Chapter 4
Advanced partial differential equation models

This final chapter addresses more complicated PDE models, including linear elasticity, viscous flow, heat transfer, porous media flow, gas dynamics, and electrophysiology. A range of classical dimensionless numbers are discussed in terms of the scaling.

4.1 The equations of linear elasticity

To the best of the authors' knowledge, it seems that mathematical models in elasticity and structural analysis are almost never non-dimensionalized. This is probably due to tradition, but the following sections will demonstrate the usefulness of scaling also in this scientific field.

We start out with the general, time-dependent elasticity PDE with variable material properties. Analysis based on scaling is used to determine under what circumstances the acceleration term can be neglected and we end up with the widely used stationary elasticity PDE. Scaling of different types of boundary conditions is also treated. At the end, we scale the equations of coupled thermo-elasticity. All the models make the assumption of small displacement gradients and Hooke's generalized constitutive law such that linear elasticity theory applies.

4.1.1 The general time-dependent elasticity problem

The following vector PDE governs deformation and stress in purely elastic materials, under the assumption of small displacement gradients:

$$\varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla((\lambda + \mu)\nabla \cdot \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u}) + \varrho \boldsymbol{f}. \tag{4.1}$$

Here, $\boldsymbol{u}$ is the displacement vector, ϱ is the density of the material, λ and μ are the Lame elasticity parameters, and $\boldsymbol{f}$ is a body force (gravity, centrifugal force, or similar).

We introduce dimensionless variables:

$$\bar{\boldsymbol{u}} = u_c^{-1}\boldsymbol{u}, \quad \bar{x} = \frac{x}{L} \quad \bar{y} = \frac{y}{L} \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{f}{t_c}.$$

Also the elasticity parameters and the density can be scaled, if they are not constants,

$$\bar{\lambda} = \frac{\lambda}{\lambda_c}, \quad \bar{\mu} = \frac{\mu}{\mu_c}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_c},$$

where the characteristic quantities are typically spatial maximum values of the functions:

$$\lambda_c = \max_{x,y,z} \lambda, \quad \mu_c = \max_{x,y,z} \mu, \quad \varrho_c = \max_{x,y,z} \varrho.$$

Finally, we scale $\boldsymbol{f}$ too (if not constant):

$$\bar{\boldsymbol{f}} = f_c^{-1}\boldsymbol{f}, \quad f_c = \max_{x,y,z,t} ||\boldsymbol{f}||.$$

Inserting the dimensionless quantities in the governing vector PDE results in

$$\frac{\varrho_c u_c}{t_c^2} \frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = L^{-2}u_c\bar{\nabla}((\lambda_c\bar{\lambda}+\mu_c\bar{\mu})\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + L^{-2}u_c\mu_c\bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \varrho_c f_c\bar{\varrho}\bar{\boldsymbol{f}}.$$

Making the terms non-dimensional gives the equation

$$\bar{\varrho}\frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = \frac{t_c^2\lambda_c}{L^2\varrho_c}\bar{\nabla}(\bar{\lambda}\bar{\nabla}\cdot\bar{u}) + \frac{t_c^2\mu_c}{L^2\varrho_c}\bar{\nabla}(\bar{\mu}\bar{\nabla}\cdot\bar{u}) + \frac{t_c^2\mu_c}{L^2\varrho_c}\bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{u}) + \frac{t_c^2 f_c}{u_c}\bar{\varrho}\bar{\boldsymbol{f}}. \quad (4.2)$$

We may choose t_c to make the coefficient in front of any of the spatial derivative terms equal unity. Here we choose the μ term, which implies

$$t_c = L\sqrt{\frac{\varrho_c}{\mu_c}}.$$

The scale for $\boldsymbol{u}$ can be chosen from an initial displacement or by making the coefficient in front of the $\bar{\boldsymbol{f}}$ term unity. The latter means

$$u_c = \mu_c^{-1}\varrho_c f_c L^2.$$

As discussed later, in Section 4.1.4, this might not be the desired u_c in applications.

The resulting dimensionless PDE becomes

$$\bar{\varrho}\frac{\partial^2\bar{\boldsymbol{u}}}{\partial\bar{t}^2} = \bar{\nabla}((\beta\bar{\lambda}+\bar{\mu})\bar{\nabla}\cdot\bar{u}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \bar{\varrho}\bar{\boldsymbol{f}}.\tag{4.3}$$

The only dimensionless parameter is

$$\beta = \frac{\lambda_c}{\mu_c}.$$

If the source term is absent, we must use the initial condition or a known boundary displacement to determine u_c.

Software. Given software for (4.1), we can simulate the dimensionless problem by setting $\varrho = \bar{\varrho}$, $\lambda = \beta\bar{\lambda}$, and $\mu = \bar{\mu}$.

4.1.2 Dimensionless stress tensor

The stress tensor $\boldsymbol{\sigma}$ is a key quantity in elasticity and is given by

$$\boldsymbol{\sigma} = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + \mu(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T).$$

This $\boldsymbol{\sigma}$ can be computed as soon as the PDE problem for $\boldsymbol{u}$ has been solved. Inserting dimensionless variables on the right-hand side of the above relation gives

$$\boldsymbol{\sigma} = \lambda_c u_c L^{-2}\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \mu_c u_c L^{-1}\bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}}+(\bar{\nabla}\bar{\boldsymbol{u}})^T)$$
$$= \mu_c u_c L^{-1}\left(\beta\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}}+(\bar{\nabla}\bar{\boldsymbol{u}})^T)\right).$$

The coefficient on the right-hand side, $\mu_c u_c L^{-1}$, has dimension of stress, since (according to the second table in Section 1.1.2) $[\mathrm{MT}^{-2}\mathrm{L}^{-1})(\mathrm{L})(\mathrm{L}^{-1})] = [\mathrm{MT}^{-2}\mathrm{L}^{-1}]$, which is the dimension of stress. The quantity $\mu_c u_c L^{-1}$ is therefore the natural scale of the stress tensor:

$$\bar{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{\sigma_c},\quad \sigma_c = \mu_c u_c L^{-1},$$

and we have the dimensionless stress-displacement relation

$$\bar{\boldsymbol{\sigma}} = \beta\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}}+(\bar{\nabla}\bar{\boldsymbol{u}})^T).\tag{4.4}$$

4.1.3 When can the acceleration term be neglected?

A lot of applications of the elasticity equation involve static or quasi-static deformations where the acceleration term $\varrho\boldsymbol{u}_{tt}$ is neglected. Now we shall see under which conditions the quasi-static approximation holds.

The further discussion will need to look into the time scales of elastic waves, because it turns out that the chosen t_c above is closely linked to the propagation speed of elastic waves in a homogeneous body without body forces. A relevant model for such waves has constant ϱ, λ, and μ, and no force term:

$$\varrho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \mu\nabla^2\boldsymbol{u}. \tag{4.5}$$

S waves. Let us take the curl of this PDE and notice that the curl of a gradient vanishes. The result is

$$\frac{\partial^2}{\partial t^2}\nabla\times\boldsymbol{u} = c_S^2\nabla^2\nabla\times\boldsymbol{u},$$

i.e., a wave equation for $\nabla\times\boldsymbol{u}$. The wave velocity is

$$c_S = \sqrt{\frac{\mu}{\varrho}}.$$

The corresponding waves are called S waves[1]. The curl of a displacement field is closely related to rotation of continuum elements. S waves are therefore rotation waves, also sometimes referred to as shear waves.

The divergence of a displacement field can be interpreted as the volume change of continuum elements. Suppose this volume change vanishes, $\nabla\cdot\boldsymbol{u} = 0$, which means that the material is incompressible. The elasticity equation then simplifies to

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c_S^2\nabla^2\boldsymbol{u},$$

so each component of the displacement field in this case also propagates as a wave with speed c_S^2. The time it takes for such a wave to travel one characteristic length L is L/c_S, i.e., $L\sqrt{\varrho/\mu}$, which is nothing but our characteristic time t_c.

P waves. We may take the divergence of the PDE instead and notice that $\nabla\cdot\nabla = \nabla^2$ so

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\boldsymbol{u} = c_P^2\nabla^2\nabla\cdot\boldsymbol{u},$$

with wave velocity

$$c_P = \sqrt{\frac{\lambda+2\mu}{\varrho}}.$$

---

[1] https://en.wikipedia.org/wiki/S-wave

That is, the volume change (expansion/compression) propagates as a wave with speed c_P. These types of waves are called P waves[2]. Other names are pressure and expansion/compression waves.

Suppose now that $\nabla \times \boldsymbol{u} = 0$, i.e., there is no rotation ("shear") of continuum elements. Mathematically this condition implies that $\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u})$ (see any book on vector calculus or Wikipedia[3]). Our model equation (4.5) then reduces to

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c_P^2 \nabla^2 \boldsymbol{u},$$

which is nothing but a wave equation for the expansion component of the displacement field, just as (4.1.3) is for the shear component.

Time-varying load. Suppose we have some time-varying boundary condition on $\boldsymbol{u}$ or the stress vector (traction), with a time scale $1/\omega$ (some oscillating movement that goes like $\sin\omega t$, for instance). We choose $t_c = 1/\omega$. The scaling now leads to

$$\gamma \frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = \bar{\nabla}((\beta\bar{\lambda} + \bar{\mu})\bar{\nabla} \cdot \bar{\boldsymbol{u}}) + \bar{\nabla} \cdot (\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \bar{\varrho}\bar{\boldsymbol{f}}.$$

where we have set

$$u_c = \mu_c^{-1} f_c L^2 \varrho_c,$$

as before, and γ is a new dimensionless number,

$$\gamma = \frac{\varrho_c L^2 \omega^2}{\mu_c} = \left(\frac{L\sqrt{\varrho_c/\mu_c}}{1/\omega}\right)^2.$$

The last rewrite shows that $\sqrt{\gamma}$ is the ratio of the time scale for S waves and the time scale for the forced movement on the boundary. The acceleration term can therefore be neglected when $\gamma \ll 1$, i.e., when the time scale for movement on the boundary is much larger than the time it takes for the S waves to travel through the domain. Since the velocity of S waves in solids is very large and the time scale correspondingly small, $\gamma \ll 1$ is very often the case in applications involving structural analysis.

4.1.4 The stationary elasticity problem

Scaling of the PDE. We now look at the stationary version of (4.1) where the $\varrho\boldsymbol{u}_{tt}$ term is removed. The first step in the scaling is just inserting the dimensionless variables:

---

[2] https://en.wikipedia.org/wiki/P-wave
[3] https://en.wikipedia.org/wiki/Vector_calculus_identities

$$0 = L^{-2}u_c\bar{\nabla}((\lambda_c\bar{\lambda}+\mu_c\bar{\mu})\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + L^{-2}u_c\mu_c\bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \varrho_c f_c\bar{\varrho}\bar{\boldsymbol{f}}\,.$$

Dividing by $L^2 u_c\mu_c$ gives

$$0 = \bar{\nabla}((\beta\bar{\lambda}+\bar{\mu})\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \frac{L^2\varrho_c f_c}{u_c\mu_c}\bar{\varrho}\bar{\boldsymbol{f}}\,.$$

Choosing $u_c = \varrho L^2 f_c/\mu_c$ leads to

$$\bar{\nabla}((\beta\bar{\lambda}+\bar{\mu})\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \bar{\varrho}\bar{\boldsymbol{f}} = 0\,. \tag{4.6}$$

A homogeneous material with constant λ, μ, and ϱ is an interesting case (this corresponds to $\mu_c = \mu$, $\lambda_c = \lambda$, $\varrho_c = \varrho$, $\bar{\varrho} = \bar{\lambda} = \bar{\mu} = 1$):

$$(1+\beta)\bar{\nabla}(\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}^2\bar{\boldsymbol{u}} + \bar{\boldsymbol{f}} = 0\,. \tag{4.7}$$

Now β is defined as

$$\beta = \frac{\lambda}{\mu} = \left(\frac{c_p}{c_s}\right)^2 - 2\,.$$

It shows that in standard, stationary elasticity, λ/μ is the only significant physical parameter.

Remark on the characteristic displacement.

The presented scaling may not be valid for problems where the geometry involves some dimensions that are much smaller than others, such as for beams, shells, or plates. Then one more length scale must be defined which gives us non-dimensional geometrical numbers. Global balances of moments and loads then determine how characteristic displacements depend on these numbers. As an example, consider a cantilever beam of length L and square-shaped cross section of width W, deformed under its own weight. From beam theory one can derive $u_c = \frac{3}{2}\varrho g L^2\delta^2/E$, where $\delta = L/W$ (g is the acceleration of gravity). If we consider E to be of the same size as λ, this implies that $\gamma \sim \delta^{-2}$. So, it may be wise to prescribe a u_c in elasticity problems, perhaps from formulas as shown, and keep γ in the PDE.

Scaling of displacement boundary conditions. A typical boundary condition on some part of the boundary is a prescribed displacement. For simplicity, we set $\boldsymbol{u} = \boldsymbol{U}_0$ for a constant vector $\boldsymbol{U}_0$ as boundary condition. With $u_c = \varrho L^2 f_c/\mu$, we get the dimensionless condition

$$\bar{\boldsymbol{u}} = \frac{\boldsymbol{U}_0}{u_c} = \frac{\mu\boldsymbol{U}_0}{\varrho L^2 f_c}\,.$$

In the absence of body forces, the expression for u_c has no meaning ($f_c = 0$), so then $u_c = |\boldsymbol{U}_0|$ is a better choice. This gives the dimensionless boundary condition

$$\bar{\boldsymbol{u}} = \frac{\boldsymbol{U}_0}{|\boldsymbol{U}_0|}\,,$$

which is the unit vector in the direction of $\boldsymbol{U}_0$. The new u_c changes the

coefficient in front of the body force term, if that term is present, to the dimensionless number

$$\delta = \frac{L^2 \varrho f_c}{\mu |\boldsymbol{U}_0|}.$$

Scaling of traction boundary conditions. The other type of common boundary condition in elasticity is a prescribed traction (stress vector) on some part of the boundary:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{T}_0,$$

where, to make it simple, we take $\boldsymbol{T}_0$ as a constant vector. From Section 4.1.2 we have a stress scale $\sigma_c = \mu u_c / L$, but we may alternatively use $|\boldsymbol{T}_0|$ as stress scale. In that case,

$$\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \frac{\boldsymbol{T}_0}{|\boldsymbol{T}_0|},$$

which is a unit vector in the direction of $\boldsymbol{T}_0$. Many applications involve large traction free areas on the boundary, on which we simply have $\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = 0$.

4.1.5 Quasi-static thermo-elasticity

Heating solids gives rise to expansion, i.e., strains, which may cause stress if displacements are constrained. The time scale of temperature changes are usually much larger than the time scales of elastic waves, so the stationary equations of elasticity can be used, but a term depends on the temperature, so the equations must be coupled to a PDE for heat transfer in solids. The resulting system of PDEs is known as the equations of *thermo-elasticity* and reads

$$\nabla((\lambda + \mu)\nabla \cdot \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u}) = \alpha \nabla T - \varrho \boldsymbol{f}, \tag{4.8}$$

$$\varrho c \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T) + \varrho \boldsymbol{f}_T, \tag{4.9}$$

where T is the temperature, α is a coefficient of thermal expansion, c is a heat capacity, κ is the heat conduction coefficient, and $\boldsymbol{f}_T$ is some heat source. The density ϱ is strictly speaking a function of T and the stress state, but a widely used approximation is to consider ϱ as a constant. Most thermo-elasticity applications have $\boldsymbol{f}_T = 0$, so we drop this term. Most applications also involve some heating from a temperature level T_0 to some level $T_0 + \Delta T$. A suitable scaling for T is therefore

$$\bar{T} = \frac{T - T_0}{\Delta T},$$

so that $\bar{T} \in [0, 1]$. The elasticity equation has already been scaled and so has the diffusion equation for T. We base the time scale on the diffusion, i.e., the thermal conduction process:

$$t_c = \varrho c L^2 / \kappa_c.$$

We imagine that κ is scaled as $\bar{\kappa} = \kappa/\kappa_c$. The dimensionless PDE system then becomes

$$\bar{\nabla}((1+\beta)\bar{\mu}\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) = \bar{\nabla}\bar{T} - \epsilon\bar{\varrho}\bar{\boldsymbol{f}}, \qquad (4.10)$$

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \bar{\nabla}\cdot(\bar{\kappa}\bar{\nabla}\bar{T}). \qquad (4.11)$$

Here we have chosen u_c such that the "heating source term" has a unit coefficient, acknowledging that this thermal expansion balances the stress terms with $\bar{\boldsymbol{u}}$. The corresponding displacement scale is

$$u_c = \frac{\alpha L\Delta T}{\mu_c}.$$

The dimensionless number in the body force term is therefore

$$\epsilon = \frac{L\varrho_c f_c}{\alpha\Delta T},$$

which measures the ratio of the body force term and the "heating source term".

A homogeneous body with constant ϱ, λ, μ, c, and κ is common. The PDE system reduces in this case to

$$\bar{\nabla}((1+\beta)\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}^2\bar{\boldsymbol{u}} = \bar{\nabla}\bar{T} - \epsilon\bar{\boldsymbol{f}}, \qquad (4.12)$$

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \bar{\nabla}^2\bar{T}. \qquad (4.13)$$

In the absence of body forces, β is again the key parameter.

The boundary conditions for thermo-elasticity consist of the conditions for elasticity and the conditions for diffusion. Scaling of such conditions are discussed in Section 3.2 and 4.1.4.

4.2 The Navier-Stokes equations

This section shows how to scale various versions of the equations governing incompressible viscous fluid flow. We start with the plain Navier-Stokes equations without body forces and progress with adding the gravity force and a free surface. We also look at scaling low Reynolds number flow and oscillating flows.

4.2.1 The momentum equation without body forces

The Navier-Stokes equations for incompressible viscous fluid flow, without body forces, take the form

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}, \tag{4.14}$$

$$\nabla\cdot\boldsymbol{u}=0. \tag{4.15}$$

The primary unknowns are the velocity $\boldsymbol{u}$ and the pressure p. Moreover, ϱ is the fluid density, and μ is the dynamic viscosity.

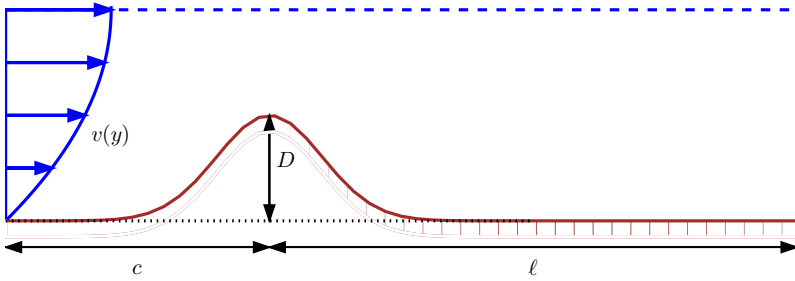**Scaling.** We start, as usual, by introducing a notation for dimensionless independent and dependent variables:

$$\bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{L},\quad \bar{z}=\frac{z}{L},\quad \bar{t}=\frac{t}{t_c},\quad \bar{\boldsymbol{u}}=\frac{\boldsymbol{u}}{u_c},\quad \bar{p}=\frac{p}{p_c},$$

where L is some characteristic distance, t_c is some characteristic time, u_c is a characteristic velocity, while p_c is a characteristic pressure. Inserted in the equations,

$$\varrho\left(\frac{u_c}{t_c}\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}}+\frac{u_c^2}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}\right)=-\frac{p_c}{L}\bar{\nabla}\bar{p}+\frac{u_c}{L^2}\mu\bar{\nabla}^2\bar{\boldsymbol{u}}, \tag{4.16}$$

$$\frac{u_c}{L}\bar{\nabla}\cdot\bar{\boldsymbol{u}}=0. \tag{4.17}$$

For the velocity it is common to just introduce some U for u_c. This U is normally implied by the problem description. For example, in the flow configuration below, with flow over a bump, we have some incoming flow with a profile $v(y)$ and U can typically be chosen as $U=\max_y v(y)$. The height of the bump influences the wake behind the bump, and is the length scale that really impacts the flow, so it is natural to set $L=D$. For numerical simulations in a domain of finite extent, $[0,c+\ell]$, c must be large enough to avoid feedback on the inlet profile, and ℓ must be large enough for the type of outflow boundary condition used. Ideally, $c,\ell\to\infty$, so none of these parameters are useful as length scales.

For flow in a channel or tube, we also have some inlet profile, e.g., $v(r)$ in a tube, where r is the radial coordinate. A natural choice of characteristic velocity is $U = v(0)$ or to let U be the average flow, i.e.,

$$U = \frac{1}{\pi R^2} \int_0^R 2\pi v(r) r\, dr,$$

if R is the radius of the tube. Other examples may be flow around a body, where there is some distant constant inlet flow $\boldsymbol{u} = U_0 \boldsymbol{i}$, for instance, and $U = U_0$ is an obvious choice. We therefore assume that the flow problem itself brings a natural candidate for U.

Having a characteristic distance L and velocity U, an obvious time measure is L/U so we set $t_c = L/U$. Dividing by the coefficient in front of the time derivative term, creates a pressure term

$$\frac{p_c}{\varrho U^2} \bar{\nabla} \bar{p}.$$

The coefficient suggest a choice $p_c = \varrho U^2$ if the pressure gradient term is to have the same size as the acceleration terms. This p_c is a very common pressure scale in fluid mechanics, arising from Bernoulli's equation

$$p + \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u} = \text{const}$$

for stationary flow.

Dimensionless PDEs and the Reynolds number. The discussions so far result in the following dimensionless form of (4.14) and (4.15):

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla}\bar{\boldsymbol{u}} = -\bar{\nabla}\bar{p} + \text{Re}^{-1}\bar{\nabla}^2\bar{\boldsymbol{u}}, \tag{4.18}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{4.19}$$

where Re is the famous *Reynolds number*,

$$\text{Re} = \frac{\varrho U L}{\mu} = \frac{U L}{\nu}.$$

The latter expression makes use of the kinematic viscosity $\nu = \mu/\varrho$. For viscous fluid flows without body forces there is hence only one dimensionless number, Re.

The Reynolds number can be interpreted as the ratio of convection and viscosity:

$$\frac{\text{convection}}{\text{viscosity}} = \frac{|\varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\mu \nabla^2 \boldsymbol{u}|} \sim \frac{\varrho U^2/L}{\mu U/L^2} = \frac{UL}{\nu} = \text{Re}.$$

(We have here used that $\nabla \boldsymbol{u}$ goes like U/L and $\nabla^2 \boldsymbol{u}$ goes like U/L^2.)

4.2.2 Scaling of time for low Reynolds numbers

As we discussed in Section 3.4 for the convection-diffusion equation, there is not just one scaling that fits all problems. Above, we used $t_c = L/U$, which is appropriate if convection is a dominating physical effect. In case the convection term $\varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is much smaller than the viscosity term $\mu \nabla^2 \boldsymbol{u}$, i.e., the Reynolds number is small, the viscosity term is dominating. However, if the scaling is right, the other terms are of order unity, and $\text{Re}^{-1} \nabla^2 \bar{\boldsymbol{u}}$ must then also be of unit size. This fact implies that $\nabla^2 \bar{\boldsymbol{u}}$ must be small, but then the scaling is not right (since a right scaling will lead to $\bar{u}$ and its derivatives around unity). Such reasoning around inconsistent size of terms clearly points to the need for other scales.

In the low-Reynolds number regime, the diffusion effect of $\nabla^2 \bar{\boldsymbol{u}}$ is dominating, and we should use a time scale based on diffusion rather than convection. Such a time scale is $t_c = L^2/(\mu/\varrho) = L^2/\nu$. With this time scale, the dimensionless Navier-Stokes equations look like

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \text{Re} \, \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} = -\bar{\nabla} p + \bar{\nabla}^2 \bar{\boldsymbol{u}}, \tag{4.20}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0. \tag{4.21}$$

As stated in the box in Section 3.4, (4.20) is the appropriate PDE for very low Reynolds number flow and suggests neglecting the convection term. If the flow is also steady, the time derivative term can be neglected, and we end up with the so-called *Stokes problem* for steady, slow, viscous flow:

$$-\bar{\nabla} p + \bar{\nabla}^2 \bar{\boldsymbol{u}} = 0, \tag{4.22}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0. \tag{4.23}$$

This flow regime is also known as *Stokes' flow* or *creeping flow*.

4.2.3 Shear stress as pressure scale

Instead of using the kinetic energy ϱU^2 as pressure scale, one can use the shear stress $\mu U/L$ (U/L reflects the spatial derivative of the velocity, which enters the shear stress expression $\mu \partial u/\partial y$). Using U as velocity scale, L/U as time scale, and $\mu U/L$ as pressure scale, results in

$$\mathrm{Re}\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}}\right) = -\bar{\nabla}\bar{p} + \bar{\nabla}^2 \bar{\boldsymbol{u}}. \tag{4.24}$$

Low Reynolds number flow now suggests neglecting both acceleration terms.

4.2.4 Gravity force and the Froude number

We now add a gravity force to the momentum equation (4.14):

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \varrho g \boldsymbol{k}, \tag{4.25}$$

where g is the acceleration of gravity, and $\boldsymbol{k}$ is a unit vector in the opposite direction of gravity. The new term takes the following form after non-dimensionalization:

$$\frac{t_c}{\varrho u_c}\varrho g \boldsymbol{k} = \frac{Lg}{U^2}\boldsymbol{k} = \mathrm{Fr}^{-2}\boldsymbol{k},$$

where Fr is the dimensionless Froude number,

$$\mathrm{Fr} = \frac{U}{\sqrt{Lg}}.$$

This quantity reflects the ratio of inertia and gravity forces:

$$\frac{|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\varrho g|} \sim \frac{\varrho U^2/L}{\varrho g} = \mathrm{Fr}^2.$$

4.2.5 Oscillating boundary conditions and the Strouhal number

Many flows have an oscillating nature, often arising from some oscillating boundary condition. Suppose such a condition, at some boundary $x = \mathrm{const}$, takes the specific form

$$\boldsymbol{u} = U\sin(\omega t)\boldsymbol{i}.$$

The dimensionless counterpart becomes

$$U\bar{\boldsymbol{u}} = U\sin(\omega\frac{L}{U}\bar{t})\boldsymbol{i},$$

if $t_c = L/U$ is the appropriate time scale. This condition can be written

$$\bar{\boldsymbol{u}} = \sin(\mathrm{St}\,\bar{t}), \tag{4.26}$$

where St is the *Strouhal number*,

$$\mathrm{St} = \frac{\omega L}{U}. \tag{4.27}$$

The two important dimensionless parameters in oscillating flows are then the Reynolds and Strouhal numbers.

Even if the boundary conditions are of steady type, as for flow around a sphere or cylinder, the flow may at certain Reynolds numbers get unsteady and oscillating. For $10^2 < \mathrm{Re} < 10^7$, steady inflow towards a cylinder will cause vortex shedding: an array of vortices are periodically shedded from the cylinder, producing an oscillating flow pattern and force on the cylinder. The Strouhal number is used to characterize the frequency of oscillations. The phenomenon, known as *von Karman vortex street*, is particularly important if the frequency of the force on the cylinder hits the free vibration frequency of the cylinder such that resonance occurs. The result can be large displacements of the cylinder and structural failure. A famous case in engineering is the failure of the Tacoma Narrows suspension bridge[4] in 1940, when wind-induced vortex shedding caused resonance with the free torsional vibrations of the bridge.

4.2.6 Cavitation and the Euler number

The dimensionless pressure in (4.18) made use of the pressure scale $p_c = \varrho U^2$. This is an appropriate scale if the pressure level is not of importance, which is very often the case since only the pressure *gradient* enters the flow equation and drives the flow. However, there are circumstances where the pressure level is of importance. For example, in some flows the pressure may become so low that the vapor pressure of the liquid is reached and that vapor cavities form (a phenomenon known as *cavitation*). A more appropriate pressure scale is then $p_c = p_\infty - p_v$, where p_∞ is a characteristic pressure level far from vapor cavities and p_v is the vapor pressure. The coefficient in front of the dimensionless pressure gradient is then

---

[4]`https://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_(1940)`

$$\frac{p_\infty - p_v}{\varrho U^2}.$$

Inspired by Bernoulli's equation $p + \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u} = \text{const}$ in fluid mechanics, a factor $\frac{1}{2}$ is often inserted in the denominator. The corresponding dimensionless number,

$$\text{Eu} = \frac{p_\infty - p_v}{\frac{1}{2}\varrho U^2}, \tag{4.28}$$

is called the *Euler number*. The pressure gradient term now reads $\frac{1}{2}\text{Eu}\,\bar{\nabla}\bar{p}$. The Euler number expresses the ratio of pressure differences and the kinetic energy of the flow.

4.2.7 Free surface conditions and the Weber number

At a free surface, $z = \eta(x,y,t)$, the boundary conditions are

$$w = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta, \tag{4.29}$$

$$p - p_0 \approx -\sigma \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right), \tag{4.30}$$

where w is the velocity component in the z direction, p_0 is the atmospheric air pressure at the surface, and σ represents the surface tension. The approximation in (4.30) is valid under small deformations of the surface.

The dimensionless form of these conditions starts with inserting the dimensionless quantities in the equations:

$$u_c \bar{w} = \frac{L}{t_c} \frac{\partial \bar{\eta}}{\partial \bar{t}} + u_c \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\eta},$$

$$p_c \bar{p} \approx -\frac{1}{L}\sigma \left(\frac{\partial^2 \bar{\eta}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\eta}}{\partial \bar{y}^2} \right).$$

The characteristic length L is usually taken as the depth of the fluid when the surface is flat. We have used $\bar{p} = (p - p_0)/p_c$ for making the pressure dimensionless. Using $u_c = U$, $t_c = L/U$, and $p_c = \varrho U^2$, results in

$$\bar{w} = \frac{\partial \bar{\eta}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\eta}, \tag{4.31}$$

$$\bar{p} \approx -\text{We}^{-1} \left(\frac{\partial^2 \bar{\eta}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\eta}}{\partial \bar{y}^2} \right), \tag{4.32}$$

where We is the *Weber number*,

$$\text{We} = \frac{\varrho U^2 L}{\sigma} \, . \tag{4.33}$$

The Weber number measures the importance of surface tension effects and is the ratio of the pressure scale ϱU^2 and the surface tension force per area, typically σ/R_x in a 2D problem, which has size σ/L.

4.3 Thermal convection

Temperature differences in fluid flow cause density differences, and since cold fluid is heavier than hot fluid, the gravity force will induce flow due to density differences. This effect is called free thermal convection and is key to our weather and heating of rooms. Forced convection refers to the case where there is no feedback from the temperature field to the motion, i.e., temperature differences do not create motion. This fact decouples the energy equation from the mass and momentum equations.

4.3.1 Forced convection

The model governing forced convection consists of the Navier-Stokes equations and the energy equation for the temperature:

$$\varrho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \varrho g \boldsymbol{k}, \tag{4.34}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.35}$$

$$\varrho c \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) = \kappa \nabla^2 T . \tag{4.36}$$

The symbol T is the temperature, c is a heat capacity, and κ is the heat conduction coefficient for the fluid. The PDE system applies primarily for liquids. For gases one may need a term $-p\nabla \cdot \boldsymbol{u}$ for the pressure work in (4.36) as well as a modified equation of continuity (4.35).

Despite the fact that ϱ depends on T, we treat ϱ as a constant ϱ_0. The major effect of the $\varrho(T)$ dependence is through the buoyancy effect caused by the gravity term $-\varrho(T)g\boldsymbol{k}$. It is common to drop this term in forced convection, and assume the momentum and continuity equations to be independent of the temperature. The flow is driven by boundary conditions (rather than density variations as in free convection), from which we can find a characteristic velocity U.

Dimensionless parameters are introduced as follows:

$$\bar{x} = \frac{x}{L}, \ t_c = \frac{L}{U}, \ \bar{u} = \frac{u}{U}, \ \bar{p} = \frac{p}{\varrho_0 U^2}, \ \bar{T} = \frac{T - T_0}{T_c}.$$

Other coordinates are also scaled by L. The characteristic temperature T_c is chosen as some range ΔT, which depends on the problem and is often given by the thermal initial and/or boundary conditions. The reference temperature T_0 is also implied by prescribed conditions. Inserted in the equations, we get

$$\varrho_0 \frac{U^2}{L} \frac{\partial \bar{u}}{\partial \bar{t}} + \varrho_0 \frac{U^2}{L} \bar{u} \cdot \bar{\nabla} \bar{u} = -\frac{\varrho_0 U^2}{L} \bar{\nabla} \bar{p} + \frac{\mu U}{L^2} \bar{\nabla}^2 \bar{u},$$

$$\frac{U}{L} \bar{\nabla} \cdot \bar{u} = 0,$$

$$\varrho_0 c \left(\frac{T_c U}{L} \frac{\partial \bar{T}}{\partial \bar{t}} + \frac{U T_c}{L} \bar{u} \cdot \bar{\nabla} \bar{T} \right) = \frac{\kappa T_c}{L^2} \bar{\nabla}^2 \bar{T}.$$

Making each term in each equation dimensionless reduces the system to

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \cdot \bar{\nabla} \bar{u} = -\bar{\nabla} \bar{p} + \mathrm{Re}^{-1} \bar{\nabla}^2 \bar{u}, \qquad (4.37)$$

$$\bar{\nabla} \cdot \bar{u} = 0, \qquad (4.38)$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u} \cdot \bar{\nabla} \bar{T} = \mathrm{Pe}^{-1} \bar{\nabla}^2 \bar{T}. \qquad (4.39)$$

The two dimensionless numbers in this system are given by

$$\mathrm{Pe} = \frac{\varrho_0 c U L}{\kappa}, \quad \mathrm{Re} = \frac{U L}{\nu} \quad \left(\nu = \frac{\mu}{\varrho_0} \right).$$

The Peclet number is here defined as the ratio of the convection term for heat $\varrho_0 c U \Delta T / L$ and the heat conduction term $\kappa U / L^2$. The fraction $\kappa / (\varrho_0 c)$ is known as the thermal diffusivity, and if this quantity is given a symbol α, we realize the relation to the Peclet number defined in Section 3.4.

4.3.2 Free convection

Governing equations. The mathematical model for free thermal convection consists of the Navier-Stokes equations coupled to an energy equation governing the temperature:

$$\varrho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \varrho g \boldsymbol{k}, \tag{4.40}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}) = 0, \tag{4.41}$$

$$\varrho c \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) = \kappa \nabla^2 T + 2\mu \varepsilon_{ij} \varepsilon_{ij}, \tag{4.42}$$

where Einstein's summation convention is implied for the $\varepsilon_{ij}\varepsilon_{ij}$ term. The symbol T is the temperature, c is a heat capacity, κ is the heat conduction coefficient for the fluid. In free convection, the gravity term $-\varrho(T)g\boldsymbol{k}$ is essential since the flow is driven by temperature differences and the fact that hot fluid rises while cold fluid falls.

For slightly compressible gas flow a term $-p\nabla \cdot \boldsymbol{u}$ may be needed in (4.42).

Heating by viscous effects. We have also included heating of the fluid due to the work of viscous forces, represented by the term $2\mu\varepsilon_{ij}\varepsilon_{ij}$, where ε_{ij} is the strain-rate tensor in the flow, defined by

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T),$$

where u_i is the velocity in direction of x_i ($i = 1, 2, 3$ measures the space directions). The term $2\mu\varepsilon_{ij}\varepsilon_{ij}$ is actually much more relevant for forced convection, but was left out in Section 4.3.2 for mathematical simplicity. Heating by the work of viscous forces is often a very small effect and can be neglected, although it plays a major role in forging and extrusion of metals where the viscosity is very large (such processes require large external forces to drive the flow). The reason for including the work by viscous forces under the heading of free convection, is that we want to scale a more complete, general mathematical model for mixed force and free convection, and arrive at dimensionless numbers that can tell if this extra term is important or not.

Relation between density and temperature. Equations (4.40) has already been made dimensionless in the previous section. The major difference is now that ϱ is no longer a constant, but a function of T. The relationship between ϱ and T is often taken as linear,

$$\varrho = \varrho_0 - \varrho_0 \beta (T - T_0),$$

where

$$\beta = -\frac{1}{\varrho} \left(\frac{\partial \varrho}{\partial t} \right)_p,$$

is known as the thermal expansion coefficient of the fluid, and ϱ_0 is a reference density when the temperature is at T_0.

The Boussinesq approximation. A very common approximation, called the *Boussinesq approximation*, is to neglect the density variations in all terms except the gravity term. This is a good approximation unless the change in ϱ is large. With the linear $\varrho(T)$ formula and the Boussinesq approximation, (4.40)-(4.42) take the form

$$\varrho_0\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} - (\varrho_0 - \varrho_0\beta(T - T_0))g\boldsymbol{k}, \qquad (4.43)$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad (4.44)$$

$$\varrho_0 c\left(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T\right) = \kappa\nabla^2 T + 2\mu\varepsilon_{ij}\varepsilon_{ij}. \qquad (4.45)$$

A good justification of the Boussinesq approximation is provided by Tritton [9, Ch. 13].

Scaling. Dimensionless variables are introduced as

$$\bar{x} = \frac{x}{L}, \quad t_c = \frac{L}{U}, \quad \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \bar{p} = \frac{p}{\varrho U^2}, \quad \bar{T} = \frac{T - T_0}{\Delta T}.$$

The dimensionless y and z coordinates also make use of L as scale. As in forced convection, we assume the characteristic temperature level T_0 and the scale ΔT are given by thermal boundary and/or initial conditions. Contrary to Sections 4.2 and 4.3.2, U is now not given by the problem description, but implied by ΔT.

Replacing quantities with dimensions by their dimensionless counterparts results in

$$\varrho_0\frac{U^2}{L}\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + \varrho_0\frac{U^2}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\frac{p_c}{L}\bar{\nabla}\bar{p} + \frac{\mu U}{L^2}\bar{\nabla}^2\bar{\boldsymbol{u}} - \varrho_0 g\boldsymbol{k} + \varrho_0\beta T_c\bar{T}g\boldsymbol{k},$$

$$\frac{U}{L}\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0,$$

$$\varrho_0 c\left(\frac{T_c U}{L}\frac{\partial\bar{T}}{\partial\bar{t}} + \frac{UT_c}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T}\right) = \frac{\kappa T_c}{L^2}\bar{\nabla}^2\bar{T} + 2\frac{\mu U}{L}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}.$$

These equations reduce to

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\bar{\nabla}\bar{p} + \mathrm{Re}^{-1}\bar{\nabla}^2\bar{\boldsymbol{u}} - \mathrm{Fr}^{-2}\boldsymbol{k} + \gamma\bar{T}\boldsymbol{k}, \qquad (4.46)$$

$$\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \qquad (4.47)$$

$$\frac{\partial\bar{T}}{\partial\bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{T} + 2\delta\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}. \qquad (4.48)$$

The dimensionless numbers, in addition to Re and Fr, are

$$\gamma = \frac{g\beta L \Delta T}{U^2}, \quad \mathrm{Pe}^{-1} = \frac{\kappa}{\varrho_0 c U L}, \quad \delta = \frac{\mu U}{L \varrho_0 c \Delta T}.$$

The γ number measures the ratio of thermal buoyancy and the convection term:

$$\gamma = \frac{\varrho_0 g \beta \Delta T}{\varrho_0 U^2 / L} = \frac{g \beta L \Delta T}{U^2}.$$

The Pe parameter is the fraction of the convection term and the thermal diffusion term:

$$\frac{|\varrho_0 \boldsymbol{u} \cdot \nabla T|}{|\kappa \nabla^2 T|} \sim \frac{\varrho_0 c U \Delta T L^{-1}}{\kappa L^{-2} \Delta T} = \frac{\varrho c U L}{\kappa} = \mathrm{Pe}.$$

The δ parameter is the ratio of the viscous dissipation term and the convection term:

$$\frac{|\mu \nabla^2 \boldsymbol{u}|}{|\varrho_0 c \boldsymbol{u} \cdot \nabla T|} \sim \frac{\mu U^2 / L^2}{\varrho_0 c U \Delta T / L} = \frac{\mu U}{L \varrho_0 c \Delta T} = \delta.$$

4.3.3 The Grashof, Prandtl, and Eckert numbers

The problem with the above dimensionless numbers is that they involve U, but U is implied by ΔT. Assuming that the convection term is much bigger than the viscous diffusion term, the momentum equation features a balance between the buoyancy term and the convection term:

$$|\varrho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u}| \sim \varrho_0 g \beta \Delta T.$$

Translating this similarity to scales,

$$\varrho_0 U^2 / L \sim \varrho_0 g \beta \Delta T,$$

gives an U in terms of ΔT :

$$U = \sqrt{\beta L \Delta T}.$$

The Reynolds number with this U now becomes

$$\mathrm{Re}_T = \frac{UL}{\nu} = \frac{\sqrt{g \beta L^3 \Delta T}}{\nu^2} = \mathrm{Gr}^{1/2},$$

where Gr is the Grashof number in free thermal convection:

$$\mathrm{Gr} = \mathrm{Re}_T^2 = \frac{g \beta L^3 \Delta T}{\nu^2}.$$

The Grashof number replaces the Reynolds number in the scaled equations of free thermal convection. We shall soon look at its interpretations, which are not as straightforward as for the Reynolds and Peclet numbers.

The above choice of U in terms of ΔT results in γ equal to unity:

$$\gamma = \frac{g\beta L\Delta T}{U^2} = \frac{g\beta L\Delta T}{g\beta L\Delta T} = 1\,.$$

The Peclet number can also be rewritten as

$$\mathrm{Pe} = \frac{\varrho c U L}{\kappa} = \frac{\mu c}{\kappa}\frac{\varrho U L}{\mu} = \mathrm{Pr}\mathrm{Re}^{-1} = \mathrm{Pr}\mathrm{Re}_T^{-1},$$

where Pr is the Prandtl number, defined as

$$\mathrm{Pr} = \frac{\mu c}{\kappa}\,.$$

The Prandtl number is the ratio of the momentum diffusivity (kinematic viscosity) and the thermal diffusivity. Actually, more detailed analysis shows that Pr reflects the ratio of the thickness of the thermal and velocity boundary layers: when $\mathrm{Pr} = 1$, these layers coincide, while $\mathrm{Pr} \ll 1$ implies that the thermal layer is much thicker than the velocity boundary layer, and vice versa for $\mathrm{Pr} \gg 1$.

The δ parameter is in free convection replaced by a combination of the Eckert number (Ec) and the Reynolds number. We have that

$$\mathrm{Ec} = \frac{U^2}{c\Delta T} = \delta\mathrm{Re}_T,$$

and consequently

$$\delta = \mathrm{Ec}\mathrm{Re}_T^{-1} = \mathrm{Ec}\mathrm{Gr}^{-1/2}\,.$$

Writing

$$\mathrm{Ec} = \frac{\varrho_0 U^2}{\varrho_0 c\Delta T},$$

shows that the Eckert number can be interpreted as the ratio of the kinetic energy of the flow and the thermal energy.

We use Gr instead of Re_T in the momentum equations and also instead of Pe in the energy equation (recall that $\mathrm{Pe} = \mathrm{Pr}\mathrm{Re} = \mathrm{Pr}\mathrm{Re}_T = \mathrm{Pr}\mathrm{Gr}^{-1/2}$). The resulting scaled system becomes

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} = -\bar{\nabla} \bar{p} + \mathrm{Gr}^{-1/2} \bar{\nabla}^2 \bar{\boldsymbol{u}} - \mathrm{Fr}^{-2} \boldsymbol{k} + \bar{T} \boldsymbol{k}, \qquad (4.49)$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \qquad (4.50)$$

$$\mathrm{Gr}^{1/2} \left(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{T} \right) = \mathrm{Pr}^{-1} \bar{\nabla}^2 \bar{T} + 2 \mathrm{Ec} \mathrm{Gr}^{-1/2} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{ij}. \qquad (4.51)$$

The Grashof number plays the same role as the Reynolds number in the momentum equation in free convection. In particular, it turns out that Gr governs the transition between laminar and turbulent flow. For example, the transition to turbulence occurs in the range $10^8 < \mathrm{Gr} < 10^9$ for free convection from vertical flat plates. Gr is normally interpreted as a dimensionless number expressing the ratio of buoyancy forces and viscous forces.

Interpretations of the Grashof number. Recall that the scaling leading to the Grashof number is based on an estimate of U from a balance of the convective and the buoyancy terms. When the viscous term dominates over convection, we need a different estimate of U, since in this case, the viscous force balances the buoyancy force:

$$\mu \nabla^2 \boldsymbol{u} \sim \varrho_0 g \beta \Delta T \quad \Rightarrow \quad \mu U / L^2 \sim \varrho_0 g \beta \Delta T,$$

This similarity suggests the scale

$$U = \frac{g \beta L^2 \Delta T}{\nu}.$$

Now,

$$\frac{|\varrho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\mu \nabla^2 \boldsymbol{u}|} \sim \frac{UL}{\nu} = \frac{g \beta L^3 \Delta T}{\nu} = \mathrm{Gr}.$$

The result means that $\mathrm{Gr}^{1/2}$ measures the ratio of convection and viscous forces when convection dominates, whereas Gr measures this ratio when viscous forces dominate.

The product of Gr and Pr is the Rayleigh number,

$$\mathrm{Ra} = \frac{g \beta L^3 \Delta T \varrho_0 c}{\nu \kappa},$$

since

$$\mathrm{GrPr} = \mathrm{Re}_T^2 \mathrm{Pr} = \frac{g \beta L^3 \Delta T}{\nu^2} \frac{\mu c}{\kappa} = \frac{g \beta L^3 \Delta T \varrho_0 c}{\nu \kappa} = \mathrm{Ra}.$$

The Rayleigh number is the preferred dimensionless number when studying free convection in horizontal layers [2, 9]. Otherwise, Gr and Pr are used.

4.3.4 Heat transfer at boundaries and the Nusselt and Biot numbers

A common boundary condition, modeling heat transfer to/from the surroundings, is

$$-\kappa\frac{\partial T}{\partial n} = h(T - T_s),\tag{4.52}$$

where $\partial/\partial n$ means the derivative in the normal direction $(\boldsymbol{n}\cdot\nabla)$, h is an experimentally determined heat transfer coefficient, and T_s is the temperature of the surroundings. Scaling (4.52) leads to

$$-\frac{\kappa\Delta t}{L}\frac{\partial\bar{T}}{\partial\bar{n}} = h(\Delta T\bar{T} + T_0 - T_s),$$

and further to

$$\frac{\partial\bar{T}}{\partial\bar{n}} = \frac{hL}{\kappa}(\bar{T} + \frac{T_s - T_0}{\Delta T}) = \delta(\bar{T} - \bar{T}_s),$$

where the dimensionless number δ is defined by

$$\delta = \frac{hL}{\kappa},$$

and $\bar{T}_s$ is simply the dimensionless surrounding temperature,

$$\bar{T}_s = \frac{T_s - T_0}{\Delta T}.$$

When studying heat transfer in a fluid, with solid surroundings, δ is known as the Nusselt number[5] Nu. The left-hand side of (4.52) represents in this case heat conduction, while the right-hand side models convective heat transfer at a boundary. The Nusselt number can then be interpreted as the ratio of convective and conductive heat transfer at a solid boundary:

$$\frac{|h(T - T_s)|}{\kappa T/L} \sim \frac{h}{\kappa/L} = \mathrm{Nu}.$$

The case with heat transfer in a solid with a fluid as surroundings gives the same dimensionless δ, but the number is now known as the Biot number[6]. It describes the ratio of heat loss/gain with the surroundings at the solid body's boundary and conduction inside the body. A small Biot number indicates that conduction is a fast process and one can use Newton's law of cooling (see Section 2.1.7) instead of a detailed calculation of the spatio-temporal temperature variation in the body.

---

[5]`https://en.wikipedia.org/wiki/Nusselt_number`
[6]`https://en.wikipedia.org/wiki/Biot_number`

Heat transfer is a huge engineering field with lots of experimental investigations that are summarized by curves relating various dimensionless numbers such as Gr, Pr, Nu, and Bi.

4.4 Compressible gas dynamics

4.4.1 The Euler equations of gas dynamics

The fundamental equations for a compressible fluid are based on balance of mass, momentum, and energy. When molecular diffusion effects are negligible, the PDE system, known as the Euler equations of gas dynamics, can be written as

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}) = 0, \tag{4.53}$$

$$\frac{\partial (\varrho \boldsymbol{u})}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u} \boldsymbol{u}^T) = -\nabla p + \varrho \boldsymbol{f}, \tag{4.54}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{u}(E + p)) = 0, \tag{4.55}$$

with E being

$$E = \varrho e + \frac{1}{2} \varrho \boldsymbol{u} \cdot \boldsymbol{u}. \tag{4.56}$$

In these equations, $\boldsymbol{u}$ is the fluid velocity, ϱ is the density, p is the pressure, E is the total energy per unit volume, composed of the kinetic energy per unit volume, $\frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}$, and the internal energy per unit volume, ϱe.

Assuming the fluid to be an ideal gas implies the following additional relations:

$$e = c_v T, \tag{4.57}$$

$$p = \varrho R T = \frac{R}{c_v}\left(E - \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}\right), \tag{4.58}$$

where c_v is the specific heat capacity at constant volume (for dry air $c_v = 717.5\,\mathrm{J\,kg^{-1}K^{-1}}$), R is the specific ideal gas constant ($R = 287.14\mathrm{Jkg^{-1}K^{-1}}$), and T is the temperature.

The common way to solve these equations is to propagate ϱ, $\varrho \boldsymbol{u}$, and E by an explicit numerical method in time for (4.53)-(4.55), using (4.58) for p.

We introduce dimensionless independent variables,

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{t}{t_c},$$

and dimensionless dependent variables,

$$\bar{u} = \frac{u}{U}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{E} = \frac{E}{E_c}.$$

Inserting these expressions in the governing equations gives

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \frac{t_c U}{L} \bar{\nabla} \cdot (\bar{\varrho} \bar{u}) = 0,$$

$$\frac{\partial (\bar{\varrho} \bar{u})}{\partial \bar{t}} + \frac{t_c U}{L} \bar{\nabla} \cdot (\bar{\varrho} \bar{u} \bar{u}^T) = -\frac{t_c p_c}{U L \varrho_c} \nabla \bar{p} + \frac{t_c f_c}{U} \bar{\varrho} \bar{f},$$

$$\frac{\partial \bar{E}}{\partial \bar{t}} + \frac{t_c U}{L E_c} \bar{\nabla} \cdot (\bar{u}(E_c \bar{E} + p_c \bar{p})) = 0,$$

$$\bar{p} = \frac{R}{c_v p_c} (E_c \bar{E} - \frac{1}{2} \varrho_c u_c \bar{\varrho} \bar{u} \cdot \bar{u}).$$

A natural choice of time scale is $t_c = L/U$. A common choice of pressure scale is $p_c = \varrho_c U^2$. The energy equation simplifies if we choose $E_c = p_c = \varrho_c U^2$. With these scales we get

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\varrho} \bar{u}) = 0,$$

$$\frac{\partial (\bar{\varrho} \bar{u})}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\varrho} \bar{u} \bar{u}^T) = -\nabla \bar{p} + \alpha \bar{\varrho} \bar{f},$$

$$\frac{\partial \bar{E}}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{u}(\bar{E} + \bar{p})) = 0,$$

$$\bar{p} = \frac{R}{c_v} (\bar{E} - \frac{1}{2} \bar{\varrho} \bar{u} \cdot \bar{u}),$$

where α is a dimensionless number:

$$\alpha = \frac{L f_c}{U^2}.$$

We realize that the scaled Euler equations look like the ones with dimension, apart from the α coefficient.

4.4.2 General isentropic flow

Heat transfer can be neglected in isentropic flow[7], and there is hence an equation of state involving only ϱ and p:

$$p = F(\varrho).$$

The energy equation is now not needed and the Euler equations simplify to

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\boldsymbol{u}\varrho) = 0, \qquad (4.59)$$

$$\varrho \frac{\partial \boldsymbol{u}}{\partial t} + \varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0. \qquad (4.60)$$

Elimination of the pressure. A common equation of state is

$$F(\varrho) = p_0 \left(\frac{\varrho}{\varrho_0} \right)^{\gamma},$$

where $\gamma = 5/3$ for air. The first step is to eliminate p in favor of ϱ so we get a system for ϱ and $\boldsymbol{u}$. To this end, we must calculate ∇p:

$$\nabla p = F'(\varrho)\nabla\varrho, \quad F'(\varrho) = c_0^2 \left(\frac{\varrho}{\varrho_0} \right)^{\gamma-1},$$

where

$$c_0 = \sqrt{\frac{\gamma p_0}{\varrho_0}}$$

is the speed of sound within the fluid in the equilibrium state (see the subsequent section). Equation (4.60) with eliminated pressure p reads

$$\varrho \frac{\partial \boldsymbol{u}}{\partial t} + \varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + c_0^2 \left(\frac{\varrho}{\varrho_0} \right)^{\gamma-1} \nabla\varrho = 0. \qquad (4.61)$$

The governing equations are now (4.59) and (4.61). Space and time are scaled in the usual way as

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{t}{t_c}.$$

The scaled dependent variables are

$$\bar{\varrho} = \frac{\varrho}{\varrho_c}, \quad \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}.$$

Then $F'(\varrho) = c_0^2 \bar{\varrho}^{\gamma-1}$.

---

[7] https://en.wikipedia.org/wiki/Isentropic_process

Inserting the dimensionless variables in the two governing PDEs leads to

$$\frac{\varrho_c}{t_c}\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \frac{\varrho_c U}{L}\bar{\nabla}\cdot(\bar{\varrho}\bar{\boldsymbol{u}}) = 0,$$

$$\frac{\varrho_c U}{t_c}\bar{\varrho}\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \frac{\varrho_c U^2}{L}\bar{\varrho}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} + \frac{\varrho_c}{L}\left(\frac{\varrho_c}{\varrho_0}\right)^{\gamma-1}c_0^2\bar{\varrho}^{\gamma-1}\bar{\nabla}\bar{\varrho} = 0.$$

The characteristic flow velocity is U so a natural time scale is $t_c = L/U$. This choice leads to the scaled PDEs

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \bar{\nabla}\cdot(\bar{\varrho}\bar{\boldsymbol{u}}) = 0, \tag{4.62}$$

$$\bar{\varrho}\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\varrho}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} + \mathrm{M}^{-2}\left(\frac{\varrho_c}{\varrho_0}\right)^{\gamma-1}\bar{\varrho}^{\gamma-1}\bar{\nabla}\bar{\varrho} = 0, \tag{4.63}$$

where the dimensionless number

$$\mathrm{M} = \frac{U}{c_0},$$

is known as the *Mach number*. The boundary conditions specify the characteristic velocity U and thereby the Mach number. Observe that (4.63) simplifies if $\varrho_c = \varrho_0$ is an appropriate choice.

4.4.3 The acoustic approximation for sound waves

Wave nature of isentropic flow with small perturbations. A model for sound waves can be based on (4.59) and (4.61), but in this case there are small pressure, velocity, and density *perturbations* from a ground state at rest where $\boldsymbol{u} = 0$, $\varrho = \varrho_0$, and $p = p_0 = F(\varrho_0)$. Introducing the perturbations $\hat{\varrho} = \varrho - \varrho_0$ and $\hat{\boldsymbol{u}}$, (4.59) and (4.61) take the form

$$\frac{\partial \hat{\varrho}}{\partial t} + \nabla\cdot(\hat{\boldsymbol{u}}(\varrho_0 + \hat{\varrho})) = 0,$$

$$(\varrho_0 + \hat{\varrho})\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + (\varrho_0 + \hat{\varrho})\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}} + c_0^2\left(1 + \frac{\hat{\varrho}}{\varrho_0}\right)^{\gamma-1}\nabla\hat{\varrho} = 0.$$

For small perturbations we can linearize this PDE system by neglecting all products of $\hat{\varrho}$ and $\hat{\boldsymbol{u}}$. Also, $1 + \hat{\varrho}/\varrho_0 \approx 1$. This leaves us with the simplified system

$$\frac{\partial \hat{\varrho}}{\partial t} + \varrho_0 \nabla \cdot \hat{\boldsymbol{u}} = 0,$$

$$\varrho_0 \frac{\partial \hat{\boldsymbol{u}}}{\partial t} + c_0^2 \nabla \hat{\varrho} = 0.$$

Eliminating $\hat{\boldsymbol{u}}$ by differentiating the first PDE with respect to t and taking the divergence of the second PDE gives a standard wave equation for the density perturbations:

$$\frac{\partial^2 \hat{\varrho}}{\partial t^2} = c_0^2 \nabla^2 \hat{\varrho}.$$

Similarly, $\hat{\varrho}$ can be eliminated and one gets a wave equation for $\hat{\boldsymbol{u}}$, also with wave velocity c_0. This means that the sound perturbations travel with velocity c_0.

Basic scaling for small wave perturbations. Let ϱ_c and u_c be characteristic sizes of the perturbations in density and velocity. The density will then vary in $[\varrho_0 - \varrho_c, \varrho_0 + \varrho_c]$. An appropriate scaling is

$$\bar{\varrho} = \frac{\varrho - \varrho_0}{\varrho_c}$$

such that $\bar{\varrho} \in [-1, 1]$. Consequently,

$$\varrho = \varrho_0 + \varrho_c \bar{\varrho} = \varrho_0 (1 + \alpha \bar{\varrho}), \quad \alpha = \frac{\varrho_c}{\varrho_0}.$$

Note that the dimensionless α is expected to be a very small number since $\varrho_c \ll \varrho_0$. The velocity, space, and time are scaled as in the previous section. Also note that ϱ_0 and p_0 are known values, but the scales ϱ_c and U are not known. Usually these can be estimated from perturbations (i.e., sound generation) applied at the boundary.

Inserting the scaled variables in (4.59) and (4.61) results in

$$\alpha \frac{\varrho_0}{t_c} \frac{\partial \bar{\varrho}}{\partial \bar{t}} + \frac{\varrho_0 U}{L} \bar{\nabla} \cdot ((1 + \alpha \bar{\varrho}) \bar{\boldsymbol{u}}) = 0,$$

$$\frac{\varrho_0 U}{t_c} (1 + \alpha \bar{\varrho}) \frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \frac{\varrho_0 U^2}{L} (1 + \alpha \bar{\varrho}) \bar{u} \cdot \bar{\nabla} \bar{u} + \alpha \frac{\varrho_0}{L} c_0^2 (1 + \alpha \bar{\varrho})^{\gamma-1} \bar{\nabla} \bar{\varrho} = 0.$$

Since we now model sound waves, the relevant time scale is not L/U but the time it takes a wave to travel through the domain: $t_c = L/c_0$. This is a much smaller time scale than in the previous section because $c_0 \gg U$ (think of humans speaking: the sound travels very fast but one cannot feel the corresponding very small flow perturbation in the air!). Using $t_c = L/u_0$ we get

$$\alpha\frac{\partial\bar{\varrho}}{\partial\bar{t}} + M\bar{\nabla}\cdot((1+\alpha\bar{\varrho})\bar{\boldsymbol{u}}) = 0,$$

$$(1+\alpha\bar{\varrho})\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + M(1+\alpha\bar{\varrho})\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} + \alpha M^{-1}(1+\alpha\bar{\varrho})^{\gamma-1}\bar{\nabla}\bar{\varrho} = 0.$$

For small perturbations the linear terms in these equations must balance. This points to M and α being of the same order and we may choose $\alpha = M$ (implying $\varrho_c = \varrho_0 M$) to obtain

$$\frac{\partial\bar{\varrho}}{\partial\bar{t}} + \bar{\nabla}\cdot((1+M\bar{\varrho})\bar{\boldsymbol{u}}) = 0,$$

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + M\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} + (1+M\bar{\varrho})^{\gamma-2}\bar{\nabla}\bar{\varrho} = 0.$$

Now the Mach number, M, appears in the nonlinear terms only. Letting $M \to 0$ we arrive at the following linearized system of PDEs

$$\frac{\partial\bar{\varrho}}{\partial\bar{t}} + \bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{4.64}$$

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + \bar{\nabla}\bar{\varrho} = 0, \tag{4.65}$$

The velocity $\boldsymbol{u}$ can be eliminated by taking the time derivative of (4.64) and the divergence of (4.65):

$$\frac{\partial^2\bar{\varrho}}{\partial\bar{t}^2} = \bar{\nabla}^2\bar{\varrho}, \tag{4.66}$$

which is nothing but a standard dimensionless wave equation with unit wave velocity. Similarly, we can eliminate ϱ by taking the divergence of (4.64) and the time derivative of (4.65):

$$\frac{\partial^2\bar{\boldsymbol{u}}}{\partial\bar{t}^2} = \bar{\nabla}^2\bar{\boldsymbol{u}}. \tag{4.67}$$

We also observe that there are no physical parameters in the scaled wave equations.

4.5 Water surface waves driven by gravity

4.5.1 The mathematical model

Provided the Weber number (see section 4.2.7) is sufficiently small, capillary effects may be omitted and water surface waves are governed by gravity.

For large Reynolds numbers, viscous effects may also be ignored (except in boundary layers close to the bottom or the surface of the fluid). The flow of an incompressible homogeneous fluid under these assumptions is governed by the Euler equations of motion on the form

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.68}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\rho} \nabla p + g\mathbf{k} = 0. \tag{4.69}$$

When the free surface position is described as $z = \eta(x,y,t)$, with z as the vertical coordinate, the boundary conditions at the surface read

$$p = p_s, \tag{4.70}$$

$$\frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta = w, \tag{4.71}$$

where p_s is the external pressure applied to the surface. At the bottom, $z = -h(x,y)$, there is the no-flux condition

$$\frac{\partial h}{\partial x} u + \frac{\partial h}{\partial y} v = -w.$$

In addition to ρ and g we assume that a typical depth h_c, a typical wavelength λ_c, and a typical surface elevation A, which then by definition is a scale for η, are the given parameters. From these we must derive scales for the coordinates, the velocity components, and the pressure.

4.5.2 Scaling

First, it is instructive to define a typical wave celerity, c_c, which must be linked to the length and time scale according to $c_c = \lambda_c/t_c$. Since there is no other given parameter that matches the mass dimension of ρ, we express c_c in terms of A, λ_c, h_c, and g. Most of the work on waves in any discipline of physics is devoted to linear or weakly nonlinear waves, and the wave celerity must be presumed to remain finite as A goes to zero (see, for instance, Section 4.4.3). Hence, we may assume that c_c must depend on g and either h_c or λ_c. Next, the two horizontal directions are equivalent with regard to scaling, implying that we have a common velocity scale, U, for u and v, a common length scale L for x and y. The obvious choice for L is λ_c, while the "vertical quantities" w and z have scales W and Z, respectively, which may differ from the horizontal counterparts. However, we assume that also the length scale Z remains finite as $A \to 0$ and hence is independent of A. This is less obvious for Z than for c_c

and t_c, but may eventually be confirmed by the existence of linear solutions when solving the equation set. From the linear part of (4.71) and (4.68) we obtain two relations between velocity and coordinate scales by demanding the non-dimensionalized terms to be of order unity

$$\frac{A}{t_c} = W, \quad \frac{U}{L} = \frac{W}{Z}. \tag{4.72}$$

These relations are indeed useful, but they do not suffice to establish the scaling.

The pressure may be regarded as the sum of a large equilibrium part, balancing gravity, and a much smaller dynamic part associated with the presence of waves. To make the latter appear in the equations we define the dynamic pressure, p_d, according to

$$p = p_s - \rho g z + p_d,$$

and the pressure scale $p_c = \rho g A$ for p_d then follows directly from the surface condition (4.70).

The equation set will be scaled according to

$$\bar{t} = \frac{t}{t_c}, \ \bar{x} = \frac{x}{L}, \ \bar{y} = \frac{y}{L}, \ \bar{z} = \frac{z}{Z}, \ \bar{\eta} = \frac{\eta}{A}, \ \bar{u} = \frac{u}{U}, \ \bar{v} = \frac{v}{U}, \ \bar{w} = \frac{w}{W}, \ \bar{p}_d = \frac{p_d}{p_c}.$$

In the further development of the scaling we focus on two limiting cases, namely deep and shallow water.

4.5.3 Waves in deep water

Deep water means that $h_c \gg \lambda_c$. Presumably the waves will not feel the bottom, and h as well as h_c are removed from our equations. The bottom boundary condition is replaced by a requirement of vanishing velocity as $z \to -\infty$. Consequently, c_c must depend upon λ_c and g, leaving us with $c_c = \sqrt{g\lambda_c}$ and $Z = \lambda_c = L$ as the only options. Then, $t_c = \sqrt{\lambda_c/g}$ and (4.72) implies $U = W = c_0 \frac{A}{\lambda_c} = \epsilon c_0$, where we have introduced the non-dimensional number

$$\epsilon = \frac{A}{\lambda_c},$$

which is the wave steepness. The equality of the horizontal and the vertical scale is consistent with the common knowledge that the particle orbits in deep water surface waves are circular.

The scaled equations are now expressed with ϵ as sole dimensionless number

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{4.73}$$

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \epsilon \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} + \bar{\nabla} \bar{p}_d = 0. \tag{4.74}$$

The surface conditions, at $z = \epsilon \eta$, become

$$\bar{p}_d = \bar{\eta}, \tag{4.75}$$

$$\frac{\partial \bar{\eta}}{\partial \bar{t}} + \epsilon \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\eta} = \bar{w}, \tag{4.76}$$

while the bottom condition is replaced by

$$\bar{\boldsymbol{u}} \to 0, \tag{4.77}$$

as $\bar{z} \to -\infty$.

4.5.4 Long waves in shallow water

Long waves imply that the wavelength is large compared to the depth: $\lambda_c \gg h_c$. In analogy with the reasoning above, we presume that the speed of the waves remains finite as $\lambda_c \to \infty$. Then, c_c must be based on g and h_c, which leads to $c_c = \sqrt{gh_c}$ and $t_c = \lambda_c/\sqrt{gh_c}$. The natural choice for the vertical length scale is now the depth; $Z = h_c$. Application of (4.72) then leads to $W = c_c A/\lambda_c$ and $U = c_c A/h_c$.

Introducing the dimensionless numbers

$$\alpha = \frac{A}{h_c}, \quad \mu = \frac{h_c}{\lambda_c},$$

we rewrite the velocity scales as

$$W = \mu \alpha c_c, \quad U = \alpha c_c.$$

We observe that $W \ll U$ for shallow water and that particle orbits must be elongated in the horizontal direction.

The equation set is now most transparently written by introducing the horizontal velocity $\bar{\boldsymbol{u}}_h = \bar{u}\mathbf{i} + \bar{v}\mathbf{j}$ and the corresponding horizontal components of the gradient operator, $\bar{\nabla}_h$:

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}}_h + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{4.78}$$

$$\frac{\partial \bar{\boldsymbol{u}}_h}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}}_h \cdot \bar{\nabla}_h \bar{\boldsymbol{u}}_h + \alpha \bar{w} \frac{\partial \bar{\boldsymbol{u}}_h}{\partial \bar{z}} + \bar{\nabla}_h \bar{p}_d = 0, \tag{4.79}$$

$$\mu^2 \left(\frac{\partial \bar{w}}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}}_h \cdot \bar{\nabla}_h \bar{w} + \alpha \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) + \frac{\partial \bar{p}_d}{\partial \bar{z}} = 0.. \tag{4.80}$$

Surface conditions, at $z = \alpha \eta$, now become

$$\bar{p}_d = \bar{\eta}, \tag{4.81}$$

$$\frac{\partial \bar{\eta}}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}}_h \cdot \bar{\nabla}_h \bar{\eta} = \bar{w}, \tag{4.82}$$

while the bottom condition is invariant with respect to the present scaling

$$\bar{\nabla}_h \cdot \bar{\boldsymbol{u}}_h = -\bar{w}. \tag{4.83}$$

An immediate consequence is that $\bar{p}_d$ remains equal to $\bar{\eta}$ throughout the water column when $\mu^2 \to 0$, which implies that the pressure is hydrostatic. The above set of equations is a common starting point for perturbation expansions in ϵ and μ^2 that lead to shallow water, KdV, and Boussinesq type equations.

4.6 Two-phase porous media flow

We consider the flow of two incompressible, immiscible fluids in a porous medium with porosity $\phi(\boldsymbol{x})$. The two fluids are referred to as the wetting[8] and non-wetting fluid. In an oil-water mixture, water is usually the wetting fluid. The fraction of the pore volume occupied by the wetting fluid is denoted by $S(\boldsymbol{x}, t)$. The non-wetting fluid then occupies $1 - S$ of the pore volume (or $(1 - S)\phi$ of the total volume). The variable $P(\boldsymbol{x}, t)$ represents the pressure in the non-wetting fluid. It is related to the pressure P_n in the non-wetting fluid through the capillary pressure $p_c = P_n - P$, which is an empirically determined function of S.

From mass conservation of the two fluids and from Darcy's law for each fluid, one can derive the following system of PDEs and algebraic relations that govern the two primary unknowns S and P:

---

[8]https://en.wikipedia.org/wiki/Wetting

$$\nabla \cdot \boldsymbol{v} = -(Q_n + Q_w), \tag{4.84}$$

$$\boldsymbol{v} = -\lambda_t \nabla P + \lambda_w p_c'(S)\nabla S + (\lambda_w \varrho_w + \lambda_n \varrho_n)g\boldsymbol{k}, \tag{4.85}$$

$$\phi \frac{\partial S}{\partial t} + f_w'(S)\boldsymbol{v} \cdot \nabla S = \nabla \cdot (h_w(S)p_c'(S)\nabla S) +$$

$$g\frac{\partial G_w}{\partial z} + f_w(Q_n + Q_w) - Q_w, \tag{4.86}$$

$$Q_w = \frac{q_w}{\varrho_w}, \tag{4.87}$$

$$Q_n = \frac{q_n}{\varrho_n}, \tag{4.88}$$

$$\lambda_w(S) = \frac{K}{\mu_w}k_{rw}(S), \tag{4.89}$$

$$\lambda_n(S) = \frac{K}{\mu_n}k_{rn}(S), \tag{4.90}$$

$$\lambda_t(S) = \lambda_w(S) + \lambda_n(S), \tag{4.91}$$

$$k_{rw}(S) = K_{wm}\left[\frac{S - S_{wr}}{1 - S_{nr} - S_{wr}}\right]^a, \tag{4.92}$$

$$k_{rn}(S) = K_{nm}\left[\frac{1 - S - S_{nr}}{1 - S_{nr} - S_{wr}}\right]^b, \tag{4.93}$$

$$f_w(S) = \frac{\lambda_w}{\lambda_t}, \tag{4.94}$$

$$G_w(S) = h_w(S)(\varrho_n - \varrho_w), \tag{4.95}$$

$$h_w(S) = -\lambda_n(S)f_w(S). \tag{4.96}$$

The permeability of the porous medium is K (usually a tensor, but here taken as a scalar for simplicity); μ_w and μ_n are the dynamic viscosities of the wetting and non-wetting fluid, respectively; ϱ_w and ϱ_n are the densities of the wetting and non-wetting fluid, respectively; q_w and q_n are the injection rates of the wetting and non-wetting fluid through wells, respectively; S_{wr} is the irreducible saturation of the wetting fluid (i.e., $S \geq S_{wr}$); S_{nr} is the corresponding irreducible saturation of the non-wetting fluid (i.e., $(1-S) \geq S_{nr}$), K_{wn} and K_{nr} are the maximum values of the relative permeabilities k_{rw} and k_{rn}, respectively, and a and b are given (Corey) exponents in the expressions for the relative permeabilities.

The two PDEs are of elliptic and hyperbolic/parabolic nature: (4.84) is elliptic since it is the divergence of a vector field, while (4.86) is parabolic ($h_w \geq 0$ because $p_c'(S) \geq 0$ and λ_n as well as f_w are positive since $k_{rn} > 0$ and $k_{rw} > 0$). Very often, p_c' is small so (4.86) is of hyperbolic nature, and S features very steep gradients that become shocks in the limit $p_c' \to 0$ and (4.86) is purely hyperbolic. A popular solution technique is based on operator splitting at each time level in a numerical scheme: (4.84) is solved with respect to P, given S, and (4.86) is solved with respect to S, given P.

The saturation S is a non-dimensional quantity, and so are ϕ, k_{rw}, k_{rn}, K_{wm}, K_{nm}, f_w, and f'_w. The quantity $\boldsymbol{v}$ is the total filtration velocity, i.e., the sum of the velocities of the wetting and non-wetting fluid. An associated velocity scale v_c is convenient to define. It is also convenient to introduce dimensionless fractions of wetting and non-wetting fluid properties:

$$\varrho \equiv \varrho_w,$$
$$\varrho_n = \varrho\alpha, \quad \alpha = \frac{\varrho_n}{\varrho_w},$$
$$\mu \equiv \mu_w,$$
$$\mu_n = \mu\beta, \quad \beta = \frac{\mu_n}{\mu_w},$$
$$Q \equiv Q_w = \frac{q_w}{\varrho},$$
$$Q_n = Q\frac{\gamma}{\alpha}, \quad \gamma = \frac{q_n}{q_w}.$$

We will benefit from making λ_w, λ_n, and λ_t dimensionless:

$$\lambda_w(S) = \frac{K}{\mu}k_{rw}(S) = \lambda_c\bar{\lambda}_w, \quad \lambda_c = \frac{K}{\mu}, \quad \bar{\lambda}_w = k_{rw},$$
$$\lambda_n(S) = \frac{K}{\mu}\beta^{-1}k_{rn}(S) = \lambda_c\beta^{-1}\bar{\lambda}_n, \quad \bar{\lambda}_n = k_{rn},$$
$$\lambda_t(S) = \lambda_w(S) + \lambda_n(S) = \lambda_c\bar{\lambda}_t, \quad \bar{\lambda}_t = \bar{\lambda}_w + \beta^{-1}\bar{\lambda}_n.$$

As we see, λ_c is the characteristic size of any "lambda" quantity, and a bar indicates as always a dimensionless variable. The above formulas imply

$$h_w(S) = -\lambda_c\beta^{-1}\bar{\lambda}_n(S)f_w(S), \quad G_w(S) = h_w(S)\varrho(\alpha-1).$$

Furthermore, we introduce dimensionless quantities by

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{v_c}, \quad \bar{P} = \frac{P}{P_c}, \quad \bar{p}_c = \frac{p_c}{P_c}.$$

Inserting the above scaled quantities in the governing PDEs results in

$$\bar{\nabla} \cdot \bar{\boldsymbol{v}} = -\frac{LQ}{v_c}(1+\alpha^{-1}\gamma), \tag{4.97}$$

$$\bar{\boldsymbol{v}} = -\frac{P_c\lambda_c}{v_cL}\bar{\lambda}_t\bar{\nabla}\bar{P} + \frac{\lambda_cP_c}{v_cL}\bar{\lambda}_w\bar{p}'_c(S)\bar{\nabla}S +$$
$$\frac{g\lambda_c\varrho}{v_c}(\bar{\lambda}_w + \alpha\beta^{-1}\bar{\lambda}_n)\boldsymbol{k}, \tag{4.98}$$

$$\phi\frac{\partial S}{\partial \bar{t}} + \frac{t_cv_c}{L}f'_w(S)\bar{\boldsymbol{v}}\cdot\bar{\nabla}S = \frac{t_cP_c\lambda_c}{L^2}\bar{\nabla}\cdot(-\beta^{-1}\bar{\lambda}_n(S)f_w(S)\bar{p}'_c(S)\bar{\nabla}S) +$$
$$\frac{t_cg}{L}\frac{\partial G_w}{\partial \bar{z}} + t_cf_wQ(1+\alpha^{-1}\gamma) - t_cQ. \tag{4.99}$$

As usual, L is taken as the characteristic length of the spatial domain. Since v_c is a velocity scale, a natural time scale is the time it takes to transport a signal with velocity v_c through the domain: $t_c = L/v_c$. The diffusion term in the equation (4.102) then gets a dimensionless fraction

$$\frac{LP_c\lambda_c}{v_cL^2}.$$

Forcing this fraction to be unity gives

$$v_c = \lambda_c\frac{P_c}{L}.$$

We realize that this is indeed a natural velocity scale if the velocity is given by the pressure term in Darcy's law. This term is K/μ times the pressure gradient:

$$\frac{K}{\mu}|\nabla P| \sim \frac{K}{\mu}\frac{P_c}{L} = \lambda_c\frac{P_c}{L} = v_c.$$

We have here dropped the impact of the relative permeabilities $\bar{\lambda}_w$ or $\bar{\lambda}_n$ since these are quantities that are less than or equal to unity.

The other term in Darcy's law is the gravity term that goes like $\lambda_c\varrho g$ (again dropping relative permeabilities). The ratio of the gravity term and the pressure gradient term in Darcy's law is an interesting dimensionless number:

$$\delta = \frac{\lambda_c\varrho g}{\lambda_cP_c/L} = \frac{L\varrho g}{P_c}.$$

This number naturally arises when we discuss the term

$$\frac{t_cg}{L}\frac{\partial G_w}{\partial \bar{z}} = -(\alpha-1)\beta^{-1}\delta(\bar{\lambda}'_n(S)f_w(S) + \bar{\lambda}_n(S)f'_w(S))\frac{\partial S}{\partial \bar{z}}$$

Introducing another dimensionless variable,

$$\epsilon = t_cQ = \frac{L^2Q}{\lambda_cP_c},$$

we can write (4.97)-(4.99) in the final dimensionless form as

$$\bar{\nabla} \cdot \bar{\boldsymbol{v}} = -\epsilon(1+\alpha^{-1}\gamma), \tag{4.100}$$

$$\bar{\boldsymbol{v}} = -\bar{\lambda}_t \bar{\nabla}\bar{P} + \bar{\lambda}_w \bar{p}'_c(S)\bar{\nabla}S + \delta(\bar{\lambda}_w + \alpha\beta^{-1}\bar{\lambda}_n)\boldsymbol{k}, \tag{4.101}$$

$$\phi\frac{\partial S}{\partial \bar{t}} + f'_w(S)\bar{\boldsymbol{v}} \cdot \bar{\nabla}S = -\bar{\nabla} \cdot (-\beta^{-1}\bar{\lambda}_n(S)f_w(S)\bar{p}'_c(S)\bar{\nabla}S) -$$

$$(\alpha-1)\beta^{-1}\delta(\bar{\lambda}'_n(S)f_w(S) + \bar{\lambda}_n(S)f'_w(S))\frac{\partial S}{\partial \bar{z}} +$$

$$\epsilon f_w(1+\alpha^{-1}\gamma) - \epsilon. \tag{4.102}$$

The eight input parameters L, q_w, q_n, μ_w, μ_n, ϱ_w, ϱ_n, and K are reduced to five dimensionless parameters α, β, γ, δ, and ϵ. There are six remaining dimensionless numbers to be set: K_{wm}, K_{nm}, S_{wr}, S_{nr}, a, and b.