Chapter 16
PQ Theory

Most quantum theories describe operators whose eigenvalues form continua of real numbers. Examples are one or more particles in some potential well, harmonic oscillators, but also bosonic quantum fields and strings. If we want to relate these to deterministic systems we could consider ontological observables that take values in the real numbers as well. There is however an other option.

One important application of the transformations described in this book could be our attempts to produce fundamental theories about Nature at the Planck scale. Here, we have the holographic principle at work, and the Bekenstein bound [5]. What these tell us is that Hilbert space assigned to any small domain of space should be finite-dimensional. In contrast, real numbers are described by unlimited sequences of digits and therefore require infinite dimensional Hilbert spaces. That's too large. One may suspect that uncertainty relations, or non-commutativity, add some blur to these numbers. In this chapter, we outline a mathematical procedure for a systematic approach.

For PQ theory, as our approach will be called, we employed a new notation in earlier work [114, 115], where not $\hbar$ but h is normalized to one. Wave functions then take the form $e^{2\pi ipx}=\epsilon^{ipx}$, where $\epsilon=e^{2\pi}\approx 535.5$. This notation was very useful to avoid factors $1/\sqrt{2\pi}$ for the normalization of wave functions on a circle. Yet we decided not to use such notation in this book, so as to avoid clashes with other discussions in the standard notation in various other chapters. Therefore, we return to the normalization $\hbar=1$. Factors $\sqrt{2\pi}$ (for normalized states) will now occur more frequently, and hopefully they won't deter the reader.

In this chapter, and part of the following ones, dynamical variables can be real numbers, indicated with lower case letters: $p, q, r, x, \dots$, they can be integers indicated by capitals: $N, P, Q, X, \dots$, or they are angles (numbers defined on a circle), indicated by Greek letters $\alpha, \eta, \kappa, \varrho, \dots$, usually obeying $-\pi<\alpha\le\pi$, or sometimes defined merely *modulo* 2π.

A real number r, for example the number $r=137.035999074\cdots$, is composed of an integer, here $R=137$, and an angle, $\varrho/2\pi=0.035999074\cdots$. In examples such as a quantum particle on a line, Hilbert space is spanned by a basis defined on the line: $\{|r\rangle\}$. In PQ theory, we regard such a Hilbert space as the product of Hilbert

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space spanned by the integers $|R\rangle$ and Hilbert space spanned by the angles, $|\varrho\rangle$. So, we have

$$\frac{1}{\sqrt{2\pi}}|r\rangle = |R,\varrho\rangle = |R\rangle|\varrho\rangle. \tag{16.1}$$

Note that *continuity* of a wave function $|\psi\rangle$ implies twisted boundary conditions:

$$\langle R+1,\varrho|\psi\rangle = \langle R,\varrho+2\pi|\psi\rangle. \tag{16.2}$$

The fractional part, or angle, is defined unambiguously, but the definition of the integral part depends on how the angle is projected on the segment $[0, 2\pi]$, or: how exactly do we round a real number to its integer value? We'll take care of that when the question arises.

So far our introduction; it suggests that we can split up a coordinate q into an integer part Q and a fractional part $\xi/2\pi$ and its momentum into an integer part $2\pi P$ and a fractional part κ. Now we claim that this can be done in such a way that both $[P, Q] = 0$ and $[\xi, \kappa] = 0$.

Let us set up our algebra as systematically as possible.

16.1 The Algebra of Finite Displacements

Let there be given an operator q_{op} with non-degenerate eigenstates $|q\rangle$ having eigenvalues q spanning the entire real line. The associated momentum operator is p_{op} with eigenstates $|p\rangle$ having eigenvalues p, also spanning the real line. The usual quantum mechanical notation, now with $\hbar = 1$, is

$$\begin{aligned} &q_{\mathrm{op}}|q\rangle = q|q\rangle; \qquad p_{\mathrm{op}}|p\rangle = p|p\rangle; \qquad [q_{\mathrm{op}}, p_{\mathrm{op}}] = i; \\ &\langle q|q'\rangle = \delta(q-q'); \qquad \langle p|p'\rangle = \delta(p-p'); \qquad \langle q|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq} \end{aligned} \tag{16.3}$$

(often, we will omit the subscript 'op' denoting that we refer to an operator, if this should be clear from the context)

Consider now the displacement operators $e^{-ip_{\mathrm{op}}a}$ in position space, and $e^{iq_{\mathrm{op}}b}$ in momentum space, where a and b are real numbers:

$$e^{-ip_{\mathrm{op}}a}|q\rangle = |q+a\rangle; \qquad e^{iq_{\mathrm{op}}b}|p\rangle = |p+b\rangle. \tag{16.4}$$

We have

$$\begin{aligned} [q_{\mathrm{op}}, p_{\mathrm{op}}] = i, \qquad & e^{iq_{\mathrm{op}}b}p_{\mathrm{op}} = (p_{\mathrm{op}} - b)e^{iq_{\mathrm{op}}b}; \\ e^{iq_{\mathrm{op}}b}e^{-ip_{\mathrm{op}}a} &= e^{-ip_{\mathrm{op}}a}e^{iab}e^{iq_{\mathrm{op}}b} \\ &= e^{-ip_{\mathrm{op}}a}e^{iq_{\mathrm{op}}b}, \quad \text{if } ab = 2\pi \times \text{ integer.} \end{aligned} \tag{16.5}$$

Let us consider the displacement operator in position space for $a = 1$. It is unitary, and therefore can be written uniquely as $e^{-i\kappa}$, where κ is a Hermitian operator with eigenvalues κ obeying $-\pi < \kappa \le \pi$. As we see in Eq. (16.4), κ also represents the momentum *modulo* 2π. Similarly, $e^{i\xi}$, with $-\pi < \xi \le \pi$, is defined to be an operator that causes a shift in the momentum by a step $b = 2\pi$. This means that $\xi/2\pi$ is the position operator q *modulo* one. We write

$$p = 2\pi K + \kappa, \qquad q = X + \xi/2\pi, \tag{16.6}$$

where both K and X are integers and κ and ξ are angles. We suggest that the reader ignore the factors 2π at first reading; these will only be needed when doing precise calculations.

16.1.1 *From the One-Dimensional Infinite Line to the Two-Dimensional Torus*

As should be clear from Eqs. (16.5), we can regard the angle κ as the generator of a shift in the integer X, and the angle ξ as generating a shift in K:

$$e^{-i\kappa}|X\rangle = |X+1\rangle, \qquad e^{i\xi}|K\rangle = |K+1\rangle. \tag{16.7}$$

Since κ and ξ are uniquely defined as generating these elementary shifts, we deduce from Eqs. (16.5) and (16.7) that

$$[\xi, \kappa] = 0. \tag{16.8}$$

Thus, consider the torus spanned by the eigenvalues of the operators κ and ξ. We now claim that the Hilbert space generated by the eigenstates $|\kappa, \xi\rangle$ is equivalent to the Hilbert space spanned by the eigenstates $|q\rangle$ of the operator q_{op}, or equivalently, the eigenstates $|p\rangle$ of the operator p_{op} (with the exception of exactly one state, see later).

It is easiest now to consider the states defined on this torus, but we must proceed with care. If we start with a wave function $|\psi\rangle$ that is continuous in q , we have twisted periodic boundary conditions, as indicated in Eq. (16.2). Here, in the ξ coordinate,

$$\begin{aligned} \langle X+1, \xi|\psi\rangle &= \langle X, \xi + 2\pi|\psi\rangle, \quad \text{or} \\ \langle \kappa, \xi + 2\pi|\psi\rangle &= \langle \kappa, \xi|e^{i\kappa}|\psi\rangle, \end{aligned} \tag{16.9}$$

whereas, since this wave function assumes X to be integer, we have strict periodicity in κ:

$$\langle \kappa + 2\pi, \xi|\psi\rangle = \langle \kappa, \xi|\psi\rangle. \tag{16.10}$$

If we would consider the same state in momentum space, the periodic boundary conditions would be the other way around, and this is why, in the expression used here, the transformations from position space to momentum space and back are non-trivial. For our subsequent calculations, it is much better to transform first to a strictly periodic torus. To this end, we introduce a phase function $\phi(\kappa, \xi)$ with the following properties:

$$\phi(\kappa, \xi + 2\pi) = \phi(\kappa, \xi) + \kappa; \qquad \phi(\kappa + 2\pi, \xi) = \phi(\kappa, \xi); \tag{16.11}$$

$$\phi(\kappa, \xi) = -\phi(-\kappa, \xi) = -\phi(\kappa, -\xi); \qquad \phi(\kappa, \xi) + \phi(\xi, \kappa) = \kappa\xi/2\pi. \tag{16.12}$$

An explicit expression for such a function is derived in Sect. 16.2 and summarized in Sect. 16.3. Here, we just note that this function suffers from a singularity at the point $(\kappa = \pm\pi, \xi = \pm\pi)$. This singularity is an inevitable consequence of the demands (16.11) and (16.12). It is a topological defect that can be moved around on the torus, but not disposed of.

Transforming $|\psi\rangle$ now with the unitary transformation

$$\langle\kappa, \xi|\psi\rangle = \langle\kappa, \xi|U(\kappa, \xi)|\tilde{\psi}\rangle; \qquad U(\kappa, \xi) = e^{i\phi(\kappa,\xi)} = e^{i\kappa\xi/2\pi - i\phi(\xi,\kappa)}, \tag{16.13}$$

turns the boundary conditions (16.9) and (16.10) both into strictly periodic boundaries for $|\tilde{\psi}\rangle$.

For the old wave function, we had $X = i\partial/\partial\kappa$, so, $q_{\text{op}} = i\partial/\partial\kappa + \xi/2\pi$. The operator p_{op} would simply be $-2\pi i\partial/\partial\xi$, assuming that the boundary condition (16.9) ensures that this reduces to the usual differential operator. Our new, transformed wave function now requires a modification of these operators to accommodate for the unusual phase factor $\phi(\kappa, \xi)$. Now our two operators become

$$q_{\text{op}} = i\frac{\partial}{\partial\kappa} + \frac{\xi}{2\pi} - \left(\frac{\partial}{\partial\kappa}\phi(\kappa, \xi)\right) = i\frac{\partial}{\partial\kappa} + \left(\frac{\partial}{\partial\kappa}\phi(\xi, \kappa)\right); \tag{16.14}$$

$$p_{\text{op}} = -2\pi i\frac{\partial}{\partial\xi} + 2\pi\left(\frac{\partial}{\partial\xi}\phi(\kappa, \xi)\right) = -2\pi i\frac{\partial}{\partial\xi} + \kappa - 2\pi\left(\frac{\partial}{\partial\xi}\phi(\xi, \kappa)\right). \tag{16.15}$$

This is how the introduction of a phase factor $\phi(\kappa, \xi)$ can restore the symmetry between the operators q_{op} and p_{op}. Note that, although ϕ is not periodic, the derivative $\partial\phi(\kappa, \xi)/\partial\xi$ is periodic, and therefore, both q_{op} and p_{op} are strictly periodic in κ and ξ (beware the reflections $\xi \leftrightarrow \kappa$ in Eqs. (16.14) and (16.15)).

We check that they obey the correct commutation rule:

$$[q_{\text{op}}, p_{\text{op}}] = i. \tag{16.16}$$

It is very important that these operators are periodic. It implies that we have no theta jumps in their definitions. If we had not introduced the phase function $\phi(\kappa, \xi)$, we would have such theta jumps and in such descriptions the matrix elements in Q, P space would be much less convergent.

The operators $i\partial/\partial\kappa$ and $-i\partial/\partial\xi$ now do not exactly correspond to the operators X and K anymore, because of the last terms in Eqs. (16.14) and (16.15). They are integers however, which obviously commute, and these we shall call Q and P. To obtain the operators q_{op} and p_{op} in the basis of the states $|Q, P\rangle$, we simply expand the wave functions in κ, ξ space in terms of the Fourier modes,

$$\langle\kappa, \xi|Q, P\rangle = \frac{1}{2\pi}e^{iP\xi - iQ\kappa}. \tag{16.17}$$

We now need the Fourier coefficients of the phase function $\phi(\kappa, \xi)$. They are given in Sect. 16.2, where we also derive the explicit expressions for the operators q_{op} and p_{op} in the Q, P basis:

$$q_{\mathrm{op}} = Q_{\mathrm{op}} + a_{\mathrm{op}}; \qquad \langle Q_1, P_1|Q_{\mathrm{op}}|Q_2, P_2\rangle = Q_1 \delta_{Q_1 Q_2} \delta_{P_1 P_2},$$
$$\langle Q_1, P_1|a_{\mathrm{op}}|Q_2, P_2\rangle = \frac{(-1)^{P+Q+1} i P}{2\pi(P^2 + Q^2)}, \tag{16.18}$$

where Q stands short for $Q_2 - Q_1$, and $P = P_2 - P_1$.

For the p operator, it is derived analogously,

$$p_{\mathrm{op}} = 2\pi P_{\mathrm{op}} + b_{\mathrm{op}}, \qquad \langle Q_1, P_1|P_{\mathrm{op}}|Q_2, P_2\rangle = P_1 \delta_{Q_1 Q_2} \delta_{P_1 P_2}; \tag{16.19}$$

$$\langle Q_1, P_1|b_{\mathrm{op}}|Q_2, P_2\rangle = \frac{(-1)^{P+Q} i Q}{P^2 + Q^2}. \tag{16.20}$$

And now for some surprise. Let us inspect the commutator, $[q_{\mathrm{op}}, p_{\mathrm{op}}]$, in the basis of the integers Q and P. We have

$$[Q_{\mathrm{op}}, P_{\mathrm{op}}] = 0; \qquad [a_{\mathrm{op}}, b_{\mathrm{op}}] = 0;$$
$$[q_{\mathrm{op}}, p_{\mathrm{op}}] = [Q_{\mathrm{op}}, b_{\mathrm{op}}] + 2\pi[a_{\mathrm{op}}, P_{\mathrm{op}}]; \tag{16.21}$$
$$\big\langle Q_1, P_1|[q_{\mathrm{op}}, p_{\mathrm{op}}]|Q_2, P_2\big\rangle = -i(-1)^{P+Q)}(1 - \delta_{Q_1 Q_2} \delta_{P_1 P_2}).$$

Here, the delta function is inserted because the commutator vanishes if $Q_1 = Q_2$ and $P_1 = P_2$. So, the commutator is not equal to i times the identity, but it can be written as

$$[q_{\mathrm{op}}, p_{\mathrm{op}}] = i\big(\mathbb{I} - |\psi_e\rangle\langle\psi_e|\big), \quad \text{where } \langle Q, P|\psi_e\rangle = (-1)^{P+Q}. \tag{16.22}$$

Apparently, there is one state $|\psi_e\rangle$ (with infinite norm), for which the standard commutator rule is violated. We encounter more such states in this book, to be referred to as *edge states*, that have to be factored out of our Hilbert space. From a physical point of view it will usually be easy to ignore the edge states, but when we do mathematical calculations it is important to understand their nature. The edge state here coincides with the state $\delta(\kappa - \pi)\delta(\xi - \pi)$, so its mathematical origin is easy to spot: it is located at the singularity of our auxiliary phase function $\phi(\kappa, \xi)$, the one we observed following Eqs. (16.11) and (16.12); apparently, we must limit ourselves to wave functions that vanish at that spot in (κ, ξ) space.

16.1.2 The States $|Q, P\rangle$ in the q Basis

As in other chapters, we now wish to identify the transformation matrix enabling us to transform from one basis to an other. Thus, we wish to find the matrix elements connecting states $|Q, P\rangle$ to states $|q\rangle$ or to states $|p\rangle$. If we can find the function $\langle q|0, 0\rangle$, which gives the wave function in q space of the state with $Q = P = 0$,

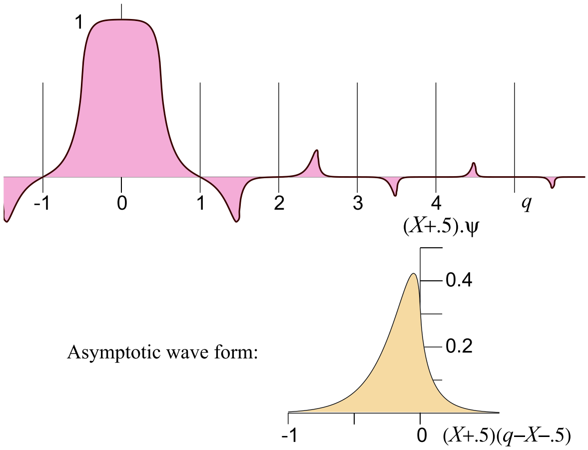

Fig. 16.1 The wave function of the state $|Q, P\rangle$ when $P = Q = 0$. Below, the asymptotic form of the little peaks far from the centre, scaled up

finding the rest will be easy. Section 16.4, shows the derivation of this wave function. In (κ, ξ) space, the state $|q\rangle$ is

$$\langle \kappa, \xi | q\rangle = e^{i\phi(\kappa, 2\pi q)}\delta(\xi - \xi_q), \tag{16.23}$$

if q is written as $q = X + \xi_q/2\pi$, and X is an integer. Section 16.4 shows that then the wave function for $P = Q = 0$ is

$$\langle q|0, 0\rangle = \tfrac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\kappa\, e^{-i\phi(\kappa, 2\pi q)}. \tag{16.24}$$

The general wave function is obtained by shifting P and Q by integer amounts:

$$\langle q|Q, P\rangle = \tfrac{1}{2\pi} e^{2\pi i P q}\int_0^{2\pi} \mathrm{d}\kappa\, e^{-i\phi(\kappa, 2\pi(q-Q))}. \tag{16.25}$$

The wave function (16.24), which is equal to its own Fourier transform, is special because it is close to a block wave, having just very small tails outside the domain $|q| < \frac{1}{2}$, see Fig. 16.1.

The wave $\langle q|Q, P\rangle$ is similar to the so-called *wavelets* [83], which are sometimes used to describe pulsed waves, but this one has two extra features. Not only is it equal to its own Fourier transform, it is also orthogonal to itself when shifted by an integer. This makes the set of waves $|Q, P\rangle$ in Eq. (16.25) an orthonormal basis.

This *PQ* formalism is intended to be used to transform systems based on integer numbers only, to systems based on real numbers, and back. The integers may be assumed to undergo switches described by a permutation operator $\mathcal{P}_{\text{op}}$. After identifying some useful expression for a Hamiltonian H_{op}, with $\mathcal{P}_{\text{op}} = e^{-iH_{\text{op}}\delta t}$, one can now transform this to a quantum system with that Hamiltonian in its new basis.

For a single *PQ* pair, constructing a deterministic model whose evolution operator resembles a realistic quantum Hamiltonian is difficult. A precise, canonical, discrete Hamiltonian formalism *is* possible in the *PQ* scheme, but it requires some more work that we postpone to Sect. 19. Interesting Hamiltonians are obtained in the multidimensional case: P_i, Q_i. Such a system is considered in Chap. 17.

16.2 Transformations in the *PQ* Theory

The following sections, with which we end Chap. 16, can be read as Appendices to Chap. 16. They contain technicalities most readers might not be interested in, but they are needed to understand the details of some features that were encountered, in particular when explicit calculations have to be done, connecting the basis sets of the real numbers q, their Fourier transforms p , the integers (Q, P) and the torus (κ, ξ).

Let us first construct a solution to the boundary conditions (16.11) and (16.12) for a phase function $\phi(\kappa, \xi)$,

$$\phi(\kappa, \xi + 2\pi) = \phi(\kappa, \xi) + \kappa; \qquad \phi(\kappa + 2\pi, \xi) = \phi(\kappa, \xi); \tag{16.26}$$

$$\phi(\kappa, \xi) = -\phi(-\kappa, \xi) = -\phi(\kappa, -\xi); \qquad \phi(\kappa, \xi) + \phi(\xi, \kappa) = \kappa\xi/2\pi. \tag{16.27}$$

At first sight, these conditions appear to be contradictory. If we follow a closed contour $(\kappa, \xi) = (0, 0) \to (2\pi, 0) \to (2\pi, 2\pi) \to (0, 2\pi) \to (0, 0)$, we pick up a term 2π. This implies that a function that is single valued everywhere does not exist, hence, we must have a singularity. We can write down an amplitude $\psi(\kappa, \xi) = re^{i\phi}$ of which this is the phase, but this function must have a zero or a pole. Let's assume it has a zero, and r is simply periodic. Then one can find the smoothest solution. If r and ϕ are real functions:

$$r(\kappa, \xi)e^{i\phi(\kappa,\xi)} = \sum_{N=-\infty}^{\infty} e^{-\pi(N-\frac{\xi}{2\pi})^2+iN\kappa}, \tag{16.28}$$

one finds that this is obviously periodic in κ, while the substitution

$$\xi \to \xi + 2\pi, \qquad N \to N + 1, \tag{16.29}$$

gives the first part of Eq. (16.26).

The sum in Eq. (16.28), which fortunately converges rapidly, is a special case of the elliptic function ϑ_3, and it can also be written as a product [31, 45]:

$$r(\kappa, \xi)e^{i\phi(\kappa,\xi)} = e^{-\frac{\xi^2}{4\pi}} \prod_{N=1}^{\infty} \big(1 - e^{-2\pi N}\big) \times \prod_{N=0}^{\infty} \big(1 + e^{\xi+i\kappa-2\pi N-\pi}\big)\big(1 + e^{-\xi-i\kappa-2\pi N-\pi}\big), \tag{16.30}$$

from which we can easily read off the zeros: they are at $(\kappa, \xi) = (2\pi N_1 + \pi, 2\pi N_2 + \pi)$. We deliberately chose these to be at the corners $(\pm\pi, \pm\pi)$ of the torus, but it does not really matter where they are; they are simply unavoidable.

The matrix elements $\langle Q_1, P_1|q_{\rm op}|Q_2, P_2\rangle$ are obtained by first calculating them on the torus. We have[1]

$$q_{\rm op} = Q_{\rm op} + a(\xi,\kappa); \quad Q_{\rm op} = i\frac{\partial}{\partial\kappa}, \quad a(\xi,\kappa) = \left(\frac{\partial}{\partial\kappa}\phi(\xi,\kappa)\right). \tag{16.31}$$

To calculate $a(\xi,\kappa)$ we can best take the product formula (16.30):

$$a(\xi,\kappa) = \sum_{N=0}^{\infty} a_N(\xi,\kappa),$$
$$a_N(\xi,\kappa) = \frac{\partial}{\partial\kappa}\Big(\arg\big(1+e^{\kappa+i\xi-2\pi(N+\frac12)}\big) + \arg\big(1+e^{-\kappa-i\xi-2\pi(N+\frac12)}\big)\Big). \tag{16.32}$$

Evaluation gives (note the interchange of κ and ξ):

$$a_N(\xi,\kappa) = \frac{\frac12\sin\xi}{\cos\xi+\cosh(\kappa-2\pi N-\pi)} + \frac{\frac12\sin\xi}{\cos\xi+\cosh(\kappa+2\pi N+\pi)}, \tag{16.33}$$

which now allows us to rewrite it as a single sum for N running from $-\infty$ to ∞ instead of 0 to ∞.

Let us first transform from the ξ basis to the P basis, leaving κ unchanged. This turns $a(\xi,\kappa)$ into an operator $a_{\rm op}$. With

$$\langle P_1|\xi\rangle\langle\xi|P_2\rangle = \tfrac{1}{2\pi}e^{iP\xi}, \quad P = P_2 - P_1, \tag{16.34}$$

we get the matrix elements of the operator $a_{\rm op}$ in the (P,κ) frame:

$$\langle P_1|a_{\rm op}(\kappa)|P_2\rangle = \sum_{N=-\infty}^{\infty} a_N(P,\kappa), \quad P \equiv P_2 - P_1, \tag{16.35}$$

and writing $z = e^{i\xi}$, we find

$$a_N(P,\kappa) = \oint\frac{{\rm d}z}{2\pi i z}z^P\frac{-\frac12 i(z-1/z)}{z+1/z+e^R+e^{-R}}, \quad R = \kappa+2\pi N+\pi,$$
$$a_N(P,\kappa) = \tfrac12\,{\rm sgn}(P)(-1)^{P-1}ie^{-|P(\kappa+2\pi N+\pi)|}, \tag{16.36}$$

where ${\rm sgn}(P)$ is defined to be ±1 if $P \gtrless 0$ and 0 if $P=0$. The absolute value taken in the exponent indeed means that we always have a negative exponent there; it originated when the contour integral over the unit circle forced us to choose a pole inside the unit circle.

Next, we find the (Q,P) matrix elements by integrating this over κ with a factor $\langle Q_1|\kappa\rangle\langle\kappa|Q_2\rangle = \frac{1}{2\pi}e^{iQ\kappa}$, with $Q = Q_2 - Q_1$. The sum over N and the integral over κ from 0 to 2π combine into an integral over all real values of κ, to obtain the remarkably simple expression

$$\langle Q_1, P_1|a_{\rm op}|Q_2, P_2\rangle = \frac{(-1)^{P+Q+1}iP}{2\pi(P^2+Q^2)}. \tag{16.37}$$

[1] With apologies for interchanging the κ and ξ variables at some places, which was unavoidable, please beware.

In Eq. (16.14), this gives for the q operator:

$$q_{op} = Q_{op} + a_{op};$$
$$\langle Q_1, P_1|q_{op}|Q_2, P_2\rangle = Q_1\delta_{Q_1Q_2}\delta_{P_1P_2} + \langle Q_1, P_1|a_{op}|Q_2, P_2\rangle. \quad (16.38)$$

For the p operator, the role of Eq. (16.31) is played by

$$p_{op} = 2\pi P + b(\kappa,\xi); \quad P = -i\frac{\partial}{\partial\xi}, \quad b(\kappa,\xi) = 2\pi\left(\frac{\partial}{\partial\xi}\phi(\kappa,\xi)\right), \quad (16.39)$$

and one obtains analogously, writing $P \equiv P_2 - P_1$,

$$p_{op} = 2\pi P_{op} + b_{op}, \qquad \langle Q_1, P_1|P_{op}|Q_2, P_2\rangle = P_1\delta_{Q_1Q_2}\delta_{P_1P_2}; \quad (16.40)$$

$$\langle Q_1, P_1|b_{op}|Q_2, P_2\rangle = \frac{(-1)^{P+Q}iQ}{P^2+Q^2}. \quad (16.41)$$

Note, in all these expressions, we have the symmetry under the combined interchange

$$p_{op} \leftrightarrow 2\pi q_{op}, \qquad P \leftrightarrow Q, \qquad X \leftrightarrow K,$$
$$\xi \leftrightarrow \kappa, \qquad i \leftrightarrow -i, \qquad 2\pi a_{op} \leftrightarrow b_{op}. \quad (16.42)$$

16.3 Resume of the Quasi-periodic Phase Function $\phi(\xi,\kappa)$

Let Q and P be integers while ξ and κ obey $-\pi < \xi < \pi$, $-\pi < \kappa < \pi$. The operators obey

$$[Q_{op}, P_{op}] = [\xi_{op}, P_{op}] = [Q_{op}, \kappa_{op}] = [\xi_{op}, \kappa_{op}] = 0. \quad (16.43)$$

Inner products:

$$\langle\kappa|Q\rangle = \tfrac{1}{\sqrt{2\pi}}e^{-iQ\kappa}, \qquad \langle\xi|P\rangle = \tfrac{1}{\sqrt{2\pi}}e^{iP\xi}. \quad (16.44)$$

The phase angle functions ϕ and $\tilde{\phi}$ are defined obeying (16.26) and (16.27), and

$$\tilde{\phi}(\xi,\kappa) = \phi(\kappa,\xi), \qquad \phi(\xi,\kappa) + \tilde{\phi}(\xi,\kappa) = \xi\kappa/2\pi. \quad (16.45)$$

When computing matrix elements, we should not take the operator ϕ itself, since it is pseudo periodic instead of periodic, so that the edge state $\delta(\kappa \pm \pi)$ gives singularities. Instead, the operator $\partial\phi(\xi,\kappa)/\partial\kappa \equiv a(\xi,\kappa)$ is periodic, so we start from that. Note that, on the (ξ,κ)-torus, our calculations often force us to interchange the order of the variables ξ and κ. Thus, in a slightly modified notation,

$$a(\xi,\kappa) = \sum_{N=-\infty}^{\infty} a_N(\xi,\kappa); \quad a_N(\xi,\kappa) = \frac{\frac{1}{2}\sin\xi}{\cos\xi + \cosh(\kappa + 2\pi(N+\frac{1}{2}))}. \quad (16.46)$$

With $P_2 - P_1 \equiv P$, we write in the (κ, P) basis

$$\begin{aligned}
&\langle \kappa_1, P_1 | a_N^{\rm op} | \kappa_2, P_2 \rangle = \delta(\kappa_1 - \kappa_2) a_N(\kappa_1, P) \\
&a_N(\kappa, P) \equiv \tfrac{1}{2\pi} \int_0^{2\pi} {\rm d}\xi\, e^{iP\xi} a_N(\xi, \kappa) = \tfrac{1}{2}\, {\rm sgn}(P)(-1)^{P-1} i e^{-|P(\kappa + 2\pi N + \pi)|}, \\
&a(\kappa, P) = \tfrac{1}{2} i (-1)^{P-1} \frac{\cosh(P\kappa)}{\sinh(P\pi)}.
\end{aligned} \tag{16.47}$$

Conversely, we can derive in the (Q, ξ) basis, with $Q_2 - Q_1 \equiv Q$:

$$\langle Q_1, \xi_1 | a_N^{\rm op} | Q_2, \xi_2 \rangle = \delta(\xi_1 - \xi_2) a_N(Q, \xi_1)$$

$$a_N(Q, \xi) = \frac{1}{4\pi} \sin\xi \int_{-\pi}^{\pi} {\rm d}\kappa \frac{e^{iQ\kappa}}{\cos\xi + \cosh(\kappa + 2\pi N + \pi)}; \tag{16.48}$$

$$a(Q, \xi) = \sum_N a_N(Q, \xi) = (-1)^Q \frac{\sinh(Q\xi)}{2\sinh(Q\pi)}$$

(the latter expression is found by contour integration).

Finally, in the (Q, P) basis, we have, either by Fourier transforming (16.47) or (16.48):

$$\langle Q_1, P_1 | a^{\rm op} | Q_2, P_2 \rangle = \frac{-i}{2\pi} (-1)^{Q+P} \frac{P}{Q^2 + P^2}. \tag{16.49}$$

The operators $q_{\rm op}$ and $p_{\rm op}$ are now defined on the torus as

$$q_{\rm op}(\kappa, \xi) = Q_{\rm op} + a(\xi, \kappa), \qquad p_{\rm op}(\kappa, \xi) = 2\pi\big(P_{\rm op} + a(\kappa, \xi)\big), \tag{16.50}$$

with $Q_{\rm op} = i\partial/\partial\kappa$, $P_{\rm op} = -i\partial/\partial\xi$. This reproduces Eqs. (16.37) and (16.41).

16.4 The Wave Function of the State $|0, 0\rangle$

We calculate the state $|q\rangle$ in the (κ, ξ) torus. Its wave equation is (see Eq. (16.14))

$$q_{\rm op}|q\rangle = i e^{i\phi(\xi,\kappa)} \frac{\partial}{\partial\kappa} \big(e^{-i\phi(\xi,\kappa)} |q\rangle\big) = q|q\rangle. \tag{16.51}$$

This equation is easy to solve:

$$\langle \kappa, \xi | q \rangle = C(\xi) e^{i\phi(\xi,\kappa) - iq\kappa}. \tag{16.52}$$

Since the solution must be periodic in κ and ξ, while we have the periodicity conditions (16.26) for ϕ, we deduce that this only has a solution if $\xi/2\pi$ is the fractional part of q, or, $q = X + \xi/2\pi$, where X is integer. In that case, we can write

$$\langle \kappa, \xi | q \rangle = C(\xi) e^{-iX\kappa - i\phi(\kappa,\xi)} = C(\xi) e^{-i\phi(\kappa, 2\pi q)}. \tag{16.53}$$

The complete matrix element is then, writing $q = X + \xi_q/2\pi$,

$$\langle \kappa, \xi | q \rangle = C e^{-i\phi(\kappa, 2\pi q)} \delta(\xi - \xi_q) \tag{16.54}$$

(note that the phase $\phi(\kappa, \xi)$ is not periodic in its second entry ξ; the entries are reversed compared to Eqs. (16.26)). The normalization follows from requiring

$$\int_0^{2\pi} \mathrm{d}\kappa \int_0^{2\pi} \mathrm{d}\xi \, \langle q_1|\kappa, \xi\rangle\langle\kappa, \xi|q_2\rangle = \delta(q_1 - q_2);$$
$$\int_{-\infty}^{\infty} \mathrm{d}q \, \langle\kappa_1, \xi_1|q\rangle\langle q|\kappa_2, \xi_2\rangle = \delta(\kappa_1 - \kappa_2)\delta(\xi_1 - \xi_2), \tag{16.55}$$

from which

$$C = 1. \tag{16.56}$$

Note that we chose the phase to be +1. As we also find elsewhere in this book, phases can be chosen freely.

Since

$$\langle\kappa, \xi|Q, P\rangle = \tfrac{1}{2\pi} e^{iP\xi - iQ\kappa}, \tag{16.57}$$

we have

$$\langle q|Q, P\rangle = \tfrac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\kappa \, e^{iP\xi - iQ\kappa + i\phi(\kappa, 2\pi q)}$$
$$= \tfrac{1}{2\pi} e^{2\pi i P q} \int_0^{2\pi} \mathrm{d}\kappa \, e^{i\phi(\kappa, 2\pi(q-Q))}. \tag{16.58}$$