Chapter 11
Introduction to Part II

Many of the technical calculations and arguments mentioned in Part I of this book, were postponed to the second part, so as to make the first part easier to read while keeping it coherent, and to give some nice firework in the second part. The price we pay for this is that there will be a number of repetitions, for which we apologize.

11.1 Outline of Part II

One of our main themes is that quantum mechanics may be viewed as a mathematical tool rather than a new theory of physical phenomena. Indeed, in condensed matter theory, several models exist where the physical setup and the questions asked are fundamentally classical, yet the calculations are performed by regarding the system as a quantum mechanical one. The *two-dimensional Ising Model* is a beautiful example of this [56].

There is no better way to illustrate our approach than by actually showing how such calculations are done. The Cogwheel Model was already introduced in Sect. 2.2. Now, in Chaps. 12.1–13, we show some more of our mathematical tools, how to construct quantum Hamiltonians and how to approach continuum limits. Here, the cogwheel model is linked to the *harmonic rotator*, but also other, notoriously 'classical' structures, such as the planetary system, are transformed into models that appear to be quantum mechanical.

The continuum limit of a single, periodic cogwheel is an important example. It approaches the ordinary quantum harmonic oscillator with the same period T. The continuum cogwheel is actually a smoothly rotating wheel. Is the classical rotating wheel equivalent to a quantum harmonic oscillator? In a sense, yes, but there are some subtleties that one has to be aware of. This is why we decided to do this limit in two steps: first transform the cogwheel into a harmonic rotator, allowing the teeth to form a representation of the group $SU(2)$, and only then consider the continuum limit. This enables us to recognize the operators x and p of a genuine harmonic oscillator already in finite cogwheels.

© The Author(s) 2016

G. 't Hooft, *The Cellular Automaton Interpretation of Quantum Mechanics*,
Fundamental Theories of Physics 185, DOI 10.1007/978-3-319-41285-6_11

Like other technical calculations elsewhere in this book, they were done on order to check the internal consistency of the systems under study. It was fun to do these calculations, but they are not intended to discourage the reader. Just skip them, if you are more interested in the general picture.

The issue of the locality of the Hamiltonian is further treated in Chap. 14. It will come up frequently in almost any deterministic model, and again the mathematics is interesting. We observe that a lot depends on the construction of the *vacuum state*. It is the state of lowest energy, and the solution of the equation "energy = lowest", generates non-localities indeed. In reality, as is well known in quantum field theories, signals will not go faster than the speed of light. What will be shown in this chapter is that there is a way to avoid non-localities when objects move around surrounded by a vacuum, provided one uses a first-quantized theory where only the center part of the energy spectrum is used. Consequently, energy can be positive or negative there. Subsequently, one introduces anti-particles, such that the negative energy states actually represent holes of antiparticles. It is nothing but Dirac's trick to ensure that the physical vacuum has lowest possible energy.

Dirac first phrased his theory for fermionic particles. Indeed, fermions are easier to understand in this respect that bosons are. Therefore, we first introduce fermions as an essential element in our models, see Chap. 15.

It so happens that Dirac's equation for the electron is well suited to demonstrate our prescription of searching for "beables" in a quantum theory. Section 15.2 also begins at an easy pace but ends up in lengthy derivations. Here also, the reader is invited to enjoy the intricate features of the 'neutrino' model, but they can just as well be skipped.

We take the simplified case of the Dirac equation for a two-component neutrino. It is fundamentally simpler than the Dirac equation for the electron. Furthermore, we assume the absence of interactions. The math starts out simple, but the result is striking: neutrinos are configurations of flat membranes, or 'sheets', rather than particles. the sheets move around classically. This is not a theory but a mathematical fact, as long as we keep mass terms and interactions out of the picture; these must be left for later.

Having observed this, we asked the question how to go from the sheet variables back to the neutrino's quantum operators such as position $\vec{x}$, momentum $\vec{p}$, and spin $\vec{\sigma}$. Here, the math does become complicated, and it is interesting as an exercise (Sects. 15.2.1 and 15.2.2). The neutrinos are ideal for the application of second quantization (Sect. 15.2.3), although, in this language, we cannot yet introduce interactions for them.

Our models, discussed in Chaps. 12–17 and 19, have in common that they are local, realistic, and based on conventional procedures in physics. They also have in common that they are limited in scope, they do not capture all features known to exist in the real world, such as all particle species, all symmetry groups, and in particular special and general relativity. The models should be utterly transparent, they indicate directions that one should look at, and, as was our primary goal, they suggest a great approach towards *interpreting* the quantum mechanical laws that are all so familiar to us.

PQ theory, Chap. 16 is a first attempt to understand links between theories based on *real* numbers and theories based on *integer*, or *discrete*, numbers. The idea is to set up a clean formalism connecting the two, so that it can be used in many instances. Chapter 16 also shows some nice mathematical features, with good use of the elliptic theta functions. The calculations look more complicated than they should be, just because we searched for an elegant mechanism relating the real line to pairs of integers on the one hand and the torus on the other, keeping the symmetry between coordinates and momenta.

In Chap. 17, we find some other interesting extensions of what was done in Chap. 16. A very straightforward argument drew our interest to String Theory and Superstring Theory. We are not strongly advocating the idea that the only way to do interesting physics at the Planck scale is to believe what string theoreticians tell us. It is not clear from our work that such theories are *the* way to go, but we do notice that our program shows remarkable links with string theory. *In the absence of interactions*, the local equations of string- and superstring theory appear to allow the construction of beables, exactly along the route that we advocate. The most striking feature exposed here, is that quantized strings, written in the usual form of continuous quantum field theories in one space, and one time dimension, map onto classical string theories that are not defined in a continuous target space, but on a space–time lattice, where the lattice spacing a is given as $a = 2\pi\sqrt{\alpha'}$.

Symmetries, discussed in Chap. 18, are difficult to understand in the CA Interpretation of quantum mechanics. However, in the CAI, symmetry considerations are as important as anywhere else in physics. Most of our symmetries are discrete, but in some cases, notably in string theory, continuous symmetries such as the Poincaré group, can be recovered.

In Chap. 19, we address the positivity problem of the Hamiltonian from a different perspective. There, the usual Hamiltonian formalism is extended to include discrete variables, again in pairs P_i, Q_i, evolving in discrete time. When we first tried to study this, it seemed like a nightmare, but it so happens that the 'discrete Hamilton formalism' comes out to be almost as elegant as the usual differential form. And indeed here, the Hamiltonian can easily be chosen to be bounded from below.

Eventually, we wish to reproduce effective laws of Nature that should take the form of today's quantum field theories. This is still quite difficult. It was the reason for setting up our procedures in a formal way, so that we will keep the flexibility to adapt our systems to what Nature seems to be telling us through the numerous ingenious experiments that have been performed. We explain some of the most important features of quantum field theory in Chap. 20. Most notably: in quantum field theories, no signal can carry useful information faster than the speed of light, and probabilities always add up to one. Quantum field theory is entirely local, in its own inimitable quantum way. These features we would like to reproduce in a deterministic quantum theory.

To set up the *Cellular Automaton Interpretation* in more detail, we first elaborate some technical issues in cellular automata in general (Chap. 21). These are not the technicalities encountered when computer programs are written for such systems;

software experts will not understand much of our analysis. This is because we are aiming at understanding how such systems may generate quantum mechanics at the very large time and distance limit, and how we may be able to connect to elementary particle physics. What we find is a beautiful expression for a quantum Hamiltonian, in terms of an expansion called the BCH expansion. Everything would have been perfect if this were a convergent expansion.

However, it is easy to see that the expansion is not convergent. We try a number of alternative approaches with some modest successes, but not all issues will be resolved, and the suspicion is aired concerning the source of our difficulties: quantum gravitational effects may be of crucial importance, while it is exactly these effects that are still not understood as well as is needed here. We do propose to use the BCH expansion for many classes of cellular automata to demonstrate how they could be used to *interpret* quantum mechanics. I know that the details are not yet quite right, but this probably has to be attributed to the simple fact that we left out lots of things, notably special and general relativity

11.2 Notation

It is difficult keep our notation completely unambiguous. In Chap. 16, we are dealing with many different types of variables and operators. When a dynamical variable is an integer there, we shall use capitals $A, B, \ldots, P, Q, \ldots$. Variables that are periodic with period 2π, or at least constrained to lie in an interval such as $(-\pi, \pi]$, are angles, mostly denoted by Greek lower case letters $\alpha, \beta, \ldots, \kappa, \theta, \ldots$, whereas real variables will most often be denoted by lower case Latin letters $a, b, \ldots, x, y, \ldots$. Yet sometimes we run out of symbols and deviate from this scheme, if it seems to be harmless to do so. For instance, indices will still be $i, j, \ldots$ for space-like vector components, $\alpha, \beta, \ldots$ for spinors and $\mu, \nu, \ldots$ for Lorentz indices. The Greek letters ψ and φ will be used for wave functions as well.

Yet it is difficult to keep our notation completely consistent; in some chapters before Chap. 16, we use the quantum numbers ℓ and m of the $SU(2)$ representations to denote the integers that earlier were denoted as k or $k - m$, and later in Chap. 16 replaced by capitals.

As in Part I, we use a super- or subscript "op" to distinguish an operator from an ordinary numerical variable. The caret ($\hat{}$) will be reserved for vectors with length one, the arrow for more general vectors, not necessarily restricted to three dimensional space. Only in Chap. 20, where norms of vectors do not arise, we use the caret for the Fourier transform of a function.

Dirac's constant $\hbar$ and the velocity of light c will nearly always be defined to be one in the units chosen. In previous work, we used a spacial symbol to denote $e^{2\pi}$ as an alternative basis for exponential functions. This would indeed sometimes be useful for calculations, when we use fractions that lie between 0 and 1, rather than angles, and it would require that we normalize Planck's original constant h rather then $\hbar$ to one, but in the present monograph we return to the more usual notation.

Concepts frequently discussed are the following:

- *discrete* variables are variables such as the integer numbers, whose possible values can be counted. Opposed to continuous variables, which are typically represented by real or complex numbers.
- *fractional* variables are variables that take values in a finite interval or on a circle. The interval may be $[0, 1)$, $[0, 2\pi)$, $(-\frac{1}{2}, \frac{1}{2}]$, or $(-\pi, \pi]$. Here, the square bracket indicates a bound whose value itself may be included, a round bracket excludes that value. A real number can always be decomposed into an integer (or discrete) number and a fractional one.
- a theory is *ontological*, or 'ontic', if it only describes 'really existing' objects; it is simply a classical theory such as the planetary system, in the absence of quantum mechanics. The theory does not require the introduction of Hilbert space, although, as will be explained, Hilbert space might be very useful. But then, the theory is formulated in terms of observables that are commuting at all times.
- a feature is *counterfactual* when it is assumed to exist even if, for fundamental reasons, it cannot actually be observed; if one would try to observe it, some other feature might no longer be observable and hence become counterfactual. This situation typically occurs if one considers the measurement of two or more operators that do not commute. More often, in our models, we shall encounter features that are not allowed to be counterfactual.
- We talk of *templates* when we describe particles and fields as solutions of Schrödinger's equation in an ontological model, as was explained in Sect. 4.3.1. Templates may be superpositions of ontic states and/or other templates, but the ontic states all form an orthonormal set; superpositions of ontic states are never ontic themselves.

11.3 More on Dirac's Notation for Quantum Mechanics

A denumerable set of states $|e_i\rangle$ is called an orthonormal basis of $\mathcal{H}$ if every state $|\psi\rangle \in \mathcal{H}$ can be approximated by a linear combination of a finite number of states $|e_i\rangle$ up to any required precision:

$$|\psi\rangle = \sum_{i=1}^{N(\varepsilon)} \lambda_i |e_i\rangle + |\varepsilon\rangle, \quad \||\varepsilon\rangle\|^2 = \langle\varepsilon|\varepsilon\rangle < \varepsilon^2, \quad \text{for any } \varepsilon > 0 \tag{11.1}$$

(a property called 'completeness'), while

$$\langle e_i|e_j\rangle = \delta_{ij} \tag{11.2}$$

(called 'orthonormality'). From Eqs. (11.1) and (11.2), one derives

$$\lambda_i = \langle e_i|\psi\rangle, \quad \sum_i |e_i\rangle\langle e_i| = \mathbb{I}, \tag{11.3}$$

where $\mathbb{I}$ is the identity operator: $\mathbb{I}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$.

In many cases, the discrete sum in Eqs. (11.1) and (11.3) will be replaced by an integral, and the Kronecker delta δ_{ij} in Eq. (11.2) by a Dirac delta function, $\delta(x^1 - x^2)$. We shall still call the states $|e_{(x)}\rangle$ a basis, although it is not denumerable.

A typical example is the set of wave functions $\psi(\vec{x})$ describing a particle in position space. They are regarded as vectors in Hilbert space where the set of delta peak wave functions $|\vec{x}\rangle$ is chosen to be the basis:

$$\psi(\vec{x}) \equiv \langle \vec{x}|\psi\rangle, \quad \langle \vec{x}|\vec{x}\,'\rangle = \delta^3\big(\vec{x} - \vec{x}\,'\big). \tag{11.4}$$

The Fourier transformation is now a simple rotation in Hilbert space, or a transition to the *momentum basis*:

$$\langle \vec{x}|\psi\rangle = \int \mathrm{d}^3\vec{p}\,\langle \vec{x}|\vec{p}\rangle\langle \vec{p}|\psi\rangle; \quad \langle \vec{x}|\vec{p}\rangle = \tfrac{1}{(2\pi)^{3/2}} e^{i\,\vec{p}\cdot\vec{x}}. \tag{11.5}$$

Many special functions, such as the Hermite, Laguerre, Legendre, and Bessel functions, may be seen as generating different sets of basis elements of Hilbert space.

Often, we use product Hilbert spaces: $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_3$, which means that states $|\phi\rangle$ in $\mathcal{H}_3$ can be seen as normal products of states $|\psi^{(1)}\rangle$ in $\mathcal{H}_1$ and $|\psi^{(2)}\rangle$ in $\mathcal{H}_2$:

$$|\phi\rangle = \big|\psi^{(1)}\big\rangle\big|\psi^{(2)}\big\rangle, \tag{11.6}$$

and a basis for $\mathcal{H}_3$ can be obtained by combining a basis in $\mathcal{H}_1$ with one in $\mathcal{H}_2$:

$$\big|e^{(3)}_{ij}\big\rangle = \big|e^{(1)}_i\big\rangle\big|e^{(2)}_j\big\rangle. \tag{11.7}$$

Often, some or all of these factor Hilbert spaces are finite-dimensional vector spaces, which of course also allow all the above manipulations.[1] We have, for example, the 2-dimensional vector space spanned by spin $\frac{1}{2}$ particles. A basis is formed by the two states $|\uparrow\rangle$ and $|\downarrow\rangle$. In this basis, the Pauli matrices $\sigma^{\mathrm{op}}_{x,y,z}$ are defined as in Part I, Eqs. (1.7). The states

$$|\rightarrow\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad |\leftarrow\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{11.8}$$

form the basis where the operator σ_x is diagonal: $\sigma^{\mathrm{op}}_x \to \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Dirac derived the words 'bra' and 'ket' from the fact that the expectation value for an operator $\mathcal{O}^{\mathrm{op}}$ can be written as the operator between brackets, or

$$\big\langle \mathcal{O}^{\mathrm{op}}\big\rangle = \langle\psi|\mathcal{O}^{\mathrm{op}}|\psi\rangle. \tag{11.9}$$

More generally, we shall often need the matrix elements of an operator in a basis $\{|e_i\rangle\}$:

$$\mathcal{O}_{ij} = \langle e_i|\mathcal{O}^{\mathrm{op}}|e_j\rangle. \tag{11.10}$$

[1] The term *Hilbert space* is often restricted to apply to infinite dimensional vector spaces only; here we will also include the finite dimensional cases.

The transformation from one basis $\{|e_i\rangle\}$ to another, $\{|e'_i\rangle\}$ is a unitary operator U_{ij}:

$$\begin{aligned} |e'_i\rangle &= \sum_j U_{ij}|e_j\rangle, \quad U_{ij} = \langle e_j|e'_i\rangle; \\ \sum_k U_{ik}U_{jk} &= \sum_k \langle e'_i|e_k\rangle\langle e_k|e'_j\rangle = \delta_{ij}. \end{aligned} \tag{11.11}$$

This will be used frequently. For instance, the Fourier transform is unitary:

$$\int \mathrm{d}^3\vec{p}\,\langle\vec{x}|\vec{p}\rangle\langle\vec{p}|\vec{x}'\rangle = \tfrac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{p}\,e^{i\vec{p}\cdot\vec{x}-i\vec{p}\cdot\vec{x}'} = \delta^3\big(\vec{x}-\vec{x}'\big). \tag{11.12}$$

The Schrödinger equation will be written as:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= -iH^{\mathrm{op}}|\psi(t)\rangle, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)| = \langle\psi(t)|iH^{\mathrm{op}}; \\ |\psi(t)\rangle &= e^{-iH^{\mathrm{op}}t}|\psi(0)\rangle, \end{aligned} \tag{11.13}$$

where H^{op} is the Hamiltonian, defined by its matrix elements $H_{ij} = \langle e_i|H^{\mathrm{op}}|e_j\rangle$.

Dirac's notation may be used to describe non-relativistic wave functions in three space dimensions, in position space, in momentum space or in some other basis, such as a partial wave expansion, it can be used for particles with spin, it can be used in many-particle systems, and also for quantized fields in solid state theory or in elementary particle theory. The transition from a Fock space notation, where the basis is spanned by states containing a fixed number N of particles (in position or in momentum space, possibly having spin as well), to a notation where the basis is spanned by the functions representing the fields of these particles, is simply a rotation in Hilbert space, from one basis into another.

Open Access This chapter is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits use, duplication, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, a link is provided to the Creative Commons license and any changes made are indicated.

The images or other third party material in this chapter are included in the work's Creative Commons license, unless indicated otherwise in the credit line; if such material is not included in the work's Creative Commons license and the respective action is not permitted by statutory regulation, users will need to obtain permission from the license holder to duplicate, adapt or reproduce the material.