Chapter 12
More on Cogwheels

12.1 The Group $SU(2)$, and the Harmonic Rotator

Let us return to the original cogwheel with N teeth, as introduced in Chap. 2, Sect. 2.2. It may be very illuminating to define the constant $\ell = (N-1)/2$, and introduce the operators L_1, L_2 and L_3 as follows ($k = 0, 1, \dots, 2\ell$ is the energy quantum number; the time step is $\delta t = 1$):

$$\begin{aligned} L_3 &= \tfrac{N}{2\pi} H_{\mathrm{op}} - \ell = k - \ell, \\ L_1 &= \tfrac{1}{2}(L_+ + L_-), \qquad L_2 = -\tfrac{1}{2} i (L_+ - L_-), \\ L_+ |k\rangle_H &= \sqrt{(k+1)(2\ell - k)}\, |k+1\rangle_H, \\ L_- |k\rangle_H &= \sqrt{k(2\ell + 1 - k)}\, |k-1\rangle_H. \end{aligned} \tag{12.1}$$

Using the quantum number $m = k - \ell = L_3$, we get the more familiar expressions for the angular momentum operators $L_a, a = 1, 2, 3$, obeying the commutation relations

$$[L_a, L_b] = i \varepsilon_{abc} L_c. \tag{12.2}$$

The original ontological states $|n\rangle_{\mathrm{ont}}$ can be obtained from the angular momentum states by means of the transformation rules (2.21) and (2.22). It is only these that evolve as ontological states. Other operators can be very useful, however. Take, for instance,

$$x = \tfrac{1}{\sqrt{\ell}} L_1, \qquad p = -\tfrac{1}{\sqrt{\ell}} L_2, \qquad [x, p] = i \big(1 - \tfrac{2\ell + 1}{2\pi \ell} H_{\mathrm{op}}\big), \tag{12.3}$$

then, for states where the energy $\langle H_{\mathrm{op}}\rangle \ll 1$, we have the familiar commutation rules for positions x and momenta p, while the relation $L_1^2 + L_2^2 + L_3^2 = L^2 = \ell(\ell+1)$ implies that, when $\langle H_{\mathrm{op}}\rangle \ll 1/\sqrt{\ell}$,

$$H \to \tfrac{2\pi}{N} \tfrac{1}{2} \big(p^2 + x^2 - 1\big), \tag{12.4}$$

which is the Hamiltonian for the harmonic oscillator (the zero point energy has been subtracted, as the lowest energy eigen state was set at the value zero). Also, at low

G. 't Hooft, *The Cellular Automaton Interpretation of Quantum Mechanics*, Fundamental Theories of Physics 185, DOI 10.1007/978-3-319-41285-6_12

values for the energy quantum number k , we see that $L_\pm$ approach the creation and annihilation operators of the harmonic oscillator (see Eqs. (12.1)):

$$L_- \to \sqrt{2\ell+1}a, \qquad L_+ \to \sqrt{2\ell+1}a^\dagger. \tag{12.5}$$

Thus, we see that the lowest energy states of the cogwheel approach the lowest energy states of the harmonic oscillator. This will be a very useful observation if we wish to construct models for quantum field theories, starting from deterministic cogwheels. The model described by Eqs. (12.1)–(12.3) will be referred to as the *harmonic rotator*. The Zeeman atom of Sect. 2.2 is a simple example with $\ell = 1$.

Note, that the spectrum of the Hamiltonian of the harmonic rotator is *exactly* that of the harmonic oscillator, except that there is an upper limit, $H_{\text{op}} < 2\pi$. By construction, the *period* $T = (2\ell + 1)\delta t$ of the harmonic rotator, as well as that of the harmonic oscillator, is exactly that of the periodic cogwheel.

The Hamiltonian that we associate to the harmonic rotator is also that for a spinning object that exhibits *precession* due to a torque force on its axis. Thus, physically, we see that an oscillator drawing circles in its (x, p) phase space is here replaced by a precessing top. At the lowest energy levels, they obey the same equations.

We conclude from this section that a cogwheel with N states can be regarded as a representation of the group $SU(2)$ with total angular momentum ℓ, and $N = 2\ell + 1$. The importance of this approach is that the representation is a unitary one, and that there is a natural ground state, the ground state of the harmonic oscillator. In contrast to the harmonic oscillator, the harmonic rotator also has an upper bound to its Hamiltonian. The usual annihilation and creation operators, a and $a^\dagger$, are replaced by L_- and L_+, whose commutator is not longer constant but proportional to L_3, and therefore changing sign for states $|k\rangle$ with $\ell < k \leq 2\ell$. This sign change assures that the spectrum is bounded from below as well as above, as a consequence of the modified algebra (12.2).

12.2 Infinite, Discrete Cogwheels

Discrete models with infinitely many states may have the new feature that some orbits may not be periodic. They then contain at least one non-periodic 'rack'. There exists a universal definition of a quantum Hamiltonian for this general case, though it is not unique. Defining the time reversible evolution operator over the smallest discrete time step to be an operator $U_{\text{op}}(1)$, we now construct the simplest Hamiltonian H_{op} such that $U_{\text{op}}(1) = e^{-iH_{\text{op}}}$. For this, we use the evolution over n steps, where n is positive or negative:

$$U_{\text{op}}(n) = U_{\text{op}}(1)^n = e^{-inH_{\text{op}}}. \tag{12.6}$$

Let us assume that the eigenvalues ω of this Hamiltonian lie between 0 and 2π. We can then consider the Hamiltonian in the basis where both $U(1)$ and H are diagonal. Write

$$e^{-in\omega} = \cos(n\omega) - i\sin(n\omega), \tag{12.7}$$

and then use Fourier transformations to derive that, if $-\pi < x < \pi$,

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(nx)}{n}. \tag{12.8}$$

Next, write $H_{\mathrm{op}} = \omega = x + \pi$, to find that Eq. (12.8) gives

$$\omega = \pi - 2\sum_{n=1}^{\infty} \frac{\sin(n\omega)}{n}. \tag{12.9}$$

Consequently, as in Eq. (2.8),

$$\begin{aligned} \omega &= \pi - \sum_{n-1}^{\infty} \frac{i}{n}\big(U(n\delta t) - U(-n\delta t)\big) \quad \text{and} \\ H_{\mathrm{op}} &= \pi - \sum_{n-1}^{\infty} \frac{i}{n}\big(U_{\mathrm{op}}(n\delta t) - U_{\mathrm{op}}(-n\delta t)\big). \end{aligned} \tag{12.10}$$

Very often, we will not be content with this Hamiltonian, as it has no eigenvalues beyond the range $(0, 2\pi)$. As soon as there are conserved quantities, one can add functions of these at will to the Hamiltonian, to be compared with what is often done with chemical potentials. Cellular automata in general will exhibit many such conservation laws. See Fig. 2.3, where every closed orbit represents something that is conserved: the label of the orbit.

In Sect. 13, we consider the other continuum limit, which is the limit $\delta t \to 0$ for the cogwheel model. First, we look at continuous theories more generally.

12.3 Automata that Are Continuous in Time

In the physical world, we have no direct indication that time is truly discrete. It is therefore tempting to consider the limit $\delta t \to 0$. At first sight, one might think that this limit should be the same as having a continuous degree of freedom obeying differential equations in time, but this is not quite so, as will be explained later in this chapter. First, in this section, we consider the strictly continuous deterministic systems. Then, we compare those with the continuum limit of discrete systems.

Consider an ontological theory described by having a continuous, multi-dimensional space of degrees of freedom $\vec{q}(t)$, depending on one continuous time variable t, and its time evolution following classical differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i(t) = f_i(\vec{q}), \tag{12.11}$$

where $f_i(\vec{q})$ may be almost any function of the variables q_j.

An example is the description of massive objects obeying classical mechanics in N dimensions. Let $a = 1, \dots, N$, and $i = 1, \dots, 2N$:

$$\begin{aligned} &\{i\} = \{a\} \oplus \{a+N\}, \qquad q_a(t) = x_a(t), \qquad q_{a+N}(t) = p_a(t), \\ &f_a(\vec{q}) = \frac{\partial H_{\mathrm{class}}(\vec{x}, \vec{p})}{\partial p_a}, \qquad f_{a+N}(\vec{q}) = -\frac{\partial H_{\mathrm{class}}(\vec{x}, \vec{p})}{\partial x_a}, \end{aligned} \tag{12.12}$$

where H_{class} is the *classical* Hamiltonian.

An other example is the quantum wave function of a particle in one dimension:

$$\{i\} = \{x\}, \qquad q_i(t) = \psi(x,t); \qquad f_i(\vec{q}) = -i H_S \psi(x,t), \tag{12.13}$$

where now H_S is the Schrödinger Hamiltonian. Note, however, that, in this case, the function $\psi(x,t)$ would be treated as an ontological object, so that the Schrödinger equation and the Hamiltonian eventually obtained will be quite different from the Schrödinger equation we start off with; actually it will look more like the corresponding *second quantized* system (see later).

We are now interested in turning Eq. (12.11) into a quantum system by changing the notation, not the physics. The ontological basis is then the set of states $|\vec{q}\rangle$, obeying the orthogonality property

$$\langle \vec{q} | \vec{q}\,' \rangle = \delta^N (\vec{q} - \vec{q}\,'), \tag{12.14}$$

where δ is now the Dirac delta distribution, and N is the dimensionality of the vectors $\vec{q}$.

If we wrote

$$\frac{\mathrm{d}}{\mathrm{d}t} \psi(\vec{q}) \stackrel{?}{=} -f_i(\vec{q}) \frac{\partial}{\partial q_i} \psi(\vec{q}) \stackrel{?}{=} -i H_{\mathrm{op}} \psi(\vec{q}), \tag{12.15}$$

where the index i is summed over, we would read off that

$$H_{\mathrm{op}} \stackrel{?}{=} -i f_i(\vec{q}) \frac{\partial}{\partial q_i} = f_i(\vec{q}) p_i, \qquad p_i = -i \frac{\partial}{\partial q_i}. \tag{12.16}$$

This, however, is not quite the right Hamiltonian because it may violate hermiticity: $H_{\mathrm{op}} \neq H_{\mathrm{op}}^\dagger$. The correct Hamiltonian is obtained if we impose that probabilities are preserved, so that, in case the Jacobian of $\vec{f}(\vec{q})$ does not vanish, the integral $\int \mathrm{d}^N \vec{q}\, \psi^\dagger(\vec{q}) \psi(\vec{q})$ is still conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t} \psi(\vec{q}) = -f_i(\vec{q}) \frac{\partial}{\partial q_i} \psi(\vec{q}) - \tfrac{1}{2} \left(\frac{\partial f_i(\vec{q})}{\partial q_i} \right) \psi(\vec{q}) = -i H_{\mathrm{op}} \psi(\vec{q}), \tag{12.17}$$

$$\begin{aligned} H_{\mathrm{op}} &= -i f_i(\vec{q}) \frac{\partial}{\partial q_i} - \tfrac{1}{2} i \left(\frac{\partial f_i(\vec{q})}{\partial q_i} \right) \\ &= \tfrac{1}{2} \big(f_i(\vec{q}) p_i + p_i f_i(\vec{q}) \big) \equiv \tfrac{1}{2} \big\{ f_i(\vec{q}), p_i \big\}. \end{aligned} \tag{12.18}$$

The $1/2$ in Eq. (12.17) ensures that the product $\psi^\dagger \psi$ evolves with the right Jacobian. Note that this Hamiltonian is Hermitian, and the evolution equation (12.11) follows immediately from the commutation rules

$$[q_i, p_j] = i \delta_{ij}; \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{O}_{\mathrm{op}}(t) = -i [\mathcal{O}_{\mathrm{op}}, H_{\mathrm{op}}]. \tag{12.19}$$

Now, however, we encounter a very important difficulty: this Hamiltonian has no lower bound. It therefore cannot be used to *stabilize* the wave functions. Without lower bound, one cannot do thermodynamics. This feature would turn our model into something very unlike quantum mechanics as we know it.

If we take $\vec{q}$ space either one-dimensional, or in some cases two-dimensional, we can make our system periodic. Then let T be the smallest positive number such that

$$\vec{q}(T) = \vec{q}(0). \tag{12.20}$$

We have consequently

$$e^{-iHT}|\vec{q}(0)\rangle = |\vec{q}(0)\rangle, \tag{12.21}$$

and therefore, on these states,

$$H_{\text{op}}|\vec{q}\rangle = \sum_{n=-\infty}^{\infty} \frac{2\pi n}{T} |n\rangle_{H\,H}\langle n|\vec{q}\rangle. \tag{12.22}$$

Thus, the spectrum of eigenvalues of the energy eigenstates $|n\rangle_H$ runs over all integers from $-\infty$ to ∞.

In the discrete case, the Hamiltonian has a finite number of eigenstates, with eigenvalues $2\pi k/(N\delta t) + \delta E$ where $k = 0, \dots, N-1$, which means that they lie in an interval $[\delta E, 2\pi/T + \delta E]$, where T is the period, and δE can be freely chosen. So here, we always have a lower bound, and the state with that energy can be called 'ground state' or 'vacuum'.

Depending on how the continuum limit is taken, we may or may not preserve this lower bound. The lower bound on the energy seems to be artificial, because all energy eigenstates look exactly alike. It is here that the $SU(2)$ formulation for harmonic rotators, handled in Sect. 12.1, may be more useful.

An other remedy against this problem could be that we demand analyticity when time is chosen to be complex, and boundedness of the wave functions in the lower half of the complex time frame. This would exclude the negative energy states, and still allow us to represent all probability distributions with wave functions. Equivalently, one could consider complex values for the variable(s) $\vec{q}$ and demand the absence of singularities in the complex plane below the real axis. Such analyticity constraints however seem to be rather arbitrary; they are difficult to maintain as soon as interactions are introduced, so they would certainly have to be handled with caution.

One very promising approach to solve the ground state problem is Dirac's great idea of second quantization: take an indefinite number of objects $\vec{q}$, that is, a Hilbert space spanned by all states $|\vec{q}^{(1)}, \vec{q}^{(2)}, \dots \vec{q}^{(n)}\rangle$, for all particle numbers n, and regard the negative energy configurations as 'holes' of antiparticles. This we propose to do in our 'neutrino model', Sect. 15.2, and in later chapters.

Alternatively, we might consider the continuum limit of a discrete theory more carefully. This we try first in the next chapter. Let us emphasize again: in general, excising the negative energy states just like that is not always a good idea, because any perturbation of the system might cause transitions to these negative energy states, and leaving these transitions out may violate unitarity.

The importance of the ground state of the Hamiltonian was discussed in Chap. 9 of Part I. The Hamiltonian (12.18) is an important expression for fundamental discussions on quantum mechanics.

As in the discrete case, also in the case of deterministic models with a continuous evolution law, one finds discrete and continuous eigenvalues, depending on whether or not a system is periodic. In the limit $\delta t \to 0$ of the discrete periodic ontological model, the eigenvalues are integer multiples of $2\pi/T$, and this is also the spectrum of the harmonic oscillator with period T, as explained in Chap. 13. The harmonic oscillator may be regarded as a deterministic system in disguise.

The more general continuous model is then the system obtained first by having a (finite or infinite) number of harmonic oscillators, which means that our system consists of many periodic substructures, and secondly by admitting a (finite or infinite) number of conserved quantities on which the periods of the oscillators depend. An example is the field of non-interacting particles; quantum field theory then corresponds to having an infinite number of oscillating modes of this field. The particles may be fermionic or bosonic; the fermionic case is also a set of oscillators if the fermions are put in a box with periodic boundary conditions. *Interacting quantum particles* will be encountered later (Chap. 19 and onwards).