

Chapter 18

Symmetries

In classical and in quantum systems, we have Noether's theorem [16, 63]:

Whenever there is a continuous symmetry in a system, there exists a conserved quantity associated with it.

Examples are conservation of momentum (translation symmetry), conservation of energy (symmetry with respect to time translations), and angular momentum (rotation symmetry). In classical systems, Noether's theorem is limited to continuous symmetries, in quantum systems, this theorem is even more universal: here also discrete symmetries have their associated conservation laws: parity $P = \pm 1$ of a system or particle (mirror symmetry), non commuting discrete quantum numbers associated with more general discrete permutations, etc. Also, in a quantum system, one can reverse the theorem:

Every conserved quantity is associated to a symmetry,

for instance isospin symmetry follows from the conservation of the isospin vector $\vec{I} = (I_1, I_2, I_3)$, baryon number conservation leads to a symmetry with respect to $U(1)$ rotations of baryonic wave functions, and so on.

18.1 Classical and Quantum Symmetries

We now claim that this more generalized Noether theorem can also be applied to classical systems, simply by attaching a basis element of Hilbert space to every state the classical system can be in. If, for instance, the evolution law $U_{t,t+\delta t}$ is independent of time t , we have a conserved energy. This energy is obtained from the eigenvalue of $U_{t,t+\delta t}$ for the smallest admissible value of δt . Now since an energy eigenstate will usually not be an ontological state of the system, this energy conservation law only emerges in our quantum procedure; it does not show up in standard classical considerations. For us, this is very important: if δt is as small as the Planck time, the energy eigenstates, all the way to the Planck energy, are superpositions of

ontological states. If, as we usually do, we limit ourselves to quantum systems with much lower energies, we are singling out a section of Hilbert space that is not represented by individual ontological states, and for this reason we should not expect recognizable classical features in the quantum systems that we are usually looking at: atoms, molecules, elementary particles.

Often, our deterministic models are based on a lattice rather than a space–time continuum. The classical space–time symmetries on a lattice are more restricted than those of a continuum. It is here that our mappings onto quantum systems may help. If we allow ontological states to have symmetry relations with superimposed states, much more general symmetry groups may be encountered. This is further illustrated in this chapter.

Since we often work with models having only finite amounts of data in the form of bits and bytes in given volume elements, we are naturally led to systems defined on a lattice. There are many ways in which points can be arranged in a lattice configuration, as is well known from the study of the arrangement of atoms in crystalline minerals. The symmetry properties of the minerals are characterized by the set of crystallographic point groups, of which there are 32 in three dimensions [72].

The simplest of these is the cubic symmetry group generated by a cubic lattice:

$$\vec{x} = (n_1, n_2, n_3), \quad (18.1)$$

where n_1, n_2 and n_3 are integers. What we call the cubic group here, is the set of all 48 orthogonal rotations including the reflections of these three integers into \pm each other (6 permutations and 2^3 signs). This group, called $O(3, \mathbb{Z})$, is obviously much smaller than the group $O(3, \mathbb{R})$ of all orthonormal rotations. The cubic group is a finite subgroup of the infinite orthogonal group.

Yet in string theory, Sect. 17.3.2, something peculiar seems to happen: even though the string theory is equivalent to a lattice model, it nevertheless appears not to lose its full orthogonal rotation symmetry. How can this be explained?

18.2 Continuous Transformations on a Lattice

Consider a classical model whose states are defined by data that can be arranged in a d dimensional cubic lattice. Rotation symmetry is then usually limited by the group $O(d, \mathbb{Z})$. If now we introduce our Hilbert space, such that every state of the classical system is a basis element of that, then we can introduce superpositions, and much more symmetry groups are possible. There are several ways now to introduce *continuous* translations and rotations.

To this end, it is, again, very instructive to do the Fourier transformation:

$$\langle \vec{x} | \psi \rangle = (2\pi)^{-d/2} \int_{|\kappa_i| < \pi} d^d \vec{\kappa} \langle \vec{\kappa} | \psi \rangle e^{i\vec{\kappa} \cdot \vec{x}}. \quad (18.2)$$

Here, $|\psi\rangle$ describes a single particle living on the lattice, but we could also take it as the operator field of a second-quantized system, as is usual in quantum field theories.

Of course, as is usual in physics notation, $\langle \vec{x} |$ are the bras in x space (where \vec{x} is Eq. (18.1), the lattice), whereas $\langle \vec{k} |$ are the bras in momentum space, where \vec{k} are continuous, and all its components κ_i obey $|\kappa_i| < \pi$.

The inverse of the Fourier transform (18.2) is:

$$\langle \vec{k} | \psi \rangle = (2\pi)^{-d/2} \sum_{\vec{x} \in \mathbb{Z}^d} \langle \vec{x} | \psi \rangle e^{-i\vec{k} \cdot \vec{x}}. \quad (18.3)$$

18.2.1 Continuous Translations

Translations over *any* distance \vec{a} can now be defined as the operation

$$\langle \vec{k} | \psi \rangle \rightarrow \langle \vec{k} | \psi \rangle e^{-i\vec{k} \cdot \vec{a}}, \quad (18.4)$$

although only if \vec{a} has integer components, this represents an ontological shift

$$\langle \vec{x} | \psi \rangle \rightarrow \langle \vec{x} - \vec{a} | \psi \rangle, \quad (18.5)$$

since \vec{x} must sit in the lattice, otherwise this would represent a non ontological state.

If \vec{a} has fractional components, the translation in x space can still be defined. Take for instance a fractional value for a_x , or, $\vec{a} = (a_x, 0, 0)$. Then

$$\begin{aligned} \langle \kappa_x | \psi \rangle &\rightarrow \langle \kappa_x | \psi \rangle e^{-i\kappa_x a_x}, & \langle x | \psi \rangle &\rightarrow \sum_{x'} \langle x' | \psi \rangle \Delta_{a_x}(x - x'), \\ \Delta_{a_x}(x_1) &= (2\pi)^{-1} \int_{-\pi}^{\pi} d\kappa_x e^{-ia_x \kappa_x + i\kappa_x x_1} = \frac{\sin \pi(x_1 - a_x)}{\pi(x_1 - a_x)}, \end{aligned} \quad (18.6)$$

where we also used Eq. (18.3) for the inverse of Eq. (18.2).

One easily observes that Eq. (18.6) reduces to Eq. (18.5) if a_x tends to an integer. Translations over a completely arbitrary vector \vec{a} are obtained as the product of fractional translations over $(a_x, 0, 0)$, $(0, a_y, 0)$ and $(0, 0, a_z)$:

$$\langle \vec{x} | \psi \rangle \rightarrow \sum_{x', y', z'} \langle \vec{x}' | \psi \rangle \Delta_{\vec{a}}(\vec{x} - \vec{x}'), \quad \Delta_{\vec{a}}(\vec{x}_1) = \Delta_{a_x}(x_1) \Delta_{a_y}(y_1) \Delta_{a_z}(z_1). \quad (18.7)$$

Notice that the kernel function $\Delta_{\vec{a}}(\vec{x}_1)$ maximizes for the values of \vec{x}_1 closest to \vec{a} , so, even for translations over fractional values of the components of \vec{a} , the translation operation involves only the components of $|\psi\rangle$ closest to the target value $\vec{x} - \vec{a}$.

The *generator* for infinitesimal translations is the operator $\vec{\eta}_{\text{op}}$. Translations over a finite distance \vec{a} can then be described by the operator $e^{i\vec{a} \cdot \vec{\eta}_{\text{op}}}$. Writing $\vec{\eta}_{\text{op}} = (\eta_x, \eta_y, \eta_z)$, and taking η_x, η_y , and η_z each to act only in one dimension, we have

$$\langle \vec{k} | \vec{\eta}_{\text{op}} | \psi \rangle = -\vec{k} \langle \vec{k} | \psi \rangle, \quad (18.8)$$

$$\langle x | e^{i\eta_x a_x} | \psi \rangle = \sum_{x'} \langle x' | \psi \rangle \frac{\sin \pi(x - x' - a_x)}{\pi(x - x' - a_x)}, \quad (18.9)$$

and when a_x is taken to be infinitesimal, while x and x' are integers, one finds

$$\langle x | (\mathbb{I} + i\eta_x a_x) | \psi \rangle = \sum_{x'} \langle x' | \psi \rangle \left(\delta_{xx'} + (1 - \delta_{xx'}) \frac{(-1)^{x-x'} (-\pi a_x)}{\pi(x-x')} \right); \quad (18.10)$$

$$\langle x | \eta_x | x \rangle = 0, \quad \langle x | \eta_x | x' \rangle = \frac{i(-1)^{x-x'}}{x-x'} \quad \text{if } x \neq x'. \quad (18.11)$$

The eigenstates $|\eta_x\rangle$ of the operator η_x can be found: $\langle x | \eta_x \rangle = e^{-i\eta_x x}$.

The expressions we found for this generator are the most natural ones but not the only possible choices; we must always remember that one may add multiples of 2π to its eigenvalues. This modifies the matrix elements (18.11) while the effects of translations over integer distances remain the same.

An important feature of our definition of fractional translations on a lattice is their commutation rules. These translations are entirely commutative (as we can deduce from the definition Eq. (18.4)):

$$[\vec{\eta}, \vec{\eta}'] = 0. \quad (18.12)$$

18.2.2 Continuous Rotations I: Covering the Brillouin Zone with Circular Regions

What can be done with translations on a lattice, can also be done for rotations, in various ways. Let us first show how to obtain a perfect general rotation operator on a lattice, in principle. Again, we start from the Fourier modes, e^{ikx} , Eq. (18.2). How do we generate arbitrary rotations?

Taking again the cubic lattice as our prototype, we immediately see the difficulty: the space of allowed values for \vec{k} is a square (in 2 dimensions) or a cube (in 3 dimensions). This square and this cube are only invariant under the discrete rotation group $O(d, \mathbb{Z})$. Therefore, rotations over other angles can at best be approximate, it seems. We illustrate the situation for a two-dimensional square lattice, but extrapolation to $d > 2$ space dimensions and/or other lattice configurations is straightforward.

The space of allowed momentum values is called the Brillouin zone, and it is the square in Fig. 18.1a. A first approximation for a rotation of the lattice by any angle φ is obtained by drawing the largest possible circle in the Brillouin zone (or the largest possible sphere in the 3 or higher dimensional case) and rotate the region inside that. The data on the remainder of the Brillouin zone, outside the circle, are ignored or replaced by zero.

This procedure perhaps looks good for the lower frequency modes, but it does not rotate everything, and it would clearly disobey the desired group properties of rotations and translations, so we must do something better with the remainder of the Brillouin zone. This is possible, see Fig. 18.1b. The rotation operator could then be defined as follows.

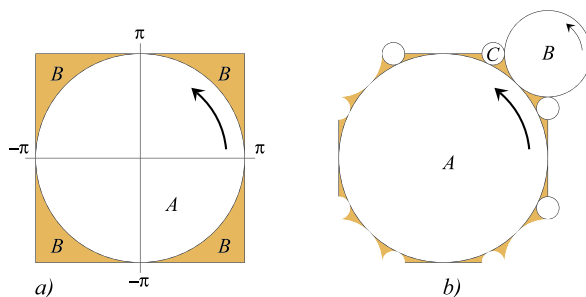


Fig. 18.1 Rotations in the Brillouin zone of a rectangular lattice. **a** We can limit ourselves to the region inside the largest circle that fits in the Brillouin zone (A). The shaded region (B) is neglected and the amplitude there replaced by zero. This is good if strongly fluctuating modes in \vec{x} space may be ignored, such as in a photograph with a rectangular grid of pixels. **b** Unitarity is restored if we also fill the remainder of the Brillouin zone also with circles, B, C , etc., the larger the better (as explained in the text), but never overlapping. In the picture, the shaded regions should also be filled with circles. The rotation operator must rotate every circle by the same angle φ (arrows)

We fill the entire Brillouin zone with circular regions, such that they completely cover the entire space without overlappings. As will be explained shortly, we prefer to keep these circles as large as possible to get the best¹ result. The action of the rotation operator will now be defined to correspond to a *rotation over the same angle φ inside all of these circles* (arrows in Figs. 18.1a and b). With “circles” we here mean circular regions, or, if $d > 2$, regions bounded by $(d - 1)$ -spheres.

This is—nearly²—the best we can do in the Brillouin zone, being the space of the Fourier vectors \vec{k} . The reason why we split the Brillouin zones into perfectly spherical regions, rather than other shapes, becomes clear if we inspect the action of this operator in \vec{x} -space: how does this operator work in the original space of the lattice sites \vec{x} ?

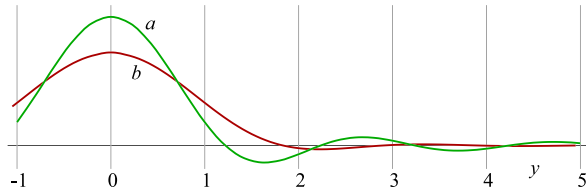
Let us first consider the action of a single circle, while the data on the rest of the Brillouin zone are replaced by zero. First take a circle (if $d = 2$) or sphere (if $d = 3$) whose centre is at the origin, and its radius is r . Projecting out this circle means that, in \vec{x} -space, a wave function $\psi(\vec{x})$ is smeared as follows:

$$\begin{aligned} \psi'(\vec{x}) &= (2\pi)^{-d} \sum_{\vec{x}'} \int_{|\vec{k}| < r} d^d \vec{k} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \psi(\vec{x}') \\ &= \sum_{\vec{x}'} \left(\frac{r}{\pi}\right)^d K_d\left(\frac{r}{\pi} |\vec{x} - \vec{x}'|\right) \psi(\vec{x}'). \end{aligned} \quad (18.13)$$

¹“Best” here means that the effect of the rotation is maximally local, as will be seen in the sequel.

²A slight complication that can be cured, is explained shortly after Eq. (18.19) on page 221.

Fig. 18.2 The function $K_d(y)$, **a** for $d = 2$, and **b** for $d = 5$



The kernel of this rotationally symmetric expression turns out to be a Bessel function:

$$K_d(y) = \frac{\pi^{\frac{d-1}{2}}}{2^d \Gamma(\frac{d+1}{2})} \int_{-1}^1 dk (1-k^2)^{\frac{d-1}{2}} e^{i\pi ky} = (2y)^{-d/2} J_{d/2}(\pi y). \quad (18.14)$$

It is a smooth function, dropping off at infinity as a power of y (see Fig. 18.2):

$$K_d(y) \longrightarrow \frac{2 \sin \pi(y + \frac{1-d}{4})}{\pi (2y)^{\frac{d+1}{2}}} \quad \text{as } y \rightarrow \infty. \quad (18.15)$$

We infer from Eq. (18.13) that, projecting out the inside of a circle with radius r in the Brillouin zone, implies smearing the data on the lattice over a few lattice sites in all directions, using the kernel $K_d(y)$. The smaller the radius r , the further out the smearing, which is why we should try to keep our circles (spheres) as large as possible.

Next, we notice that most of the circles in Fig. 18.1b are off-centre. A displacement by a vector $\vec{\kappa}_1$ in the Brillouin zone corresponds to a multiplication in configuration space by the exponent $e^{i\vec{\kappa}_1 \cdot \vec{x}}$. Projecting out a circle with radius r and its origin on the spot $\vec{\kappa}_1$ in the Brillouin zone, means dividing the wave function $\psi(\vec{x})$ by the exponent $e^{i\vec{\kappa}_1 \cdot \vec{x}}$, smearing it with the kernel $K_d(y)$, then multiplying with the exponent again (thus, we bring the circle to the origin, project out the centralized circle, then move it back to where it was). This amounts to smearing the original wave function with the modified kernel

$$K_d(y, \vec{\kappa}_1) = K_d(y) e^{i\vec{\kappa}_1 \cdot (\vec{x} - \vec{x}')} , \quad y = \frac{r}{\pi} |\vec{x} - \vec{x}'|. \quad (18.16)$$

If we add the projections of all circles with which we covered the Brillouin zone, the total effect should be that we recover the original wave function on the lattice.

And now we can rotate. Rotating a circle (r, κ_1) in the Brillouin zone over any angle φ has exactly the same effect as (1) finding the smeared wave function using the kernel $K_d(y) e^{-i\vec{\kappa}_1 \cdot \vec{x}'}$, rotating the resulting continuous function over the angle φ in \vec{x} -space, and then multiplying with the exponential $e^{i\vec{\kappa}_1 \cdot \vec{x}}$. If we add together the effects of all circles, we get the rotation operator. If we want the effect of an orthogonal rotation Ω in \vec{x} -space, then this results in

$$\psi'(\vec{x}) = \sum_{\vec{x}'} \sum_i K_d\left(\frac{r_i}{\pi} |\Omega \vec{x} - \vec{x}'|\right) e^{i\vec{\kappa}_i \cdot (\vec{x} - \vec{x}')} \psi(\vec{x}'), \quad (18.17)$$

where the index i counts the circles covering the Brillouin zone.

The transformations described in this subsection form a perfectly acceptable rotation group, converging to the usual rotations in the continuum limit. This can easily be seen by noting that the continuum case is dominated by the small values of $\vec{\kappa}$, which are all in the primary circle. The other circles also rotate the wave functions to the desired location, but they only move along the rapidly oscillating parts, while the vectors $\vec{\kappa}_1$ stay oriented in the original direction.

The desired group properties of this operator follow from the fact that the circles cover the Brillouin zone exactly once.

$$\Omega_3 = \Omega_1 \Omega_2. \quad (18.18)$$

Of course, the operation (18.17), to be referred to as $R(\Omega)$, is not quite an ordinary rotation. If $T(\vec{a})$ is the translation over a non-lattice vector \vec{a} as described in Sect. 18.2.1, then

$$R(\Omega)T(\vec{a}) \neq T(\Omega\vec{a})R(\Omega), \quad (18.19)$$

and furthermore, if Ω is chosen to be one of the elements of the crystal group of the lattice, $R(\Omega)$ does still not coincide with Ω itself. This latter defect can be cured, but we won't go into these details.

The best feature of this rotation operator is that it appears to act really locally in \vec{x} -space, spreading the lattice points only slightly with the Bessel function kernels (18.14), but it also has disadvantages: it will be extremely difficult to construct some deterministic evolution law that respects this transformation as a symmetry. For this reason, we now consider other continuous transformation prescriptions that yield rotations.

18.2.3 Continuous Rotations 2: Using Noether Charges and a Discrete Subgroup

In a deterministic theory then, we wish to identify an evolution law that respects our symmetries. This requires a different choice for the definitions of the symmetries involved. To this end, we enter Noether's theorem, as it was introduced at the beginning of this chapter. For example, symmetry under time translations is associated to the conservation of energy, translation symmetry is associated to momentum conservation, and rotation symmetry leads to the conservation of angular momentum. We refer to these conserved quantities as *Noether charges*. All these conserved charges are observable quantities, and therefore, if we wish to investigate them in a quantum theory that we relate to a deterministic system, then this deterministic system should also exhibit observable quantities that can be directly related to the Noether charges.

In the PQ formalism, the Noether charges for translation symmetry are built in, in a sense. Translations in the Q_i variables are associated to quantities p_i , of which the integer parts P_i are ontological observables. Only the fractional parts generate

translations, which then must be integer steps on the Q lattice. We need both components of the momentum. If the lattice length is small, the quanta of the integer parts of the momenta are large. Planets have very large momenta, cold atoms have very small momenta. Large momenta also are sources of gravitational fields, and as such directly observable. What about the transition region? It happens to be in a very familiar domain—the Planck unit of momentum is ~ 6.5 kg m/sec. Momentum in that domain must be a mixture of the P observables and the Q displacement operators, whereas in ordinary physics we notice nothing special in that domain.

In the case of angular momentum, we may note that angular momentum is quantized anyway. Can we associate an ontological observable (beable) to angular momentum? Not so easily, because angular momentum consists of non-commuting components. At best we will have ontological quantized variables playing the role of “the classical parts” of angular momentum, supplemented by quantum degrees of freedom (changeables) that restore the commutation rules. We observe that, for small particles, angular momentum is only partly observable; sometimes it is a beable, sometimes a changeable. For large systems, angular momentum is observable, with some margin of error.

This leads us to consider the following structure—and indeed we will have to use similar methods *whenever* a symmetry group becomes large, meaning that it has very many elements. Since angular momenta are non commutative, they cannot be quite ontological, but their ‘classical parts’ must be. Therefore, we assume that the total angular momenta operators J_i can be written as follows:

$$J_i = L_i + \lambda_i, \quad [J_i, J_j] = i\varepsilon_{ijk}J_k, \quad (18.20)$$

where L_i represents the expectation values of J_i in all ontological states, so that L_i are beables. The λ_i represent the remainder, and their expectation values in ontological states vanish:

$$\langle \text{ont} | \lambda_i | \text{ont} \rangle = 0 \quad \text{for each ontological state } | \text{ont} \rangle. \quad (18.21)$$

The following subsection will show an explicit procedure to obtain L_i and λ_i .

18.2.4 Continuous Rotations 3: Using the Real Number Operators p and q Constructed Out of P and Q

If our theory is defined on a lattice, there is another great way to recover many of the symmetries of the continuum case, by using the PQ trick as it was exposed in Sect. 16. We saw that string theory, Sect. 17.3, was re-written in such a way that the string moves on a lattice in target space, where the lattice basically describes the integer parts of the coordinates, while the space in between the lattice sites actually correspond to the eigenstates of the displacement operators for the momentum variables P . Together, they form a continuum, and since the entire system is equivalent to the continuum string theory, it also shares all continuous translation and rotation symmetries with that theory.

By allowing the application of this mechanism, string theory appears to be more powerful than theories of point particles; the commutation rules for the operators in target space are fundamentally different, and string theory allows target space to be in a high number of dimensions.

Thus, in the PQ formalism, we now use the continuum definition of angular momentum. Consider the wave function of a single particle in three space dimensions, so that it lives on the product of three P , Q lattices. These lattices generate the three quantum coordinates q_i . Its Hilbert space \mathcal{H} is the product space of three Hilbert spaces \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 .

Write, as in Eqs. (16.18) and (16.19),

$$q_i^{\text{op}} = Q_i + a_i^{\text{op}}, \quad p_i^{\text{op}} = 2\pi P_i + b_i^{\text{op}}, \quad (18.22)$$

so that the angular momentum operator is (in the 3-dimensional case)

$$J_i = \varepsilon_{ijk} q_j^{\text{op}} p_k^{\text{op}} = \varepsilon_{ijk} (2\pi Q_j P_k + 2\pi a_j^{\text{op}} P_k + Q_j b_k^{\text{op}} + a_j^{\text{op}} b_k^{\text{op}}). \quad (18.23)$$

Since the expectation values of a_i^{op} and b_i^{op} vanish in the ontological states, $|\text{ont}\rangle = |\vec{P}, \vec{Q}\rangle$, and since the last term will be $\leq \mathcal{O}(2\pi)$, we can identify L_i with the first term:

$$L_i \approx 2\pi \varepsilon_{ijk} Q_j P_k. \quad (18.24)$$

Note, that the L_i are quantized in multiples of 2π rather than one, as one might have expected, so Eq. (18.24) cannot hold exactly.

Let us now inspect the modifications on the commutation rules of these angular momentum operators caused by the edge states. In each of the three Hilbert spaces \mathcal{H}_i , $i = 1, 2, 3$, we have Eq. (16.22), while the operators of one of these Hilbert spaces commute with those of the others. Writing the indices explicitly:

$$[q_1, p_1] = i\mathbb{I}_2\mathbb{I}_3(\mathbb{I}_1 - |\psi_e^1\rangle\langle\psi_e^1|), \quad [q_1, p_2] = 0, \quad \text{and cyclic permutations,} \quad (18.25)$$

where \mathbb{I}_i are the identity operators in the i th Hilbert space, and $|\psi_e^i\rangle$ are the edge states on the i th P , Q lattice. One then easily derives that the three angular momentum operators J_i defined in the usual way, Eq. (18.23), obey the commutation rules

$$[J_1, J_2] = iJ_3\mathbb{I}_1\mathbb{I}_2(\mathbb{I}_3 - |\psi_e^3\rangle\langle\psi_e^3|), \quad \text{and cyclic permutations.} \quad (18.26)$$

The importance of this result is that now we observe that the operator J_3 only acts in Hilbert spaces 1 and 2, but is proportional to the identity in \mathcal{H}_3 (since J_3 contains only q_1, q_2, p_1 , and p_2). So the projection operator for the edge state $|\psi_e^3\rangle$ commutes with J_3 . This implies that, if we limit ourselves to states that are orthogonal to the edge states, they will also rotate to states orthogonal to the edge states. In this subspace of Hilbert space the rotations act normally. And we think that this is remarkable, because certainly the ‘‘ontological’’ basis defined on the six-dimensional \vec{P} , \vec{Q} lattice has no built-in continuous rotation invariance at all.

18.2.5 Quantum Symmetries and Classical Evolution

In previous subsections it was observed that, when we project classical models on Hilbert spaces, new symmetries may emerge. These are symmetry transformations that map classical states onto superpositions of states. A few examples were shown.

None of our procedures are fool proof. In the special case to be discussed next, we study time translation invariance. As stated earlier, we might split the energy E into a classical part (δE) and a quantum part (the generator of discrete time translations, the Hamiltonian H that lies in the interval $[0, 2\pi/\delta t)$). However, this would suggest that we can only measure energies with $2\pi/\delta t$ as our margin of error. That cannot be right: if δt is the Planck time, then the energy quantum is the Planck energy, E_{Planck} , which is about 543 kiloWatt-hours; yet we pay our electricity bills per kiloWatt-hour, and those bills are certainly ontological. Mutations in our DNA profiles might require only a couple of electronVolts to take place, and these might be crucial for our genetically inherited identities; an electronVolt is about 10^{-28} times the Planck energy. Even that may have to be (mostly) ontological.

Of course we are primarily interested in symmetries that are symmetries of the evolution operator. The cogwheel model, Sect. 2.2.1, for instance has the classical symmetry of rotations over N steps, if N is the number of cogwheel position states. But if we go to the energy eigenstates $|k\rangle_H$, $k = 0, \dots, N-1$ (Eqs. (2.21) and (2.22)), we see that, there, a translation over n teeth corresponds to multiplication of these states as follows:

$$|k\rangle_H \rightarrow e^{2\pi i k n / N} |k\rangle_H. \quad (18.27)$$

Since these are eigenstates of the Hamiltonian, this multiplication commutes with H and hence the symmetry is preserved by the evolution law.

We now found out that we can enlarge the symmetry group by choosing the multiplication factors in frequency space

$$|k\rangle_H \rightarrow e^{2\pi i k \alpha / N} |k\rangle_H. \quad (18.28)$$

where α now may be any real number, and this also corresponds to a translation in time over the real number α . This enhances the symmetry group from the group of the cyclic permutations of N elements to the group of the continuous rotations of a circle.

18.2.6 Quantum Symmetries and Classical Evolution 2

An other rather trivial yet interesting example of a symmetry that is enlarged if we apply our quantum constructions, occurs in a simple cellular automaton in any number d of space dimensions. Consider the Boolean variables $\sigma(\vec{x}, t) = \pm 1$ distributed over all *even* sites in a lattice space-time, that is, over all points $(\vec{x}, t) = (x_1, \dots, x_d, t)$ with x_i and t all integers, and $x_1 + \dots + x_d + t = \text{even}$.

Let the evolution law be

$$\sigma(\vec{x}, t+1) = \left(\prod_{i=1}^d \sigma(\vec{x} + \vec{e}_i, t) \sigma(\vec{x} - \vec{e}_i, t) \right) \sigma(\vec{x}, t-1), \quad (18.29)$$

where \vec{e}_i are the unit vectors in the i th direction in d dimensional space. Or: the product of the data on all direct space–time neighbours of any odd site (\vec{x}, t) is $+1$. This law is manifestly invariant under time reversal, and we see that it fixes all variables if the data are given on a Cauchy surface consisting of two consecutive layers in time $t, t-1$. The classical model has the manifest translation symmetry over vectors $\delta x = (a_1, \dots, a_d, \tau)$ with $\sum_i a_i + \tau$ even.

Now let us introduce Hilbert space, and consider the *odd* lattice sites. On these odd sites, we define the action of changeables $\sigma_1(\vec{x}_1, t_1)$ as follows:

The data on the time frame $t = t_1$, are kept unchanged;
 on the time frame $t = t_1 - 1$, only $\sigma(\vec{x}_1, t_1 - 1)$ changes sign, and all others remain unchanged;
 consequently, according to the evolution law, also on the time frame $t = t_1 + 1$, only $\sigma(\vec{x}_1, t_1 + 1)$ changes sign, all others stay the same.

The reason for the notation σ_1 is that in a basis of Hilbert space where $\sigma(\vec{x}_1, t_1 - 1) = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, our new operator is $\sigma_1(\vec{x}, t_1) = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as in the Pauli matrices.

Now, checking how the action of $\sigma_1(\vec{x}, t)$ propagates through the lattice, we observe that

$$\sigma_1(\vec{x}, t+1) = \left(\prod_{i=1}^d \sigma_1(\vec{x} + \vec{e}_i, t) \sigma_1(\vec{x} - \vec{e}_i, t) \right) \sigma_1(\vec{x}, t-1), \quad (18.30)$$

where now the vector (\vec{x}, t) is even, while in Eq. (18.29) they were odd. Thus the product of the changeables $\sigma_1(\vec{x}', t')$ that are direct space–time neighbours of an even site (\vec{x}, t) is also one.

Since we recovered the same evolution law but now on the sites that before were empty, our translation symmetry group now has twice as many elements. Now, we can perform a translation over a vector, whose sum of components is odd, but the states in Hilbert space then have to undergo a transformation; at every site:

$$|\psi(\vec{x}, t)\rangle \rightarrow U_{\text{op}} |\psi(\vec{x}, t)\rangle, \quad U_{\text{op}} \sigma_1 U_{\text{op}}^{-1} = \sigma_3; \quad U_{\text{op}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (18.31)$$

Since $U^2 = 1$, this is actually a reflection. This means that the succession of two odd translations gives an even translation without further phase changes.

This simple model shows how the introduction of Hilbert space may enhance the symmetry properties of a theory. In this case it also implies that the Brillouin zone for momentum space becomes twice as large (see Fig. 18.3). A quantum physicist living in this world will not be able to distinguish the even sites from the odd ones.

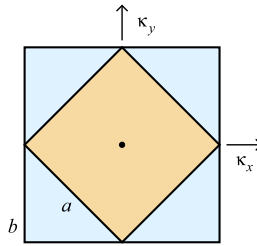


Fig. 18.3 **a** The Brillouin zone for the lattice momentum $\vec{\kappa}$ of the ontological model described by Eq. (18.29), in two space dimensions, with data only on the even lattice sites (*smaller square*, tilted by 45°), and **b** the Brillouin zone for the Hilbert space description of this model (*larger square*)

18.3 Large Symmetry Groups in the CAI

We end this chapter with a general view of large symmetry groups, such as translations in space and in time, and the Lorentz group. They have infinite numbers of group elements. Now we imagine our automaton models to have discretized amounts of information spread over space and time. How can we have infinite and/or continuous symmetry groups act on them?

Our impression from the previous results is that the conventional symmetry generators, as used in quantum theories, will be operators that always consist of *combinations* of beables and changeables: the Noether charges, such as angular momentum, energy and momentum, will have classical limits that are perfectly observable, hence they are beables; yet quantum mechanically, the operators do not commute, and so there must also be changeable parts.

The beable parts will be conjugated to the tiniest symmetry operations such as very tiny translations and rotations. These are unlikely to be useful as genuine transformations among the ontological data—of course they are not, since they must commute with the beables.

The changeable parts of these operators are not ontological observables as they do not commute. The *PQ* formalism, elaborated in Sect. 16, is a realization of this concept of splitting the operators: here, both in position space and in momentum space, the integer parts of the translation operators are beables, the fractional parts are changeables. The continuous translation operators p^{op} consist of both ingredients. We suspect that this will have to become a general feature of all large symmetry groups, in particular the Hamiltonian itself, and this is what we shall attempt to implement in the next chapter 19.

Open Access This chapter is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, duplication, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, a link is provided to the Creative Commons license and any changes made are indicated.

The images or other third party material in this chapter are included in the work's Creative Commons license, unless indicated otherwise in the credit line; if such material is not included in the work's Creative Commons license and the respective action is not permitted by statutory regulation, users will need to obtain permission from the license holder to duplicate, adapt or reproduce the material.