathrm{sa}}$ one has (5.95).

2. A map $\alpha : B(H) \to B(H)$ is a unital positive linear bijection iff it is a Jordan automorphism.

Proof. 1. An affine bijection $\mathsf{K} : \mathscr{D}(H) \to \mathscr{D}(H)$ induces an isomorphism

$$\mathsf{K}^* : A_b(\mathscr{D}(H)) \to A_b(\mathscr{D}(H)); \tag{5.96}$$
$$f \mapsto f \circ \mathsf{K}, \tag{5.97}$$

which is evidently unital, positive, and isometric. Consequently, by Proposition 5.17, K^* corresponds to some isomorphism $\alpha : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$, which necessarily shares the properties of being unital, positive, and isometric; this follows abstractly from the proposition, but may also be verified directly from (5.95). Conversely, such a map α yields a map K directly by (5.95); to see this, we identify $\mathscr{D}(H)$ with the normal state space of $B(H)$ through $\rho \leftrightarrow \omega$, as usual, cf. (2.33), and note that $\mathsf{K}\omega$ is the state defined by $(\mathsf{K}\omega)(a) = \omega(\alpha(a))$, or briefly $\mathsf{K}\omega = \omega \circ \alpha$. This is often written as $\mathsf{K} = \alpha^*$, and for future reference we write

$$\alpha^* \omega(a) = \omega(\alpha(a)). \tag{5.98}$$

2. The nontrivial direction of the proof (i.e. positive etc. $\Rightarrow$ Jordan) is based on a number of facts from operator theory:

   a. Unital positive linear maps maps on $B(H)_{\mathrm{sa}}$ preserve $\mathscr{P}(H)$, cf. (2.164).
   b. Any two projections e and f are orthogonal ($ef = 0$) iff $e + f \leq 1_H$ (easy).
   c. Any $a \in B(H)_{\mathrm{sa}}$ is a norm-limit of finite sums of the kind $\sum_i \lambda_i e_i$, where $\lambda_i \in \mathbb{R}$ and the e_i are mutually orthogonal projections (this follows from the spectral theorem for bounded self-adjoint operators in the form of Theorem B.104)
   d. Any unital positive linear map $\alpha : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$ is continuous. Since

$$-\|a\| \cdot 1_H \leq a \leq -\|a\| \cdot 1_H \quad (a \in B(H)_{\mathrm{sa}}), \tag{5.99}$$

   by (C.83), applying the positive map α and using $\alpha(1_H) = 1_H$ yields

$$-\|a\| \cdot 1_H \leq \alpha(a) \leq -\|a\| \cdot 1_H.$$

   This is possible only if $\|\alpha(a)\| \leq \|a\|$, and hence α is continuous with norm bounded by $\|\alpha\| \leq 1$. In fact, since a is unital we have $\|\alpha\| = 1$.

Therefore, any unital positive linear map α preserves orthogonality of projections, so if $a = \sum_i \lambda_i e_i$ (finite sum), then

$$\alpha(a^2) = \alpha\left(\sum_i \lambda_i^2 e_i\right) = \sum_i \lambda_i^2 \alpha(e_i) = \sum_{i,j} \lambda_i \lambda_j \alpha(e_i)\alpha(e_j) = \alpha(a)^2, \tag{5.100}$$

since $e_i e_j = \delta_{ij} e_j$ and by the above comment also $\alpha(e_i)\alpha(e_j) = \delta_{ij}\alpha(e_j)$. By continuity of α, this property extends to arbitrary $a \in B(H)_{\mathrm{sa}}$. Finally, since

$$a \circ b = \tfrac{1}{2}((a+b)^2 - a^2 - b^2), \tag{5.101}$$

preserving squares as in (5.100) implies preserving the Jordan product $\circ$. $\square$

We now turn to the equivalence between Ludwig symmetries and Jordan ones.

Proposition 5.21. *There is an isomorphism of groups between:*

- *The group of affine order isomorphism* $\mathsf{L} : \mathscr{E}(H) \to \mathscr{E}(H)$*;*
- *The group of Jordan automorphisms* $\mathsf{J} : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$.

Proof. Since L is an order isomorphism, it satisfies $\mathsf{L}(0) = 0$ (as well as $\mathsf{L}(1_H) = 1_H$), since 0 is the bottom element of $\mathscr{E}(H)$ as a poset (and 1_H is its the top element). As in the proof of Lemma 5.11, one shows that this property plus convexity implies $\mathsf{L}(ta) = t\mathsf{L}(a)$ and $\mathsf{L}(a+b) = \mathsf{L}(a) + \mathsf{L}(b)$ whenever defined. Defining J by

$$\mathsf{J}(0) = 0; \tag{5.102}$$
$$\mathsf{J}(a) = s \cdot \mathsf{L}(a/s) \ (a > 0, s \geq \|a\|); \tag{5.103}$$
$$\mathsf{J}(a) = -\mathsf{J}(-a) \ (a < 0), \tag{5.104}$$

where $a > 0$ means $a \geq 0$ and $a \neq 0$, and $a < 0$ means $-a \geq 0$ and $a \neq 0$, once again the reasoning near the end of the proof of Lemma 5.11 shows that J is linear; it is a untital order-preserving bijection by construction. Hence J is a Jordan automorphism by Lemma 5.20.2 Of course, instead of (5.104) one could equivalently have defined J on general $a \in B(H)_{\mathrm{sa}}$ by $\mathsf{J}(a) = \mathsf{J}(a_+) - \mathsf{J}(a_-)$, using the (by now hopefully familiar) decomposition $a = a_+ - a_-$ with $a_\pm \geq 0$ and $a_+ a_- = 0$.

Conversely, once again using Lemma 5.20.2, a Jordan automorphisms (5.11) preserves order as well as the unit, so that the inequality $0 \leq a \leq 1_H$ characterizing $a \in \mathscr{E}(H)$ is preserved, i.e., $0 \leq \mathsf{J}(a) \leq 1_H$. Thus J preserves $\mathscr{E}(H)$, where it preserves order. Convexity is obvious, since $\mathsf{L} = \mathsf{J}_{|\mathscr{E}(H)}$ comes from a linear map. □

The equivalence between Jordan's Theorem and von Neumann's Theorem (provided $\dim(H) \geq 3$) hinges on the following corollary of Gleason's Theorem (cf. §D.1).

Corollary 5.22. *Let* $\dim(H) > 2$. *Then an isomorphism* N *of* $\mathscr{P}(H)$ *as an orthocomplemented lattice has a unique extension to a linear map* $\alpha : B(H)_{\mathrm{sa}} \to B(H)_{\mathrm{sa}}$*, which is (automatically) invertible, unital, and positive.*

Proof. According to Lemma D.2, N preserves all suprema in $\mathscr{P}(H)$. Since we have $\sum_i e_i = \bigvee e_i$ for any family of mutually orthogonal projections and since N by definition preserves the orthocomplementation $e^\perp = 1 - e$ and hence preserves orthogonality of projections, we may compute

$$\mathsf{N}\left(\sum_i e_i\right) = \mathsf{N}\left(\bigvee_i e_i\right) = \bigvee_i \mathsf{N}(e_i) = \sum_i \mathsf{N}(e_i). \tag{5.105}$$

Consequently, for any normal state ω on $B(H)$, the map $e \mapsto \omega \circ \mathsf{N}(e)$ is a probability measure on $\mathscr{P}(H)$, which by Gleason's Theorem has a unique linear extension to $B(H)$ and hence *a fortiori* to $B(H)_{\mathrm{sa}}$. We use this in order to define α, as follows.

First, let $a \in B(H)_{\mathrm{sa}}$ and suppose $a = \sum_j \lambda_j f_j$ for some *finite* family (f_j) of projections (not necessarily orthogonal), and some $\lambda_j \in \mathbb{R}$. Then $\sum_j \lambda_j \mathsf{N}(f_j)$ is independent of the particular decomposition of a that has been chosen, so we may put

$$\alpha(a) = \sum_j \lambda_j \mathsf{N}(f_j). \tag{5.106}$$

To see this, put $a = \sum_{j'} \lambda'_{j'} f'_{j'}$ and hence $\alpha'(a) = \sum_{j'} \lambda'_{j'} \mathsf{N}(f'_{j'})$, and suppose $\alpha'(a) \neq \alpha(a)$. By (B.477) there exists a normal state ω such that $\omega(\alpha'(a)) \neq \omega(\alpha(a))$; indeed, each element of $B_1(H)$ is a linear combination of at most four density operators, so that each normal linear functional on $B(H)$ is a linear combination of at most four normal states. But since $\omega \circ \mathsf{N}$ is linear, this implies $\omega \circ \mathsf{N}(a) \neq \omega \circ \mathsf{N}(a)$, which is a contradiction. Hence $\alpha'(a) = \alpha(a)$ and accordingly, (5.106) is well defined. Because it is independent of the decomposition of a into projections, α is linear: if $a = \sum_j \lambda_j f_j$ and $a' = \sum_{j'} \lambda'_{j'} f'_{j'}$, then $a + a' = \sum_j \lambda_j f_j + \sum_{j'} \lambda'_{j'} f'_{j'}$, so that

$$\mathsf{N}(a+a') = \mathsf{N}\left(\sum_j \lambda_j f_j + \sum_{j'} \lambda'_{j'} f'_{j'}\right) = \sum_j \lambda_j \mathsf{N}(f_j) + \sum_{j'} \lambda'_{j'} \mathsf{N}(f'_{j'}) = \mathsf{N}(a) + \mathsf{N}(a').$$

Similarly, for any $t \in \mathbb{R}$ we have

$$\mathsf{N}(ta) = \mathsf{N}\left(\sum_j t\lambda_j f_j\right) = \sum_j t\lambda_j \mathsf{N}(f_j) = t\sum_j \lambda_j \mathsf{N}(f_j) = t\mathsf{N}(a).$$

We may now extend α to all of $B(H)_{\mathrm{sa}}$ by continuity. Indeed, according to the spectral theorem in the form (B.326), the set of all operators of the form $a = \sum_j \lambda_j f_j$ with all f_j mutually orthogonal (so that a is given by its spectral resolution) is norm-dense in $B(H)_{\mathrm{sa}}$. Applying (5.106), and noting that $\|a\| = \sup_j |\lambda_j|$, we may estimate

$$\|\alpha(a)\| = \|\sum_j \lambda_j \mathsf{N}(f_j)\| \leq \sup_j\{|\lambda_j|\} \|\sum_j \mathsf{N}(f_j)\| \leq \|a\|,$$

since the $\mathsf{N}(f_j)$ are mutually orthogonal and hence sum to some projection, which has norm 1 (unless $a = 0$). For general $a \in B(H)_{\mathrm{sa}}$, we may therefore define N by $\mathsf{N}(a) = \lim_n \mathsf{N}(a_n)$, where each a_n is of the above (spectral) form and $\|a_n - a\| \to 0$.

To prove that α is positive, we show that $\alpha(a) \geq 0$ whenever $a \geq 0$. As in the preceding step, initially suppose that $a = \sum_j \lambda_j f_j$ has a finite spectral resolution. Then $a \geq 0$ iff $\lambda_j \geq 0$ for each j, and hence $\alpha(a) \geq 0$ by (5.106), since by orthogonality of the $\mathsf{N}(f_j)$ this equation states the spectral resolution of $\alpha(a)$. Now if $a_n \geq 0$ and $a_n \to a$ (in norm), then $\langle \psi, a_n \psi \rangle \to \langle \psi, a\psi \rangle$, which must remain positive, so that $a \geq 0$. Hence positivity of α on all of $B(H)_{\mathrm{sa}}$ follows by continuity.

Finally, α inherits invertibility from N, and it is unital by (5.105), taking $e_i = |\upsilon_i\rangle\langle\upsilon_i|$ for some basis (υ_i) of H (or using the fact that it preserves $\top = 1_H$). □

Subsequently, we use Lemma 5.20 to further extend α by complex linearity to a Jordan isomorphism of $B(H)$; see Definition 5.1.

Finally, the equivalence between weak Jordan symmetries and Bohr symmetries follows from Hamhalter's Theorem 9.4, whereas Theorem 9.7 strengthens this to an equivalence between Jordan symmetries and Bohr symmetries. The proof of these theorems does not seem to simplify in the special case at hand, i.e. $A = B(H)$.

5.4 Proof of Jordan's Theorem

In view of the equivalence between the six parts of Theorem 5.4, we only need to prove one of them. In the literature, one only finds proofs of Jordan's Theorem and of Wigner's Theorem, and we present each of these (surprisingly but instructively, these proofs look completely different). We start with ***Jordan's Theorem***:

Theorem 5.23. *Any Jordan automorphism* $\mathsf{J}_{\mathbb{C}}$ *of* $B(H)$ *is given by either*

$$\mathsf{J}_{\mathbb{C}}(a) = \alpha_u(a) \equiv uau^*, \tag{5.107}$$

where u is unitary (and is determined by $\mathsf{J}_{\mathbb{C}}$ *up to a phase), or by*

$$\mathsf{J}_{\mathbb{C}}(a) = \alpha'_u(a) \equiv ua^*u^*, \tag{5.108}$$

where u is anti-unitary (and is determined by $\mathsf{J}_{\mathbb{C}}$ *up to a phase, too).*

The difficult part of the proof is Theorem C.175, which implies:

Proposition 5.24. *A Jordan automorphism* α *of* $B(H)$ *is either an automorphism or an anti-automorphism.*

Recall that an ***automorphism*** of $B(H)$ is a linear bijection $\alpha : B(H) \to B(H)$ that satisfies $\alpha(a^*) = \alpha(a)^*$ and $\alpha(ab) = \alpha(a)\alpha(b)$; an ***anti-automorphism***, on the other hand, satisfies the first property whilst the latter is replaced by $\alpha(ab) = \alpha(b)\alpha(a)$. Clearly, both automorphisms and anti-automorphisms are Jordan automorphisms. Granting this result, we may deal with the two cases separately.

Proposition 5.25. *Any automorphism* $\alpha : B(H) \to B(H)$ *takes the form* $\alpha = \alpha_u$*, see* (5.107)*, where* $u : H \to H$ *is unitary, uniquely determined by* α *up to a phase.*

The proof uses the following lemmas. The first follows from Theorem C.62.4.

Lemma 5.26. *If* $\alpha : B(H) \to B(H)$ *is an automorphism and* $a \in B(H)$*, then*

$$\|\alpha(a)\| = \|a\|. \tag{5.109}$$

Lemma 5.27. *If* $\alpha : B(H) \to B(H)$ *is an automorphism and* $e \in B(H)$ *is a one-dimensional projection, then so is* $\alpha(e)$.

Proof. It should be obvious that automorphisms α preserve projections e (whose defining properties are $e^2 = e^* = e$). Furthermore, α preserves order, i.e., if $a \geq 0$ (in that, as always, $\langle \psi, a\psi \rangle \geq 0$ for each $\psi \in H$, or, equivalently, $a = b^*b$), then $\alpha(a) \geq 0$ (this is clear from the second way of expressing positivity). Consequently, if $a \leq b$ (in that $b - a \geq 0$), then $\alpha(a) \leq \alpha(b)$. We notice that if we define $e \leq f$ iff $eH \subseteq fH$, then $e \leq f$ iff $e \leq f$ as self-adjoint operators (in that $\langle \psi, e\psi \rangle \leq \langle \psi, f\psi \rangle$ for each $\psi \in H$); see Proposition C.170. With respect to the ordering $\leq$ the one-dimensional projections e are ***atomic***, in the sense that $0 \leq e$ (but $e \neq 0$) and if $0 \leq f \leq e$, then either $f = 0$ or $f = e$. Now automorphisms of the projection lattice $B(H)$ restrict to isomorphisms of $\mathscr{P}(H)$, which preserve atoms (as these are intrinsically defined by the partial order). □

We are now ready for the (constructive!) proof of Proposition 5.25.

Proof. For some fixed unit vector $\chi \in H$, take the corresponding one-dimensional projection e_χ and define a new unit vector φ (up to a phase) by

$$e_\varphi = \alpha^{-1}(e_\chi). \tag{5.110}$$

Now any $\psi \in H$ may be written as $\psi = a\varphi$, for some $a \in B(H)$. Attempt to define an operator u by $u\psi = \alpha(a)\chi$, i.e.,

$$ua\varphi = \alpha(a)\chi. \tag{5.111}$$

This looks dangerously ill-defined, since many different operators a may give rise to the same ψ. Fortunately, we may compute

$$\begin{aligned}\|a\varphi\|_H &= \|ae_\varphi\varphi\|_H = \|ae_\varphi\|_{B(H)} = \|\alpha(ae_\varphi)\|_{B(H)} \\ &= \|\alpha(a)\alpha(e_\varphi)\|_{B(H)} = \|\alpha(a)e_\chi\|_{B(H)} = \|\alpha(a)\chi\|_H \\ &= \|ua\varphi\|_H,\end{aligned}$$

so that if $a\varphi = b\varphi$, then $\alpha(a)\chi = \alpha(b)\chi$ and hence u is well defined. By this computation u is also isometric and since it is clearly surjective, it is unitary. The property $\alpha(a) = uau^*$ is equivalent to $ua = \alpha(a)u$, which in turn is equivalent to $uab\varphi = \alpha(a)ub\varphi$ for any $b \in B(H)$, which by definition of u is the same as

$$\alpha(ab)\chi = \alpha(a)\alpha(b)\chi. \tag{5.112}$$

But this holds by virtue of α being an automorphism. Finally, all arbitrariness in u lies in the lack of uniqueness of φ given its definition (5.110). □

Proposition 5.28. *Any antiautomorphism $\alpha : B(H) \to B(H)$ takes the form $\alpha = \alpha_u$, cf.* (5.108)*, where $u : H \to H$ is anti-unitary, uniquely determined by α up to a phase.*

Proof. Pick an arbitrary anti-unitary operator $J : H \to H$ and define

$$\begin{aligned}\beta &: B(H) \to B(H); \\ \beta(a) &= Ja^*J^*.\end{aligned} \tag{5.113}$$

Then $\alpha \circ \beta$ is an automorphism, to which Proposition 5.25 applies, so that

$$\alpha \circ \beta(a) = \tilde{u}a\tilde{u}^*, \tag{5.114}$$

for some unitary $\tilde{u}$. Hence

$$\alpha(a) = \alpha(\beta \circ \beta^{-1}(a)) = \alpha \circ \beta(J^*a^*J) = \tilde{u}J^*a^*J\tilde{u}^*,$$

so that $\alpha(a) = ua^*u^*$ with $u = \tilde{u}J^*$.

The precise lack of uniqueness of u is inherited from the unitary case. □

5.5 Proof of Wigner's Theorem

We recall ***Wigner's Theorem***, i.e. Theorem 5.4.1:

Theorem 5.29. *Each bijection* $\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H)$ *that satisfies*

$$\mathrm{Tr}(\mathsf{W}(e)\mathsf{W}(f)) = \mathrm{Tr}(ef), \ \ (e, f \in \mathscr{P}_1(H)), \tag{5.115}$$

is given by $\mathsf{W}(e) = ueu^* \equiv \alpha_u(e)$, *where the operator u is either unitary or anti-unitary, and is uniquely determined by* W *up to a phase.*

The problem is to lift a given map $\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H)$ that satisfies (5.115) to either a unitary or an anti-unitary map $u : H \to H$ such that

$$\mathsf{W}(e_\psi) = e_{u\psi} = ue_\psi u^*. \tag{5.116}$$

Suppose $\mathsf{W}(e_\psi) = e_{\psi'}$. Since $e_{z\psi} = e_\psi$ for any $z \in \mathbb{T}$, and likewise for $e_{\psi'}$, this means that $u\psi = z\psi'$ for some $z \in \mathbb{T}$; the problem is to choose the z's coherently all over the unit sphere of H. There are many proofs in the literature, of which the following one—partly based on an earlier proof by Bargmann (1964)—has the advantage of making at least the construction of u explicit (at the cost of opaque proofs of some crucial lemma's). We assume $\dim(H) > 2$, since $H = \mathbb{C}^2$ has already been covered.

Fix unit vectors $\psi \in H$ and $\psi' \in \mathsf{W}(e_\psi)H$; clearly, ψ' is unique up to multiplication by $z \in \mathbb{T}$, whose choice turns out to completely determine u (i.e., the ambiguity in ψ' is the only one in the entire construction). For a modest start, we put

$$u\psi = \psi'. \tag{5.117}$$

Lemma 5.30. *If* $V \subset H$ *is a k-dimensional subspace (where* $k < \infty$*), then there is a unique k-dimensional linear subspace* $V' \subset H$ *with the following property:*

For all unit vectors $\psi \in H$, *we have* $\psi \in V$ *iff* $\mathsf{W}(e_\psi)H \subset V'$.

Proof. Pick a basis $(\upsilon_1, \dots, \upsilon_k)$ of V and find unit vectors $\upsilon'_i \in H$ such that $\upsilon'_i \in \mathsf{W}(e_{\upsilon_i})H$, $i = 1, \dots, k$. Then, using (5.115) we compute

$$|\langle \upsilon'_i, \upsilon'_j \rangle|^2 = \mathrm{Tr}(e_{\upsilon'_i} e_{\upsilon'_j}) = \mathrm{Tr}(\mathsf{W}(e_{\upsilon_i})\mathsf{W}(e_{\upsilon_j})) = \mathrm{Tr}(e_{\upsilon_i} e_{\upsilon_j}) = |\langle \upsilon_i, \upsilon_j \rangle|^2 = \delta_{ij},$$

so that the vectors $(\upsilon'_1, \dots, \upsilon'_k)$ form an orthonormal set and hence form a basis of their linear span V'. Now, as mentioned below (B.214), we have $\psi \in V$ iff $\sum_{i=1}^k |\langle \upsilon_i, \psi \rangle|^2 = 1$ and similarly $\psi' \in V'$ iff $\sum_{i=1}^k |\langle \upsilon'_i, \psi' \rangle|^2 = 1$. Since W preserves transition probabilities, a computation similar to one just given yields

$$\sum_{i=1}^k |\langle \upsilon_i, \psi \rangle|^2 = \sum_{i=1}^k |\langle \upsilon'_i, \psi' \rangle|^2, \tag{5.118}$$

so that both sides do or do not equal unity, and hence $\psi \in V$ iff $\psi' \in V'$. □

Wigner's Theorem for $H = \mathbb{C}^2$ (i.e. Theorem 5.7) implies:

Lemma 5.31. *If V and V' are related as in Lemma 5.30, and*

$$\dim(V) = \dim(V') = 2, \tag{5.119}$$

then there is a unitary or anti-unitary operator $u_V : V \to V'$ such that

$$\mathsf{W}(e) = u_V e u_V^*, \tag{5.120}$$

for any one-dimensional projection $e \in \mathscr{P}_1(V)$, where $\mathscr{P}_1(V) \subset \mathscr{P}_1(H)$ consists of all $e \in \mathscr{P}_1(H)$ with $eH \subset V$. Moreover, u_V is unique up to a phase.

Proof. A choice of basis for both V and V' gives unitary isomorphisms $u : V \stackrel{\cong}{\to} \mathbb{C}^2$ and $u' : V' \stackrel{\cong}{\to} \mathbb{C}^2$, which jointly induce a map

$$\mathsf{W}' \equiv u' \mathsf{W} u^{-1} : \mathscr{P}_1(\mathbb{C}^2) \to \mathscr{P}_1(\mathbb{C}^2). \tag{5.121}$$

This maps satisfies the hypotheses of Wigner's Theorem in $d = 2$, and so it is (anti-) unitarily induced as $\mathsf{W}' = \alpha_v$, where $v : \mathbb{C}^2 \to \mathbb{C}^2$ is (anti-) unitary. Then the operator $u_V = (u')^{-1} v u$ does the job; its lack of uniqueness stems entirely from v. □

Lemma 5.32. *Given a Wigner symmetry W, the ensuing operator u_V is either unitary or anti-unitary for all two-dimensional subspaces $V \subset H$ (simultaneously).*

Proof. We first design a "unitarity test" for W. Define a function

$$T : \mathscr{P}_1(H) \times \mathscr{P}_1(H) \times \mathscr{P}_1(H) \to \mathbb{C}; \tag{5.122}$$

$$T(e, f, g) = \mathrm{Tr}\,(efg), \tag{5.123}$$

$$T(e_{\psi_1}, e_{\psi_2}, e_{\psi_3}) = \langle \psi_1, \psi_2 \rangle \langle \psi_2, \psi_3 \rangle \langle \psi_3, \psi_1 \rangle. \tag{5.124}$$

Let $V \subset H$ be two-dimensional and pick an orthonormal basis (υ_1, υ_2). Define

$$\chi_1 = \upsilon_1,\;\; \chi_2 = (\upsilon_1 - \upsilon_2)/\sqrt{2},\;\; \chi_3 = (\upsilon_1 - i\upsilon_2)/\sqrt{2}. \tag{5.125}$$

A simple computation then shows that

$$T(e_{\chi_1}, e_{\chi_2}, e_{\chi_3}) = \tfrac{1}{4}(1 + i). \tag{5.126}$$

It follows from (5.124) that for u unitary and v anti-unitary, we have

$$T(e_{u\psi_1}, e_{u\psi_2}, e_{u\psi_3}) = T(e_{\psi_1}, e_{\psi_2}, e_{\psi_3}); \tag{5.127}$$

$$T(e_{v\psi_1}, e_{v\psi_2}, e_{v\psi_3}) = \overline{T(e_{\psi_1}, e_{\psi_2}, e_{\psi_3})}. \tag{5.128}$$

Eq. (5.126) implies that if $\mathsf{W} : V \to V'$ is (anti-) unitarily implemented, we have

$$T(\mathsf{W}(e_{\chi_1}), \mathsf{W}(e_{\chi_2}), \mathsf{W}(e_{\chi_3})) = T(e_{u\chi_1}, e_{u\chi_2}, e_{u\chi_3}) = \tfrac{1}{4}(1 \pm i), \tag{5.129}$$

with a plus sign if u is unitary and a minus sign if u is anti-unitary. Now take a second pair $(\tilde{V},\tilde{V}')$ as above, and pick a basis $(\tilde{\upsilon}_1,\tilde{\upsilon}_2)$ of $\tilde{V}$, with associated vectors $(\tilde{\chi}_1,\tilde{\chi}_2,\tilde{\chi}_3)$, as in (5.125). Suppose $u : V \to V'$ implementing W is unitary, whereas $\tilde{u} : \tilde{V} \to \tilde{V}'$ implementing W is anti-unitary. It then follows from (5.129) that

$$T(\mathsf{W}(e_{\chi_1}),\mathsf{W}(e_{\chi_2}),\mathsf{W}(e_{\chi_3})) = T(e_{u\chi_1},e_{u\chi_2},e_{u\chi_3}) = \tfrac{1}{4}(1+i); \tag{5.130}$$
$$T(\mathsf{W}(e_{\tilde{\chi}_1}),\mathsf{W}(e_{\tilde{\chi}_2}),\mathsf{W}(e_{\tilde{\chi}_3})) = T(e_{\tilde{u}\tilde{\chi}_1},e_{\tilde{u}\tilde{\chi}_2},e_{\tilde{u}\tilde{\chi}_3}) = \tfrac{1}{4}(1-i). \tag{5.131}$$

In view of (C.637), the following expression defies a metric d on $\mathscr{P}_1(H)$:

$$d(e_\psi,e_\varphi) = \|\omega_\psi - \omega_\varphi\| = \|e_\psi - e_\varphi\|_1 = 2\sqrt{1-|\langle\varphi,\psi\rangle|^2}, \tag{5.132}$$

with respect to which both W and T are continuous (the latter with respect to the product metric on $\mathscr{P}_1(H)^3$, of course). Let $t \mapsto (\upsilon_1(t),\upsilon_2(t))$ be a continuous path of orthonormal vectors (i.e., in $H \times H$), with associated vectors $(\chi_1(t),\chi_2(t),\chi_3(t))$, as in (5.125). Then the function $f(t) = T(\mathsf{W}(\chi_1(t)),\mathsf{W}(\chi_2(t)),\mathsf{W}(\chi_3(t)))$ is continuous, and by (5.129) it can only take the values $\frac{1}{4}(1\pm i)$. Hence $f(t)$ must be constant. However, taking a path such that $(\upsilon_1(0),\upsilon_2(0)) = (\upsilon_1,\upsilon_2)$ and $(\upsilon_1(1),\upsilon_2(1)) = (\tilde{\upsilon}_1,\tilde{\upsilon}_2)$, gives $f(0) = \frac{1}{4}(1+i)$ and $f(1) = \frac{1}{4}(1-i)$, which is a contradiction. □

Lemma 5.33. *Wigner's Theorem holds for three-dimensional Hilbert spaces.*

Proof. Let $(\upsilon_1,\upsilon_2,\upsilon_3)$ be some basis of of H (like the usual basis of $H = \mathbb{C}^3$). We first show that if W is the identity if restricted to both $\mathrm{span}(\upsilon_1,\upsilon_2)$ and $\mathrm{span}(\upsilon_1,\upsilon_3)$, then W is the identity on H altogether. To this end, take $\psi = \sum_i c_i\upsilon_i$, initially with $c_1 \in \mathbb{R}\backslash\{0\}$. Take a unit vector $\psi' \in \mathsf{W}(e_\psi)$, with $\psi = \sum_i c_i'\upsilon_i$. By the first assumption on W we have $|\langle\upsilon,\psi'\rangle| = |\langle\upsilon,\psi\rangle|$ for any unit vector $\upsilon \in \mathrm{span}(\upsilon_1,\upsilon_2)$. Taking

$$\upsilon = \upsilon_1, \qquad \upsilon = \upsilon_2, \qquad \upsilon = (\upsilon_1+\upsilon_2)/\sqrt{2}, \qquad \upsilon = (\upsilon_1+i\upsilon_2)/\sqrt{2}, \tag{5.133}$$

gives the equations

$$|c_1'| = |c_1|, \quad |c_2'| = |c_2|, \quad |c_1'+c_2'| = |c_1+c_2|, \quad |c_1'-ic_2'| = |c_1-ic_2|, \tag{5.134}$$

respectively. By a choice of phase we may and will assume $c_1' = c_1$, in which case the only solution is $c_2 = c_2'$ (geometrically, the solution c_2' lies in the intersection of three different circles in the complex plane, which is either empty or consists of a single point). Similarly, the second assumption on W gives $c_3 = c_3'$, whence $\psi' = \psi$. The case $c_1 = 0$ may be settled by a straightforward limit argument, since inner products (and hence their absolute values) are continuous on $H \times H$.

Given a Wigner symmetry $\mathsf{W} : \mathscr{P}_1(H) \to \mathscr{P}_1(H)$, we now construct u as follows.

1. Fix a basis $(\upsilon_1,\upsilon_2,\upsilon_3)$ with "image" $(\upsilon_1',\upsilon_2',\upsilon_3')$ under W, i.e, $\mathsf{W}(e_{\upsilon_i}) = e_{\upsilon_i'}$.
2. The unitarity test in the proof of Lemma 5.32 settles if the operators should be chosen to be unitary or anti-unitary; for simplicity we assume the unitary case.
3. Define a unitary $u_1 : H \to H$ by $u_1\upsilon_i' = \upsilon_i$ for $i = 1,2,3$, and subsequently define $\mathsf{W}_1 = \alpha_{u_1} \circ \mathsf{W}$, which (being the composition of two Wigner symmetries)

is a Wigner symmetry. Clearly, $\mathsf{W}_1(e_{\upsilon_i}) = e_{\upsilon_i}$ $(i = 1,2,3)$, so that W_1 maps $\mathscr{P}_1(H_{(12)})$ to itself, where $H_{(12)} \equiv \mathrm{span}(\upsilon_1, \upsilon_2)$. Hence Lemma 5.31 gives a unitary map $\tilde{u}_1 : H_{(12)} \to H_{(12)}$ such that the restriction of W_1 to $H_{(12)}$ is $\alpha_{\tilde{u}_1}$.

4. Define a unitary $u_2 : H \to H$ by $u_2 = \tilde{u}_1^{-1}$ on $H_{(12)}$ and $u_2\upsilon_3 = \upsilon_3$, followed by the Wigner symmetry $\mathsf{W}_2 = \alpha_{u_2} \circ \mathsf{W}_1$. By construction, $\mathsf{W}_2(e_{\upsilon_i}) = e_{\upsilon_i}$ for $i = 1,2,3$) (W_2 is even the identity on $\mathscr{P}_1(H_{(12)})$), so that W_2 maps $\mathscr{P}_1(H_{(13)})$ to itself, where $H_{(13)} \equiv \mathrm{span}(\upsilon_1, \upsilon_3)$. Hence the restriction of W_2 to $H_{(13)}$ is implemented by a unitary $\tilde{u}_2 : H_{(13)} \to H_{(13)}$, whose phase may be fixed by requiring $\tilde{u}_2\upsilon_1 = \upsilon_1$.
5. Similarly to u_2, we define $u_3 : H \to H$ by $u_3 = \tilde{u}_2^{-1}$ on $H_{(13)}$ and $u_3\upsilon_2 = \upsilon_2$, so that u_3 is the identity on $H_{(12)}$. Of course, we now define a Wigner symmetry
$$\mathsf{W}_3 = \alpha_{u_3} \circ \mathsf{W}_2 = \alpha_{u_3} \circ \alpha_{u_2} \circ \alpha_{u_1} \circ \mathsf{W}, \tag{5.135}$$
which by construction is the identity on both $\mathscr{P}_1(H_{(12)})$ and $\mathscr{P}_1(H_{(13)})$, and so by the first part of the proof it must be the identity on all of $\mathscr{P}_1(H)$. Hence
$$\mathsf{W} = \alpha_{u_1^{-1}} \circ \alpha_{u_2^{-1}} \circ \alpha_{u_3^{-1}} = \alpha_u \qquad (u = u_1^{-1}u_2^{-1}u_3^{-1}).$$
□

Lemma 5.34. *As in Lemma 5.30, if* $\dim(V) = \dim(V') = 3$*, then there is a unitary or anti-unitary operator* $u_V : V \to V'$ *such that* $\mathsf{W}(e) = u_V e u_V^*$ *for any* $e \in \mathscr{P}_1(V)$*,*

Proof. Given Lemma 5.33, the proof is practically the same as for Lemma 5.31. □

We now finish the proof of Wigner's Theorem. We assume that the outcome of Lemma 5.32 is that each u_V is unitary; the anti-unitary case requires obvious modifications of the argument below. The first step is, of course, to define $u(\lambda\psi) = \lambda u\psi$, $\lambda \in \mathbb{C}$ (so this would have been $\overline{\lambda}u\psi$ in the anti-unitary case). Let $\varphi \in H$ be linearly independent of ψ and consider the two-dimensional space V spanned by ψ and φ. Define $u(\varphi) = u_V\varphi$. With (5.117), this defines u on all of H. To prove that u is linear, take φ_1 and φ_2 linearly independent of each other and of ψ, so that the linear span V_3 of ψ, φ_1, and φ_2 is three-dimensional. Let V_i be the two-dimensional linear span of ψ and φ_i, $i = 1,2$. Then $u\varphi_i = u_{V_i}\varphi_i$, where the phase of u_{V_i} is fixed by (5.117). Let $w : V_3 \to V_3'$ be the unitary that implements W according to Lemma 5.33.2, with phase determined by (5.117). Since u_{V_1} and u_{V_2} and w are unique up to a phase and this phase has been fixed for each in the same way, we must have $u_{V_1} = w_{|V_1}$ and $u_{V_2} = w_{|V_2}$. Finally, we have V_{12} spanned by ψ and $\varphi_1 + \varphi_2$, and by the same token, $u_{V_{12}} = w_{|V_{12}}$. Now w is unitary and hence linear, so
$$\begin{aligned} u(\varphi_1 + \varphi_2) &= u_{V_{12}}(\varphi_1 + \varphi_2) = w(\varphi_1 + \varphi_2) = w(\varphi_1) + w(\varphi_2) \\ &= u_{V_1}(\varphi_1) + u_{V_2}(\varphi_2) = u(\varphi_1) + u(\varphi_2), \end{aligned}$$
since this is how u was defined. Since each u_V is unitary, so is u, and similarly it is easy to verify that u implements W, because each u_V does so. □

5.6 Some abstract representation theory

Since all symmetries we have considered (named after Wigner, Kadison, Jordan, Ludwig, von Neumann, and Bohr) are implemented by either unitary or anti-unitary operators, which are determined (by the given symmetry) only up to a phase $z \in \mathbb{T}$, the quantum-mechanical symmetry group $\mathscr{G}^H$ of a Hilbert space H is given by

$$\mathscr{G}^H = (U(H) \cup U_a(H))/\mathbb{T}, \tag{5.136}$$

where $U(H)$ is the group of unitary operators on H, and $U_a(H)$ is the set of anti-unitary operators on H; the latter is not a group (since the product of two anti-unitaries is unitary) but their union is. Furthermore, $\mathbb{T}$ is identified with the normal subgroup $\mathbb{T} \equiv \mathbb{T} \cdot 1_H = \{z \cdot 1_H \mid z \in \mathbb{T}\}$ of $U(H) \cup U_a(H)$ (and also of $U(H)$) consisting of multiples of the unit operators by a phase; thus the quotient $\mathscr{G}^H$ is a group.

The fact that $\mathscr{G}^H$ rather than $U(H)$ is the symmetry group of quantum mechanics has profound consequences (one of which is our very existence), which we will study from §5.10 onwards. However, this material relies on the theory of "ordinary" (i.e., non-projective) unitary representations, which we therefore review first.

Namely, let G be a group. In mathematics, the natural kind of action of G on a Hilbert space H is a ***unitary representation***, i.e., a homomorphism

$$u : G \to U(H), \tag{5.137}$$

so that $u(x)^{-1} = u(x^{-1}) = u(x)^*$ and $u(x)u(y) = u(xy)$, which imply $u(e) = 1_H$.

As to the possible continuity properties of unitary representations in case that G is a *topological* group (i.e., a group G that is also a topological space, such that group multiplication $G \times G \mapsto G$ and inverse $G \to G$ are continuous), one should equip $U(H)$ with the *strong* operator topology (as opposed to the norm topology).

Proposition 5.35. *If $u : x \mapsto u(x)$ is a unitary representation of some locally compact group G on a Hilbert space H, then the following conditions are equivalent:*

1. *The map $G \times H \to H$, $(x, \psi) \mapsto u(x)\psi$, is continuous;*
2. *The map $G \to U(H)$, $x \mapsto u(x)$, is continuous in the strong topology on $U(H)$.*

Proof. Strong continuity means that if $x_\lambda \to x$ in G, then for each $\psi \in H$ we have $\|(u(x_\lambda) - u(x))\psi\| \to 0$. This is clearly implied by the first kind of continuity, giving $1 \Rightarrow 2$, so let us prove the nontrivial converse. Suppose $x_\lambda \to x$ and $\psi_\mu \to \psi$; since G is locally compact, x has a compact neighborhood K and we may assume that each $x_\lambda \in K$. If u is strongly continuous, then for any $\varphi \in H$ the set $\{u(y)\varphi, y \in K\}$ is compact in H and hence bounded. The Banach–Steinhaus Theorem B.78 gives boundedness of the corresponding operator norms, that is, $\{\|u(y)\|, y \in K\} < C_K$ for some $C_K > 0$. We now estimate

$$\|u(x_\lambda)\psi_\mu - u(x)\psi\| \leq \|u(x_\lambda)\psi_\mu - u(x_\lambda)\psi\| + \|(u(x_\lambda) - u(x))\psi\|.$$

The first term vanishes as $\psi_\mu \to \psi$ since it is bounded by $C_K\|\psi_\mu - \psi\|$, whereas the second vanishes as $x_\lambda \to x$ by the (assumed) strong continuity of u. □

Since the first kind of continuity is the usual one for group actions, this justifies the choice of strong continuity as the natural one for unitary representations (to which a pragmatic point may be added: norm continuity is quite rare for unitary representations on infinite-dimensional Hilbert spaces). Things further simplify under mild restrictions on G and H, which are satisfied in all examples of physical interest.

Proposition 5.36. *If H is separable and G is second countable locally compact (sclc), then each of the two continuity conditions in Proposition 5.35 is in turn equivalent to* **weak measurability** *of u, in that for each $\varphi, \psi \in H$ the function*

$$x \mapsto \langle \varphi, u(x)\psi \rangle$$

from G to $\mathbb{C}$ is (Borel) measurable.

Proof. This spectacular result is due to von Neumann, who more generally proved that a measurable homomorphism between sclc groups is continuous. This implies the claim: first, if H is separable, then the group $U(H)$ is sclc in its weak operator topology, so that if the map $G \to U(H)$, $x \mapsto u(x)$ is weakly measurable, then it is continuous in the weak topology on $U(H)$. Second, for any Hilbert space, weak (operator) continuity of a unitary representation implies strong continuity (so that, given the trivial converse, weak and strong continuity of unitary group representations are equivalent). We only prove this last claim: for $x, y \in G$, we compute

$$\begin{aligned} \|(u(y) - u(x))\psi\| &= \|u(x)\psi\|^2 + \|u(y)\psi\|^2 - \langle u(x)\psi, u(y)\psi \rangle - \langle u(y)\psi, u(x)\psi \rangle \\ &= 2\|\psi\|^2 - \langle \psi, u(x^{-1}y)\psi \rangle - \langle \psi, u(y^{-1}x)\psi \rangle, \end{aligned}$$

Weak continuity obviously implies that the function $x \mapsto \langle \psi, u(x)\psi \rangle$ is continuous at the identity $e \in G$, so if $y = x_\lambda \to x$, then $\|(u(x_\lambda) - u(x))\psi\| \to 0$. □

In view of this, it is hardly a restriction for a unitary representation of a locally compact group on a Hilbert space to be continuous in the sense of Proposition 5.35, so we always assume this in what follows. Furthermore, any group we consider is locally compact, so this will be a standing assumption, too. An important consequence of this assumption is the existence of a translation-invariant measure on G.

Theorem 5.37. *Each locally compact group G has a canonical nonzero (outer regular Borel) measure μ, called* **Haar measure***, which is left-invariant in that*

$$\int_G d\mu(x)\, L_y f(x) = \int_G d\mu(x)\, f(x), \tag{5.138}$$

for each $f \in C_c(G)$ and $y \in G$, where the **left translation** *L_y of f by y is defined by*

$$L_y f(x) = f(y^{-1}x). \tag{5.139}$$

This measure is unique up to scalar multiplication. Moreover, if G is compact, then:

1. *μ is finite and hence can be normalized to a probability measure, i.e.,*

$$\mu(G) = 1. \tag{5.140}$$

2. *μ is also right-invariant in that*

$$\int_G d\mu(x)\, R_y f(x) = \int_G d\mu(x)\, f(x), \tag{5.141}$$

where the **right translation** *R_y of f by $y \in G$ is defined by*

$$R_y f(x) = f(xy). \tag{5.142}$$

3. *μ is invariant under inversion, in that*

$$\int_G d\mu(x)\, f(x^{-1}) = \int_G d\mu(x)\, f(x). \tag{5.143}$$

Existence is due to Haar and uniqueness was first proved by von Neumann. One often writes $dx \equiv d\mu(x)$ for Haar measure. Here are some examples:

- For $G = \mathbb{R}^n$, Haar measure equals Lebesgue measure μ_L (up to a constant); eqs. (5.139) and (5.141) state the familiar translation invariance of μ_L.
- For $G = \mathbb{T}$, we have

$$\int_{\mathbb{T}} d\mu(z) f(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, f(e^{i\theta}). \tag{5.144}$$

- For $G = GL_n(\mathbb{R})$ with $X = (x_{ij})$, we have

$$d\mu(X) = \prod_{i,j=1}^{m} dx_{ij} |\det(X)|^{-n}, \tag{5.145}$$

  which for $G = SL_n(\mathbb{R})$ of course simplifies to $d\mu(X) = \prod_{i,j} dx_{ij}$.

Definition 5.38. *A unitary representation u of a group G on a Hilbert space H is* **irreducible** *if the only closed subspaces K of H that are stable under $u(G)$ (in the sense that if $\psi \in K$, then $u(x)\psi \in K$ for all $x \in G$) are either $K = H$ or $K = \{0\}$.*

We will often need two important results about irreducibility. The first is ***Schur's Lemma***, in which the commutant S' of some subset $S \subset B(H)$ is defined by

$$S' = \{a \in B(H) \mid ab = ba \forall b' \in S\}. \tag{5.146}$$

Lemma 5.39. *A unitary representation u of a group G is irreducible iff*

$$u(G)' = \mathbb{C} \cdot 1, \tag{5.147}$$

i.e., if $au(x) = u(x)a$ for each $x \in G$ implies $a = \lambda \cdot 1_H$ for some $\lambda \in \mathbb{C}$.

This follows from Theorem C.90, of which the above lemma is a special case: take $A = u(G)'' \equiv (u(G)')'$. The second is part of the ***Peter–Weyl Theorem***.

Theorem 5.40. *Irreducible representations of compact groups are finite-dimensional.*

Proof. We first reduce the situation to the unitary case: if $\langle\cdot,\cdot,\rangle'$ is the given inner product on H, we define a new inner product $\langle\cdot,\cdot,\rangle$ by averaging with respect to Haar measure $dx \equiv d\mu(x)$, i.e.,

$$\langle\psi,\varphi\rangle = \int_G dx\, \langle u(x)\psi, u(x)\varphi\rangle. \tag{5.148}$$

Using (5.141), it is easy to verify that this new inner product makes u unitary.

So let $u : G \to u(H)$ be an irreducible unitary representation. For each unit vector $\varphi \in H$ and $x \in G$, we define the following projection and its G-average:

$$e_{u(x)\varphi} = |u(x)\varphi\rangle\langle u(x)\varphi|, \tag{5.149}$$

$$W_\varphi = \int_G dx\, e_{u(x)\varphi}. \tag{5.150}$$

The ***Weyl operator*** (5.150) is initially defined as a quadratic form by

$$\langle\psi_1, W_\varphi\psi_2\rangle = \int_G dx\, \langle\psi_1, e_{u(x)\varphi}\psi_2\rangle. \tag{5.151}$$

The integral exists because the integrand is continuous and bounded, defining a *bounded* quadratic form by the estimate $|\langle\psi_1, W_\varphi\psi_2\rangle| \leq \|\psi_1\|\|\psi_2\|$, where we assumed (5.140) and used $\|e_{u(x)\varphi}\| = 1$, as (5.149) is a nonzero projection. Thus the operator W_φ may be reconstructed from its matrix elements (5.151), cf. Proposition B.79. It is easy to verify that $[W_\varphi, u(y)] = 0$ for each $y \in G$, so that Schur's Lemma yields $W_\varphi = \lambda_\varphi \cdot 1_H$ for some $\lambda_\varphi \in \mathbb{C}$. Hence $\langle\psi, W_\varphi\psi\rangle = \lambda_\varphi\|\psi\|^2$, in other words,

$$\int_G dx\, |\langle\psi, u(x)\varphi\rangle|^2 = \lambda_\varphi\|\psi\|^2. \tag{5.152}$$

If we now interchange φ and ψ and use (5.143) we find $\lambda_\varphi\|\psi\|^2 = \lambda_\psi\|\varphi\|^2$, so that, taking ψ to be a unit vector, too, since ψ and φ are arbitrary we obtain $\lambda_\varphi = \lambda_\psi \equiv \lambda$, where in fact $\lambda > 0$, as follows by taking $\psi = \varphi$ in (5.152). Finally, take n orthonormal vectors $(\upsilon_1, \ldots, \upsilon_n)$ in H, so that also $(u(x)\upsilon_1, \ldots, u(x)\upsilon_n)$ are orthonormal (since $u(x)$ is unitary), upon which Bessel's inequality (B.212) gives

$$\sum_{i=1}^{n} |\langle\psi, u(x)\upsilon_i\rangle|^2 \leq \|\psi\|^2. \tag{5.153}$$

Integrating both sides over G, taking $\|\psi\| = 1$, and using (5.140) gives

$$\sum_{i=1}^{n} \int_G dx\, |\langle\psi, u(x)\upsilon_i\rangle|^2 \leq 1. \tag{5.154}$$

On the other hand, summing (5.152) over i simply yields $n\lambda$, whence $n\lambda \leq 1$, for any $n \leq \dim(H)$. Since $\lambda > 0$ this forces $\dim(H) < \infty$. □

5.7 Representations of Lie groups and Lie algebras

We now assume that G is a ***Lie group***; as in §3.3, for our purposes we may restrict ourselves to *linear* Lie groups, i.e. closed subgroups of $GL_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $u : G \to U(H)$ be a unitary representation of a Lie group G on some Hilbert space H (assumed strongly continuous). If H is finite-dimensional, the following operation is unproblematic: for $A \in \mathfrak{g}$ (i.e. the Lie algebra of G) we define an operator

$$u'(A) \ : \ H \to H; \tag{5.155}$$

$$u'(A) = \frac{d}{dt} u\left(e^{tA}\right)_{|t=0}. \tag{5.156}$$

This gives a linear map $u' : \mathfrak{g} \to B(H)$, which satisfies

$$[u'(A), u'(B)] = u'([A,B]); \tag{5.157}$$

$$u'(A)^* = -u'(A). \tag{5.158}$$

Note that physicists use Planck's constant $\hbar > 0$ and like to write

$$\pi(A) = i\hbar u'(A), \tag{5.159}$$

so that one has the following commutation relations and self-adjointness condition:

$$[\pi(A), \pi(B)] = i\hbar\pi([A,B]); \tag{5.160}$$

$$\pi(A)^* = \pi(A). \tag{5.161}$$

If one knows that $u' : \mathfrak{g} \to B(H)$ comes from $u : G \to U(H)$, one conversely has

$$u(e^A) = e^{u'(A)} = e^{-\frac{i}{\hbar}\pi(A)}. \tag{5.162}$$

More generally, we call a map $\rho : \mathfrak{g} \to B(H)$ (where $H \cong \mathbb{C}^n$ remains finite-dimensional, so that $\rho : \mathfrak{g} \to M_n(\mathbb{C})$), a ***skew-adjoint*** representation of $\mathfrak{g}$ on H if

$$[\rho(A), \rho(B)] = \rho([A,B]); \tag{5.163}$$

$$\rho(A)^* = -\rho(A). \tag{5.164}$$

The property of irreducibility of such a representation $\rho : \mathfrak{g} \to B(H)$ is defined in the same way as for groups, namely that the only linear subspaces of $H \cong \mathbb{C}^n$ that are stable under $\rho(\mathfrak{g})$ are $\{0\}$ and H. Equivalently, by Schur's Lemma, $\rho(\mathfrak{g})$ is irreducible iff the only operators that commute with all $\pi(A)$ are multiples of the unit operator. If $\rho = u'$ for some unitary representation $u(G)$, it is easy to see that u is irreducible iff u' is irreducible. In view of this, it is a reasonable strategy to try and construct irreducible unitary representations $u(G)$ by starting, as it were, from $u'(\mathfrak{g})$. More precisely, if ρ is some (irreducible) skew-adjoint representation of $\mathfrak{g}$, we may ask if there is a (necessarily irreducible) unitary representation $u(G)$ such that $\rho = u'$. Writing $\exp(\rho)$ for u, one would therefore hope that

$$u\left(e^{A}\right) \equiv e^{\rho}\left(e^{A}\right) = e^{\rho(A)}, \tag{5.165}$$

as in (5.162). Note that if G is connected, then ρ duly defines $u(x)$ for each $x \in G$ through (5.165), since by Lie theory every element x of a connected Lie group is a finite product $x = \exp(A_1)\cdots\exp(A_n)$ of exponentials of elements $(A_1,\ldots,A_n)$ of $\mathfrak{g}$.

In general, this hope is in vain, since although each operator $\exp(A)$ is unitary, the representation property $u(x)u(y) = u(xy)$ may fail for global reasons. For example, if $G = SO(3)$, then $\mathfrak{g} \cong \mathbb{R}^3$, with basis (J_1, J_2, J_3), as in (3.66). Define an *a priori* linear map $\rho : \mathfrak{g} \to M_2(\mathbb{C})$ by linear extension of

$$\rho(J_k) = -\tfrac{1}{2} i\sigma_k, \tag{5.166}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices (5.42), so that physicists would write

$$\pi(J_k) = \tfrac{1}{2}\hbar\sigma_k, \tag{5.167}$$

cf. (5.159). This is easily checked to give a skew-adjoint representation of $\mathfrak{g}$, but it does not exponentiate to a unitary representation of $SO(3)$: as already mentioned after Proposition 5.46, if $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$, then a rotation $R_\theta(\mathbf{u})$ around the $\mathbf{u}$-axis by an angle $\theta \in [0, 2\pi]$ is represented by

$$u(R_\theta(\mathbf{u})) = \cos(\theta/2) \cdot 1_2 + i\sin(\theta/2)\mathbf{u} \cdot \sigma. \tag{5.168}$$

Consequently, $u(R_\pi(\mathbf{u})) = i\mathbf{u} \cdot \sigma$, so that $u(R_\pi(\mathbf{u}))^2 = -1_2$, although within $SO(3)$ one has $R_\pi(\mathbf{u})^2 = e$, the unit of $SO(3)$, so that $u(R_\pi(\mathbf{u}))^2 \neq u(R_\pi(\mathbf{u})^2)$.

However, ρ does exponentiate to a representation of $SU(2)$, which happens to be the universal covering group of $SO(3)$. This is typical of the general situation, which we state without proofs. We first need a refinement of ***Lie's Third Theorem***:

Theorem 5.41. *Let G be a connected Lie group G with Lie algebra $\mathfrak{g}$. There exists a simply connected Lie group $\tilde{G}$, unique up to isomorphism, such that:*

- *The Lie algebra of $\tilde{G}$ is $\mathfrak{g}$.*
- *$G \cong \tilde{G}/D$, where D is a discrete normal subgroup of the center of $\tilde{G}$.*
- *$D \cong \pi_1(G)$, i.e., the fundamental group of G, which is therefore abelian.*

For example, for $G = SO(3)$ we have $\tilde{G} = SU(2)$ and $D = \mathbb{Z}_2$, cf. Proposition 5.46.

Theorem 5.42. *Let G_1 and G_2 be Lie groups, with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, and suppose that G_1 is simply connected. Then every Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ comes from a unique Lie group homomorphism $\Phi : G_1 \to G_2$ through $\varphi = \Phi'$, where (realizing G_1 and G_2 as matrices)*

$$\Phi'(X) = \frac{d}{dt}\Phi\left(e^{tX}\right)_{|t=0}. \tag{5.169}$$

Let H be a finite-dimensional Hilbert space, so that $B(H) \cong M_n(\mathbb{C})$, where $n = \dim(H)$, and take $U(H) \cong U_n(\mathbb{C})$ to be the group of all unitary matrices on $\mathbb{C}^n$. The Lie algebra $\mathfrak{u}_n(\mathbb{C})$ of $U_n(\mathbb{C})$ consists of all skew-adjoint $n \times n$ complex matrices. Since irreducibility is preserved under the correspondence $u(G) \leftrightarrow u'(\mathfrak{g})$, we infer:

Corollary 5.43. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Any finite-dimensional skew-adjoint representation $\pi : \mathfrak{g} \to \mathfrak{u}_n(\mathbb{C})$ of $\mathfrak{g}$ comes from a unique unitary representation $u(G)$ through* (5.156)*, in which case we have*

$$e^{u'(A)} = u\left(e^A\right) \quad (A \in \mathfrak{g}). \tag{5.170}$$

Thus there is a bijective correspondence between finite-dimensional unitary representations of G and finite-dimensional skew-adjoint representations of $\mathfrak{g}$. In particular, if G is compact, this specializes to a bijective correspondence between unitary irreducible representations of G and skew-adjoint irreducible representations of $\mathfrak{g}$.

If $G \cong \tilde{G}/D$ is connected but not simply connected, then a finite-dimensional skew-adjoint representation $\rho : \mathfrak{g} \to B(H)$ exponentiates to a unitary representation $u : G \to U(H)$ iff the representation $\exp(\rho) : \tilde{G} \to U(H)$ *is trivial on D.*

For example, $G = SO(3)$, the last condition is satisfied for the irreducible representations with integer spins $j \in \mathbb{N}$ (as well as for $j = 0$), see §5.8.

A similar construction is possible when H is infinite-dimensional, except for the fact that the derivative in (5.156) may not exist. For example, $G = \mathbb{R}$ has its canonical regular representation on $H = L^2(\mathbb{R})$, defined by $u(a)\psi(x) = \psi(x-a)$, in which case (5.159) gives some multiple of the momentum operator $-i\hbar d/dx$. This operator is unbounded and hence is not defined on all of H, see also §5.11 and §5.12. As in Stone's Theorem 5.73, this problem is solved by finding a suitable domain in H on which the underlying limit, taken strongly, does exist. This is the ***Gårding domain***

$$D_G = \left\{u^{\int}(f)\psi, f \in C_c^{\infty}(G), \psi \in H\right\}, \tag{5.171}$$

where for each $f \in C_c^{\infty}(G)$ (or even $f \in L^1(G)$) the operator $u^{\int}(f)$ is defined by

$$u^{\int}(f) = \int_G dx\, f(x)u(x). \tag{5.172}$$

Like the derivative u', this integral is most easily defined weakly, i.e., the (bounded) operator $u^{\int}(f)$ is initially defined as a bounded quadratic form

$$Q(\varphi, \psi) = \int_G dx\, f(x)\langle\varphi, u(x)\psi\rangle, \tag{5.173}$$

from which the operator $u^{\int}(f)$ may be reconstructed as in Proposition B.79. Note that the function $x \mapsto \langle\varphi, u(x)\psi\rangle$ is in $C_b(G)$, so that the integral (5.173) exists.

It can be shown that D_G is dense in H, as well as ***invariant*** under $u'(\mathfrak{g})$, in the sense that if $\psi \in D_G$, then $u'(A)\psi \in D_G$ for any $A \in \mathfrak{g}$. Furthermore, for each $\varphi \in D_G$ the function $x \mapsto u(x)\varphi$ from G to H is smooth (if G is unimodular this property even characterizes D_G). The commutation relations (5.157) then hold on D_G, but the equalities (5.164) do not: one has to choose between (5.157) and (5.164), since the latter holds for the closure of each $\pi(A)$ (i.e., each $i\rho(A)$ is essentially self-adjoint on D_G), whose domain however depends on A: there is no common domain on which each $i\rho(A)$ is self-adjoint *and* the commutation relations (5.157) hold.

5.8 Irreducible representations of $SU(2)$

One of the most important groups in quantum physics is $SU(2)$, both as an internal symmetry group—e.g. of the Heisenberg model of ferromagnetism, of the weak nuclear interaction, and possibly also of (loop) quantum gravity—and as a spatial symmetry group in disguise (all projective unitary representations of $SO(3)$ come from unitary representations of $SU(2)$, preserving irreducibility, cf. Corollary 5.61). In this section we review the well-known classification and construction of its unitary irreducible representations. Since $SU(2)$ is compact, by Theorem 5.40 all its unitary irreducible representations are finite-dimensional. Since $G = SU(2)$ is also simply connected, by Corollary 5.43 its irreducible finite-dimensional (unitary) representations u bijectively correspond to the irreducible finite-dimensional skew-adjoint representations $\rho = u'$ of its Lie algebra $\mathfrak{g}$. Hence our job is to find the latter.

We already encountered the basis (3.66) of the Lie algebra $\mathfrak{so}(3) \cong \mathbb{R}^3$ of $SO(3)$; the corresponding basis of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is (S_1, S_2, S_3), where

$$S_k = -\tfrac{1}{2} i \sigma_k, \tag{5.174}$$

and the σ_k are the Pauli matrices given in (5.42); linear extension of the map $J_k \mapsto S_k$ defines an isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$. These matrices satisfy

$$[S_i, S_j] = \varepsilon_{ijk} S_k, \tag{5.175}$$

where ε_{ijk} is the totally anti-symmetric symbol with $\varepsilon_{123} = 1$ etc., so that (5.175) comes down to $[S_1, S_2] = S_3$, $[S_3, S_1] = S_2$, and $[S_2, S_3] = S_1$. By linearity, finding ρ is the same as finding $n \times n$ matrices

$$L_k = i\rho(S_k) \tag{5.176}$$

that satisfy

$$[L_i, L_j] = i\varepsilon_{ijk} L_k, \tag{5.177}$$

i.e., $[L_1, L_2] = iL_3$, etc., and

$$L_k^* = L_k. \tag{5.178}$$

It turns out to be convenient to introduce the ***ladder operators***

$$L_\pm = L_1 \pm iL_2, \tag{5.179}$$

with ensuing commutation relations

$$[L_3, L_\pm] = \pm L_\pm; \tag{5.180}$$

$$[L_+, L_-] = 2L_3. \tag{5.181}$$

Furthermore, we define the ***Casimir operator***

$$C = L_1^2 + L_2^2 + L_3^2, \tag{5.182}$$

which, crucially, commutes with each L_k, i.e.,

$$[C, L_k] = 0 \ (k = 1, 2, 3). \tag{5.183}$$

By Schur's lemma, in any irreducible representation we therefore must have

$$C = c \cdot 1_H, \tag{5.184}$$

where $c \in \mathbb{R}$ (in fact, $c \geq 0$). We will also use the additional algebraic relations

$$L_+L_- = C - L_3(L_3 - 1_H); \tag{5.185}$$

$$L_-L_+ = C - L_3(L_3 + 1_H). \tag{5.186}$$

The simple idea is now to diagonalize L_3, which is possible as $L_3^* = L_3$. Hence

$$H = \bigoplus_{\lambda \in \sigma(L_3)} H_\lambda, \tag{5.187}$$

where $\sigma(L_3)$ is the spectrum of L_3 (which in this finite-dimensional case consists of its eigenvalues), and H_λ is the eigenspace of L_3 for eigenvalue λ (i.e., if $\upsilon \in H_\lambda$, then $L_3\upsilon = \lambda\upsilon$). The structure of (5.187) in irreducible representations is as follows.

Lemma 5.44. *Let $\rho : \mathfrak{su}(2) \to B(H)$ be a finite-dimensional skew-adjoint irreducible representation, so that* (5.177) *holds. Then the spectrum $\sigma(L_3)$ of the self-adjoint operator $L_3 = i\rho(S_3)$ is given by*

$$\sigma(L_3) = \{-j, -j+1, \cdots, j-1, j\}. \tag{5.188}$$

If (5.187) *is the spectral decomposition of H relative to L_3, then:*

1. *The subspace H_λ is one-dimensional for each $\lambda \in \sigma(L_3)$;*
2. *For $\lambda < j$ the operator L_+ maps H_λ to $H_{\lambda+1}$, whereas $L_+ = 0$ on H_j;*
3. *For $\lambda > -j$ the operator L_- maps H_λ to $H_{\lambda-1}$, whereas $L_- = 0$ on H_{-j}.*

Proof. For any $\lambda \in \sigma(L_3)$ and nonzero $\upsilon_\lambda \in H_\lambda$, we have:

- either $\lambda + 1 \in \sigma(L_3)$ and $L_+\upsilon_\lambda \in H_{\lambda+1}$ (as a nonzero vector);
- or $L_+\upsilon_\lambda = 0$.

Indeed, (5.180) gives $L_3(L_+\upsilon_\lambda) = (\lambda + 1)L_+\upsilon_\lambda$, which immediately yields the claim. Similarly, either $\lambda - 1 \in \sigma(L_3)$ and $L_-\upsilon_\lambda \in H_{\lambda-1}$, or $L_-\upsilon_\lambda = 0$. Now let $\lambda_0 = \min \sigma(L_3)$ be the smallest eigenvalue of L_3, and pick some $0 \neq \upsilon_{\lambda_0} \in H_{\lambda_0}$. Since H is finite-dimensional by assumption, there must be some $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $L_+^{k+1}\upsilon_{\lambda_0} = 0$, whereas all vectors $L_+^l\upsilon_{\lambda_0}$ for $l = 0, \ldots, k$ are nonzero (and lie in H_{λ_0+l}). With c defined as in (5.184), it then follows from (5.185) - (5.186) that

$$c - \lambda_0(\lambda_0 - 1) = 0; \tag{5.189}$$

$$c - (\lambda_0 + k)(\lambda_0 + k + 1) = 0. \tag{5.190}$$

These relations imply $\lambda_0 = -k/2$, so that by the above bullet points we also have

$$\{-k/2, -k/2+1, \ldots, k/2-1, k/2\} \subseteq \sigma(L_3). \tag{5.191}$$

To prove equality, as in (5.188), consider the vector space

$$H' = \mathbb{C} \cdot \upsilon_{\lambda_0} \oplus \mathbb{C} \cdot L_+ \upsilon_{\lambda_0} \oplus \cdots \oplus L_+^{k-1} \upsilon_{\lambda_0} \oplus L_+^k \upsilon_{\lambda_0} \subseteq H; \tag{5.192}$$

this is just the subspace of H with basis $(\upsilon_{\lambda_0}, L_+ \upsilon_{\lambda_0}, \ldots, L_+^{k-1} \upsilon_{\lambda_0}, L_+^k \upsilon_{\lambda_0})$. By the previous arguments following from (5.180), we see that the operators L_+ and L_- never leave H', and the same is trivially true for L_3. Therefore, if ρ is irreducible, then we must have $H' = H$ (and conversely). All claims of the lemma are now trivially verified on H'. □

It should be clear from this proof that the actions of L_+, L_-, and L_3 (and hence of all elements of $\mathfrak{su}(2)$) on $H' = H$) are fixed, so that ρ is determined by its dimension

$$\dim(H) = 2j+1, \tag{5.193}$$

from which it follows that j can only take the values $0, 1/2, 1, 3/2, \ldots$.

It remains to fix an inner product on H' in which ρ is skew-adjoint, i.e., in which $L_3^* = L_3$ and $L_+^* = L_-$ (which implies that $L_1^* = L_1$ and $L_2^* = L_2$, which jointly imply $\rho(X^*) = -\rho(X)$ for any $X \in \mathfrak{g}$). This may be done in principle by starting with any inner product, integrating ρ to a unitary representation of $SU(2)$, and using the construction explained at the beginning of the proof of Theorem 5.40. In practice, it is easier to just calculate: take $H = \mathbb{C}^n$ with $n = 2j+1$, standard inner product, and standard orthonormal basis (u_l), labeled as $l = 0, 1, \ldots, 2j)$. Then put

$$L_3 u_l = (l-j) u_l; \tag{5.194}$$
$$L_+ u_l = \sqrt{(l+1)(n-l-1)}\, u_{l+1}; \tag{5.195}$$
$$L_- u_l = \sqrt{l(n-l)}\, u_{l-1}. \tag{5.196}$$

Note that (5.195) is even formally correct for $l = 2j$, since in that case $n - 2j - 1 = 0$, and similarly, (5.196) formally holds even for $l = 0$. The commutation relations (5.180) - (5.181) as well as the above conditions for skew-adjointness may be explicitly verified, from which it follows that for any prescribed dimension (5.193) we have found a skew-adjoint realization of ρ. Clearly, $u_l = \upsilon_{l-j}$.

In view of Theorem 5.40 and Corollary 5.43 we have therefore proved:

Theorem 5.45. *Up to unitary equivalence, any (unitary) irreducible representation of $SU(2)$ is completely determined by its dimension $n = \dim(H)$, and any dimension $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ occurs. Furthermore, if j is the number in* (5.188)*, we have*

$$n = 2j+1. \tag{5.197}$$

Physicists typically label these irreducible representations by j (called the ***spin*** of the given representation) rather than by n, or even by $c = j(j+1)$, cf. (5.184).

Corollary 5.43 shows that one may pass from $\rho(\mathfrak{su}(2))$ to a unitary representation $u(SU(2))$, of which one may give a direct realization. For $j \in \mathbb{N}_0/2$, define H_j as the complex vector space of all homogeneous polynomials p in two variables $z = (z_1, z_2)$ of degree $2j$. A basis of H_j is given by $(z_1^{2j}, z_1^{2j-1}z_2, \dots, z_1 z_2^{2j-1}, z_2^{2j})$, which has $2j+1$ elements. So $\dim(H_j) = 2j+1$. Then consider the map

$$D_j : SU(2) \to B(H_j); \tag{5.198}$$

$$D_j(u)f(z) = f(zu). \tag{5.199}$$

Clearly,

$$D_j(e)f(z) = f(z \cdot 1_2) = f(z)), \tag{5.200}$$

so $D_j(e) = 1$, and

$$D_j(u)D_j(v)f(z) = D_j(v)f(zu) = f(zuv) = D_j(uv)f(z),$$

so $D_j(u)D_j(v) = D_j(uv)$. Hence D_j is a representation of $SU(2)$.

We now compute $L_3 = -\frac{1}{2}iS_3$ on this space. From (5.156) with $u \rightsquigarrow D_j$, we have

$$L_3 = -\tfrac{1}{2}iD'_j \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\tfrac{1}{2}i\frac{d}{dt}D_j \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}_{t=0}, \tag{5.201}$$

so that

$$L_3 f(z) = -\tfrac{1}{2}i\frac{d}{dt}f(e^{it}z_1, e^{-it}z_2)_{t=0} = \tfrac{1}{2}\left(z_1\frac{\partial f(z)}{\partial z_1} - z_2\frac{\partial f(z)}{\partial z_2}\right). \tag{5.202}$$

Similarly, we obtain

$$L_+ f(z) = z_1\frac{\partial f(z)}{\partial z_2}; \tag{5.203}$$

$$L_- f(z) = z_2\frac{\partial f(z)}{\partial z_1}. \tag{5.204}$$

Hence $f_{2j}(z) = z_1^{2j}$ gives $L_3 f_{2j} = jf_{2j}$, and $f_0(z) = z_2^{2j}$ gives $L_3 f_0 = -jf_0$. In general, $f_l(z) = z_1^l z_2^{2j-l}$ spans the eigenspace H_λ of L_3 with eigenvalue $\lambda = -j+l$. Since $l = 0, 1, \dots, 2j$, this confirms (5.188), as well as the fact that the corresponding eigenspaces are all one-dimensional. The rest is easily checked, too, except for the unitarity of the representation, for which we refer to the proof of Theorem 5.40.

Finally, we return to $SO(3)$. Either explicit exponentiation (5.165), as done for $j = 1/2$ in (5.168), or the above construction of D_j, allows one to verify the crucial condition stated in Corollary 5.43, namely that $D_j(\delta) = 1_{H_j}$ for $\delta \in D = \mathbb{Z}_2$, which comes down to $D_j(-1_2) = 1_{H_j}$. This is easily seen to be the case iff $j \in \mathbb{N}_0$.

Corollary 5.46. *Up to unitary equivalence, each unitary irreducible representation of $SO(3)$ is completely fixed by its dimension $n = 2j+1$, where $j \in \mathbb{N}_0$ (so that $n = 1$ for spin-0, $n = 3$ for spin-1, $n = 5$ for spin-2, ...), and each such dimension occurs.*

5.9 Irreducible representations of compact Lie groups

Because of its importance for the classical-quantum correspondence (cf. §7.1) we first reformulate the main result of the previous section (i.e. the classification the irreducible representations of $SU(2)$) and on that basis generalize this result to arbitrary compact Lie groups. This gives a classification of great simplicity and beauty.

We already encountered the coadjoint representation (3.100) of a Lie group G on $\mathfrak{g}^*$, given by $(x\cdot\theta)(A)=\theta(x^{-1}Ax)$, where $x\in G$, $\theta\in\mathfrak{g}^*$, $A\in\mathfrak{g}$. The orbits under this action are called ***coadjoint orbits***. If $G=SO(3)$, we have $\mathfrak{g}\cong\mathbb{R}^3$ under the map

$$\mathbf{x}\cdot\mathbf{J}\equiv\sum_{k=1}^{3}x_xJ_i\mapsto(x_1,x_2,x_3)\equiv\mathbf{x}, \tag{5.205}$$

where the matrices J_k are given in (3.66). Hence also $\mathfrak{g}^*\cong\mathbb{R}^3$ under the map

$$\theta\mapsto\left((\theta_1,\theta_2,\theta_3):\mathbf{x}\mapsto\sum_{k=1}^{3}\theta_kx_k\right). \tag{5.206}$$

Writing $R\in SO(3)$ for a generic element $x\in G$, analogously to (5.44), we can compute the adoint action $R:A\mapsto RAR^{-1}$, seen as an action on $\mathbb{R}^3$, through

$$R(\mathbf{x}\cdot\mathbf{J})R^{-1}=(R\mathbf{x})\cdot\mathbf{J}. \tag{5.207}$$

Using the fact that the angular momentum matrices transform as vectors, i.e.,

$$RJ_iR^{-1}=\sum_j R_{ji}J_j, \tag{5.208}$$

we find that the adjoint action of $SO(3)$ on $\mathfrak{g}$, seen as $\mathbb{R}^3$, is its defining action. In general, if $\mathfrak{g}\cong\mathbb{R}^n$ and also $\mathfrak{g}^*\cong\mathbb{R}^n$ under the usual pairing of $\mathbb{R}^n$ and $\mathbb{R}^n$ through the Euclidean inner product, the coadjoint action of G on $\mathfrak{g}^*$, seen as an action on $\mathbb{R}^n$, is given by the inverse transpose of the adjoint action on $\mathfrak{g}\cong\mathbb{R}^n$. For $SO(3)$ we have $(R^{-1})^T=R$, so the coadjoint action of $SO(3)$ on $\mathbb{R}^3$ is just its defining action, too, and hence the coadjoint orbits are the 2-spheres S_r with radius $r\geq 0$.

Turning to $SU(2)$, we now make the identification of $\mathfrak{g}^*$ with $\mathbb{R}^3$ slightly differently, namely by replacing the 3×3 real matrices J_i in (5.205) by the 2×2 matrices S_i in (5.174), but the computation is similar: using (5.44) - (5.45), we find that the coadjoint action of $u\in SU(2)$ on $\mathbb{R}^3$ is given by the defining action of $\tilde{\pi}(u)\in SO(3)$, cf. (5.46). It follows that the coadjoint orbits for $SU(2)$ are the same as for $SO(3)$.

Returning to general Lie groups G for the moment, assumed connected for simplicity, we take some coadjoint orbit $\mathscr{O}\subset\mathfrak{g}^*$, fix a point $\theta\in\mathscr{O}$ (so that $\mathscr{O}=G\cdot\theta\equiv G\theta$), and look at the stabilizer G_θ and its Lie algebra $\mathfrak{g}_\theta$. Since the derivative Ad' of the adjoint action Ad of G on $\mathfrak{g}$—defined as in (5.156)—is given by

$$\mathrm{Ad}'(A):B\mapsto[A,B], \tag{5.209}$$

it follows that the "infinitesimal stabilizer" $\mathfrak{g}_\theta$ is given by

$$\mathfrak{g}_\theta = \{A \in \mathfrak{g} \mid \theta([A,B]) = 0 \forall B \in \mathfrak{g}\}. \tag{5.210}$$

Consequently, the restriction of $\theta : \mathfrak{g} \to \mathbb{R}$ to $\mathfrak{g}_\theta \subset \mathfrak{g}$ is a Lie algebra homomorphism (where $\mathbb{R}$ is obviously endowed with the zero Lie bracket). Consider a ***character*** $\chi : G_\theta \to \mathbb{T}$, which is the same thing as a one-dimensional unitary representation of G_θ. If we regard $\mathbb{T}$ as a closed subgroup of $GL_1(\mathbb{C})$, its Lie algebra $\mathfrak{t}$ is given by $i\mathbb{R} \subset M_1(\mathbb{C}) = \mathbb{C}$. It is conventional (at least among physicists) to take $-i$ as the basis element of $\mathfrak{t}$, so that $\mathfrak{t} \cong \mathbb{R}$ under $-it \leftrightarrow t$, so that the exponential map $\exp : \mathfrak{t} \to \mathbb{T}$ (which is the usual one), seen as a map from $\mathbb{R}$ to $\mathbb{T}$, is given by $t \mapsto \exp(-it)$. Defining the derivative $\chi' : \mathfrak{g}_\theta \to \mathbb{C}$ as in (5.156), it follows that actually $\chi' : \mathfrak{g}_\theta \to i\mathbb{R}$, so that $i\chi'$ maps $\mathfrak{g}_\theta$ to $\mathbb{R}$ and is a Lie algebra homomorphism.

Definition 5.47. *Let G be a connected Lie group. A coadjoint orbit $\mathscr{O} \subset \mathfrak{g}^*$ is called* **integral** *if for some (and hence all) $\theta \in \mathscr{O}$ one has $\theta_{|\mathfrak{g}_\theta} = i\chi'$ for some character $\chi : G_\theta \to \mathbb{T}$, i.e., if there is a character χ such that for each $A \in \mathfrak{g}_\theta$ one has*

$$\theta(A) = i\frac{d}{dt}\chi\left(e^{tA}\right)_{|t=0}. \tag{5.211}$$

In the simplest case where $G = \mathbb{T}$, the coadjoint action on $\mathfrak{t}^*$ is evidently trivial, so that $G_\theta = G = \mathbb{T}$ for any $\theta \in \mathfrak{t}^* \cong \mathbb{R}$. Furthermore, any character on $\mathbb{T}$ takes the form $\chi_n(z) = z^n$, where $n \in \mathbb{Z}$, cf. (C.351). As explained above, if $\mathfrak{t} \cong \mathbb{R}$ and hence also $\mathfrak{t}^* \cong \mathbb{R}$, the identification of $\lambda \in \mathfrak{t}^*$ with $\lambda \in \mathbb{R}$ is made by $\lambda(-i) \leftrightarrow \lambda$, where $-i \in \mathfrak{t}$. If $\chi = \chi_n$, the right-hand side of (5.211) evaluated at $A = -i$ equals n, so that (5.211) holds iff $\theta = n$ for some $n \in \mathbb{Z}$. Thus the integral coadjoint orbits in $\mathfrak{t}^*$ are the integers $\mathbb{Z} \subset \mathbb{R}$. Similarly, if $G = \mathbb{T}^d$, the characters are elements of $\mathbb{Z}^d$, as in

$$\chi_{(n_1,\dots,n_d)}(z_1,\dots,z_d) = z_1^{n_1}\cdots z_d^{n_d}, \tag{5.212}$$

and the integral coadjoint orbits in $\mathfrak{g}^* \cong \mathbb{R}^d$ are the points of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$.

For $G = SU(2)$ we take a coadjoint orbit $S^2_r \subset \mathbb{R}^3$ and fix $\theta_r = (0,0,r)$. If $r = 0$, then $G_\theta = G$ and (5.211) holds for the trivial character $\chi \equiv 1$, so the orbit $\{(0,0,0)\}$ is integral. Let $r > 0$. Then $G_{\theta_r} \equiv G_r$ consist of the pre-image of $SO(2)$ in $SU(2)$ under the projection $\tilde{\pi}$ in (5.46), where $SO(2) \subset SO(3)$ is the group of rotations around the z-axis. This is the abelian group

$$T = \{\mathrm{diag}(z,\overline{z}) \mid z \in \mathbb{T}\}. \tag{5.213}$$

This group is isomorphic to $\mathbb{T}$ under $\mathrm{diag}(z,\overline{z}) \mapsto z$ and hence its characters are given by $\chi_n(\mathrm{diag}(z,\overline{z})) = z^n$, where $n \in \mathbb{Z}$. The identification $\mathfrak{g}^* \cong \mathbb{R}^3$ is made by identifying $\theta \in \mathfrak{g}^*$ with $(\theta_1,\theta_2,\theta_3)$, where $\theta_1 = \theta(S_i)$. Putting $A = S_3$ in (5.211), see (5.174), therefore gives $r = n/2$ for some $n \in \mathbb{N}$. We conclude that the coadjoint orbits for $SU(2)$ are given by the two-spheres $S^2_r \subset \mathbb{R}^3$ with $r \in \mathbb{N}_0/2$.

Similarly, for $G = SO(3)$ the stabilizer of $(0,0,r)$ is $SO(2) \cong \mathbb{T}$ itself, and putting $A = J_3$ in (5.211) one finds that the coadjoint orbits are the spheres S^2_r with $r \in \mathbb{N}_0$.

For any (Lie) group G, let the ***unitary dual*** $\hat{G}$ be the set whose elements are equivalence classes of unitary irreducible representations of G, where we say:

Definition 5.48. *Two unitary representations $u_i : G \to U(H_i)$, $i = 1,2$, are* **equivalent** *if there is unitary $v : H_1 \to H_2$ such that $u_2(x) = vu_1(x)v^*$ for each $x \in G$.*

The examples $G = \mathbb{T}^d$ as well as for $G = SU(2)$ now suggest the following theorem:

Theorem 5.49. *If G is a compact connected Lie group, then the unitary dual $\hat{G}$ is parametrized by the set of integral coadjoint orbits in $\mathfrak{g}^*$.*

Furthermore, there is an explicit (geometric) procedure to a construct an irreducible representation $u_{\mathscr{O}}$ corresponding to such an orbit, namely by the method of *geometric quantization*. We will not explain this method, which would require some reasonably advanced differential geometry, but instead we outline the connection between coadjoint orbits and the well-known ***method of the highest weight***.

Let G be a compact connected Lie group and pick a maximal torus $T \subset G$. Let

$$W_T = N(T)/T \tag{5.214}$$

be the corresponding ***Weyl group***, where $N(T)$ is the normalizer of T in G (i.e., $x \in N(T)$ iff $xzx^{-1} \in T$ for each $z \in T$). Note that all maximal tori in compact connected Lie groups are conjugate, so that the specific choice of T is irrelevant.

For example, for $SU(2)$ we take (5.213), in which case $N(T)$ is generated by T and $\sigma_1 \in SU(2)$, so that $W \cong \mathfrak{S}_2$, i.e., the permutation group on two variables. In general the Weyl group inherits the adjoint action of $N(T)$ on T, so that W_T acts on T and hence also acts on $\mathfrak{t}$ and $\mathfrak{t}^*$; for $SU(2)$ the action of the nontrivial element of W_T, i.e., image $[\sigma_1]$ of $\sigma_1 \in N(T)$ in $N(T)/T)$, on T is given by

$$[\sigma_1](\text{diag}(z,\bar{z})) = \text{diag}(\bar{z},z), \tag{5.215}$$

so that its action on $\mathbb{T} \cong T$ is $z \mapsto \bar{z}$, which gives rise to actions $A \mapsto -A$ of W_T on $\mathfrak{t}$ and hence $\lambda \mapsto -\lambda$ of W_T on $\mathfrak{t}^*$. This is a special case of the following bijection:

$$\mathfrak{g}^*/G \cong \mathfrak{t}^*/W_T, \tag{5.216}$$

where the G-action on $\mathfrak{g}^*$ is the coadjoint one; globally, one has $G/\text{Ad}(G) \cong T/W_T$.

Indeed, for $SU(2)$ the left-hand side of (5.216) is the set of spheres S_r^2 in $\mathbb{R}^3$, $r \geq 0$, whereas the right-hand side is $\mathbb{R}/\mathfrak{S}_2$ (where $\mathfrak{S}_2$ acts on $\mathbb{R}$ by $\theta \mapsto -\theta$).

In general, a given coadjoint orbit $\mathscr{O} \subset \mathfrak{g}^*$ defines a Weyl group orbit $\mathscr{O}_W$ in $\mathfrak{t}^*$ as follows: $\mathscr{O}$ contains a point θ for which $T \subseteq G_\theta$, and we take $\mathscr{O}_W$ to be the orbit through $\theta_{|\mathfrak{t}}$. Conversely, any G-invariant inner product on $\mathfrak{g}$ induces a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp, \tag{5.217}$$

which yields an extension of $\lambda \in \mathfrak{t}^*$ to $\theta_\lambda \in \mathfrak{g}^*$ that vanishes on $t^\perp$. Let $\Lambda \subset \mathfrak{t}^*$ be the set of integral elements in $\mathfrak{t}^*$ (as explained after Definition 5.47). Elements of Λ are called ***weights***. Theorem 5.51 below gives a parametrization

$$\hat{G} \cong \Lambda/W_T, \tag{5.218}$$

which, restricting (5.216) to the integral part $\Lambda \subset \mathfrak{t}^*$, implies Theorem 5.49.

Instead of with the quotient Λ/W_T, one may prefer to work with Λ itself, as follows: we say that $\lambda \in \mathfrak{t}^*$ is ***regular*** if $w \cdot \lambda$ for $w \in W_T$ iff $w = e$; this is the case iff $\lambda = \theta_{|\mathfrak{t}}$ with $G_\theta = T$. For $SU(2)$ all weights $\lambda \in \mathbb{Z}$ are regular except $\lambda = 0$. The set $\mathfrak{t}^*_r$ of regular elements of $\mathfrak{t}^*$ falls apart into connected components C, called ***Weyl chambers***, which are mapped into each other by W_T. For $SU(2)$ one has $\mathfrak{t}^* = (-\infty, 0) \cup (0, \infty)$, so that the Weyl chambers are $(-\infty, 0)$ and $(0, \infty)$.

One picks an arbitrary Weyl chamber C_d (for $SU(2)$ this is $(0, \infty)$) and forms

$$\Lambda_d = \Lambda \cap C_d^-, \tag{5.219}$$

where C_d^- is the closure of C_d in $\mathfrak{t}^*$. Elements of Λ_d are called ***dominant weights***. For each element of Λ/W_T there is a unique dominant weight representing it in Λ, so that instead of (5.218) we may also write what Theorem 5.51 actually gives, viz.

$$\hat{G} \cong \Lambda_d. \tag{5.220}$$

To explain this in some detail, we need further preparation. Any (unitary) representation $u : G \to U(H)$ on some finite-dimensional Hilbert space H restricts to T, and since T is abelian, we may simultaneously diagonalize all operators $u(z)$, $z \in T$. The operators $iu'(A)$, where $A \in \mathfrak{t}$, commute as well, so that we may decompose

$$H = \bigoplus_{\mu \in \Lambda_H} H_\mu, \tag{5.221}$$

where $\Lambda_H \subset \Lambda$ contains the weights that occur in $u_{|T}$, so that for each $\psi \in H_\mu$,

$$u(z)\psi = \chi_\mu(z)\psi \ \ (z \in T); \tag{5.222}$$

$$iu'(Z)\psi = \mu(Z)\psi \ \ (Z \in \mathfrak{t}), \tag{5.223}$$

where the character $\chi_\mu : T \to \mathbb{T}$ corresponding to the weight $\mu \in \Lambda$ is defined as in (5.212) with $\mu = (n_1, \ldots, n_d)$ and $z = (z_1, \ldots, z_d) \in T \cong \mathbb{T}^d$, where $d = \dim(T)$. For example, we have seen that the irreducible representations $D_j(SU(2))$ on $H_j \cong \mathbb{C}^{2j+1}$ contains weights in $\Lambda_j = \{-j, -j+1, \ldots, j-1, j\}$, where $j \in \mathbb{N}_0/2$.

In particular, take $H = \mathfrak{g}_\mathbb{C}$ with some G-invariant inner product, cf. (5.148), and take $u = \mathrm{Ad}$, given by $\mathrm{Ad}(x)B = xBx^{-1}$, so that $\mathrm{Ad}'(A)(B) = [A, B]$, extended from $\mathfrak{g}$ to $\mathfrak{g}_\mathbb{C}$: we write $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ and hence put $\mathrm{Ad}'(A)(B + iC) = [A, B] + i[A, C]$, where $A, B, C \in \mathfrak{g}$. We assume that the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_\mathbb{C}$ is obtained from a real inner product on $\mathfrak{g}$ by complexification. This inner product on $\mathfrak{g}$ may be restricted to $\mathfrak{t} \subset \mathfrak{g}$ and hence induces an inner product on $\mathfrak{t}^*$, also denoted by $\langle \cdot, \cdot \rangle$. For example, if G is semi-simple (like $SU(2)$), one may take the inner product on $\mathfrak{g}$ and hence on $\mathfrak{g}_\mathbb{C}$ to be the Cartan–Killing form $\langle A, B \rangle = -\frac{1}{2}\mathrm{Tr}\,(\mathrm{Ad}'(A)\mathrm{Ad}'(B))$, which is nondegenerate because G is semi-simple, and positive definite since G is compact. For $SU(2)$ or $SO(3)$ this gives the usual inner product on $\mathbb{R}^3$ and $\mathbb{C}^3$.

Definition 5.50. *The* **roots** *of* $\mathfrak{g}$ *are the* nonzero *weights of the adjoint representation* $u = \mathrm{Ad}$ *on* $H = \mathfrak{g}_{\mathbb{C}}$. *That is, writing* $\Delta \subset \Lambda$ *for the set of roots, we have* $\alpha \in \Delta$ *iff* $\alpha : \mathfrak{t} \to \mathbb{R}$ *is not identically zero and there is some* $E_\alpha \in \mathfrak{g}_{\mathbb{C}}$ *such that for each* $Z \in \mathfrak{t}$,

$$i[Z, E_\alpha] = \alpha(Z)E_\alpha, \tag{5.224}$$

cf. (5.223). *Furthermore, subject to the choice of a preferred Weyl chamber* C_d *in* $\mathfrak{t}_r^*$, *we say* $\alpha \in \Delta$ *is* **positive**, *denoted by* $\alpha \in \Delta^+$, *if* $\langle \alpha, \lambda \rangle > 0$ *for each* $\lambda \in C_d$.

Since $\langle \alpha, \lambda \rangle$ is real and nonzero for each $\alpha \in \Delta$ and $\lambda \in C_d$, one has either $\alpha \in \Delta^+$ or $-\alpha \in \Delta^+$, i.e., $\alpha \in \Delta^- = -\Delta^+$. Since $\mathfrak{t}$ is maximal abelian in $\mathfrak{g}$, it can also be shown that each root is nondegenerate. Writing $\mathfrak{g}_\alpha = \mathbb{C} \cdot E_\alpha$, this gives a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha. \tag{5.225}$$

For $G = SU(2)$, the single generator of $\mathfrak{t}$ is S_3, and taking $E_\pm = i(S_1 \pm iS_2)$, we see from (5.180) that $i[S_3, E_\pm] = \pm E_\pm$. Hence the roots are $\alpha_\pm$, given by $\alpha_\pm(S_3) = \pm 1$, and with $(0, \infty)$ as the Weyl chamber of choice, the root α_+ is the positive one.

We now define a partial ordering $\leq$ on Λ by putting $\mu \leq \lambda$ iff $\lambda - \mu = \sum_i n_i \alpha_i$ for some $n_i \in \mathbb{N}_0$ and $\alpha_i \in \Delta^+$. This brings us to the ***theorem of the highest weight***:

Theorem 5.51. *Let G be a connected compact Lie group. There is a parametrization* $\hat{G} \cong \Lambda_d$, *such that any unitary irreducible representation* $u_\lambda : G \to H_\lambda$ *in the class* $\lambda \in \hat{G}$ *defined by a given dominant weight* $\lambda \in \Lambda_d$ *has the following properties:*

1. H_λ *contains a unit vector* υ_λ, *unique up to a phase, such that*

$$iu'_\lambda(Z)\upsilon_\lambda = \lambda(Z)\upsilon_\lambda \ \ (Z \in \mathfrak{t}); \tag{5.226}$$

$$iu'_\lambda(E_\alpha)\upsilon_\lambda = 0 \ \ (\alpha \in \Delta^+). \tag{5.227}$$

2. Any other weight μ *occurring in H, cf.* (5.221), *satisfies* $\mu \leq \lambda$ *and* $\mu \neq \lambda$.

The crucial point is that eqs. (5.226) - (5.227) imply

$$\theta_\lambda(A) = i\langle \upsilon_\lambda, u'_\lambda(A)\upsilon_\lambda \rangle \ \ (A \in \mathfrak{g}), \tag{5.228}$$

where $\theta_\lambda \in \mathfrak{g}^*$ was defined after (5.217) by $\lambda \in \Lambda_d \subset \mathfrak{t}^*$. Since each operator $u_\lambda(x)$ is unitary, each vector $u_\lambda(x)\upsilon_\lambda$ is a unit vector, so we may form the G-orbit

$$\mathscr{O}'_\lambda = \{|u_\lambda(x)\upsilon_\lambda\rangle\langle u_\lambda(x)\upsilon_\lambda|, x \in G\} \tag{5.229}$$

through $|\upsilon_\lambda\rangle\langle\upsilon_\lambda|$ in the space $\mathscr{P}_1(H_\lambda)$ of all one-dimensional projections on H_λ. Denoting the coadjoint orbit $G \cdot \theta_\lambda \subset \mathfrak{g}^*$ by $\mathscr{O}_\lambda$, where $\lambda = (\theta_\lambda)_{|\mathfrak{t}}$, the map

$$x \cdot \theta_\lambda \mapsto |u_\lambda(x)\upsilon_\lambda\rangle\langle u_\lambda(x)\upsilon_\lambda|, \tag{5.230}$$

is a G-equivariant diffeomorphism (in fact, a symplectomorphism) from $\mathscr{O}_\lambda$ to $\mathscr{O}'_\lambda$. This amplifies Theorem 5.49 by making the the bijective correspondence between the set Λ_d of dominant weights and the set of integral coadjoint orbits explicit.

5.10 Symmetry groups and projective representations

Despite the power and beauty of unitary group representations in *mathematics*, in the context of e.g. Wigner's Theorem we have seen that in *physics* one should look at homomorphisms $x \mapsto \mathsf{W}(x)$, where $\mathsf{W}(x)$ is a symmetry of $\mathscr{P}_1(H)$. In view of Theorems 5.4, this is equivalent to considering a *single* homomorphism $h : G \mapsto \mathscr{G}^H$, cf. (5.136). To simplify the discussion, we now drop $U_a(H)$ from consideration and just deal with the connected component $\mathscr{G}_0^H = U(H)/\mathbb{T}$ of the identity. This restriction may be justified by noting that in what follows we will only deal with symmetries given by *connected Lie groups*, which have the property that each element is a product of squares $x = y^2$. In that case, $h(x) = h(y)^2$ is always a square and hence it cannot lie in the component $U_a(H)/\mathbb{T}$ (the anti-unitary case does play a role as soon as *discrete* symmetries are studied, such as time inversion, parity, or charge conjugation). Thus in what follows we will study continuous homomorphisms

$$h : G \to U(H)/\mathbb{T}, \tag{5.231}$$

where $U(H)/\mathbb{T}$ has the quotient topology inherited from the strong operator topology on $U(H)$, as explained above. Since it is inconvenient to deal with such a quotient, we try to lift h to some map (5.137) where, in terms of the canonical projection

$$\pi : U(H) \to U(H)/\mathbb{T}, \tag{5.232}$$

which is evidently a group homomorphism, we have

$$\pi \circ u = h. \tag{5.233}$$

This can be done by choosing a cross-section s of π, that is, a measurable map

$$s : U(H)/\mathbb{T} \to U(H), \tag{5.234}$$

or (this doesn't matter much) a map $s : h(G)/\mathbb{T} \to U(H)$, such that

$$\pi \circ s = \mathrm{id}. \tag{5.235}$$

Given h, such a cross-section s yields a map $u : G \to U(H)$ through

$$u = s \circ h; \tag{5.236}$$

in particular, $\pi(u(x)) = h(x)$. Such a lift often loses the homomorphism property, *though in a controlled way*, as follows. Since different choices of s must differ by a phase, and h is a homomorphism of groups, there must be a function

$$c : G \times G \to \mathbb{T} \tag{5.237}$$

such that

$$u(x)u(y) = c(x,y)u(xy) \ (x,y \in G). \tag{5.238}$$

Indeed, since π and h are homomorphisms, we may compute

$$\begin{aligned}\pi(u(x)u(y)u(xy)^{-1}) &= \pi(s(h(x))\pi(s(h(y))\pi(s(h(xy)))^{-1} \\ &= h(xy)h(xy)^{-1} = h(e_G) = e_{U(H)/\mathbb{T}}.\end{aligned}$$

Hence $u(x)u(y)u(xy)^{-1} \in \pi^{-1}(e_{U(H)/\mathbb{T}}) = \mathbb{T}\cdot 1_H$, which yields (5.238), or, more directly,

$$c(x,y)\cdot 1_H = u(x)u(y)u(xy)^*. \tag{5.239}$$

Associativity of multiplication in G and the homomorphism property of h yield

$$c(x,y)c(xy,z) = c(x,yz)c(y,z), \tag{5.240}$$

and if we impose the natural requirement $u_e = 1_H$, we also have

$$c(e,x) = c(x,e) = 1. \tag{5.241}$$

Definition 5.52. *A function $c : G\times G \to \mathbb{T}$ satisfying* (5.240) *and* (5.241) *is called a* **multiplier** *or* **C@2-cocycle** *on G (in the topological case one requires c to be Borel measurable, and for Lie groups it should in addition be smooth* near the identity*). The set of such multipliers, seen as an abelian group under (pointwise) operations in $\mathbb{T}$, is denoted by $Z^2(G,\mathbb{T})$. If c takes the form*

$$c(x,y) = \frac{b(xy)}{b(x)b(y)}, \tag{5.242}$$

where $b : G \to \mathbb{T}$ satisfies $b(e) = 1$ (and is measurable and smooth near e as appropriate), then c is called a **2-coboundary** *or an* **exact multiplier**. *The set of trivial multipliers forms a (normal) subgroup $B^2(G,\mathbb{T})$ of $Z^2(G,\mathbb{T})$, and the quotient*

$$H^2(G,\mathbb{T}) = \frac{Z^2(G,\mathbb{T})}{B^2(G,\mathbb{T})} \tag{5.243}$$

is called the **second cohomology group** *of G with coefficients in $\mathbb{T}$.*

The reason 2-coboundaries and the ensuing group $H^2(G,\mathbb{T})$ are interesting for our problem is as follows. Given a map $x \mapsto u(x)$ from G to $U(H)$ with (5.238), suppose we change $u(x)$ to $u(x)' = b(x)u(x)$. The associated multiplier then changes to

$$c'(x,y) = \frac{b(x)b(y)}{b(xy)}c(x,y), \tag{5.244}$$

in that $u(x)'u(y)' = c'(x,y)u'_{xy}$. In particular, a multiplier of the form (5.242) may be removed by such a transformation, and is accordingly called ***exact***.

Proposition 5.53. *If $H^2(G,\mathbb{T})$ is trivial, then any multiplier can be removed by modifying the lift u of h, and the ensuing map $u' : G \to U(H)$ is a homomorphism and hence a unitary representation of G on H. In that case, any homomorphism $G \to U(H)/\mathbb{T}$ comes from a unitary representation $u : G \to U(H)$ through* (5.233).

This is true by construction. By the same token, if $H^2(G,\mathbb{T})$ is non-trivial, then G will have projective representations that cannot be turned into ordinary ones by a change of phase (for it can be shown that any multiplier $c \in Z^2(G,\mathbb{T})$ is realized by some projective representation). Thus it is important to compute $H^2(G,\mathbb{T})$ for any given (physically relevant) group G, and see what can be done if it is non-trivial.

To this end we present the main results of practical use. In order to state one of the main results (Whitehead's Lemma), we need to set up a cohomology theory for $\mathfrak{g}$ (which we only need with trivial coefficients). Let $C^k(\mathfrak{g},\mathbb{R})$ be the abelian group of all k-linear totally antisymmetric maps $\varphi : \mathfrak{g}^k \to \mathbb{R}$, with ***coboundary maps***

$$\delta^{(k)} : C^k(\mathfrak{g},\mathbb{R}) \to C^{k+1}(\mathfrak{g},\mathbb{R}); \tag{5.245}$$

$$(X_0,X_1,\ldots,X_k) \mapsto \sum_{i<j=1}^{k+1} (-1)^{i+j}\varphi([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k), \tag{5.246}$$

where the hat means that the corresponding entry is omitted. For example, we have

$$\delta^{(1)}\varphi(X_0,X_1) = -\varphi([X_0,X_1]);$$
$$\delta^{(2)}\varphi(X_0,X_1,X_2) = -\varphi([X_0,X_1],X_2)+\varphi([X_0,X_2],X_1)-\varphi([X_1,X_2],X_0).$$

These maps satisfy "$\delta^2 = 0$", or, more precisely,

$$\delta^{(k+1)}\circ\delta^{(k)} = 0, \tag{5.247}$$

and hence we may define the following abelian groups:

$$B^k(\mathfrak{g},\mathbb{R}) = \mathrm{ran}(\delta^{(k-1)}); \tag{5.248}$$
$$Z^k(\mathfrak{g},\mathbb{R}) = \ker(\delta^{(k)}); \tag{5.249}$$
$$H^k(\mathfrak{g},\mathbb{R}) = \frac{Z^k(\mathfrak{g},\mathbb{R})}{B^k(\mathfrak{g},\mathbb{R})}. \tag{5.250}$$

Note that $B^k(\mathfrak{g},\mathbb{R}) \subseteq H^k(\mathfrak{g},\mathbb{R})$ because of (5.247). In particular, for $k=2$ the group $Z^2(\mathfrak{g},\mathbb{R})$ of all ***2-cocycles*** on $\mathfrak{g}$ consists of all bilinear maps $\varphi : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ that satisfy

$$\varphi(X,Y) = -\varphi(Y,X); \tag{5.251}$$
$$\varphi(X,[Y,Z])+\varphi(Z,[X,Y])+\varphi(Y,[Z,X]) = 0, \tag{5.252}$$

and its subgroup $B^2(\mathfrak{g},\mathbb{R})$ of all ***2-coboundaries*** comprises all φ taking the form

$$\varphi(X,Y) = \theta([X,Y]),\ \theta\in\mathfrak{g}^*. \tag{5.253}$$

For example, for $\mathfrak{g}=\mathbb{R}$ any antisymmetric bilinear map $\varphi : \mathbb{R}^2 \to 0$ is zero, so that

$$H^2(\mathbb{R},\mathbb{R}) = 0. \tag{5.254}$$

This has nothing to with the fact that the Lie bracket on $\mathfrak{g}$ vanishes. Indeed, $\mathfrak{g} = \mathbb{R}^2$ does admit a unique nontrivial 2-cocycle, given by (half) the symplectic form, i.e.,

$$\varphi_0((p,q),(p',q')) = \tfrac{1}{2}(pq' - qp'). \tag{5.255}$$

Since $B^2(\mathbb{R}^2,\mathbb{R}) = 0$, this cannot be removed, hence (5.255) generates $H^2(\mathbb{R}^2,\mathbb{R})$:

$$H^2(\mathbb{R}^2,\mathbb{R}) \cong \mathbb{R}. \tag{5.256}$$

As far as cohomology is concerned, each Lie group and each Lie algebra has its own story, although in some cases a group of stories may be collected into a single narrative. As a case in point, a Lie algebra $\mathfrak{g}$ is called ***simple*** when it has no proper ideals, and ***semi-simple*** when it has no commutative ideals. A Lie algebra is semi-simple iff it is a direct sum of simple Lie algebras. If a Lie group G is (semi-) simple, then so is its Lie algebra $\mathfrak{g}$. A basic result, often called ***Whitehead's Lemma***, is:

Lemma 5.54. *If $\mathfrak{g}$ is semi-simple, then $H^2(\mathfrak{g},\mathbb{R}) = 0$.*

Proof. The key point is that $C^k(\mathfrak{g},\mathbb{R})$ is a $\mathfrak{g}$-module under the action

$$(X_0 \cdot \varphi)(X_1,\ldots,X_k) = -\sum_{i=1}^{k} \varphi(X_1,\ldots,[X_0,X_i],\ldots,X_k). \tag{5.257}$$

For $k = 2$, a simple computation shows that

$$\begin{aligned}(X_0 \cdot \varphi)(X_1,X_2) &= -\varphi([X_0,X_1],X_2) - \varphi(X_1,[X_0,X_2]) \\ &= \delta^{(2)}\varphi(X_0,X_1,X_2) - \delta^{(1)}\varphi(X_0,-)(X_1,X_2),\end{aligned} \tag{5.258}$$

where at fixed X_0, the map $\varphi(X_0,-)$ is seen as an element of $C^1(\mathfrak{g},\mathbb{R})$. This show that $\mathfrak{g}$ maps both $B^2(\mathfrak{g},\mathbb{R})$ and $Z^2(\mathfrak{g},\mathbb{R})$ onto itself. Indeed, if $\varphi = \delta^{(1)}\chi$, then the first term in (5.258) vanishes because $\delta^{(2)} \circ \delta^{(1)} = 0$, cf. (5.247), so that the right-hand side of (5.258) takes the form $\delta^{(1)}(\cdots)$ and hence lies in $B^2(\mathfrak{g},\mathbb{R})$. Similarly, if $\delta^{(2)}\varphi = 0$, then $\delta^{(2)}(X_0 \cdot \varphi) = 0$. We now use the fact that if $\mathfrak{g}$ is semi-simple, then any finite-dimensional module is completely reducible. Consequently, as a $\mathfrak{g}$-module, $Z^2(\mathfrak{g},\mathbb{R})$ must decompose as $Z^2(\mathfrak{g},\mathbb{R}) = B^2(\mathfrak{g},\mathbb{R}) \oplus V$, where V is some $\mathfrak{g}$-module. Hence if $\varphi \in V$, then $X_0 \cdot \varphi \in V$. Since $\varphi \in Z^2(\mathfrak{g},\mathbb{R})$, the first term in (5.258) vanishes, whilst the second term lies in $B^2(\mathfrak{g},\mathbb{R})$. Since $V \cap B^2(\mathfrak{g},\mathbb{R}) = \{0\}$, we therefore have $X_0 \cdot \varphi = 0$, and hence $\delta^{(1)}\varphi(X_0,-)(X_1,X_2) = 0$, which gives $\varphi(X_0,[X_1,X_2]) = 0$, for all $X_0,X_1,X_2 \in \mathfrak{g}$. At this point we use another implication of the semi-simplicity of $\mathfrak{g}$, namely $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. It follows that $\varphi = 0$, whence $V = \{0\}$, from which $Z^2(\mathfrak{g},\mathbb{R}) = B^2(\mathfrak{g},\mathbb{R})$, or, in other words, $H^2(\mathfrak{g},\mathbb{R}) = 0$. □

Theorem 5.55. *Let G be a connected and simply connected Lie group. Then*

$$H^2(G,\mathbb{T}) \cong H^2(\mathfrak{g},\mathbb{R}). \tag{5.259}$$

Proof. This is really a conjunction of two isomorphisms:

$$H^2(G,\mathbb{T}) \cong H^2(G,\mathbb{R}); \tag{5.260}$$
$$H^2(G,\mathbb{R}) \cong H^2(\mathfrak{g},\mathbb{R}), \tag{5.261}$$

where $\mathbb{R}$ is the usual additive group, and $Z^2(G,\mathbb{R})$, $B^2(G,\mathbb{R})$, and hence $H^2(G,\mathbb{R})$ are defined analogously to $Z^2(G,\mathbb{T})$ etc. The first isomorphism is simply induced by

$$Z^2(G,\mathbb{R}) \mapsto Z^2(G,\mathbb{T}); \tag{5.262}$$
$$\Gamma(x,y) \mapsto e^{i\Gamma(x,y)} \equiv c(x,y), \tag{5.263}$$

which preserves exactness and induces an isomorphism in cohomology (but note that (5.262) - (5.263) may not itself define an isomorphism).

The isomorphism (5.261) is induced at the cochain level, too. Given a cocycle $\varphi \in Z^2(G,\mathbb{R})$, we construct a new Lie algebra $\mathfrak{g}_\varphi$ (called a ***central extension*** of $\mathfrak{g}$) by taking $\mathfrak{g}_\varphi = \mathfrak{g} \oplus \mathbb{R}$ as a vector space, equipped though with the unusual bracket

$$[(X,v),(Y,w)] = ([X,Y],\varphi(X,Y)); \tag{5.264}$$

the condition $\varphi \in Z^2(G,\mathbb{R})$ guarantees that this is a Lie bracket. Furthermore, $\mathfrak{g}_\varphi$ is isomorphic (as a Lie algebra) to a direct sum iff $\varphi \in B^2(\mathfrak{g},\mathbb{R})$; indeed, if (5.253) holds, then $(X,v) \mapsto (X, v+\theta(X))$ yields the desired isomorphism $\mathfrak{g}_\varphi \to \mathfrak{g} \oplus \mathbb{R}$.

By Lie's Third Theorem, there is a connected and simply connected Lie group G_φ (again called a ***central extension*** of G), with Lie algebra $\mathfrak{g}_\varphi$, As a manifold, $G_\varphi = G \times \mathbb{R}$, but the group laws are given, in terms of a function $\Gamma : G \times G \to \mathbb{R}$, by

$$(x,v)\cdot(y,w) = (xy, v+w+\Gamma(x,y)); \tag{5.265}$$
$$(x,v)^{-1} = (x^{-1}, -v-\Gamma(x,x^{-1})). \tag{5.266}$$

The group axioms then imply (indeed, they are equivalent to) the condition $\Gamma \in Z^2(G,\mathbb{R})$. Furthermore, two such extensions G_φ and G'_φ are isomorphic iff the corresponding cocycles Γ and Γ' are related by (5.244), and in particular, $\Gamma \in B^2(G,\mathbb{R})$ iff G_φ is isomorphic (as a Lie group) to a direct product $G \times \mathbb{R}$, which in turn is the case iff $\varphi \in B^2(\mathfrak{g},\mathbb{R})$. Conversely, given $\Gamma \in Z^2(G,\mathbb{R})$, we define the central extension G_φ by (5.265) - (5.266), to find that the associated Lie algebra $\mathfrak{g}_\varphi$ takes the above form, defining $\varphi \in B^2(\mathfrak{g},\mathbb{R})$ through (5.264). Explicitly,

$$\varphi(X,Y) = \frac{d}{ds}\frac{d}{dt}\left[\Gamma\left(e^{tX},e^{sY}\right)\right]_{|s=t=0} - (X \leftrightarrow Y). \tag{5.267}$$

Lie's Third Theorem thus implies that the map $\varphi \leftrightarrow \Gamma$ (which is not necessarily a bijection) descends to an isomorphism $H^2(\mathfrak{g},\mathbb{R}) \to H^2(G,\mathbb{R})$ in cohomology. □

Given (5.254), Theorem 5.55 immediately gives

$$H^2(\mathbb{R},\mathbb{T}) = 0. \tag{5.268}$$

In particular, if $\mathbb{R}$ is the relevant symmetry group, which is the case e.g. with time translation, by Proposition 5.53 we may restrict ourselves to unitary representations.

Once again, this has nothing to do with abelianness or topological triviality of $\mathbb{R}$. Indeed, for $G = \mathfrak{g} = \mathbb{R}^2$, the ***Heisenberg cocycle*** (5.255) comes from the multiplier

$$c_0((p,q),(p',q')) = e^{i(pq'-qp')/2}, \tag{5.269}$$

where $\mathbb{R}^2$ is seen as the group of translations in the *phase space* $\mathbb{R}^2$ of a particle moving on $\mathbb{R}$. Accordingly, this multiplier is realized by the following projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$:

$$u(p,q)\psi(x) = e^{-ipq/2}e^{ixp}\psi(x-q). \tag{5.270}$$

If $\mathbb{R}^2$ is the *configuration space* of some particle, and the group $\mathbb{R}^2$ produces translations in the latter (i.e., of *position*), then the appropriate unitary representation would rather be on $L^2(\mathbb{R}^2)$ and would have trivial multiplier, viz.

$$u(q_1,q_2)\psi(x_1,x_2) = \psi(x_1-q_1,x_2-q_2). \tag{5.271}$$

Similarly, $G = \mathbb{R}^2$, now seen as generating translations of *momentum* in the phase space $\mathbb{R}^4$ of the latter example would appropriately be represented on $L^2(\mathbb{R}^2)$ as

$$u(q_1,q_2)\psi(x_1,x_2) = e^{i(x_1q_1+x_2q_2)}\psi(x_1,x_2). \tag{5.272}$$

Corollary 5.56. *Let G be a connected and simply connected semi-simple Lie group. Then $H^2(G,\mathbb{T})$ is trivial.*

Here we say that a Lie group is ***simple*** when it has no proper *connected* normal subgroups, and ***semi-simple*** if it has no proper connected normal *abelian* subgroups. For example, the "classical Lie groups" of Weyl are semi-simple, including $SO(3)$ and $SU(2)$, which are even simple (note that the latter does have a *discrete* normal subgroup, namely its center $\{\pm 1_2\} \cong \mathbb{Z}_2$). Also, products of simple Lie groups are semi-simple. However, Corollary 5.56 does not apply to $SO(3)$, which is semi-simple but not simply connected. Here the relevant general result is:

Theorem 5.57. *Let G be a connected Lie group with $H^2(\mathfrak{g},\mathbb{R}) = 0$. Then*

$$H^2(G,\mathbb{T}) \cong \widehat{\pi_1(G)}. \tag{5.273}$$

We need some background (cf. §C.15). For any abelian (topological) group A, the set

$$\hat{A} = \mathrm{Hom}(A,\mathbb{T}) \tag{5.274}$$

consists of all (continuous) homomorphisms (also called ***characters***) $\chi : A \to \mathbb{T}$; these are just the irreducible (and hence necessarily one-dimensional) unitary representations of A. This set is a group under the obvious pointwise operations

$$\chi_1\chi_2(a) = \chi_1(a)\chi_2(a); \tag{5.275}$$

$$\chi^{-1}(a) = \chi(a)^{-1}. \tag{5.276}$$

As such, the group $\hat{A}$ is called the ***(Pontryagin) dual*** of A; the ***Pontryagin Duality Theorem*** states that $\hat{\hat{A}} \cong A$. Using Theorem 5.57 and Theorem 5.41, this gives

$$H^2(SO(3), \mathbb{T}) = \mathbb{Z}_2. \tag{5.277}$$

We now use Theorem 5.41 as a lemma to prove Theorem 5.57:

Proof. We first state the map $\widehat{\pi_1(G)} \to H^2(G, \mathbb{T})$ that will turn out to be an isomorphism. Assuming Theorem 5.41, pick a (Borel measurable) cross-section

$$\tilde{s} : G \to \tilde{G} \tag{5.278}$$

of the canonical projection

$$\tilde{\pi} : \tilde{G} \to G = \tilde{G}/D. \tag{5.279}$$

As always, this means that $\tilde{\pi} \circ \tilde{s} = \mathrm{id}_G$, and $\tilde{s}$ is supposed to be smooth near the identity, and chosen such that $\tilde{s}(e_G) = e_{\tilde{G}}$, where e_G and $e_{\tilde{G}}$ are the unit elements of G and $\tilde{G}$, respectively. Given a character $\chi \in \widehat{\pi_1(G)}$, define $c_\chi : G \times G \to \mathbb{T}$ by

$$c_\chi(x,y) = \chi(\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}). \tag{5.280}$$

This makes sense: $\tilde{\pi}$ is a homomorphism, so that (cf. the computation below (5.238))

$$\tilde{\pi}(\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}) = \tilde{\pi}(\tilde{s}(x))\tilde{\pi}(\tilde{s}(y))\tilde{\pi}(\tilde{s}(xy))^{-1} = xy(xy)^{-1} = e_G,$$

and hence $\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}) \in \ker(\tilde{\pi}) = D$ (where we identify D with $\pi_1(G)$, cf. Theorem 5.41). Furthermore, tedious computations show that (5.240) and (5.241) hold, so that $c_\chi \in Z^2(G, \mathbb{T})$. Different choices of $\tilde{s}$ lead to equivalent 2-cocycles c, and hence by taking the cohomology class $[c_\chi]$ of c_χ we obtain an injective map

$$\widehat{\pi_1(G)} \to H^2(G, \mathbb{T}); \tag{5.281}$$

$$\chi \mapsto [c_\chi]. \tag{5.282}$$

To prove surjectivity of this map, let $c \in Z^2(G, \mathbb{T})$ and define $\tilde{c} : \tilde{G} \times \tilde{G} \to \mathbb{T}$ by

$$\tilde{c}(\tilde{x}, \tilde{y}) = c(\tilde{\pi}(x), \tilde{\pi}(y)). \tag{5.283}$$

Conversely, we may recover c from $\tilde{c}$ and some cross-section $\tilde{s} : G \to \tilde{G}$ of $\tilde{\pi}$ by

$$c(x,y) = \tilde{c}(\tilde{s}(x), \tilde{s}(y)). \tag{5.284}$$

It follows that $\tilde{c} \in Z^2(\tilde{G}, \mathbb{T})$. Theorem 5.55 implies that $H^2(\tilde{G}, \mathbb{T})$ is trivial, so that

$$\tilde{c}(\tilde{x}, \tilde{y}) = \tilde{b}(\tilde{x}\tilde{y})/\tilde{b}(\tilde{x})\tilde{b}(\tilde{y}), \tag{5.285}$$

for some function $\tilde{b} : \tilde{G} \to \mathbb{T}$ satisfying $\tilde{b}(\tilde{e}) = 1$. From (5.241), i.e., $c(e,x) = 1$, we infer that if $\tilde{x} = \delta \in D$, so that $\tilde{\pi}(\delta) = e$, then $\tilde{c}(\delta, \tilde{y}) = 1$, and hence

$$\tilde{b}(\delta\tilde{y}) = \tilde{b}(\delta)\tilde{b}(\tilde{y}). \tag{5.286}$$

Taking $\tilde{x}$ and $\tilde{y}$ both in D, we see that $\tilde{b}_{|D}$ is a character, which we call χ. Hence

$$\begin{aligned} c(x,y) &= \frac{\tilde{b}(\tilde{s}(x)\tilde{s}(y))}{\tilde{b}(\tilde{s}(x))\tilde{b}(\tilde{s}(y))} = \frac{\tilde{b}(\tilde{s}(xy))}{\tilde{b}(\tilde{s}(x))\tilde{b}(\tilde{s}(y))} \cdot \frac{\tilde{b}(\tilde{s}(x)\tilde{s}(y))}{\tilde{b}(\tilde{s}(xy))} \\ &= \frac{\tilde{b}(\tilde{s}(xy))}{\tilde{b}(\tilde{s}(x))\tilde{b}(\tilde{s}(y))} \cdot c_\chi(x,y), \end{aligned} \tag{5.287}$$

since, using (5.286) with $\delta \rightsquigarrow \tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}$ and $\tilde{y} \rightsquigarrow \tilde{s}(xy)$, we have

$$\begin{aligned} \frac{\tilde{b}(\tilde{s}(x)\tilde{s}(y))}{\tilde{b}(\tilde{s}(xy))} &= \frac{\tilde{b}(\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}\tilde{s}(xy))}{\tilde{b}(\tilde{s}(xy))} = \tilde{b}(\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}) \\ &= \chi(\tilde{s}(x)\tilde{s}(y)\tilde{s}(xy)^{-1}) = c_\chi(x,y). \end{aligned}$$

Thus $[c] = [c_\chi]$, and hence the map (5.281) - (5.282) is surjective. □

Definition 5.58. *In the situation and notation of Theorem 5.41, a unitary representation $\tilde{u} : \tilde{G} \to U(H)$ is called* **admissible** *if $\tilde{u}(D) \subset \mathbb{T} \cdot 1_H$.*

In that case, there is obviously a character $\chi \in \hat{D}$ such that for each $\delta \in D$ we have

$$\tilde{u}(\delta) = \chi(\delta) \cdot 1_H. \tag{5.288}$$

Unitary irreducible representations are admissible, since Schur's Lemma implies that, since D lies in the center of $\tilde{G}$, its image $\tilde{u}(D)$ consists of multiples of the unit.

If $\tilde{u}$ is admissible, we obtain a homomorphism (5.231) by means of

$$h = \pi \circ \tilde{u} \circ \tilde{s}, \tag{5.289}$$

where $\tilde{s}$ is any cross-section of $\tilde{\pi}$, cf. (5.278) - (5.279). Note that different choices $\tilde{s}, \tilde{s}'$ are related by $\tilde{s}'(x) = \tilde{s}(x)\delta(x)$, where $\delta : G \to D$ is some function, so that

$$h'(x) = \pi(\tilde{u}(\tilde{s}'(x))) = \pi(\tilde{u}(\tilde{s}(x))\tilde{u}(\delta(x))) = \pi(\tilde{u}(\tilde{s}(x)))\pi(\delta(x) \cdot 1_H) = h(x).$$

Theorem 5.59. *1. If G is a connected Lie group with $H^2(\mathfrak{g}, \mathbb{R}) = 0$, any homomorphism $h : G \to U(H)/\mathbb{T}$ as in* (5.231) *comes from some admissible unitary representation $\tilde{u}$ of $\tilde{G}$ by* (5.289)*. If H is separable, then h is continuous iff $\tilde{u}$ is.*

2. Moreover, if $\tilde{u}(\tilde{G})$ is **super-admissible** *in that $\tilde{u}(\delta) = 1_H$ for each $\delta \in D$, then $u = \tilde{u} \circ \tilde{s}$ is a unitary representation of G, in which case $h = \pi \circ u$ therefore comes from a unitary representation of G itself.*

Proof. Given such a homomorphism h, pick a cross-section $s : U(H)/\mathbb{T} \to U(H)$, as in (5.234), with associated 2-cocycle c on G given by (5.239). By Theorem 5.57 and its proof, we may assume (possibly after redefining s) that there exists a character $\chi \in \hat{D}$ and a cross-section (5.278) such that $c = c_\chi$, cf. (5.280). We then define

$$\tilde{u} : \tilde{G} \to B(H); \tag{5.290}$$
$$\tilde{x} \mapsto \chi(\tilde{x} \cdot (\tilde{s} \circ \tilde{\pi}(\tilde{x}))^{-1}) u(\tilde{\pi}(\tilde{x})). \tag{5.291}$$

Simple computations then show that $\tilde{x} \cdot (\tilde{s} \circ \tilde{\pi}(\tilde{x}))^{-1} \in D$ (i.e., the center of $\tilde{G}$), that (5.288) holds, that each operator $\tilde{u}(\tilde{x})$ is unitary, that the group homomorphism properties $\tilde{u}(\tilde{x})\tilde{u}(\tilde{y}) = \tilde{u}(\tilde{x}\tilde{y})$ and $\tilde{u}(\tilde{e}) = 1_H$ hold, and that (5.289) is valid. As to the last equation, since π removes the term with χ in (5.291), and $u = s \circ h$, we have

$$\pi \circ \tilde{u} \circ \tilde{s}(x) = \pi \circ s \circ h \circ \tilde{\pi} \circ \tilde{s}(x) = h(x),$$

since $\pi \circ s = \mathrm{id}$ (on $U(H)/\mathbb{T}$) and $\tilde{\pi} \circ \tilde{s} = \mathrm{id}$ (on G).

If $\tilde{u}(\delta) = 1_H$ for each $\delta \in D$, then $c_\chi = 1$ from (5.280), so that $u(x)u(y) = u_{xy}$ by (5.238). If s preserves units, or, equivalently, if $h_e = 1_H$, as we always assume, we see that u is a unitary representation of G. In this case, (5.291) simply reads $\tilde{u} = s \circ h \circ \tilde{\pi}$. This immediately yields $\tilde{u} = u \circ \tilde{\pi}$, which in turn gives $u = \tilde{u} \circ \tilde{s}$.

Finally, even if h is continuous, it is *a priori* unclear if $\tilde{u}$ is, since the cross-sections s and $\tilde{s}$ appearing in the above construction typically fail to be continuous. Fortunately, since they are assumed measurable, there is no question about measurability of $\tilde{u}$, and if H is separable, continuity follows from Proposition 5.36. □

Corollary 5.60. *If G is a connected Lie group with covering group $\tilde{G}$, the formulae*

$$\tilde{u} = u \circ \tilde{\pi}; \tag{5.292}$$
$$u = \tilde{u} \circ \tilde{s}, \tag{5.293}$$

where $\tilde{s} : G \to \tilde{G}$ is any cross-section of the covering map $\tilde{\pi} : \tilde{G} \to G$, give a bijective correspondence between (continuous) super-admissible unitary representations $\tilde{u}$ of $\tilde{G}$ and (continuous) unitary representations u of G, preserving irreducibility.

Corollary 5.61. *Any homomorphism $h : SO(3) \to U(H)/\mathbb{T}$ as in* (5.231) *comes from an admissible unitary representation $\tilde{u}$ of $SU(2)$ by* (5.289). *Moreover, h comes from a unitary representation $u = \tilde{u} \circ \tilde{s}$ of $SO(3)$ itself iff $\tilde{u}$ is trivial on the center $\mathbb{Z}_2$.*

In particular, if h is irreducible, it must come from the unitary irreducible representations $\tilde{u} = D_j$, where $j = 0, \frac{1}{2}, 1, \ldots$ is the (half-) integer **spin** *label. Then $D_j(SU(2))$ is super-admissible iff j is integral, in which case it defines a unitary irreducible representation of $SO(3)$.*

Indeed, the assumption $H^2(\mathfrak{g}, \mathbb{R}) = 0$ in Theorem 5.59 is satisfied for $SO(3)$ because of Whitehead's Lemma 5.54. The case where $H^2(\mathfrak{g}, \mathbb{R}) \neq 0$ occurs e.g. for the Galilei group (cf. §7.6). It can be shown that $H^2(\mathfrak{g}, \mathbb{R})$ has finitely many generators, for which one finds pre-images $(\varphi_1, \ldots, \varphi_M)$ in $Z^2(\mathfrak{g}, \mathbb{R})$, with corresponding elements $(\Gamma_1, \ldots, \Gamma_M)$ of $Z^2(\tilde{G}, \mathbb{R})$, cf. the proof of Theorem 5.55. Of these, a subset $(\Gamma_1, \ldots, \Gamma_N)$, $N \leq M$, satisfies the relation $\Gamma_i(\delta, \tilde{x}) = \Gamma_i(\tilde{x}, \delta)$ for any $\delta \in D$ (cf. Theorem 5.41) and $\tilde{x} \in \tilde{G}$. This yields a map $\Gamma : \tilde{G} \times \tilde{G} \to \mathbb{R}^N$ given by $\Gamma(\tilde{x}, \tilde{y}) = (\Gamma_1(\tilde{x}, \tilde{y}), \ldots, \Gamma_N(\tilde{x}, \tilde{y}))$, which in turn equips the set

$$\check{G} = \tilde{G} \times \mathbb{R}^N, \tag{5.294}$$

with a group multiplication $(\tilde{x},v)\cdot(\tilde{y},w)=(\tilde{x}\tilde{y},v+w+\Gamma(\tilde{x},\tilde{y}))$. We then have the following generalization of Theorem 5.59, in which a unitary representation u of $\check{G}$ is called ***admissible*** if $u(\delta,v)\in\mathbb{T}\cdot 1_H$ for any $\delta\in D$ and $v\in\mathbb{R}^N$.

Theorem 5.62. *Let G be a connected Lie group, and H a separable Hilbert space. Then any continuous homomorphism $h:G\to U(H)/\mathbb{T}$ comes from some admissible continuous unitary representation $\tilde{u}$ of $\check{G}$.*

As we only apply this to the Galilei group (where $N=1$), basically only for illustrative purposes, we omit the proof. The correct (and natural) notion of equivalence of projective representations is as follows: we say that two such homomorphisms $h_i:G\to U(H_i)/\mathbb{T}$, $i=1,2$ are ***equivalent*** if there is a unitary $w:H_1\to H_2$ such that

$$\mathrm{Ad}_w(h_1(x))=h_2(x),\ x\in G, \tag{5.295}$$

where $\mathrm{Ad}_w:U(H_1)/\mathbb{T}\to U(H_2)/\mathbb{T}$ is the map $[u]\mapsto[vuv^*]$, which is well defined (here $[u]$ is the equivalence class of $u\in U(H)$ in $U(H)/\mathbb{T}$ under $u\sim zu$, $z\in\mathbb{T}$).

This induces the following notion for $\check{G}$: two admissible unitary representations $\tilde{u}_1,\tilde{u}_2$ of $\tilde{G}$ on Hilbert spaces H_1,H_2 are ***equivalent*** if there is a unitary $w:H_1\to H_2$ *and* a map $b:\check{G}\to\mathbb{T}$ such that $wu_1(\check{x})w^*=b(\check{x})u_2(\check{x})$, for any $\check{x}\in\check{G}$. It can be shown that such a map b always comes from a character $\chi:\tilde{G}\to\mathbb{T}$ through $b(\tilde{x},v)=\chi(\tilde{x})$.

To close this long and difficult section, in relief it should be mentioned that the above theory vastly simplifies if H is finite-dimensional. By Theorem 5.40, this is true, for example, if G is compact and u is irreducible. Suppose $u:G\to U(H)$ is merely a projective unitary representation of G, so that instead of (5.157) one has

$$[u'(X),u'(Y)]=u'([X,Y])+i\varphi(X,Y)\cdot 1_H, \tag{5.296}$$

where φ is given by (5.267). Taking the trace yields

$$\varphi(X,Y)=\frac{i}{n}\mathrm{Tr}\,(u'([X,Y])), \tag{5.297}$$

where $n=\dim(H)<\infty$. We may define a linear function $\theta:\mathfrak{g}\to\mathbb{R}$ by

$$\theta(X)=\frac{i}{n}\mathrm{Tr}\,(u'(X)), \tag{5.298}$$

so that $\varphi(X,Y)=\theta([X,Y])$, cf. (5.253), and hence we may remove φ by redefining

$$\tilde{u}'(X)=u'(X)+i\theta(X)\cdot 1_H, \tag{5.299}$$

which satisfies (5.157) - (5.158). Hence by Corollary 5.43 the map $\tilde{u}'$ exponentiates to a unitary representation $\tilde{u}$ of the universal covering group $\tilde{G}$ of G; it should be checked from the values of $\tilde{u}$ on D if $\tilde{u}$ also defines a unitary representation of G. This argument shows that *finite-dimensional* projective unitary representations of Lie groups always come from unitary representations of the covering group.

5.11 Position, momentum, and free Hamiltonian

The three basic operators of non-relativistic quantum mechanics are position, denoted q, momentum, p, and the free Hamiltonian h_0. Assuming for simplicity that the particle moves in one dimension, these are informally given on $H = L^2(\mathbb{R})$ by

$$q\psi(x) = x\psi(x); \tag{5.300}$$

$$p\psi(x) = -i\hbar\frac{d}{dx}\psi(x); \tag{5.301}$$

$$h_0\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x), \tag{5.302}$$

where m is the mass of the particle under consideration. We put $\hbar = 1$ and $m = 1/2$.

The issue is that these operators are unbounded; see §B.13. In general, quantum-mechanical observables are supposed to be represented by self-adjoint operators, and examples like (5.300) - (5.302) show that these may not be bounded. The Hellinger–Toeplitz Theorem B.68 then shows that it makes no sense to try and extend the above expressions to all of $L^2(\mathbb{R})$, so we have to live with the fact that some crucial operators $a : D(a) \to H$ are merely defined on a dense subspace $D(a) \subset H$.

Each such operator has an ***adjoint*** $a^* : D(a^*) \to H$, whose domain $D(a^*) \subset H$ consists of all $\psi \in H$ for which the functional $\varphi \mapsto \langle\psi, a\varphi\rangle$ is bounded on $D(a)$, and hence (since $D(a)$ is dense in H) can be extended to all of H by continuity through the unique "Riesz–Fréchet vector" χ for which $\langle\psi, a\varphi\rangle = \langle\chi, \varphi\rangle$. Writing $\chi = a^*\psi$, for each $\psi \in D(a^*)$ and $\varphi \in D(a)$ we therefore have

$$\langle a^*\psi, \varphi\rangle = \langle\psi, a\varphi\rangle. \tag{5.303}$$

Assuming that $D(a)$ is dense in H, we say that a is ***self-adjoint***, written $a^* = a$, if

$$\langle a\varphi, \psi\rangle = \langle\varphi, a\psi\rangle, \tag{5.304}$$

for each $\psi, \varphi \in D(a)$ *and* $D(a^*) = D(a)$. A self-adjoint operator a is automatically ***closed***, in that its graph $G(a) = \{(\psi, a\psi) \mid \psi \in D(a)\}$ is a closed subspace of the Hilbert space $H \oplus H$ (indeed, the adjoint of any densely defined operator is closed, see Proposition B.72). In practice, self-adjoint operators often arise as closures of ***essentially self-adjoint*** operators a, which by definition satisfy $a^{**} = a^*$. Equivalently, such an operator is ***closable***, in that the closure of its graph is the graph of some (uniquely defined) operator, called the ***closure*** a^- of a, and furthermore this closure is self-adjoint, so that $a^- = a^*$. If a is closable, the domain $D(a^-)$ of its closure consists of all $\psi \in H$ for which there exists a sequence (ψ_n) in $D(a)$ such that $\psi_n \to \psi$ *and* $a\psi_n$ converges, on which we define a^- by $a^-\psi = \lim_n a\psi_n$.

The simplest case is the position operator.

Theorem 5.63. *The operator q is self-adjoint on the domain*

$$D(q) = \{\psi \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} dx x^2 |\psi(x)|^2 < \infty\}. \tag{5.305}$$

See Proposition B.73 for the proof. To give a convenient domain of essential self-adjointness (also for the other two operators), we need a little distribution theory.

Definition 5.64. *The* **Schwartz space** $\mathscr{S}(\mathbb{R})$ *(whose elements are* **functions of rapid decrease***) consist of all smooth function $f:\mathbb{R}\to\mathbb{C}$ for which each expression*

$$\|f\|_{n,m} = \sup\{|x^n f^{(m)}(x)|, x\in\mathbb{R}\}, \tag{5.306}$$

where $f^{(m)}$ is the m'th derivative of f, is finite. The topology of $\mathscr{S}(\mathbb{R})$ is given by saying that a sequence (or net) f_λ converges to f iff $\|f_\lambda - f\|_{n,m}\to 0$ for all $n,m\in\mathbb{N}$.

Each $\|\cdot\|_{n,m}$ happens to be a norm, but positive definiteness is nowhere used in the theory below (which therefore works for families of ***seminorms***, which satisfy the axioms of a norm expect perhaps for positive definiteness). Since there are countably many such (semi)norms defining the topology, we may equivalently say that $\mathscr{S}(\mathbb{R})$ is a metric space defined by

$$d(f,g) = \sum_{n,m=0}^{\infty} 2^{-n}\frac{\|f-g\|_{n,m}}{1+\|f-g\|_{n,m}}. \tag{5.307}$$

Indeed, $\mathscr{S}(\mathbb{R})$ is complete in this metric. A typical element is $f(x)=\exp(-x^2)$.

Definition 5.65. *A* **tempered distribution** *is a continuous linear map $\varphi:\mathscr{S}(\mathbb{R})\to\mathbb{C}$. The space of all such maps, equipped with the topology of pointwise convergence (i.e., $\varphi_\lambda\to\varphi$ iff $\varphi_\lambda(f)\to\varphi(f)$ for each $f\in\mathscr{S}(\mathbb{R})$) is denoted by $\mathscr{S}'(\mathbb{R})$.*

It can be shown that (because of the metrizability of $\mathscr{S}(\mathbb{R})$) continuity is the same as sequential continuity, i.e., some linear map $\varphi:\mathscr{S}(\mathbb{R})\to\mathbb{C}$ belongs to $\mathscr{S}'(\mathbb{R})$ iff $\lim_N \varphi(f_N)=\varphi(f)$ for each convergent *sequence* $f_N\to f$ in $\mathscr{S}(\mathbb{R})$. Like $\mathscr{S}(\mathbb{R})$, the tempered distributions $\mathscr{S}'(\mathbb{R})$ form a (locally convex) ***topological vector space***, that is, a vector space with a topology in which addition and scalar multiplication are continuous. The topology of $\mathscr{S}'(\mathbb{R})$ is given by a family of seminorms, namely $\|\varphi\|_f = |\varphi(f)|$, $f\in\mathscr{S}(\mathbb{R})$, and hence a simple way to prove that $\varphi\in\mathscr{S}'(\mathbb{R})$ is to find some (n,m) for which $|\varphi(f))|\leq C\|f\|_{n,m}$ for each $f\in\mathscr{S}(\mathbb{R})$, since in that case $f_N\to f$, which means that $\|f_N - f\|_{n,m}\to 0$ for *all* $n,m\in\mathbb{N}$, certainly implies that $\varphi(f_N)\to\varphi(f)$, so that φ is continuous. For example, the evaluation maps δ_x defined by $\delta_x(f)=f(x)$ are continuous (take $n=m=0$). Similarly, each finite measure on $\mathbb{R}$ defines a tempered distribution. Taking the $(0,m)$ seminorm shows that the maps $f\mapsto f^{(m)}(x)$ for fixed $m\in\mathbb{N}$ and $x\in\mathbb{R}$ are tempered distributions.

A less obvious example (defining a so-called ***Gelfand triple***) is as follows:

Proposition 5.66. *We have continuous dense inclusions*

$$\mathscr{S}(\mathbb{R})\subset L^2(\mathbb{R})\subset\mathscr{S}'(\mathbb{R}), \tag{5.308}$$

where the second inclusion identifies $\varphi\in L^2(\mathbb{R})$ with the map

$$f\mapsto\langle\overline{\varphi},f\rangle = \int_{\mathbb{R}} dx\,\varphi(x)f(x). \tag{5.309}$$

Proof. As vector spaces, the first inclusion is obvious. For $f \in \mathscr{S}(\mathbb{R})$ we estimate

$$\|f\|_2^2 = \int_{\mathbb{R}} dx\,|f(x)| \cdot |f(x)| \leq \|f\|_1 \|f\|_\infty; \tag{5.310}$$

$$\begin{aligned}\|f\|_1 &= \int_{\mathbb{R}} dx\, \frac{(1+x^2)|f(x)|}{1+x^2} \leq \int_{\mathbb{R}} dy\, \frac{1}{1+y^2}\, \|(1+m_{x^2})f\|_\infty \\ &\leq \pi(\|f\|_{0,0} + \|f\|_{2,0}),\end{aligned} \tag{5.311}$$

so that, noting that $\|\cdot\|_{0,0} = \|\cdot\|_\infty$, we have

$$\|f\|_2^2 \leq \pi(\|f\|_\infty + \|f\|_{2,0})\|f\|_\infty. \tag{5.312}$$

Hence $f_\lambda \to f$ in $\mathscr{S}(\mathbb{R})$, which incorporates the conditions $\|f_\lambda - f\|_{0,0} \to 0$ and $\|f_\lambda - f\|_{2,0} \to 0$, implies $\|f_\lambda - f\|_2 \to 0$. This shows that the first inclusion in (5.308) is continuous. Density may be proved in two steps. First, take some fixed positive function $h \in C_c^\infty(-1,1)$ with the property $\int dx\, h(x) = 1$, and define $h_n(x) = nh(nx)$, so that informally $h_n \in C_c^\infty(\mathbb{R})$ converges to a δ-function as $n \to \infty$. For each $\psi \in L^2(\mathbb{R})$, we consider the convolution $h_n * \psi$, where for suitable f, g,

$$f * g(x) \equiv \int_{\mathbb{R}} dy\, f(x-y)g(y). \tag{5.313}$$

Then $h_n * \psi \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and, from elementary analysis, $\|h_n * \psi - \psi\| \to 0$.

Second, for $\psi \in C_c(\mathbb{R})$, the functions $h_n * \psi$ lie in $C_c^\infty(\mathbb{R})$ and hence in $\mathscr{S}(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ by Theorem B.30, for $\psi \in L^2(\mathbb{R})$ and $\varepsilon > 0$ we can find $\varphi \in C_c(\mathbb{R})$ such that $\|\psi - \varphi\| < \varepsilon/2$, and (as just shown) find n such that $\|\varphi - \varphi_n\| < \varepsilon/2$, whence $\|\psi - \varphi_n\| < \varepsilon$. This proves that $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

The second inclusion is continuous by Cauchy–Schwarz, which gives

$$|\varphi(f)| \leq \|\varphi\|_2 \|f\|_2,$$

to be combined with (5.312). It should be noted that also the second inclusion in (5.308) is indeed an injection, i.e., that $\varphi(f) = 0$ for each $f \in \mathscr{S}(\mathbb{R})$ implies $\varphi = 0$ in $L^2(\mathbb{R})$; this is true because $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, plus the standard fact that, in any Hilbert space H, if $\langle \varphi, f \rangle = 0$ for all f in some dense subspace of H, then $\varphi = 0$. Finally, the fact that $L^2(\mathbb{R})$ is dense in the seemingly huge space $\mathscr{S}'(\mathbb{R})$ follows from the even more remarkable fact that $\mathscr{S}(\mathbb{R})$ is dense in $\mathscr{S}'(\mathbb{R})$. On top of the functions h_n just defined, also employ a function $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(x) = 1$ on $(-1,1)$, and define $\chi_n(x) = \chi(x/n)$, so that informally $\lim_{n\to\infty} \chi(x) = 1$ (as opposed to the h_n, which converge to a δ-function as $n \to \infty$). If for any $g \in \mathscr{S}(\mathbb{R})$ and any $\varphi \in \mathscr{S}'(\mathbb{R})$ we define $g\varphi$ as the distribution that maps $f \in \mathscr{S}(\mathbb{R})$ to $\varphi(fg)$, and similarly define $g * \varphi$ as the distribution that maps f to $\varphi(g * f)$, we may define a sequence of distributions $\varphi_n = h_n * (\chi_n \varphi)$. From the point of view of (5.308), these correspond to functions $\varphi_n \in \mathscr{S}(\mathbb{R})$ in the sense that $\varphi_n(f) = \int dx\, \varphi_n(x) f(x)$, where $f \in \mathscr{S}(\mathbb{R})$. Using similar analysis as above, it then follows that for any $f \in \mathscr{S}(\mathbb{R})$ we have $\varphi_n(f) \to \varphi(f)$, so that $\varphi_n \to \varphi$ in $\mathscr{S}'(\mathbb{R})$. □

For our purposes, the point of all this is that we can define generalized derivatives of (tempered) distributions, and hence, because of (5.308), of functions in $L^2(\mathbb{R})$.

Definition 5.67. *For $\varphi \in \mathscr{S}(\mathbb{R})'$ and $m \in \mathbb{N}$, the m'th* **generalized derivative** $\varphi^{(m)}$ *is defined by*

$$\varphi^{(m)}(f) = (-1)^m \varphi(f^{(m)}). \tag{5.314}$$

The idea is that under (5.308) this is an identity if $\varphi \in \mathscr{S}(\mathbb{R})$ (partial integration). Like the constructions at the end of the proof of Proposition 5.66, this is a special case of a more general construction: whenever we have a continuous linear map $T : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$, we obtain a dual continuous linear map $T' : \mathscr{S}(\mathbb{R})' \to \mathscr{S}(\mathbb{R})'$ defined by $T'\varphi = \varphi \circ T$, i.e.,

$$(T'\varphi)(f) = \varphi(T(f)). \tag{5.315}$$

Sometimes a slight change in the definition (as in (5.314), or as in the Fourier transform below) is appropriate so that the restriction of T' to $\mathscr{S}(\mathbb{R})$ coincides with T.

Theorem 5.68. *The momentum operator $p = -id/dx$ is self-adjoint on the domain*

$$D(p) = \{\psi \in L^2(\mathbb{R}) \mid \psi' \in L^2(\mathbb{R})\}, \tag{5.316}$$

where the derivative ψ' is taken in the distributional sense (i.e., letting $\psi \in \mathscr{S}'(\mathbb{R})$).

Proof. We first show that p is symmetric, or $p \subseteq p^*$. This comes down to

$$\langle \psi', \varphi \rangle = -\langle \psi, \varphi' \rangle, \tag{5.317}$$

for each $\psi, \varphi \in D(p)$, where both derivates are "generalized". The most elegant proof (though perhaps not the shortest) uses the Sobolev space $H^1(\mathbb{R})$, which equals $D(p)$ as a vector space, now equipped, however, with the new inner product

$$\langle \psi, \varphi \rangle_{(1)} = \langle \psi, \varphi \rangle + \langle \psi', \varphi' \rangle, \tag{5.318}$$

with both inner products on the right-hand side in $L^2(\mathbb{R})$; the associated norm is

$$\|\psi\|^2_{(1)} = \|\psi\|^2 + \|\psi'\|^2. \tag{5.319}$$

Similar to the Gelfand triple (5.308), we have dense continuous inclusions

$$\mathscr{S}(\mathbb{R}) \subset H^1(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}), \tag{5.320}$$

with analogous proof. All we need for Theorem 5.68 is the first inclusion of the triple (5.320): for $\psi \in H^1(\mathbb{R})$ we now have $h_n * \psi \in C^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ as well as $h_n * \psi \to \psi$ in $H^1(\mathbb{R})$, both of which follow from the L^2-case plus the identity

$$(h_n * \psi)' = h_n * \psi'. \tag{5.321}$$

Using the same cutoff function χ as in the L^2 case, we have $\chi_n \psi \to \psi$ and $\chi_n' \psi \to 0$ in $L^2(\mathbb{R})$, so that $(\chi_n \psi)' \to \psi'$ in $L^2(\mathbb{R})$ and hence $\chi_n \psi \to \psi$ also in $H^1(\mathbb{R})$.

Furthermore, the functions $\psi_n = h_n * (\chi_n \psi)$ lie in $C_c^\infty(\mathbb{R})$ and hence in $\mathscr{S}(\mathbb{R})$; using the above facts we obtain $\psi_n \to \psi$ in $H^1(\mathbb{R})$. In sum, for each $\psi \in H^1(\mathbb{R})$ we can find a sequence (ψ_n) in $\mathscr{S}(\mathbb{R})$ such that $\psi_n \to \psi$ *and* $\psi_n' \to \psi'$ in $L^2(\mathbb{R})$. Hence

$$\langle \psi, \varphi' \rangle = \lim_n \langle \psi_n, \varphi' \rangle = -\lim_n \langle \psi_n', \varphi \rangle = -\langle \psi', \varphi' \rangle. \tag{5.322}$$

For the converse, let $\psi \in D(p^*)$, so that by definition for each $\varphi \in D(p)$ we have

$$\langle p^* \psi, \varphi \rangle = \langle \psi, p\varphi \rangle = -i\langle \psi, \varphi' \rangle. \tag{5.323}$$

Since $\mathscr{S}(\mathbb{R}) \subset D(p)$, this is true in particular for each $\varphi \in \mathscr{S}(\mathbb{R})$, in which case the right-hand side equals $-i\psi'(\varphi)$, where the derivative is distributional. But this equals $\langle p^* \psi, \varphi \rangle$ and so the distribution $-i\psi'$ is given by taking the inner product with $p^* \psi \in L^2(\mathbb{R})$. Hence $-i\psi' = p^* \psi \in L^2(\mathbb{R})$, and in particular $\psi' \in L^2(\mathbb{R})$, so that $\psi \in D(p)$. This proves that $D(p^*) \subseteq D(p)$, and since from the first step we have the oppositie inclusion, we find $D(p^*) = D(p)$ and $p^* = p$. □

For the free Hamiltonian $h_0 = -\Delta$ with $\Delta = d^2/dx^2$, we similarly have:

Theorem 5.69. *The free Hamiltonian $h_0 = -\Delta$ is self-adjoint on the domain*

$$D(\Delta) = \{\psi \in L^2(\mathbb{R}) \mid \psi'' \in L^2(\mathbb{R})\}, \tag{5.324}$$

where the double derivative ψ'' is taken in the distributional sense.

Although this may be proved in an analogous way, such proofs are increasingly burdensome if the number of derivatives gets higher. It is easier to use the Fourier transform (which also provided an alternative way of proving Theorem 5.68).

Theorem 5.70. *The formulae*

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x); \tag{5.325}$$

$$\check{f}(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikx} f(k), \tag{5.326}$$

are rigorously defined on $\mathscr{S}(\mathbb{R})$, $L^2(\mathbb{R})$, and $\mathscr{S}'(\mathbb{R})$, and provide continuous isomorphisms of each of these spaces. Furthermore, (5.326) *is inverse to* (5.325)*, i.e.*

$$\check{\hat{f}} = \hat{\check{f}} = f, \tag{5.327}$$

so that we may (and often do) write $\hat{f} = \mathscr{F}(f)$ and $\check{f} = \mathscr{F}^{-1}(f)$, or $f = \mathscr{F}^{-1}(\hat{f})$.

In all three cases we have the identities (in a distributional sense if appropriate)

$$\mathscr{F}(x^n f^{(m)})(k) = (id/dk)^n (ik)^m \mathscr{F}(f)(k). \tag{5.328}$$

Finally, as a map $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Fourier transform is unitary, so that

$$\langle \hat{\psi}, \hat{\varphi} \rangle = \langle \psi, \varphi \rangle. \tag{5.329}$$

See §C.15 for further discussion. For example, we have

$$D(p) = \{\psi \in L^2(\mathbb{R}) \mid k \cdot \hat{\psi}(k) \in L^2(\mathbb{R})\}; \tag{5.330}$$
$$D(\Delta) = \{\psi \in L^2(\mathbb{R}) \mid k^2 \cdot \hat{\psi}(k) \in L^2(\mathbb{R})\}. \tag{5.331}$$

Thus we may now reformulate Theorems 5.68 and 5.69 as follows:

Theorem 5.71. *The momentum operator p is self-adjoint on the domain* (5.330). *The free Hamiltonian $h_0 = -\Delta$ is self-adjoint on the domain* (5.331).

Proof. Denoting multiplication by x^n by the symbol k^n, we have

$$p = \mathscr{F}^{-1} k \mathscr{F}; \tag{5.332}$$
$$\Delta = -\mathscr{F}^{-1} k^2 \mathscr{F}. \tag{5.333}$$

Hence the theorem follows from Proposition B.73 and unitarity of the Fourier transform $\mathscr{F}$ (plus the little observation that if $a = a^*$ on $D(a) \subset H$ and $u : H \to K$ is unitary, then $b = uau^*$ is self-adjoint on $D(b) = uD(a) \subset K$). □

Much is known about regularity properties of functions in such domains, e.g.,

$$D(p) \subset C_0(\mathbb{R}); \tag{5.334}$$
$$D(\Delta) \subset C_0^{(1)}(\mathbb{R}). \tag{5.335}$$

These are the most elementary cases of the famous ***Sobolev Embedding Theorem***.

If $\psi \in D(p)$, then $k \mapsto (1+k^2)^{1/2}\hat{\psi}(k)$ is in $L^2(\mathbb{R})$, so applying Hölder's inequality (B.15) with $p = q = 2$ to $f(k) = (1+k^2)^{1/2}\hat{\psi}(k)$ and $g(k) = (1+k^2)^{-1/2}$, which is in $L^2(\mathbb{R})$, too, gives $\hat{\psi} \in L^1(\mathbb{R})$. The Riemann–Lebesgue Lemma (see §C.15) then yields $\psi \in C_0(\mathbb{R})$. To prove (5.335), one uses $(1+k^2)$ rather than its square root.

Finally, we give a common domain of essential self-adjointness for q, p, and h_0.

Proposition 5.72. *The operators q, p, and h_0 are essentially self-adjoint on $\mathscr{S}(\mathbb{R})$.*

Proof. We see from (5.332) that the cases of p and q are similar, so we only explain the case of q. Denoting the operator of multiplication by x on the domain $\mathscr{S}(\mathbb{R})$ by q_0, as in the proof of Proposition B.73 it is easy to see that $D(q_0^*) = D(q)$. Fourier-transforming, the fact that $\mathscr{S}(\mathbb{R})$ is dense in $H^1(\mathbb{R})$ (cf. the proof of Theorem 5.68) shows that $D(q_0^-) = D(q)$,so that $D(q_0^*) = D(q_0^-)$. The actions of q_0^* and q_0^- obviously being given by multiplication by x in both cases, we have $q_0^* = q_0^-$.

The proof for h_0 is similar; in the second step we now use the fact that $\mathscr{S}(\mathbb{R})$ is dense in $H^2(\mathbb{R})$, defined as $D(\Delta)$, as in (5.324), but now seen as a Hilbert space in the inner product $\langle \psi, \varphi \rangle_{(2)} = \langle \psi, \varphi \rangle + \langle \psi'', \varphi'' \rangle$, with corresponding norm given by $\|\psi\|_{(2)}^2 = \|\psi\|^2 + \|\psi''\|^2$. This is proved just as in the case of a single derivative. □

We also say that $\mathscr{S}(\mathbb{R})$ is a ***core*** for the operators in question. For example, the canonical commutation relations $[q, p] = i\hbar \cdot 1_H$ rigorously hold on this domain.

5.12 Stone's Theorem

We now come to a central result on symmetries in quantum mechanics "explaining" the Hamiltonian. Recall that a continuous unitary representation of $\mathbb{R}$ (as an additive group) on a Hilbert space H is a map $t \mapsto u_t$, where $t \in \mathbb{R}$ and each $u_t \in B(H)$ is unitary, such that the associated map $\mathbb{R} \times H \to H$, $(t, \psi) \mapsto u_t\psi$, is continuous, and

$$u_s u_t = u_{s+t}, \; s,t \in \mathbb{R}; \tag{5.336}$$

$$u_0 = 1_H; \tag{5.337}$$

$$\lim_{t\to 0} u_t\psi = \psi \;\; (t \in \mathbb{R}, \, \psi \in H). \tag{5.338}$$

These conditions imply

$$\lim_{t\to s} u_t\psi = u_s\psi \;\; (s,t \in \mathbb{R}, \psi \in H). \tag{5.339}$$

Note that according to Proposition 5.36 continuity may be replaced by weak measurability. Probably the simplest nontrivial example is given by $H = L^2(\mathbb{R})$ and

$$u_t\psi(x) = \psi(x-t). \tag{5.340}$$

To prove (5.338), we use a routine $\varepsilon/3$ argument. We first prove (5.338) for $\psi \in C_c(\mathbb{R})$, where it is elementary in the sup-norm, i.e., $\lim_{t\to 0} \|u_t\psi - \psi\|_\infty = 0$ by continuity and hence (given compact support) uniform continuity of ψ. But then the (ugly) estimate $\|\psi\|_2^2 \leq |K| \|\psi\|_\infty$, where $K \subset \mathbb{R}$ is any compact set containing the support of ψ, also yields $\lim_{t\to 0} \|u_t\psi - \psi\|_2 = 0$. Hence for $\varepsilon > 0$ we may find $\delta > 0$ such that $\|u_t\psi - \psi\|_2 < \varepsilon/3$ whenever $|t| < \delta$. For general $\psi' \in H$, we find $\psi \in C_c(\mathbb{R})$ such that $\|\psi - \psi'\| < \varepsilon/3$, and, using unitarity of u_t, estimate

$$\begin{aligned}\|u_t\psi' - \psi'\| &\leq \|u_t\psi' - u_t\psi\| + \|u_t\psi - \psi\| + \|\psi - \psi'\| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

In the context of quantum mechanics, physicists formally write

$$u_t = e^{-ita}, \tag{5.341}$$

where a is usually thought of as the Hamiltonian of the system, although in the previous example it is rather the momentum operator. In any case, we avoid the notation h instead of a here, partly in order to rightly suggest far greater generality of the construction and partly to avoid confusion with the notation in §B.21; if h *is* the Hamiltonian, one would have $a = h/\hbar$ in (5.341). Mathematically speaking, if a is self-adjoint, eq. (5.341) is rigorously defined by Theorem B.158, where

$$e_t(x) = \exp(-itx). \tag{5.342}$$

Conversely, given a continuous unitary representation $t \mapsto u_t$ of $\mathbb{R}$ on H, one may attempt to define an operator a by specifying its domain and action by

$$D(a) = \left\{ \psi \in H \mid \lim_{s\to 0} \frac{u_s - 1}{s} \psi \text{ exists} \right\}; \tag{5.343}$$

$$a\psi = i \lim_{s\to 0} \frac{u_s - 1}{s} \psi \ (\psi \in D(a)). \tag{5.344}$$

Stone's Theorem makes this rigorous, and even turns the passage from the generator a to the unitary group $t \mapsto u_t$ (and back) into a bijective correspondence.

Theorem 5.73. *1. If $a : D(a) \to H$ is self-adjoint, the map $t \mapsto u_t$ defined by* (5.341)*, which is rigorously defined by Proposition B.159 with* (5.342)*, defines a continuous unitary representation of $\mathbb{R}$ on H.*
2. Conversely, given such a representation, the operator a defined by (5.343) - (5.344) *is self-adjoint; in particular, $D(a)$ is dense in H.*
3. These constructions are mutually inverse.

Proof. We use the setting of §B.21, so that b is the bounded transform of a.

1. Eqs. (5.336) - (5.337) are immediate from Theorem B.158, which also yields unitarity of each operator u_t. To prove (5.338) we first take $\varphi \in C_c^*(b)H$, which means that φ is a finite linear combinations of vectors of the type $\varphi = h(a)\psi$, where $h \in C_c(\sigma(a))$ and $\psi \in H$. Using (5.342) and (B.573), we have

$$\|u_t\varphi - \varphi\| \leq \|e_t h - h\|_\infty \|\psi\| \leq \|h\|_\infty \|e_t - 1_K\|_\infty^{(K)} \|\psi\|, \tag{5.345}$$

   where K is the (compact) support of h in $\sigma(b)$. Since the exponential function is uniformly convergent on any compact set, this gives $\lim_{t\to 0} \|u_t\varphi - \varphi\| = 0$. Taking finite linear combinations of such vectors φ gives the same result for any $\varphi \in C_c^*(b)H$ (with an extra step this could have been done on $C_0^*(b)H$, too). Thus for $\varepsilon > 0$ we can find $\delta > 0$ so that $\|u_t\varphi - \varphi\| < \varepsilon/3$ whenever $|t| < \delta$. For general $\psi' \in H$, we find $\varphi \in C_0^*(b)H$ such that $\|\varphi - \psi'\| < \varepsilon/3$, and estimate

$$\begin{aligned} \|u_t\psi' - \psi'\| &\leq \|u_t\psi' - u_t\varphi\| + \|u_t\varphi - \varphi\| + \|\varphi - \psi'\| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

   since $\|u_t\psi' - u_t\varphi\| = \|\psi' - \varphi\|$ by unitarity of u_t. This is equivalent to (5.338).
2. For any $\psi \in H$ and $n \in \mathbb{N}$, define $\psi_n \in H$ by

$$\psi_n = n \int_0^\infty ds\, e^{-ns} u_s \psi, \tag{5.346}$$

   either as a Riemann-type integral (whose approximants converge in norm) or as a functional $\varphi \mapsto n \int_0^\infty ds\, e^{-ns} \langle u_s\psi, \varphi\rangle$, which is obviously continuous and hence is represented by a unique vector $\psi_n \in H$. Then simple computations show that

$$\lim_{s\to 0} \frac{u_s - 1}{s} \psi_n = n(\psi_n - \psi),$$

so that $\psi_n \in D(a)$. The proof that $\psi_n \to \psi$ starts with the elementary estimate

$$\|\psi_n - \psi\| \leq n \int_0^\infty ds\, e^{-ns} \|u_s\psi - \psi\|,$$

in which we split up the $\int_0^\infty$ as $\int_0^\delta \cdots + \int_\delta^\infty \cdots$, where $\delta > 0$. Using strong continuity of the map $t \mapsto u_t$, i.e., (5.338), for any n the first integral vanishes as $\delta \to 0$. In the second integral we estimate $\|u_s\psi - \psi\| \leq 2\|\psi\|$ and take the limit $n \to \infty$. Thus $\psi_n \to \psi$, so that $D(a)$ is dense in H.
To prove self-adjointness of a, we need a tiny variation on Theorem B.93:

Lemma 5.74. *Let a be symmetric. Then a is self-adjoint (i.e. $a^* = a$) iff*

$$\mathrm{ran}(a+i) = \mathrm{ran}(a-i) = H. \tag{5.347}$$

Proof. We only need the implication from (5.347) to $a^* = a$ (but the converse immediately follows from Theorem B.93). So assume (5.347). For given $\psi \in D(a^*)$ there must then be a $\varphi \in H$ such that $(a^* - i)\psi = (a-i)\varphi$. Since a is symmetric, we have $D(a) \subset D(a^*)$, so $\psi - \varphi \in D(a^*)$, and $(a^* - i)(\psi - \varphi) = 0$. But $\ker(a^* - i) = \mathrm{ran}(a+i)^\perp$, so $\ker(a^* - i) = 0$. Hence $\psi = \varphi$, and in particular $\psi \in D(a)$ and hence $D(a^*) \subset D(a)$. Since we already know the opposite inclusion, we have $D(a^*) = D(a)$. Given symmetry, this implies $a^* = a$. □

Continuing the proof of Theorem 5.73.2, symmetry of a easily follows from its definition, combined with the property $u_t^* = u_t^{-1} = u_{-t}$. Indeed, for $\psi, \varphi \in D(a)$, the weak limit $s \to 0$ below exists by definition of $D(a)$, cf. (5.343), whence:

$$\langle \varphi, a\psi \rangle = i \lim_{s \to 0} \langle \varphi, \frac{u_s - 1}{s} \psi \rangle = -i \lim_{s \to 0} \langle \frac{u_{-s} - 1}{-s} \varphi, \psi \rangle = \langle a\varphi, \psi \rangle.$$

To prove that $\mathrm{ran}(a-i) = H$, we compute $(a-i)\psi_1 = -i\psi$, with ψ_1 defined by (5.346) with $n = 1$. The property $\mathrm{ran}(-i) = H$ is proved in a similar way: now define $\tilde{\psi}_1 = \int_{-\infty}^0 ds\, e^s u_s \psi$ and obtain $(a+i)\tilde{\psi}_1 = i\psi$. Thus Lemma 5.74 applies.

3. Bijectivity has two directions: $a \mapsto u_t \mapsto a$ and $u_t \mapsto a \mapsto u_t$.

   - Given a and hence (5.341) defining u_t, we change notation from a to a' in (5.343) - (5.344) and need to show that $a' = a$. Denoting the restriction of a to the domain $C_c^*(b)$ by a_0, we first show that $a_0 \subseteq a'$. The technique to prove this is similar to the argument around (5.345). We initially assume that $\varphi \in D(a_0) = C_c^*(b)H$ takes the form $\varphi = h(a)\psi$ for some $h \in C_c(\sigma(a))$ and $\psi \in H$. Just a trifle more complicated than (5.345), using (5.342), (B.573), and unitarity of u_t, we estimate:

$$\begin{aligned}\left\| \frac{u_{t+s}\varphi - u_t\varphi}{s} + ia_0 u_t \varphi \right\| &\leq \left\| \frac{e_s h - h}{s} + i \cdot \mathrm{id}_{\sigma(T)} h \right\|_\infty \|\psi\| \\ &\leq \left\| \frac{e_s - 1_K}{s} + i \cdot \mathrm{id}_K \right\|_\infty^{(K)} \|h\|_\infty \|\psi\|,\end{aligned}$$

so that by definition of the (strong) derivative we obtain

$$\frac{du_t}{dt}\varphi = \lim_{s\to 0}\frac{u_{t+s}\varphi - u_t\varphi}{s} = -iau_t\varphi, \tag{5.348}$$

initially for any φ of the said form $h(a)\psi$, and hence, taking finite sums, for any $\varphi \in D(a_0)$. The existence of this limit shows that, on the assumption $\psi \in D(a_0)$, we have $\psi \in D(a')$, and we also see that $a' = a$ on $D(a_0)$, or, in other words, that $a_0 \subseteq a'$. Since a' is self-adjoint (by part 2 of the theorem) and hence closed, we have $a_0^- \subseteq a'$. Since a_0 is essentially self-adjoint by Theorem B.159, this gives $a \subseteq a'$. Taking adjoints reverses the inclusion, and since both operators are self-adjoint this gives $a = a'$.

- Given u_t and hence (5.343) - (5.344) defining a, we change notation from u_t to u'_t in (5.341) and need to show that $u'_t = u_t$. Indeed, let

$$\psi_t = u_t\psi, \tag{5.349}$$

and similarly $\psi'_t = u'_t\psi$. If $\psi \in D(a)$, then by definition of a we have

$$i\frac{d\psi_t}{dt} = i\lim_{s\to 0}\frac{u_{t+s}-u_t}{s}\psi = i\lim_{s\to 0}\frac{u_s - 1_H}{s}u_t\psi = a\psi_t, \tag{5.350}$$

which also shows that $\psi_t \in D(a)$. Similarly, $id\psi'_t/dt = a\psi'_t$, so that ψ_t and ψ'_t satisfy the same differential equation with the same initial condition

$$\psi^{(0)} = (\psi^{(0)})' = \psi.$$

Now consider $\hat{\psi}_t = \psi_t - \psi'_t$, which once again satisfies the same equation (i.e., $id\hat{\psi}_t/dt = a\hat{\psi}_t$), but this time with initial condition $\hat{\psi}_0 = \psi^{(0)} - (\psi^{(0)})' = \psi - \psi = 0$. The key point is that *any* solution $\hat{\psi}_t$ of this equation has the property $\|\hat{\psi}_t\| = \|\hat{\psi}_0\|$ for any $t \in \mathbb{R}$, since by symmetry of a,

$$\frac{d}{dt}\|\hat{\psi}_t\|^2 = \frac{d}{dt}\langle\hat{\psi}_t,\hat{\psi}_t\rangle = -i(\langle\hat{\psi}_t, a\hat{\psi}_t\rangle - \langle a\hat{\psi}_t,\hat{\psi}_t\rangle) = 0.$$

For our *specific* $\hat{\psi}_t$ we have $\|\hat{\psi}_0\| = 0$ and hence $\psi_t = \psi'_t$, that is, $u'_t = u_t$. □

Corollary 5.75. *With $t \mapsto u_t$ and a defined and related as in Theorem 5.73, if $\psi \in D(a)$, for each $t \in \mathbb{R}$ the vector ψ_t defined by* (5.349) *lies in $D(a)$ and satisfies*

$$a\psi_t = i\frac{d\psi_t}{dt}, \tag{5.351}$$

whence $t \mapsto \psi_t$ is the unique solution of (5.351) *with initial value $\psi^{(0)} = \psi$.*

This follows from the proof of part 3 of Theorem 5.73. With $a = h/\hbar$ (as above), this is just the famous ***time-dependent Schrödinger equation***

$$h\psi_t = i\hbar\frac{d\psi_t}{dt}. \tag{5.352}$$

Notes

§5.1. **Six basic mathematical structures of quantum mechanics**

Wigner's Theorem was first stated by von Neumann and Wigner (1928), but the first proof appeared in Wigner (1931). See Bonolis (2004) and Scholz (2006) for some history. Instead of working with $\mathscr{P}_1(H)$ with the bilinear trace form expressing the transition probabilities, one may also formulate and prove Wigner's Theorem in terms of the projective Hilbert space $\mathbb{P}H$ equipped with the Fubini–Study metric, in which case the relevant symmetries may be defined geometrically as isometries. See Freed (2012) for this proof, as well as Brody & Hughston (2001) for the underlying geometry. Kadison's Theorem may be traced back from Kadison (1965). See also Moretti (2013). Ludwig symmetries go back to Ludwig (1983); see also Kraus (1983). Our approach to von Neumann symmetries was inspired by Hamhalter (2004), and has a large pedigree in quantum logic. Bohr symmetries were introduced in Landsman & Lindenhovius (2016), where Theorem 5.4.6 was also proved.

§5.2. **The case** $H = \mathbb{C}^2$

This material is partly based on Simon (1976). The covering map (5.46) has a nice geometric description: if $\Sigma = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, we have the well-known stereographic projection

$$S^2 \stackrel{\cong}{\to} \Sigma; \tag{5.353}$$

$$(x,y,z) \mapsto \frac{x+iy}{1-z}. \tag{5.354}$$

If $u \in SU(2)$ is given by (5.43), then the associated Möbius transformation

$$z \mapsto \frac{\alpha z + \beta}{-\overline{\beta} z + \overline{\alpha}}$$

is a bijection of Σ, whose associated transformation of S^2 is the rotation $R = \tilde{\pi}(u)$.

§5.3. **Equivalence between the six symmetry theorems**

Most proofs may be also found in Cassinelli et al (2004) or Moretti (2013).

§5.4. **Proof of Jordan's Theorem**

Our proof of Jordan's Theorem is taken from Bratteli & Robinson (1987); see also Thomsen (1982) for a simplification of the purely algebraic step (which we delegated to Theorem C.175), originally proved by Jacobson & Rickart (1950).

§5.5. **Proof of Wigner's Theorem**

There are many proofs of Wigner's Theorem, none of them really satisfactory (in this respect the situation is similar to Gleason's Theorem). Our proof follows Simon (1976), who in turn relies on Bargmann (1964) and Hunziker (1972). The proof in Cassinelli et al (2004) seems cleaner, but their proof of the additivity of their operator T_ω is not easy to follow. For a geometric approach see Freed (2012).

If $\dim(H) \geq 3$, the conclusion of Wigner's Theorem follows if W merely preserves orthogonality (Uhlhorn, 1963). See also Cassinelli et al (2004). This, in turn, has been generalized in various directions, e.g. to indefinite inner product spaces (Molnár, 2002) as well as to certain Banach spaces, where one says that x is orthogonal to y if for all $\lambda \in \mathbb{C}$ one has $\|x+\lambda y\| \geq \|x\|$ (Blanco & Turnšek, 2006).

§5.6. **Some abstract representation theory**

Among numerous books on representation theory, our personal favourite is Barut & Raçka (1977), and also Gaal (1973) and Kirillov (1976) are classics at least for the abstract theory. An interesting recent paper on the unitary group on infinite-dimensional Hilbert space is Schottenloher (2013).

§5.7. **Representations of Lie groups and Lie algebras**

This section was inspired by Hall (2013) and Knapp (1988). For Lie's Third Theorem, see, for example, Duistermaat & Kolk (2000), §1.14. To obtain Theorem 5.41, consider the canonical projection $\tilde{\pi} : \tilde{G} \to G$ and define $D = \tilde{\pi}^{-1}(\{e\})$. This is a discrete normal subgroup of $\tilde{G}$, and it is an easy fact that a discrete normal subgroup of any connected topological group must lie in its center. Note that a discrete subgroup of the center of $\tilde{G}$ is automatically normal.

The exponentiation problem for skew-adjoint representations of $\mathfrak{g}$ is considerably more complicated than in finite dimension. Let H be an infinite-dimensional Hilbert space with dense subspace D and let $\rho : \mathfrak{g} \to L(D,H)$ be a linear map, where $L(D,H)$ is the space of linear maps from L to H. We say that ρ is a ***skew-adjoint representation*** of $\mathfrak{g}$ if *(i):* D is invariant under $u'(\mathfrak{g})$, *(ii):* the commutation relations (5.157) hold on D, and *(i):* each $i\rho(A)$ is essentially self-adjoint on D. For example, we have seen that if $u : G \to U(H)$ is a unitary representation, then the construction $\rho(A) = u'(A)$, defined on the Gårding domain $D = D_G$, fits the bill. Conversely, additional conditions are needed for ρ to exponentiate to a unitary representation. The best-known of those is ***Nelson's criterion***: if, given a skew-adjoint representation $\rho : \mathfrak{g} \to L(D,H)$, the ***Nelson operator*** or ***Laplacian*** $\Delta = \sum_{k=1}^{\dim(\mathfrak{g})} \rho(T_k)^2$ is essentially self-adjoint on D, then ρ exponentiates to a unitary representation of $\tilde{G}$ (with additional remarks similar to those in Corollary 5.43).

§5.8. **Irreducible representations of** $SU(2)$

§5.9. **Irreducible representations of compact Lie groups**

See e.g. Knapp (1988), Simon (1996), and Deitmar (2005), and innumerable other books. This material ultimately goes back to (É.) Cartan and Weyl.

§5.10. **Symmetry groups and projective representations**

See Varadarajan (1985), Tuynman & Wiegerinck (1987), Landsman (1998a), Cassinelli et al (2004), and Hall (2013). For different proofs of Theorem 5.59 (Bargmann, 1954) see Simms (1971) and Cassinelli et al (2004). Leaving out the anti-unitary symmetries is a pity; see e.g. Freed & Moore and Roberts (2016).

§5.11. **Position, momentum, and free Hamiltonian**

§5.12. **Stone's Theorem**

See Reed & Simon (1972), Schmüdgen (2012), Moretti (2013), Hall (2013), and many other books. Our proof of part 1 of Theorem 5.73 is original.