Simultaneous Orthogonal Planarity

Patrizio Angelini[1], Steven Chaplick[2], Sabine Cornelsen[3], Giordano Da Lozzo[4(✉)], Giuseppe Di Battista[4], Peter Eades[5], Philipp Kindermann[6], Jan Kratochvíl[7], Fabian Lipp[2], and Ignaz Rutter[8]

[1] Universität Tübingen, Tübingen, Germany
angelini@informatik.uni-tuebingen.de
[2] Universität Würzburg, Würzburg, Germany
{steven.chaplick,fabian.lipp}@uni-wuerzburg.de
[3] Universität Konstanz, Konstanz, Germany
sabine.cornelsen@uni-konstanz.de
[4] Roma Tre University, Rome, Italy
{dalozzo,gdb}@dia.uniroma3.it
[5] The University of Sydney, Sydney, Australia
peter@it.usyd.edu.au
[6] FernUniversität in Hagen, Hagen, Germany
philipp.kindermann@fernuni-hagen.de
[7] Charles University, Prague, Czech Republic
honza@kam.mff.cuni.cz
[8] Karlsruhe Institute of Technology, Karlsruhe, Germany
rutter@kit.edu

Abstract. We introduce and study the OrthoSEFE-k problem: Given k planar graphs each with maximum degree 4 and the same vertex set, do they admit an OrthoSEFE, that is, is there an assignment of the vertices to grid points and of the edges to paths on the grid such that the same edges in distinct graphs are assigned the same path and such that the assignment induces a planar orthogonal drawing of each of the k graphs? We show that the problem is NP-complete for $k \geq 3$ even if the shared graph is a Hamiltonian cycle and has sunflower intersection and for $k \geq 2$ even if the shared graph consists of a cycle and of isolated vertices. Whereas the problem is polynomial-time solvable for $k = 2$ when the union graph has maximum degree five and the shared graph is biconnected. Further, when the shared graph is biconnected and has sunflower intersection, we show that every positive instance has an OrthoSEFE with at most three bends per edge.

1 Introduction

The input of a simultaneous embedding problem consists of several graphs $G_1 = (V, E_1), \ldots, G_k = (V, E_k)$ on the same vertex set. For a fixed drawing style $\mathcal{S}$, the simultaneous embedding problem asks whether there exist drawings $\Gamma_1, \ldots, \Gamma_k$

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of $G_1, \ldots, G_k$, respectively, in drawing style $\mathcal{S}$ such that for any i and j the restrictions of Γ_i and Γ_j to $G_i \cap G_j = (V, E_i \cap E_j)$ coincide.

The problem has been most widely studied in the setting of topological planar drawings, where vertices are represented as points and edges are represented as pairwise interior-disjoint Jordan arcs between their endpoints. This problem is called Simultaneous Embedding with Fixed Edges or SEFE-k for short, where k is the number of input graphs. It is known that SEFE-k is NP-complete for $k \geq 3$, even in the restricted case of *sunflower instances* [25], where every pair of graphs shares the same set of edges, and even if such a set induces a star [3]. On the other hand, the complexity for $k = 2$ is still open. Recently, efficient algorithms for restricted instances have been presented, namely when (i) the shared graph $G_\cap = G_1 \cap G_2$ is biconnected [4,18] or a star-graph [4], (ii) $G_\cap$ is a collection of disjoint cycles [12], (iii) every connected component of $G_\cap$ is either subcubic or biconnected [10,25], (iv) G_1 and G_2 are biconnected and $G_\cap$ is connected [13], and (v) $G_\cap$ is connected and the input graphs have maximum degree 5 [13]; see the survey by Bläsius et al. [11] for an overview.

For planar straight-line drawings, the simultaneous embedding problem is called Simultaneous Geometric Embedding and it is known to be NP-hard even for two graphs [17]. Besides simultaneous intersection representation for, e.g., interval graphs [13,19] and permutation and chordal graphs [20], it is only recently that the simultaneous embedding paradigm has been applied to other fundamental planarity-related drawing styles, namely simultaneous level planar drawings [2] and RAC drawings [5,7].

We continue this line of research by studying simultaneous embeddings in the planar *orthogonal* drawing style, where vertices are assigned to grid points and edges to paths on the grid connecting their endpoints [28]. In accordance with the existing naming scheme, we define OrthoSEFE-k to be the problem of testing whether k input graphs $\langle G_1, \ldots, G_k \rangle$ admit a simultaneous planar orthogonal drawing. If such a drawing exists, we call it an OrthoSEFE of $\langle G_1, \ldots, G_k \rangle$. Note that it is a necessary condition that each G_i has maximum degree 4 in order to obtain planar orthogonal drawings. Hence, in the remainder of the paper we assume that all instances have this property. For instances with this property, at least when the shared graph is connected, the problem SEFE-2 can be solved efficiently [13]. However, there are instances of OrthoSEFE-2 that admit a SEFE but not an OrthoSEFE; see Fig. 1(a).

Unless mentioned otherwise, all instances of OrthoSEFE-k and SEFE-k we consider are sunflower. Notice that instances with $k = 2$ are always sunflower. Let $\langle G_1 = (V, E_1), G_2 = (V, E_2) \rangle$ be an instance of OrthoSEFE-2. We define the *shared graph* (resp. the *union graph*) to be the graph $G_\cap = (V, E_1 \cap E_2)$ (resp. $G_\cup = (V, E_1 \cup E_2)$) with the same vertex set as G_1 and G_2, whose edge set is the intersection (resp. the union) of the ones of G_1 and G_2. Also, we call the edges in $E_1 \cap E_2$ the *shared edges* and we call the edges in $E_1 \setminus E_2$ and in $E_2 \setminus E_1$ the *exclusive edges*. The definitions of *shared graph*, *shared edges*, and *exclusive edges* naturally extend to sunflower instances for any value of k.

Fig. 1. (a) A negative instance of OrthoSEFE-2. Shared edges are black, while exclusive edges are red and blue. The red edges require 270° angles on different sides of C. Thus, the blue edge (u, v) cannot be drawn. Note that the given drawing is a SEFE-2. (b) Examples of side assignments for the exclusive edges incident to degree-2 vertices of $G_\cap$: orthogonality constraints are satisfied at v_4 and v_5, while they are violated at v_3. (Color figure online)

One main issue is to decide how degree-2 vertices of the shared graph are represented. Note that, in planar topological drawings, degree-2 vertices do not require any decisions as there exists only a single cyclic order of their incident edges. In the case of orthogonal drawings there are, however, two choices for a degree-2 vertex: It can either be drawn *straight*, i.e., it is incident to two angles of 180°, or *bent*, i.e., it is incident to one angle of 90° and to one angle of 270°. If v is a degree-2 vertex of the shared graph with neighbors u and w, and two exclusive edges e, e', say of G_1, are incident to v and are embedded on the same side of the path uvw, then v must be bent, which in turn implies that also every exclusive edge of G_2 incident to v has to be embedded on the same side of uvw as e and e'. In this way, the two input graphs of OrthoSEFE-2 interact via the degree-2 vertices. It is the difficulty of controlling this interaction that marks the main difference between SEFE-k and OrthoSEFE-k. To study this interaction in isolation, we focus on instances of OrthoSEFE-2 where the shared graph is a cycle for most of the paper. Note that such instances are trivial yes-instances of SEFE-k (provided the input graphs are all planar).

Contributions and Outline. In Sect. 2, we provide our notation and we show that the existence of an OrthoSEFE of an instance of OrthoSEFE-k can be described as a combinatorial embedding problem. In Sect. 3, we show that OrthoSEFE-3 is NP-complete even if the shared graph is a cycle, and that OrthoSEFE-2 is NP-complete even if the shared graph consists of a cycle plus some isolated vertices. This contrasts the situation of SEFE-k where these cases are polynomially solvable [4,9,18,25]. In Sect. 4, we show that OrthoSEFE-2 is efficiently solvable if the shared graph is a cycle and the union graph has maximum degree 5. Finally, in Sect. 5, we extend this result to the case where the shared graph is biconnected (and the union graph still has maximum degree 5). Moreover, we show that any positive instance of OrthoSEFE-k whose shared graph is biconnected admits an OrthoSEFE with at most three bends per edge. We close with some concluding remarks and open questions in Sect. 6.

Complete proofs can be found in the full version of the paper [1].

2 Preliminaries

We will extensively make use of the Not-All-Equal 3-Sat (Nae3Sat) problem [24, p.187]. An instance of Nae3Sat consists of a 3-CNF formula ϕ with variables $x_1, \ldots, x_n$ and clauses $c_1, \ldots, c_m$. The task is to find a Nae *truth assignment*, i.e., a truth assignment such that each clause contains both a true and a false literal. Nae3Sat is known to be NP-complete [26]. The *variable–clause graph* is the bipartite graph whose vertices are the variables and the clauses, and whose edges represent the membership of a variable in a clause. The problem Planar Nae3Sat is the restriction of Nae3Sat to instances whose variable–clause graph is planar. Planar Nae3Sat can be solved efficiently [22,27].

Embedding Constraints. Let $\langle G_1, \ldots, G_k \rangle$ be an OrthoSEFE-k instance. A *SEFE* is a collection of embeddings $\mathcal{E}_i$ for the G_i such that their restrictions on $G_\cap$ are the same. Note that in the literature, a SEFE is often defined as a collection of drawings rather than a collection of embeddings. However, the two definitions are equivalent [21]. For a SEFE to be realizable as an OrthoSEFE it needs to satisfy two additional conditions. First, let v be a vertex of degree 2 in $G_\cap$ with neighbors u and w. If in any embedding $\mathcal{E}_i$ there exist two exclusive edges incident to v that are embedded on the same side of the path uvw, then any exclusive edge incident to v in any of the $\mathcal{E}_j \neq \mathcal{E}_i$ must be embedded on the same side of the path uvw. Second, let v be a vertex of degree 3 in $G_\cap$. All exclusive edges incident to v must appear between the same two edges of $G_\cap$ around v. We call these the *orthogonality constraints*. See Fig. 1(b).

Theorem 1. *An instance $\langle G_1, \ldots, G_k \rangle$ of* OrthoSEFE-k *has an OrthoSEFE if and only if it admits a SEFE satisfying the orthogonality constraints.*

For the case in which the shared graph is a cycle C, we give a simpler version of the constraints in Theorem 1, which will prove useful in the remainder of the paper. By the Jordan curve theorem, a planar drawing of cycle C divides the plane into a bounded and an unbounded region – the *inside* and the *outside* of C, which we call the *sides* of C. Now the problem is to assign the exclusive edges to either of the two sides of C so that the following two conditions are fulfilled.

Planarity Constraints. Two exclusive edges of the same graph must be drawn on different sides of C if their endvertices alternate along C.

Orthogonality Constraints. Let $v \in V$ be a vertex that is adjacent to two exclusive edges e_i and e'_i of the same graph G_i, $i \in \{1, \ldots, k\}$. If e_i and e'_i are on the same side of C, then all exclusive edges incident to v of all graphs $G_1, \ldots, G_k$ must be on the same side as e_i and e'_i.

Note that this is a reformulation of the general orthogonality constraints. Further, the orthogonality constraints also imply that if e_i and e'_i are on different sides of C, then for each graph G_j that contains two exclusive edges e_j and e'_j incident to v, with $j \in \{1, \ldots, k\}$, e_j and e'_j must be on different sides of C.

The next theorem follows from Theorem 1 and from the following two observations. First, for a sunflower instance $\langle G_1, \ldots, G_k \rangle$ whose shared graph is a cycle, any collection of embeddings is a SEFE [21]. Second, the planarity constraints are necessary and sufficient for the existence of an embedding of G_i [6].

Theorem 2. *An instance of* OrthoSEFE-k *whose shared graph is a cycle* C *has an OrthoSEFE if and only if there exists an assignment of the exclusive edges to the two sides of* C *satisfying the planarity and orthogonality constraints.*

3 Hardness Results

We show that OrthoSEFE-k is NP-complete for $k \geq 3$ for instances with sunflower intersection even if the shared graph is a cycle, and for $k = 2$ even if the shared graph consists of a cycle and isolated vertices.

Theorem 3. OrthoSEFE-k *with* $k \geq 3$ *is NP-complete, even for instances with sunflower intersection in which (i) the shared graph is a cycle and (ii)* $k-1$ *of the input graphs are outerplanar and have maximum degree* 3*.*

Proof sketch. The membership in NP directly follows from Theorem 2. To prove the NP-hardness, we show a polynomial-time reduction from the NP-complete problem Positive Exactly-Three Nae3Sat [23], which is the variant of Nae3Sat in which each clause consists of exactly three unnegated literals.

Let $x_1, x_2, \ldots, x_n$ be the variables and let $c_1, c_2, \ldots, c_m$ be the clauses of a 3-CNF formula ϕ of Positive Exactly-Three Nae3Sat. We show how to construct an equivalent instance $\langle G_1, G_2, G_3 \rangle$ of OrthoSEFE-3 such that G_1 and G_2 are outerplanar graphs of maximum degree 3. We refer to the exclusive edges in G_1, G_2, and G_3 as red, blue, and green, respectively; refer to Fig. 2.

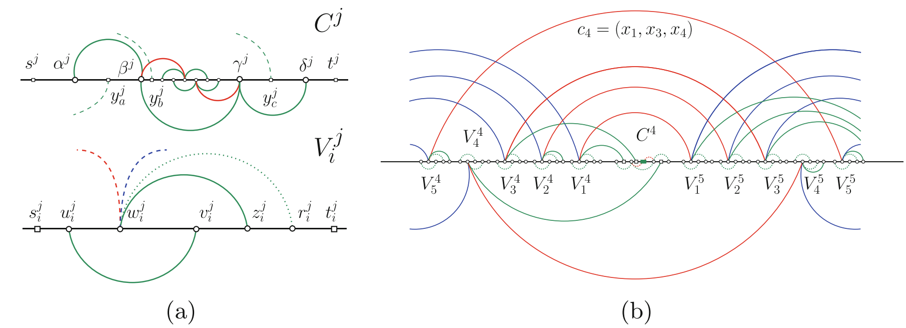


Fig. 2. (a) A clause gadget C_j (top) and a variable-clause gadget V_i^j (bottom); solid edges belong to the gadgets, dotted edges are optional, and dashed edges are transmission edges. (b) Illustration of instance $\langle G_1, G_2, G_3 \rangle$, focused on a clause c_4. Black edges belong to the shared graph $G_\cap$. The red, blue, and green edges are the exclusive edges of G_1, G_2, and G_3, respectively. (Color figure online)

For each clause c_j, $j = 1, \dots, m$, we create a *clause gadget* C^j as in Fig. 2(a) (top). For each variable x_i, $i = 1, \dots, n$, and each clause c_j, $j = 1, \dots, m$, we create a *variable-clause gadget* V_i^j as in Fig. 2(a) (bottom). Observe that the (dotted) green edge $\{w_i^j, r_i^j\}$ in a variable-clause gadget is only part of V_i^j if x_i does not occur in c_j. Otherwise, there is a green edge $\{w_i^j, y_x^j\}$ connecting w_i^j to one of the three vertices y_a^j, y_b^j, or y_c^j (dashed stubs) in the clause gadget. Observe that these three *variable-clause edges* per clause can be realized in such a way that there exist no planarity constraints between pairs of them. In Fig. 2(b), the variable-clause gadgets V_1^4, V_3^4, V_4^4 are incident to variable-clause edges, while V_2^4 and V_5^4 contain edges $\{w_2^4, r_2^4\}$ and $\{w_5^4, r_5^4\}$, respectively.

The gadgets are ordered as indicated in Fig. 2(b). The variable-clause gadgets V_i^j, with $i = 1, \dots, n$, always precede the clause gadget V^j, for any $j = 1, \dots, m$. Further, if j is odd, then the gadgets $V_1^j, \dots, V_n^j$ appear in this order, otherwise they appear in reversed order $V_n^j, \dots, V_1^j$. Finally, V_i^j and V_i^{j+1}, for $i = 1, \dots, n$ and $j = 1, \dots, m-1$, are connected by an edge $\{w_i^j, w_i^{j+1}\}$, which is blue if j is odd and red if j is even. We call these edges *transmission edges*.

Assume $\langle G_1, G_2, G_3 \rangle$ admits an OrthoSEFE. Planarity constraints and orthogonality constraints guarantee three properties: (i) If the edge $\{u_i^j, v_i^j\}$ is inside C, then so is $\{u_i^{j+1}, v_i^{j+1}\}$, $i = 1, \dots, n$, $j = 1, \dots, m-1$. This is due to the fact that, by the planarity constraints, the two green edges incident to w_i^j lie on the same side of C and hence, by the orthogonality constraints, the two transmission edges incident to w_i^j also lie on this side. We call $\{u_i^1, v_i^1\}$ the *truth edge* of variable x_i. (ii) Not all the three green edges $a = \{\alpha^j, \beta^j\}$, $b = \{\beta^j, \gamma^j\}$, and $c = \{\gamma^j, \delta^j\}$ lie on the same side of C. Namely, the two red edges of the clause gadget C^j must lie on opposite sides of C because of the interplay between the planarity and the orthogonality constraints in the subgraph of C^j induced by the vertices between β^j and γ^j. Hence, if edges a, b, and c lie on the same side of C, then the orthogonality constraints at either β^j or γ^j are not satisfied. (iii) For each clause $c_j = (x_a, x_b, x_c)$, edge $a = \{\alpha^j, \beta^j\}$ lies on the same side of C as the truth edge of x_a. This is due to the planarity constraints between each of these two edges and the variable-clause edge $\{w_a^j, y_a^j\}$. Analogously, edge b (edge c) lies on the same side as the truth edge of x_b (of x_c). Hence, setting $x_i = \mathsf{true}$ ($x_i = \mathsf{false}$) if the truth edge of x_i is inside C (outside C) yields a NAE3SAT truth assignment that satisfies ϕ.

The proof for the other direction is based on the fact that assigning the truth edges to either of the two sides of C according to the NAE3SAT assignment of ϕ also implies a unique side assignment for the remaining exclusive edges that satisfies all the orthogonality and the planarity constraints.

It is easy to see that G_1 and G_2 are outerplanar graphs with maximum degree 3, and that the reduction can be extended to any $k > 3$. $\square$

In the following, we describe how to modify the construction in Theorem 3 to show hardness of ORTHOSEFE-2. We keep only the edges of G_1 and G_3. Variable-clause gadgets and clause gadgets remain the same, as they are composed only of edges belonging to these two graphs. We replace each transmission edge in G_2 by a *transmission path* composed of alternating green and red edges, starting and ending with a red edge. This transformation allows these paths to traverse the transmission edges of G_1 and the variable-clause edges of G_3 without introducing crossings between edges of the same color. It is easy to see that the properties described in the proof of Theorem 3 on the assignments of the exclusive edges to the two sides of C also hold in the constructed instance, where transmission paths take the role of the transmission edges.

Theorem 4. OrthoSEFE-2 *is NP-complete, even for instances* $\langle G_1, G_2 \rangle$ *in which the shared graph consists of a cycle and a set of isolated vertices.*

4 Shared Graph is a Cycle

In this section, we give a polynomial-time algorithm for instances of OrthoSEFE-2 whose shared graph is a cycle and whose union graph has maximum degree 5 (Theorem 5). In order to obtain this result, we present an efficient algorithm for more restricted instances (Lemma 1) and give a series of transformations (Lemmas 2–3) to reduce any instance with the above properties to one that can be solved by the algorithm in Lemma 1.

Lemma 1. OrthoSEFE-2 *is in P for instances* $\langle G_1, G_2 \rangle$ *such that the shared graph C is a cycle and G_1 is an outerplanar graph with maximum degree* 3.

Proof. The algorithm is based on a reduction to Planar Nae3Sat, which is in P [22,27]. First note that, since G_1 is outerplanar, there exist no two edges in E_1 alternating along C. Hence, there are no planarity constraints for G_1.

We now define an auxiliary graph H with vertex set $E_2 \setminus E_1$ and edges corresponding to pairs of edges alternating along C; see Fig. 3(a). W.l.o.g. we may assume that H is bipartite, since G_2 would not meet the planarity constraints otherwise [6]. Let $\mathcal{B}$ be the set of connected components of H, and for each component $B \in \mathcal{B}$, fix a partition B_1, B_2 of B into independent sets (possibly

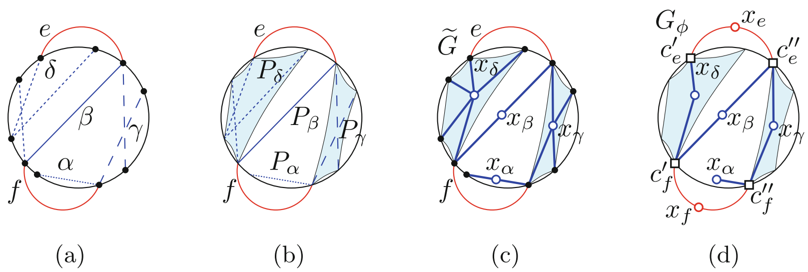


Fig. 3. (a) Instance $\langle G_1, G_2 \rangle$ satisfying the properties of Lemma 1, where the edges in E_2 belonging to the components α, β, γ, and δ of H have different line styles. (b) Polygons for the components of H. (c) Graph $\widetilde{G}$. (d) Variable–clause graph G_ϕ. (Color figure online)

$B_2 = \emptyset$ in case of a singleton B). Note that in any inside/outside assignment of the exclusive edges of G_2 that meets the planarity constraints, for every $B \in \mathcal{B}$, all edges of B_1 lie on one side of C and all edges of B_2 lie on the other side.

Draw the cycle C as a circle in the plane. For a component $B \in \mathcal{B}$, let P_B be the polygon inscribed into C whose corners are the endvertices in V of the edges in E_2 corresponding to the vertices of B; refer to Fig. 3(b). If B only contains one vertex (i.e., one edge of G_2), we consider the digon P_B as the straight-line segment connecting the vertices of this edge. If B has at least two vertices, we let P_B be open along its sides, i.e., it will contain the corners and all inner points (in Fig. 3(b) we depict this by making the sides of P_B slightly concave). One can easily show that, for any two components $B, D \in \mathcal{B}$, their polygons P_B, P_D may share only some of their corners, but no inner points. Hence, the graph $\widetilde{G}$ obtained by placing a vertex x_B inside the polygon P_B, for $B \in \mathcal{B}$, making x_B adjacent to each corner of P_B and adding the edges E_1, is planar; see Fig. 3(c).

We construct a formula ϕ with variables x_B, $B \in \mathcal{B}$, such that ϕ is NAE-satisfiable if and only if $\langle G_1, G_2 \rangle$ admits an inside/outside assignment meeting all planarity and orthogonality constraints. The encoding of the truth assignment will be such that x_B is true when the edges of B_1 are inside C and the edges of B_2 are outside, and x_B is false if the reverse holds. Every assignment satisfying the planarity constraints for G_2 defines a truth-assignment in the above sense.

Let $e = (v, w)$ be an exclusive edge of E_1 and let e_v^1, e_v^2 (e_w^1, e_w^2) be the exclusive edges of E_2 incident to v (to w, respectively); we assume that all such four edges of E_2 exist, the other cases being simpler. Let $B(u, i)$ be the component containing the edge e_u^i, for $u \in \{v, w\}$ and $i \in \{1, 2\}$. Define the literal ℓ_u^i to be $x_{B(u,i)}$ if $e_u^i \in B_1(u, i)$ and $\neg x_{B(u,i)}$ if $e_u^i \in B_2(u, i)$. With our interpretation of the truth assignment, an edge e_u^i is inside C if and only if ℓ_u^i is true. Now, for the assignment to meet the orthogonality constraints, if $\ell_v^1 = \ell_v^2$, say both are true, then e must be assigned inside C as well, which would cause a problem if and only if $\ell_w^1 = \ell_w^2 = \mathsf{false}$. Hence, the orthogonality constraints are described by NAE-satisfiability of the clauses $c_e = (\ell_v^1, \ell_v^2, \neg\ell_w^1, \neg\ell_w^2)$, for each $e \in E_1$. To reduce to NAE3SAT, we introduce a new variable x_e for each edge $e \in E_1 \setminus E_2$ and replace the clause c_e by two clauses $c_e' = (\ell_v^1, \ell_v^2, x_e)$ and $c_e'' = (\neg x_e, \neg\ell_w^1, \neg\ell_w^2)$. A planar drawing of the variable–clause graph G_ϕ of the resulting formula ϕ is obtained from the planar drawing $\widetilde{\Gamma}$ of $\widetilde{G}$ (see Figs. 3(c) and 3(d)) by (i) placing each variable x_B, with $B \in \mathcal{B}$, on the point where vertex x_B lies in $\widetilde{\Gamma}$, (ii) placing each variable x_e, with $e \in E_1$, on any point of edge e in $\widetilde{\Gamma}$, (iii) placing clauses c_e' and c_e'', for each edge $e = (v, w) \in E_1$, on the points where vertices v and w lie in $\widetilde{\Gamma}$, respectively, and (iv) drawing the edges of G_ϕ as the corresponding edges in $\widetilde{\Gamma}$. This implies that G_ϕ is planar and hence we can test the NAE-satisfiability of ϕ in polynomial time [22,27]. $\square$

The next two lemmas show that we can use Lemma 1 to test in polynomial time any instance of ORTHOSEFE-2 such that $G_\cap$ is a cycle and each vertex $v \in V$ has degree at most 3 in either G_1 or G_2.

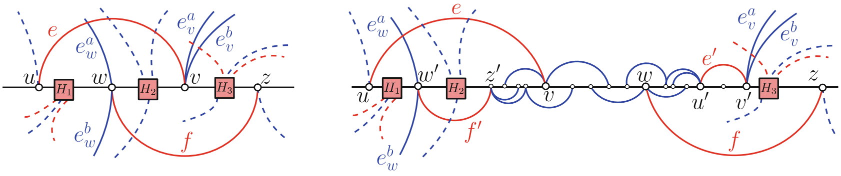


Fig. 4. Instances (left) $\langle G_1, G_2\rangle$ and (right) $\langle G'_1, G'_2\rangle$ for the proof of Lemma 2. Edges of $G_\cap$ ($G'_\cap$) are black. Exclusive edges of G_1 (G'_1) are red and those of G_2 (G'_2) are blue. (Color figure online)

Lemma 2. *Let* $\langle G_1, G_2\rangle$ *be an instance of* ORTHOSEFE-2 *whose shared graph is a cycle and such that* G_1 *has maximum degree* 3*. It is possible to construct in polynomial time an equivalent instance* $\langle G^*_1, G^*_2\rangle$ *of* ORTHOSEFE-2 *whose shared graph is a cycle and such that* G^*_1 *is outerplanar and has maximum degree* 3*.*

Proof sketch. We construct an equivalent instance $\langle G'_1, G'_2\rangle$ of ORTHOSEFE-2 such that $G'_\cap$ is a cycle, G'_1 has maximum degree 3, and the number of pairs of edges in G'_1 that alternate along $G'_\cap$ is smaller than the number of pairs of edges in G_1 that alternate along $G_\cap$. Repeatedly applying this transformation yields an equivalent instance $\langle G^*_1, G^*_2\rangle$ satisfying the requirements of the lemma.

Consider two edges $e = (u, v)$ and $f = (w, z)$ of G_1 such that u, w, v, z appear in this order along cycle $G_\cap$ and such that the path $P_{u,z}$ in $G_\cap$ between u and z that contains v and w has minimal length. If G_1 is not outerplanar, then the edges e and f always exist. Figure 4 illustrates the construction of $\langle G'_1, G'_2\rangle$.

By the choice of e and f, and by the fact that G_1 has maximum degree 3, there is no exclusive edge in G_1 with one endpoint in the set H_2 of vertices between w and v, and the other one not in H_2. Further, observe that in an OrthoSEFE of $\langle G'_1, G'_2\rangle$ edges f and f' (edges e and e') must be on the same side. Further, e and f must be in different sides of $G'_\cap$. It can be concluded that $\langle G'_1, G'_2\rangle$ has an OrthoSEFE if and only if $\langle G_1, G_2\rangle$ has an OrthoSEFE. □

The proof of the next lemma is based on the replacement illustrated in Fig. 5. Afterwards, we combine these results to present the main result of the section.

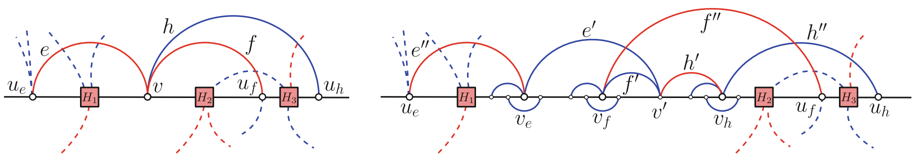


Fig. 5. Illustration of the transformation for the proof of Lemma 3 to reduce the number of vertices incident to two exclusive edges in G_1. Edges e', f' of G_2 and h' of G_1 (right) take the role of edges e, f of G_1 and h of G_2 (left), respectively. Thus, the orthogonality constraints at v' are equivalent to those at v. (Color figure online)

Lemma 3. *Let $\langle G_1, G_2\rangle$ be an instance of* OrthoSEFE-2 *whose shared graph is a cycle and whose union graph has maximum degree* 5*. It is possible to construct in polynomial time an equivalent instance $\langle G_1^*, G_2^*\rangle$ of* OrthoSEFE-2 *whose shared graph is a cycle and such that graph G_1^* has maximum degree* 3*.*

Theorem 5. OrthoSEFE-2 *can be solved in polynomial time for instances whose shared graph is a cycle and whose union graph has maximum degree* 5*.*

5 Shared Graph is Biconnected

We now study OrthoSEFE-k for instances whose shared graph is biconnected. In Theorem 6, we give a polynomial-time Turing reduction from instances of OrthoSEFE-2 whose shared graph is biconnected to instances whose shared graph is a cycle. In Theorem 7, we give an algorithm that, given a positive instance of OrthoSEFE-k such that the shared graph is biconnected together with a SEFE satisfying the orthogonality constraints, constructs an OrthoSEFE with at most three bends per edge.

We start with the Turing reduction, i.e., we develop an algorithm that takes as input an instance $\langle G_1, G_2\rangle$ of OrthoSEFE-2 whose shared graph $G_\cap = G_1 \cap G_2$ is biconnected and produces a set of $O(n)$ instances $\langle G_1^1, G_2^1\rangle, \ldots, \langle G_1^h, G_2^h\rangle$ of OrthoSEFE-2 whose shared graphs are cycles. The output is such that $\langle G_1, G_2\rangle$ is a positive instance if and only if all instances $\langle G_1^i, G_2^i\rangle$, $i = 1, \ldots, h$, are positive. The reduction is based on the SEFE testing algorithm for instances whose shared graph is biconnected by Bläsius et al. [9,10], which can be seen as a generalized and unrooted version of the one by Angelini et al. [4].

We first describe a preprocessing step. Afterwards, we give an outline of the approach of Bläsius et al. [10] and present the Turing reduction in two steps. We assume familiarity with SPQR-trees [15,16]; for formal definitions, see [1].

Lemma 4. *Let $\langle G_1, G_2\rangle$ be an instance of* OrthoSEFE-2 *whose shared graph is biconnected. It is possible to construct in polynomial time an equivalent instance $\langle G_1^*, G_2^*\rangle$ whose shared graph is biconnected and such that each endpoint of an exclusive edge has degree* 2 *in the shared graph.*

We continue with a brief outline of the algorithm by Bläsius et al. [10]. First, the algorithm computes the SPQR-tree $\mathcal{T}$ of the shared graph. To avoid special cases, $\mathcal{T}$ is augmented by adding S-nodes with only two virtual edges such that each P-node and each R-node is adjacent only to S-nodes and Q-nodes. Then, necessary conditions on the embeddings of P-nodes and R-nodes are fixed up to a flip following some necessary conditions. Afterwards, by traversing all S-nodes, a global 2SAT formula is produced whose satisfying assignments correspond to choices of the flips that result in a SEFE. We refine this approach and show that we can choose the flips independently for each S-node, which allows us to reduce each of them to a separate instance, whose shared graph is a cycle.

We now describe the algorithm of Bläsius et al. [10] in more detail. Consider a node μ of $\mathcal{T}$. A *part* of skel(μ) is either a vertex of skel(μ) or a virtual edge of skel(μ), which represents a subgraph of G. An exclusive edge e has an *attachment* in a part x of skel(μ) if x is a vertex that is an endpoint of e or if x is a virtual edge whose corresponding subgraph contains an endpoint of e. An exclusive edge e of G_1 or of G_2 is *important* for μ if its endpoints are in different parts of skel(μ). It is not hard to see that, to obtain a SEFE, the embedding of the skeleton skel(μ) of each node μ has to be chosen such that for each exclusive edge e the parts containing the attachments of e share a face. It can be shown that any embedding choice for P-nodes and R-nodes that satisfies these conditions can, after possibly flipping it, be used to obtain a SEFE [4, Theorem 1]. The proof does not modify the order of exclusive edges around degree-2 vertices of $G_\cap$, and therefore applies to OrthoSEFE-2 as well.

Now, let μ be an S-node. Let ε be a virtual edge of skel(μ), G_ε be the subgraph represented by ε, and ν be the corresponding neighbor of μ in the SPQR-tree of G. An *attachment* of ν with respect to μ is an interior vertex of G_ε that is incident to an important edge e for μ. If ν has such an attachment, then it is a P- or R-node. It is a necessary condition on the embedding of G_ε that each attachment x with respect to μ must be incident to a face incident to the virtual edge twin(ε) of skel(ν) representing μ, and that their clockwise circular order together with the poles of ε is fixed up to reversal [10, Lemma 8].

For the purpose of avoiding crossings in skel(μ), we can thus replace each virtual edge ε that does not represent a Q-node by a cycle C_ε containing the attachments of ε with respect to μ and the poles of ε in the order O_ε. We keep only the important edges of μ. Altogether, this results in an instance $\langle G_1^\mu, G_2^\mu \rangle$ of SEFE modeling the requirements for skel(μ); see Figs. 6(a) and 6(b).

Lemma 5. *Let $\langle G_1, G_2 \rangle$ be an instance of* OrthoSEFE-2 *whose shared graph is biconnected. Then $\langle G_1, G_2 \rangle$ admits an OrthoSEFE if and only if all instances $\langle G_1^\mu, G_2^\mu \rangle$ admit an OrthoSEFE.*

Next, we transform a given instance $\langle G_1^\mu, G_2^\mu \rangle$ of OrthoSEFE-2 as above into an equivalent instance $\langle \overline{G_1^\mu}, \overline{G_2^\mu} \rangle$ whose shared graph is a cycle. Let C_{ε_i} be the cycles corresponding to the neighbor ν_i, $i = 1, \dots, k$ of μ in $\langle G_1^\mu, G_2^\mu \rangle$. To

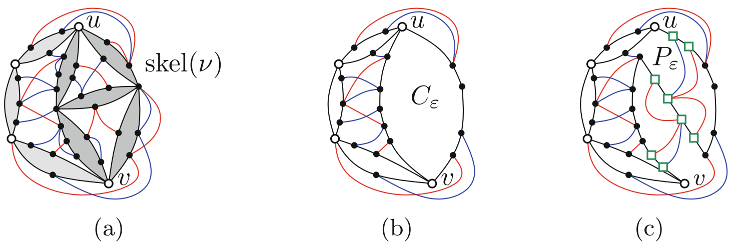


Fig. 6. (a) Skeleton of an S-node μ in which the R-node ν corresponding to the virtual edge $\varepsilon = (u, v)$ is expanded to show its skeleton. (b) Replacing ε with cycle C_ε. (c) Replacing C_ε with path P_ε; vertices $a_1, a_2, x_1, \dots, x_4, b_1, b_2$ are green boxes. (Color figure online)

obtain the instance $\langle \overline{G_1^\mu}, \overline{G_2^\mu} \rangle$, we replace each cycle C_{ε_i} with poles u and v by a path P_{ε_i} from u to v that first contains two special vertices a_1, a_2 followed by the clockwise path from u to v (excluding the endpoints), then four special vertices $x_1, \dots, x_4$, then the counterclockwise path from u to v (excluding the endpoints), and finally two special vertices b_1, b_2 followed by v. In addition to the existing exclusive edges (note that we do not remove any vertices), we add to G_1 the exclusive edges (a_2, x_3), (x_1, x_3), (x_2, x_4), (x_2, b_1), and to G_2 the exclusive edges (a_1, x_3) and (x_2, b_2) to G_2; see Fig. 6(c).

The above reduction together with the next lemma implies the main result.

Lemma 6. *$\langle G_1^\mu, G_2^\mu \rangle$ admits an OrthoSEFE if and only if $\langle \overline{G_1^\mu}, \overline{G_2^\mu} \rangle$ does.*

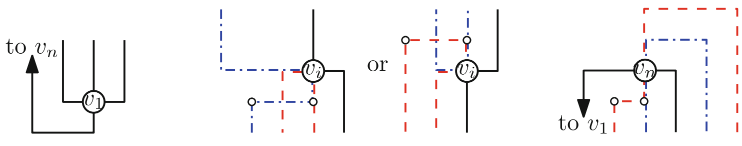
Theorem 6. OrthoSEFE-2 *when the shared graph is biconnected is polynomial-time Turing reducible to* OrthoSEFE-2 *when the shared graph is a cycle. Also, the reduction does not increase the maximum degree of the union graph.*

Corollary 1. OrthoSEFE-2 *can be solved in polynomial time for instances whose shared graph is biconnected and whose union graph has maximum degree* 5.

Observe that, from the previous results, it is not hard to also obtain a SEFE satisfying the orthogonality constraints, if it exists. We show how to construct an orthogonal geometric realizations of such a SEFE.

Theorem 7. *Let $\langle G_1, \dots, G_k \rangle$ be a positive instance of* OrthoSEFE-k *whose shared graph is biconnected. Then, there exists an OrthoSEFE $\langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$ of $\langle G_1, \dots, G_k \rangle$ in which every edge has at most three bends.*

Proof sketch. We assume that a SEFE satisfying the orthogonality constraints is given. We adopt the method of Biedl and Kant [8]. We draw the vertices with increasing y-coordinates with respect to an s-t-ordering [14] $v_1, \dots, v_n$ on the shared graph. We choose the face to the left of (v_1, v_n) as the outer face of the union graph. The edges will bend at most on y-coordinates near their incident vertices and are drawn vertically otherwise. Figure 7 indicates how the ports are assigned. We make sure that an edge may only leave a vertex to the bottom if it is incident to v_n or to a neighbor with a lower index. Thus, there are exactly three bends on $\{v_1, v_n\}$. Any other edge $\{v_i, v_j\}$, $1 \leq i < j \leq n$ has at most one bend around v_i and at most two bends around v_j. $\square$



(a) around v_1 (b) around $v_2, \dots, v_{n-1}$ (c) around v_n

Fig. 7. Constructing a drawing with at most three bends per edge. (Color figure online)

6 Conclusions and Future Work

In this work, we introduced and studied the problem OrthoSEFE-k of realizing a SEFE in the orthogonal drawing style. While the problem is already NP-hard even for instances that can be efficiently tested for a SEFE, we presented a polynomial-time testing algorithm for instances consisting of two graphs whose shared graph is biconnected and whose union graph has maximum degree 5. We have also shown that any positive instance whose shared graph is biconnected can be realized with at most three bends per edge.

We conclude the paper by presenting a lemma that, together with Theorem 6, shows that it suffices to only focus on a restricted family of instances to solve the problem for all instances whose shared graph is biconnected.

Lemma 7. *Let* $\langle G_1, G_2 \rangle$ *be an instance of* OrthoSEFE-2 *whose shared graph* $G_\cap$ *is a cycle. An equivalent instance* $\langle G_1^*, G_2^* \rangle$ *of* OrthoSEFE-2 *such that (i) the shared graph* $G_\cap^*$ *is a cycle, (ii) graph* G_1^* *is outerplanar, and (iii) no two degree-*4 *vertices in* G_1^* *are adjacent, can be constructed in polynomial time.*

References

1. Angelini, P., Chaplick, S., Cornelsen, S., Da Lozzo, G., Di Battista, G., Eades, P., Kindermann, P., Kratochvíl, J., Lipp, F.: Simultaneous Orthogonal Planarity. ArXiv e-prints, abs/1608.08427 (2016)
2. Angelini, P., Da Lozzo, G., Di Battista, G., Frati, F., Patrignani, M., Rutter, I.: Beyond level planarity. In: Hu, Y., Nöllenburg, M. (eds.) GD 2016. LNCS, vol. 9801, pp. 482–495. Springer, Heidelberg (2016)
3. Angelini, P., Da Lozzo, G., Neuwirth, D.: Advancements on SEFE and partitioned book embedding problems. Theoret. Comput. Sci. **575**, 71–89 (2015)
4. Angelini, P., Di Battista, G., Frati, F., Patrignani, M., Rutter, I.: Testing the simultaneous embeddability of two graphs whose intersection is a biconnected or a connected graph. J. Discrete Algorithms **14**, 150–172 (2012)
5. Argyriou, E.N., Bekos, M.A., Kaufmann, M., Symvonis, A.: Geometric RAC simultaneous drawings of graphs. J. Graph Algorithms Appl. **17**(1), 11–34 (2013)
6. Auslander, L., Parter, S.V.: On embedding graphs in the sphere. J. Math. Mech. **10**(3), 517–523 (1961)
7. Bekos, M.A., van Dijk, T.C., Kindermann, P., Wolff, A.: Simultaneous drawing of planar graphs with right-angle crossings and few bends. J. Graph Algorithms Appl. **20**(1), 133–158 (2016)
8. Biedl, T., Kant, G.: A better heuristic for orthogonal graph drawings. Comput. Geom. **9**(3), 159–180 (1998)
9. Bläsius, T., Karrer, A., Rutter, I.: Simultaneous embedding: edge orderings, relative positions, cutvertices. In: Wismath, S., Wolff, A. (eds.) GD 2013. LNCS, vol. 8242, pp. 220–231. Springer, Heidelberg (2013). doi:10.1007/978-3-319-03841-4_20
10. Bläsius, T., Karrer, A., Rutter, I.: Simultaneous embedding: Edge orderings, relative positions, cutvertices. ArXiv e-prints, abs/1506.05715 (2015)
11. Bläsius, T., Kobourov, S.G., Rutter, I.: Simultaneous embedding of planar graphs. In: Tamassia, R., (ed.) Handbook of Graph Drawing and Visualization. CRC Press (2013)

12. Bläsius, T., Rutter, I.: Disconnectivity and relative positions in simultaneous embeddings. Comput. Geom. **48**(6), 459–478 (2015)
13. Bläsius, T., Rutter, I.: Simultaneous PQ-ordering with applications to constrained embedding problems. ACM Trans. Algorithms **12**(2), 16 (2016)
14. Brandes, U.: Eager *st*-ordering. In: Möhring, R., Raman, R. (eds.) ESA 2002. LNCS, vol. 2461, pp. 247–256. Springer, Heidelberg (2002). doi:10.1007/3-540-45749-6_25
15. Di Battista, G., Tamassia, R.: On-line maintenance of triconnected components with SPQR-trees. Algorithmica **15**(4), 302–318 (1996)
16. Di Battista, G., Tamassia, R.: On-line planarity testing. SIAM J. Comput. **25**(5), 956–997 (1996)
17. Estrella-Balderrama, A., Gassner, E., Jünger, M., Percan, M., Schaefer, M., Schulz, M.: Simultaneous geometric graph embeddings. In: Hong, S.-H., Nishizeki, T., Quan, W. (eds.) GD 2007. LNCS, vol. 4875, pp. 280–290. Springer, Heidelberg (2008). doi:10.1007/978-3-540-77537-9_28
18. Haeupler, B., Jampani, K.R., Lubiw, A.: Testing simultaneous planarity when the common graph is 2-connected. J. Graph Algorithms Appl. **17**(3), 147–171 (2013)
19. Jampani, K.R., Lubiw, A.: Simultaneous interval graphs. In: Cheong, O., Chwa, K.-Y., Park, K. (eds.) ISAAC 2010. LNCS, vol. 6506, pp. 206–217. Springer, Heidelberg (2010). doi:10.1007/978-3-642-17517-6_20
20. Jampani, K.R., Lubiw, A.: The simultaneous representation problem for chordal, comparability and permutation graphs. J. Graph Algorithms Appl. **16**(2), 283–315 (2012)
21. Jünger, M., Schulz, M.: Intersection graphs in simultaneous embedding with fixed edges. J. Graph Algorithms Appl. **13**(2), 205–218 (2009)
22. Moret, B.M.E.: Planar NAE3SAT is in P. ACM SIGACT News **19**(2), 51–54 (1988)
23. Moret, B.M.E.: Theory of Computation. Addison-Wesley-Longman, Reading (1998)
24. Papadimitriou, C.H.: Computational Complexity. Academic Internet Publ., London (2007)
25. Schaefer, M.: Toward a theory of planarity: Hanani-Tutte and planarity variants. J. Graph Algorithms Appl. **17**(4), 367–440 (2013)
26. Schaefer, T.J.: The complexity of satisfiability problems. In: Lipton, R.J., Burkhard, W.A., Savitch, W.J., Friedman, E.P., Aho, A.V. (eds.), STOC 1978, pp. 216–226. ACM (1978)
27. Shih, W., Wu, S., Kuo, Y.: Unifying maximum cut and minimum cut of a planar graph. IEEE Trans. Comput. **39**(5), 694–697 (1990)
28. Tamassia, R.: On embedding a graph in the grid with the minimum number of bends. SIAM J. Comput. **16**(3), 421–444 (1987)