Chapter 2
Formulation of Some *NSF* Unsteady Initial-Boundary Value Problems

In the present chapter, using the results of Sects. 1.1–1.3 of Chap. 1, the reader can find *first* a mathematical formulation of a typical, initial-boundary value, unsteady NSF problem in the case of a *thermally* (calorically) *perfect gas*. Then, a more specific NSF problem of this kind is formulated for an *expansible/dilatable* liquid under the influence of temperature, with a specific equation of state; this is use to show the emergence of the working models for classical *Bénard thermal convection, heated from below*. The corresponding NSF problems are also formulated for *nonlinear acoustics* and *atmospheric motions*.

We also mention the case of so-called *Navier-Stokes* (NS) *isentropic*, *viscous*, compressible fluid flow, *a physically inconsistent case,* mainly considered by mathematicians, in their *pure rigorous investigations using abstract nonlinear functional analysis*!

Concerning *large horizontal scale atmospheric motions*, the *gravity term* means that one must consider the *influence of the Coriolis force* in the momentum equation for the velocity vector as observed in the *rotating earth frame*, and one must also employ *spherical coordinates.* In this case we are confronted with a *very stiff*, *complicated, starting system of dimensionless equations,* but this system allows one to use the RAM approach to derive various reduced working models.

It is important to note that *the equations are not sufficient for the applications of the RAM approach and deconstruction analysis* to the NSF system of equations. One must also formulate *physically convenient initial and boundary conditions*. Indeed, the equations given below govern the flow of a fluid—they are the same equations whether the flow is, for example, over an Airbus A380 (cruising speed 912 km/h) through a subsonic wind tunnel or past a windmill. The flow fields are quite different for these cases, even though the governing equations are the same.

Why? Where does the difference enter? The answer is through the boundary conditions, which are quite different for these two examples. The boundary conditions, and sometimes the initial conditions (with given data), dictate the particular solutions to be obtained from the governing equations (see Sect. 2.5). The above remark seems trivial, but unfortunately, in mathematically rigorous investigations,

R.Kh. Zeytounian, *Challenges in Fluid Dynamics*,
DOI 10.1007/978-3-319-31619-2_2

the role of the associated conditions is often not given serious consideration by mathematicians who have the strange opinion of a futility concerning these conditions! Some complementary remarks are made in Sect. 2.7.

Concerning the RAM approach, the reader can find the "*mathematics*" of this approach in Chap. 6 of our *companion 2012* book, but in Chap. 3 of the present book, I have decided, as explained in the preface, to provide some complementary enlightenment!

2.1 The Case of a Thermally Perfect Gas: Typical NSF Equations

For a *thermally perfect gas* (identified as *calorically perfect,* for simplicity), we first write the following equation of state for the specific internal energy e:

$$e = \mathrm{Cv}\,T, \tag{2.1}$$

where Cv is the specific heat for a specific volume. With the various results given in Chap. 1, we are now in a position to write the following full unsteady NSF equations for a thermally perfect gas, where we assume that the kinematic viscosity $\nu = \mu/\rho$ and also the coefficient of thermal conductivity k are both constant coefficients. Hence, we have the following NSF system of equations governing the fluid flow for the four unknown functions $\mathbf{u}$, ρ, p, and T:

$$\rho D\mathbf{u}/Dt = \rho \boldsymbol{f} - \nabla p + \left\{\nabla[\lambda(\nabla.\mathbf{u})] + \nabla.\left[2\mu\underline{\mathbf{D}}\right]\right\}, \tag{2.2a}$$

$$D\rho/Dt + \rho\,\mathrm{div}\mathbf{u} = 0, \tag{2.2b}$$

$$\rho \mathrm{Cv}\, DT/Dt + p\,\mathrm{div}\mathbf{u} = k\nabla^2 T + \Phi, \tag{2.2c}$$

$$p = R\rho T, \tag{2.2d}$$

where $R = \mathrm{Cp} - \mathrm{Cv}$ is the gas constant and Cp the specific heat for p constant and

$$\Phi = \lambda(\nabla.\mathbf{u})^2 + 2\mu\underline{\mathbf{D}} : \underline{\mathbf{D}}. \tag{2.2e}$$

The ratio $\gamma = \mathrm{Cp}/\mathrm{Cv}$ may be taken as unity for a pressure near normal values, and λ is the second viscosity coefficient (often assumed *constant*).

The above system of equations (2.2a–d), with (2.2e), is just the full unsteady/evolution (in time), typical NSF classical system of *aerodynamics* equations (often written with $\boldsymbol{f} = 0$).

For *atmospheric* motions, when dry atmospheric air is treated simply as a perfect gas, one must consider *gravity* $\mathbf{g} = -g\mathbf{k}$ as an external force in place of $\boldsymbol{f}$ in (2.2a), with the unit vector $\mathbf{k}$ in the vertical direction opposite to the gravity vector $\mathbf{g}$. For large horizontal scale atmospheric motions, when the NSF equations are written in a system of spherical coordinates (see Sect. 3.4), in a coordinate frame rotating with the earth, one must use an equation for the *relative velocity* taking into account the *Coriolis force,* in place of equation (2.2a). In a such a case, as done in Sect. 3.4, the RAM approach allows us to derive a multiplicity of working non-*ad hoc* models simulating various interesting and useful atmospheric motions. These are sketched on Fig. 6.3 of Sect. 6.3.

On the other hand, if we take into account (2.1), with (2.2e), and the above formulas for R and γ, then in place of equation (2.2d) for the pressure p, we can use the following equation of state:

$$p = \rho(\gamma - 1)e,\ \gamma = \frac{C_p}{C_v}. \tag{2.2f}$$

We also observe that a rather more complicated formulation of the NSF equations for a compressible, viscous, and heat conducting Newtonian fluid flow is possible when, instead of the equation of state, we introduce the *Helmholtz* or *Gibbs free energy* formulations in thermodynamics (mentioned in Sect. 1.4).

2.2 The Case of an Expansible Liquid

The equation of state for various liquids is usually taken to give the density ρ as a function of T and p:

$$\rho = \rho(T,\ p). \tag{2.3a}$$

In Rayleigh <1>, a pioneering paper entitled "*On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side* », instead of (2.3a), it is assumed that

$$\rho \approx \rho(T),\ \text{with} - (1/\rho)\ d\rho/dT = \alpha(T), \tag{2.3b}$$

where α (T) is the volume/thermal expansion coefficient, and measurements show rather *smaller values* for the above coefficient α(T) for liquids than the value 1/T appropriate to a thermally perfect gas. In reality, rather than the state equation (2.3a), it is usually assumed (see for instance [25], cited in "Introduction") that the expansible liquid can be described by the following *approximate* (*truncated*) law:

$$\rho = \rho_d[1 - a_d(T - T_d)\ +\ \chi_A(p - p_A)], \tag{2.4a}$$

where ρ_d, T_d, and p_A are some constant values for the density, temperature, and pressure, and χ_A , a_d are respectively the so-called *isothermal compressibility* coefficient:

$$\chi_A = (1/\rho_d)\,[\partial\rho/\partial p]_A, \tag{2.4b}$$

and the *coefficient of thermal expansion at* ρ_d and T_d:

$$\alpha_d = -(1/\rho_d)\,[\partial\rho/\partial T]_d. \tag{2.4c}$$

In his 1916 paper <1>, Rayleigh wrote: "*Bénard worked with a very thin layer, in his 1900/1901 experiments* [5], cited in "Introduction"." In the highly relevant monograph by *Chandrasekhar (1981),* the author explains that *a depth of only about 1 mm is considered on a level metallic plate; this layer was usually free, and being in contact with the air was at a lower temperature*—various liquids were employed, and the *layer rapidly resolved itself into a number of cells*.

More precisely, Bénard found that when the temperature of the lower surface was *gradually increased*, at a *certain instant* (*bifurcation*—*see* Chap. 10 *of* [14], cited in "Introduction"), the *layer became reticulated* and revealed its *dissection into cells*. He further noticed that there were motions *inside the cells*: of *ascension* at the centre, and of *descension* at the boundaries with the adjoining cells.

With the specific enthalpy h, the energy equation is

$$\rho Dh/Dt - Dp/Dt = k\nabla^2 T + \Phi. \tag{2.5}$$

In our 2009 monograph [25], cited in "Introduction", devoted to *convection in fluids*, the reader can find a *unified RAM approach* to the three main convection cases: shallow *thermal* (Rayleigh-Bénard), *deep thermal* (à la Zeytounian), and *thermocapillary* (Bénard-Marangoni) convections.

Concerning the equation for the (absolute) temperature T associated with the above full equation of state (2.3a) for the liquids, we have:

$$\big((Cp)/\gamma\big)\rho\, DT/Dt = (\alpha_d/\chi_A)T\,(\nabla.\mathbf{u}) = k\nabla^2 T + \Phi. \tag{2.6}$$

Hence, the starting NSF equations for an expansible liquid are the two Navier-Stokes dynamic equations (2.2a) and (2.2b), with the equation of state (2.3a) and equation (2.6) for the temperature, where Φ is given by (2.2e).

2.3 Navier-Stokes (NS) Barotropic Compressible Equations

In the case of *NS equations, only a reduced system of two equations is considered,* rather than the full NSF equations: the dynamic equation (2.2a) for $\mathbf{u}$, where λ and μ are both assumed constant and designated by λ_O and μ_O, with continuity equation (2.2b) for ρ, and also a simplified specifying equation between p and ρ, for a so-called elastic/barotropic compressible and viscous fluid. We thus have:

$$\rho \mathrm{D}\mathbf{u}/\mathrm{Dt} = \rho \mathbf{f} - \nabla p + \mu_O \nabla^2 \mathbf{u} + (\lambda_O + \mu_O)\nabla[\nabla.\mathbf{u}], \tag{2.7a}$$

$$\mathrm{D}\rho/\mathrm{Dt} + \rho \nabla.\mathbf{u} = 0, \tag{2.7b}$$

$$\mathrm{p} = \mathrm{P}(\rho), \tag{2.7c}$$

which forms a simplified closed system of three equations for **u**, ρ, and p. This is the so-called Navier-Stokes compressible, but isentropic, system of equations, a system often used by "pure mathematicians" in mathematical fluid dynamics, as in Lions (1998)!

The above NS system of three equations (2.7a)–(2.7c) is a *correction,* disregarding viscosity, of the Euler system (1.20b,c) given in Sect. 1.2, with the above specific equation (2.7c) in the place of the equation of state $p = \exp(S/Cv)\rho^{\gamma}$, and without the equation DS/Dt = 0 for the specific entropy. These compressible non-viscous equations for an *elastic fluid* in an adiabatic/barotropic flow, first considered by Euler in 1755, constitute, *in contrast, a very consistent non-viscous fluid flow model,* with many interesting and varied applications. Unfortunately, a barotropic and compressible fluid flow, when assumed viscous, as in the above NS system of equations (2.7a)–(2.7c), does not have, on the contrary, any physical reality, and obviously the various rigorous results obtained by mathematicians using these NS equations are of little interest!

Viscosity always generates entropy—baroclinity—the typical case being atmospheric motion for dry air, treated as a trivariate thermally perfect gas, as in Sects. 2.6 and 2.4. This was noticed by Leray (1994) in relation to rigorous mathematical abstract results derived for low Mach number asymptotics by P.L. Lions. But despite Leray's critical observation, 5 years later, in a survey (!) by Desjardins and Lin (1999), this isentropic viscous, physically inappropriate model was discussed once again (!), while the critical paper by *Leray* (1994) was curiously ignored.

2.4 The Case of Nonlinear Acoustics

As a *dominant acoustic* system of equations *à la NSF* for the velocity vector **u** and the thermodynamic perturbations, π, ω, and θ, such that

$$\mathrm{p} = 1 + \mathrm{M}\pi, \ \ \rho = 1 + \mathrm{M}\omega, \ \mathrm{T} = 1 + \mathrm{M}\theta, \tag{2.8}$$

and *neglecting* the terms proportional to $O(M^2)$—see, for instance, pp. 17–18 of my book 2006 [24], cited in "Introduction"—we obtain the following system of equations, assuming that the bulk viscosity is zero, i.e., $\lambda + (2/3)\mu = 0$ according to the Stokes relation:

$$\partial\omega/\partial t + \nabla.\mathbf{u} = -M[\nabla.(\omega\mathbf{u})], \tag{2.9a}$$

$$\partial\mathbf{u}/\partial t + (1/\gamma)\nabla\pi = (1/Re_{ac})\{\nabla^2\mathbf{u} + (1/3)\nabla(\nabla.\mathbf{u})\} - M[\omega\partial\mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u}], \tag{2.9b}$$

$$\partial\theta/\partial t + (\gamma-1)\nabla.\mathbf{u} = (\gamma/PrRe_{ac})\nabla^2\theta + \gamma(\gamma-1)(M/Re_{ac})\{(1/2)\underline{\mathbf{D}}:\underline{\mathbf{D}} + (1/3)(\nabla.\mathbf{u})^2\} - M[\omega\partial\theta/\partial t + (\mathbf{u}.\nabla)\theta + (\gamma-1)\pi\nabla.\mathbf{u}], \tag{2.9c}$$

$$\pi = -(\theta+\omega) + M\theta\omega. \tag{2.9d}$$

The above "dominant" equations (2.9a)–(2.9d) are the main starting point for the derivation of various model equations in *nonlinear acoustics*, considered as a branch of fluid dynamics by Crighton [6], cited in "Introduction", Lighthill, *1954*, and also Coulouvrat <2>.

For the frequencies and media commonly used in nonlinear acoustics, the acoustic Reynolds number Re_{ac} in the above equations (2.9b) and (2.9c) is always very large compared to unity. This means that the (continuous) medium is weakly dissipative at the frequency chosen in the most common experimental situations, and the above system of equations (2.9a)–(2.9d) is quite accurate enough. As a consequence, the above three unsteady equations turn out to be perturbation system of equations with two small parameters M and $1/Re_{ac}$. But neglecting the terms proportional to $O(M^2)$, the above system makes sense only if

$$M^2 << 1/Re_{ac}. \tag{2.10}$$

In Sect. 8.9, the reader can find a very short derivation of the famous KZK parabolic single equation model for low acoustic Mach number incorporating *nonlinearity*, *dissipation,* and *diffraction* effects. This KZK amplitude equation is in fact a *generalised form* of the well-known *one-dimensional Burgers equation,* derived in 1948, illustrating the *theory of turbulence,* when the parameters linked with the relative orders of magnitude of diffraction and nonlinearity in the KZK equation *are zero*.

More particularly, concerning the *asymptotic derivation of the KZK model* in Sect. 8.9, the main idea is basically that the *3D acoustic field is locally plane, so that the nonlinear wave propagates in the same way as a linear plane wave over a few wavelengths*. In this case, the *wave profile or amplitude is significantly altered* only at *large distances* from the source, i.e., *in the far field!*

2.5 Initial-Boundary Value Problem for the Typical NSF Equations

The boundary conditions, and sometimes initial conditions, dictate the particular solutions to be obtained from the governing NSF equations formulated above. When we consider a fluid flow governed by the typical NSF system of equations, e.g., by the three evolution equations (2.2a)–(2.2c), with the equation of state (2.2d) for a thermally perfect gas, it is first necessary to have a complete set of initial conditions, with *given data*, $\mathbf{u}^\circ$, ρ°, T°**,** for $\mathbf{u}$, ρ, T:

$$t = 0 : \mathbf{u} = \mathbf{u}^\circ(\mathbf{x}), \rho = \rho^\circ(\mathbf{x}) \text{and} \, T = T^\circ(\mathbf{x}). \tag{2.11a}$$

The typical case is obviously linked with weather forecasting and the crucial question: what will the weather be like tomorrow and for the next few days! But, these initial conditions (2.11a) are also necessary in various unsteady fluid dynamics problems—see, for instance, the problem considered in Sect. 8.8.

The case when *at time* $t = t_{ac}$ an *accident/disaster* takes place during the unsteady evolution of a fluid flow with time is also a case for which the RAM approach can be useful! A spectacular, but very dramatic example is the explosion of the space shuttle Challenger on its tenth launch, on January 28, 1986. This was indeed a frightful disaster. The main cause was the failure of the aft joint seal in the right solid rocket boosters due to the cold weather, leading to a combustion gas leak through the aft field joint of the right solid rocket motor, initiated at or shortly after ignition, which eventually weakened and/or penetrated the external tank, causing structural break-up and loss of the space shuttle during STS Mission 51-L.

When our RAM approach is applied with the above initial conditions, another intriguing aspect which appears as a consequence of the filtering of short compressible acoustic waves during the transition from compressible to incompressible fluid flow, characterize the singular nature of the unsteady low Mach number limiting process near the initial time. Section 3.4 gives some information concerning this filtering and its relation with matching, a basic concept of the RAM approach!

Concerning the boundary conditions, we need to distinguish the case of a fluid flow *extending to infinity (external flow)* in space-time, for which we set various conditions on $\mathbf{u}$ and T when $|\mathbf{x}| \to \infty$. In the case of a *solid body moving* with a given velocity $\mathbf{u}_{w,}$ in the general case of an *NSF problem*, we set the following no-slip condition (a viscous fluid *grips the wall* Γ):

$$\mathbf{u} = \mathbf{u}_w, \ \text{on}\,\Gamma, \ \text{when}\ (|\mathbf{x}|,\ t) \in \text{body}. \tag{2.11b}$$

On the other hand, in the conductive case, $k > 0$ in (2.2c), a boundary condition has to be imposed on the temperature T. In particular, we can require the following *thermal condition*:

$$\mathrm{T} = \mathrm{T_w} + \chi\,\Theta, \ \ \text{on}\,\Gamma, \tag{2.11c}$$

where the scalar $\chi > 0$ is a given constant—a measure—for the wall temperature field Θ (often *in a bounded region on the wall* Γ), and $\mathrm{T_w}$ is a given constant temperature.

The unsteady character of the *typical NSF initial-boundary value problem,* (2.2a)–(2.2d) *with* (2.11a)–(2.11c), as formulated above, is linked mainly with the change in the wall temperature field Θ with time t, for some evolution problem. In various applications of this problem, the role of the time t in the wall temperature field Θ must be made more precise in the above thermal condition (2.11c)! This is precisely the case in the above-mentioned problem linked with the crash of the space shuttle Challenger.

The well-posedness of the above typical initial-boundary value (*I-BV*) *NSF fluid flow problem* follows to some extent from properly formulated initial value conditions (2.11a) and boundary conditions (2.11b) and (2.11c). *Unfortunately*, *rigorous* and valuable mathematical results concerning the existence and uniqueness of this above *I-BV NSF fluid flow problem, i.e., ones that can actually be applied, seem to me outside the scope of "pure" mathematically rigorous investigations by mathematicians.*

Despite many and varied *rigorous papers* published recently by Elsevier B.V. (editors S. Friedlander and D. Serre) in "*Handbook of Mathematical Fluid Dynamics*" during the years 2002 to 2003, 2005 and 2007, it is *very difficult* to extract any helpful result from among them.

In the framework of the present book, the initial-boundary value NSF fluid flow problem (2.2a)–(2.2d) *with* (2.11a)–(2.11c) *formulated above is a typical working NSF problem and is designed for use with the RAM approach, when this is rewritten in dimensionless form, as is the case* Sect. 3.2*, and also in* Sect. 3.4 *for atmospheric motions*. Hence, our main objective is the "*deconstruction*" of the *I-BV NSF fluid flow problems formulated above,* first by *rewriting* in *dimensionless form* and then, thanks to the appearance of various *non-dimensional parameters* (numbers) in this *dimensionless (I-BV) NSF problem,* by *application of the RAM* approach.

In Chap. 6*,* such a deconstruction is achieved for various useful fluid flows, and in Figs. 6.1–6.4, the resulting family of working models are indicated for $Re \gg 1$, $M \ll 1$, *meteo-fluid dynamics*, and the *Bénard convection* problem, respectively. The RAM approach is also applied systematically to various fluid dynamics problems in Sects. 8.1–8.9.

2.6 The Rotating Earth and Its Atmosphere as a Continuum

This rather long section is motivated mainly by my long-standing interest in modelling atmospheric motions and the publication of two books: [22], cited in "Introduction" and <3>, a course <4>, and also various papers, e.g., [21], cited in "Introduction" <5>, <6>, and <7>.

The critical review by *Fred R. Payne* (from the Aerospace Engineering University of Texas) in Appl. Mech Rev. 46(42) B29, 1993, of my 1991 book <3>, published by Springer (Heidelberg) under the title "*Meteorological Fluid Dynamics*", describes our approach to atmospheric flows from the point of view of a fluid dynamicist:

> It is apparent that the author expended a considerable effort to make the material in this book accessible to non-specialists. The author has in mind a sequence of models leading to: "...a complete and consistent rational modeling of atmospheric phenomena...in the future." His goal of a: "return of meteorology to the family of fluid mechanics", is both admirable and essential, somewhat like Prandtl's, in 1904, enabling the reunion of (inviscid) aerodynamics and "practical fluids" (hydraulics) after a century of separation.
>
> Appendix 1 is quite a good survey of matched asymptotic expansions (MAE) for singular perturbation problems prevalent in boundary layers but a new prospective user will need to peruse some cited background books.
>
> I found Chaps. 7–10 of most interest. This book seems more suited to individual researchers seeking entry, or as a tutorial by a teacher with some PhD candidates in the area, rather than as a conventional text for a lecture course. Its many examples will be useful in a variety of settings; it is also suited to self-study by an advanced student.
>
> The teacher will need to provide some bridging material, both computational and physical insights, in the early chapters. Chapters 7–9 are written in a more expository style and are essentially self-contained; these chapters make a good text for a one-semester course and would require minimal amplification. As the book stands, it is a sketch for a first course and rather complete for a second.
>
> This is a stimulating book.
>
> Most first-time readers will likely make copious comments upon the margins. It is a member of a select subset, needful of study by specialists in fluid mechanics, turbulence, atmospheric dynamics and modeling, and finite-dimensional dynamic systems.

I feel that my "*Meteorological Fluid Dynamics*" is a good preparation for reading the more ambitious monograph "*Asymptotic Modeling of Atmospheric Flows*", despite the fact that it was published a little earlier in 1990 [22], cited in "Introduction". Concerning this book, *Huijun Yang* of the University of Chicago wrote the following in the book review of SIAM Rev (33, Dec. 1991, pp. 672–673):

> This Asymptotic Modeling of Atmospheric Flows, is rather a monograph in which the author has set forth what are, for the most part, his own results and this is particularly true of Chaps. 7–13. In the book, the author viewed meteorology as a fluid mechanics discipline.
>
> Therefore, he used singular perturbation methods as his main tools in the entirety of the book...The book consists of the author's more than 25 years work. In the thirty-two references of his own work, fewer than one third were published in English, with the rest in Russian or French.
>
> Throughout the book, the reader can strongly feel the influence of Soviet works on the author. However, the author does have his own boundary layer treatment, and well

posedness and ill posedness of the system are very important problems facing researchers today in atmospheric sciences and other related sciences. The reader will find some valuable information on these issues…

The mathematically consistent treatment of the subject does give this book a unique place on shelves of libraries and offices of researchers…

This book is very different from recent books on the market (i.e., Holton (1979), Gill (1982), Haltiner and Williams (1980), Pedlosky (1987), and Yang (1990)).

I recommend that researchers in atmospheric dynamics and numerical weather prediction read this book to have an alternative view of deriving atmospheric flow models. Researchers in theoretical fluid mechanics might also be interested to see how singular perturbation methods can be used in atmospheric sciences.

The book may be used as supplemental material for courses like numerical weather prediction or atmospheric dynamics.

However, I do not think it is a suitable textbook for a regular class: as the author said in his preface—I am well aware that this book is very personal, one might even say impassioned. Unfortunately, in France, and for instance by Roger Teman, the very no-adequate book of Pedlosky (1987) is systematically used in various papers in journal (Nonlinearity), with J.L. Lions and S.Wang (1991, 1992) devoted to atmospheric motions, which are not any practical interest for the modeling the atmospheric motions!

2.6.1 The Rotating Earth

Concerning *meteo fluid dynamics*, one must first observe that the earth revolves about its axis once every 23 H 56 min and 4 s, or a total of 86,164 s. The frequency of rotation or the angular velocity of the earth is

$$\Omega_O = 2\pi/86164 = 7.292 \times 10^{-5}\mathrm{rad/s}, \tag{2.12a}$$

and the radius of the earth at a latitude of $\varphi = 45°$ is $a_O = 6370.1$ km. The true gravitational acceleration owing to the pull of the earth, on the surface and at a geographic latitude of $\phi = 45°$, is

$$|\mathbf{f}| = 9.\ 82357\ \mathrm{m}/s^2, \tag{2.12b}$$

and therefore, it will be assumed here that the body force, $\rho \boldsymbol{f}$, in the momentum equation is the true gravitational force, where ρ is the atmospheric density.

To distinguish the experiences of a fixed and a rotating observer, let a subscript a denote quantities referred to an absolute, inertial frame of reference, and a subscript r, quantities referred to a frame rotating with the angular velocity of the earth, $\boldsymbol{\Omega} = \Omega_O\,\mathbf{e}$ relatively to the absolute frame. Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denote respectively the unit vectors pointing east, north, and vertically upward. Then,

$$\mathbf{e} = \mathbf{k}\sin\varphi + \mathbf{j}\cos\varphi. \tag{2.12c}$$

If $\boldsymbol{\gamma}_r$ is the relative acceleration and $\boldsymbol{\gamma}_a$ the absolute acceleration, we can write the following formula:

$$\boldsymbol{\gamma}_{\mathrm{a}} = \boldsymbol{\gamma}_{\mathrm{r}} + 2\Omega \times \mathbf{v}_{\mathrm{r}} - (\Omega_{\mathrm{O}})^2 \mathbf{x}_{\perp}, \tag{2.12d}$$

where the subscript $\perp$ denotes the equatorial component of $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \vec{\boldsymbol{\Omega}} \times (\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{x}})$. For the gravitational acceleration, we write

$$\mathbf{g} = \mathbf{f} + (\Omega_{\mathrm{O}})^2 \mathbf{x}_{\perp}, \text{such that} |\mathbf{g}| = g = |\mathbf{f}| - (\Omega_{\mathrm{O}})^2 |\mathbf{x}_{\perp}|, \tag{2.12e}$$

and $g = 9.8066$ m/s^2 at $\varphi = 45°$. The Froude number Fr—a measure of the significance of the gravitational acceleration (the force of gravity)—is defined by

$$\mathrm{Fr}_{\mathrm{L0}} = \mathrm{U}_0/(g\mathrm{L}_0), \tag{2.12f}$$

where $|\mathbf{U}| = \mathrm{U}_0$, $\mathbf{U}$ is the characteristic constant reference velocity, and L_0 is a characteristic length.

If $\mathbf{u}$ denotes the relative velocity $\mathbf{v}_{\mathrm{r}}$, the terme $\boldsymbol{\Omega} \times \mathbf{u}$ is an apparent acceleration known as the Coriolis acceleration, which exists only if there is motion relative to a moving frame such as the earth. For this Coriolis acceleration we have

$$2\Omega \times \mathbf{u} = (2\Omega_{\mathrm{O}}\cos\varphi\, \mathrm{w} - 2\Omega_{\mathrm{O}}\sin\varphi\, \mathrm{v})\mathbf{i} + 2\Omega_{O}\sin\varphi\, \mathrm{u}\mathbf{j} - 2\Omega_{\mathrm{O}}\cos\varphi\, \mathrm{u}\mathbf{k}, \tag{2.13a}$$

when u, v, and w are the components of $\mathbf{u}$. The symbol $\mathrm{f} = 2\Omega_{\mathrm{O}} \sin\varphi$ is the local Coriolis parameter, and we can write, in place of (2.13a), the following relation:

$$2\Omega \times \mathbf{u} = \mathrm{f}(\mathrm{u}\mathbf{j} - \mathrm{v}\mathbf{i}) + \mathrm{d}\mathrm{f}/\mathrm{d}\varphi(\mathrm{w}\mathbf{i} - \mathrm{u}\mathbf{k}). \tag{2.13b}$$

The importance of the Coriolis acceleration in relation to the inertial forces is given by the Rossby number Ro, which is defined as

$$\mathrm{Ro} = \mathrm{U}_0/\mathrm{L}_0\mathrm{f}_0, \text{with } \mathrm{f}_0 = 2\Omega_{\mathrm{O}} \sin\varphi_{\mathrm{O}}, \text{ where } \varphi_{\mathrm{O}} = \text{Constant}. \tag{2.13c}$$

In fact, $\mathrm{f}_0 \approx 10^{-4}$ in mid-latitudes, and for $\mathrm{Ro} > > 1$, the Coriolis force is likely to cause only a slight modification of the pattern of atmospheric motion, but when $\mathrm{Ro} \ll 1$, the effects of the Coriolis force are likely to be dominant! In fact, for synoptic-scale atmospheric motions, we have: $\mathrm{L}_0 \cong 10^6$ m and $\mathrm{Ro} \cong 10^{-1}$, but for the meso- (or regional) scale atmospheric flows, we have $\mathrm{L}_0 \cong 10^5$ m and $\mathrm{Ro} \cong 1$. Finally, for the case of local-scale atmospheric process, we have rather $\mathrm{L}_0 \cong 10^4$ m and $\mathrm{Ro} \cong 10$!

In the present book, we prefer to use, instead of the Rossby number Ro, the *parameter* (*taking into account my Moscovite (1957–1966) scientific meteo adventure with Il'ya Afanas'evich Kibel*) known as the *Kibel number*:

$$\mathrm{Ki} = (1/\mathrm{f}_0)/\mathrm{t}_0, \tag{2.14}$$

where t_0 is a characteristic reference time. If t_0 is the advective time scale $\mathrm{L}_0/\mathrm{U}_0$, this Kibel number is identical to the Rossby number Ro.

2.6.2 The Atmosphere as a Continuum

From a mathematical point of view of a continuum, we postulate that properties at any one point can be expressed in terms of properties at a neighboring point—this is because these properties and their derivatives are assumed continuous in their spatial variations. *From the continuum mechanics point of view, the atmosphere is a thin layer of air—a gaseous mixture—surrounding the surface of the earth, which remains attached to the earth by the pull of gravity and extends about seven miles upwards from the earth's surface.*

The layer nearest to the earth's surface, characterized by a linear temperature decrease with increasing altitude, is called the troposphere, and it is most influenced by energy transfer through radiation, evaporation, condensation, and convection. However, it is no easy matter to account for all these four effects. The troposphere also represents the limit within which conventional air flight takes place, and also within which man-made pollution from industrial activities is principally confined, and it is where most cloud formation occurs. Dynamically speaking, the troposphere is stable, but those portions of the layer nearest the surface of the earth are often unstable.

The air in the atmosphere is a Newtonian fluid, and dry air is governed by the law of perfect gases with (2.1) and (2.2d). If the relative velocities *are small*, the *pressure* will be only *slightly disturbed* from the value it would have in the absence of motion, $p_{st}(z_{st})$, defined by the relations

$$\mathrm{d}p_{st}(z_{st})/\mathrm{d}z_{st} + g\rho_{st} = 0, \text{with } \rho_{st}(z_{st}) = p_{st}(z_{st})/RT_{st}(z_{st}), \tag{2.15a}$$

in a basic, so-called *standard atmosphere* in which fluctuations due to motion occur. This basic standard atmospheric state is assumed known, although in fact its determination from first principles requires at least the consideration of mechanisms such as radiative transfer in the atmosphere. But in rather simple cases we have, from the first law of thermodynamics, for the standard temperature $T_{st}(z_{st})$, a *function only of the standard altitude* z_{st}, the following equation:

$$k(T_{st})\, \mathrm{d}T_{st}(z_{st})/\mathrm{d}z_{st} + R_{st}(T_{st}) = 0, \tag{2.15b}$$

with

$$\mathrm{d}R_{st}(z_{st})/\mathrm{d}z_{st} = \rho_{st} Q_{st}(T_{st}). \tag{2.15c}$$

Equation (2.15c) gives $R_{st}(T_{st})$, where $Q_{st}(T_{st})$ is the rate of heat supply per unit mass by radiative heat transfer and the scalar $k_{st}(T_{st})$ is the standard coefficient of *thermal conductivity*.

For our purposes here, it is sufficient to assume that $Q_{st}(T_{st})$ is a known function of T_{st} and also that the influence of the rate of heat by radiative heat transfer on the atmospheric motions is the main factor in the determination of T_{st}. By doing this, we consider only a mean standard heat source and ignore variations thereof! As a consequence, the *reference quantities* are the values for the standard state at ground level: $p_{st}(0)$, $\rho_{st}(0)$, and $T_{st}(0)$. In this case, from (2.15a), the following non-dimensional parameter—the *Boussinesq* number—appears:

$$\mathrm{Bo} = gH_0/(p_{st}(0)/\rho_{st}(0)) = H_0/(RT_{st}(0)/g), \tag{2.16a}$$

where H_0 is a characteristic length scale for vertical atmospheric motions, while the *characteristic altitude scale for the standard altitude* z_{st} is

$$H_{st} = RT_{st}(0)/g \tag{2.16b}$$

In fact, *Bo is the ratio of two vertical length scales*. It follows from (2.16a) that, between the parameters Froude (Fr), Mach (M), Boussinesq (Bo), and the ratio ε—the so-called *hydrostatic parameter*—the following relation holds:

$$(\mathrm{Fr}_{L0})^2 = \gamma\varepsilon\, M^2/\mathrm{Bo} \;\text{ or }\gamma\,[M/\mathrm{Fr}_{H0}]^2 = \mathrm{Bo}, \tag{2.16c}$$

where

$$\varepsilon = H_0/L_0,\ \text{and}\ M = U_0/\sqrt{}C_0,\ \ \text{with}\, C_0 = \gamma R T_{st}(0). \tag{2.16d − f}$$

On the other hand, $N_{st}(z_{st})$ defined by

$$N_{st}{}^2(z_{st}) = [g/(T_{st})]\{[(\gamma-1)g/\gamma R] + dT_{st}/dz_{st}\}, \tag{2.16g}$$

is called the *Brunt-Väisälä frequency* or the *natural frequency of oscillations of a vertical column of "standard" atmospheric mass during a small displacement from its equilibrium position—the standard atmosphere being statically stable when* N_{st} is *real*! Below, the existence of characteristic scales is exploited by the introduction of non-dimensional quantities denoted by primes, and we have

$$z = H_0 z',\ z_{st} = H_{st} z'_{st},\ \text{ and }\ T_{st} = T_{st}(0)T'_{st}. \tag{2.16h}$$

We can then write the following dimensionless equation in place of (2.16g):

$$a_{st}N'_{st}{}^2\left(z'_{st}\right) = [\mathrm{Bo}/T'_{st}]\left\{[(\gamma-1)/\gamma] + dT'_{st}/dz'_{st}\right\}, \tag{2.17a}$$

where a_{st} is a *dimensionless measure of the standard stability*. We can also derive an *interesting relation between* z_{st} *and* z', namely:

$$z_{st} = \mathrm{Bo}\, z', \tag{2.17b}$$

which plays a decisive role in the justification (*à la Zeytounian (1974)*) of the famous *Boussinesq equations*. Concerning these *Boussinesq (shallow) equations*, and also *Zeytounian (deep) equations,* see Sects. 2.6.3 and 2.6.4 for their *asymptotic derivation from the Euler equations*, via the *RAM approach*.

Finally, in the asymptotics of atmospheric motions, two parameters also play an important role: the *Reynolds number* $\mathrm{Re} = \mathrm{U_0 L_0}/\nu_0$, which shows the *importance* of the *inertia relative* to the *viscosity*, and the *Ekman numbe*r:

$$\mathrm{Ek} = \mathrm{Ro}/\mathrm{Re} = \left(1/\mathrm{L_0}^2\right)[\nu_0/f_0], \tag{2.17c}$$

which is a measure of the *ratio* of the *frictional* and *Coriolis* forces.

We observe that an important feature of *large synoptic-scale atmospheric motions* is that both the *Kibel and Ekman numbers are small*. A typical value of Ek in the earth's troposphere is 10^{-3}, when for the eddy viscosity we choose $\nu_0 = 5\ \mathrm{m^2/s}$.

Except in the immediate vicinity of the equator, Ki is usually a small parameter ($\ll 1$), if the characteristic time scale $\mathrm{t_0} \gg 10^4$ s, which is the case for synoptic-scale motions.

In any realistic atmospheric situation, M $\ll 1$*, and the synoptic meteorological situation corresponds to*

$$\varepsilon \sim 10^{-2} \text{and Bo} << 1. \tag{2.17d}$$

But, frequently, for the *prediction of atmospheric phenomena at regional (meso) and local scales*, one may assume that

$$\mathrm{Bo} << 1, \text{ but } \varepsilon \sim 1, \tag{2.17e}$$

as is typically the case for *lee waves*, arising *downstream of a mountain*.

For further details concerning the physical nature of the atmosphere, the reader is referred to books *Houghton (1977*) and *Scorer (1978). It must be kept in mind that modelling, i.e., the translation of a complex physical situation into correctly expressed mathematical terms, has at the present time become very important in many fields in the realm of scientific research.*

2.6.3 Shallow Boussinesq Equations

To derive the *shallow Boussinesq equations from the Euler equations*, these Euler equations must be written in dimensionless form. First we take into account (2.15a) for the *standard atmosphere*, and with $p_{\mathrm{st}}(0)$, $\rho_{\mathrm{st}}(0)$, and $T_{\mathrm{st}}(0)$ we introduce non-dimensional quantities (dropping the primes) such that

$$p = p_{st}(z_{st})\ [1\ +\pi],\quad \rho = \rho_{st}(z_{st})\ [1\ +\omega],\quad T\ = T_{st}(z_{st})\ [1 + \theta]. \tag{2.18a}$$

With (2.18a), for dimensionless velocity components ($\mathbf{v} = (u, v), w$), and thermodynamic perturbations, π, ρ, and θ, we derive the following *dimensionless, emerging equations*:

$$\begin{aligned}
&[1 + \omega]\ \mathrm{St}\ D\mathbf{v}/Dt\ + T_{st}(z_{st})\left[1/\gamma M^2\right]\ \nabla\pi = 0,\\
&[1 + \omega]\ \mathrm{St}\ Dw/Dt\ +\ \left[1/\gamma M^2\right]\{T_{st}(z_{st})\partial\pi/\partial z\\
&\qquad\qquad - [1 + \omega]\ \mathrm{Bo}\ \ \theta\} = 0,\\
&\pi =\ \omega + [1 + \omega]\ \ \theta,\\
&\mathrm{St}\ D\omega/Dt\ +\ [1\ +\omega]\ (\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)\\
&\qquad = \ [1 + \omega]\ \mathrm{Bo}\ [1/T_{st}(z_{st})]\{1 - \Gamma_{st}(z_{st})\}w,\\
&\mathrm{St}\ D\theta/Dt\ -\ [(\gamma -\ 1)/\gamma]\ \mathrm{St}\ D\pi/Dt\\
&\qquad +\ [1\ +\pi]\ \mathrm{Bo}\ [1/T_{st}(z_{st})]\{[(\gamma -\ 1)/\gamma] - \Gamma_{st}(z_{st})\}w = 0.
\end{aligned} \tag{2.18b}$$

Again we stress that the above dimensionless Euler meteo equations (2.18b) for $\mathbf{v} = (u, v)$, w, and (π, ω, θ), are a set of *exact consequences of the traditional Euler equations*, and this remark is very important for the consistent derivation of the Boussinesq model equations below!

Our approach via the RAM approach goes from the full exact equations to simplified/reduced model equations—and this is the only way to derive relevant and worthwhile reduced/simplified, non-ad hoc working model equations! As *leading-order* equations, these Boussinesq equations are derived via the RAM approach from the above Euler system of equations (2.18b), if we assume that

$$M << 1 \text{ and } \mathrm{Bo} << 1,\ \text{such that } \mathrm{Bo}/M = B^* = O(1), \tag{2.18c}$$

and use the associated asymptotic expansions

$$\begin{aligned}
(u, v, w) &= (u_{0B}, v_{0B}, w_{0B}) + \ldots,\\
(\omega, q) &= M\ (\omega_{1B}, \theta_{1B}) + \ldots,\\
\pi &= M^2\ \ \pi_{2B} + ,
\end{aligned} \tag{2.18d}$$

and also the limiting process:

$$\lim_{M\to o}[\text{with } t,\ x,\ \ y,\ z,\ \text{and}\ \ \mathrm{St},\ \ \gamma,\ \ B^* \text{ fixed}]. \tag{2.18e}$$

The above conditions (2.18c)–(2.18e) are obtained after a brief investigation of the various degeneracies of the Euler equations (2.18b) when $M \to 0$ with t, x, y, z, and St, γ, and B^* fixed! In this case, it appears that one must assume that the *Boussinesq number Bo is also a small parameter* such that the *similarity relation with* $B^* = O(1)$, in (2.18c), is satisfied!

From the above conditions (2.18c)–(2.18e), we derive the following Boussinesq model equations for the functions: u_{oB}, v_{oB}, w_{oB}, ω_{1B}, θ_{1B}, and π_{2B} in place of the Euler equations (2.18b) (written without primes):

$$\begin{aligned}
&\mathrm{St}\,\mathrm{D}\mathbf{v}_{0B}/\mathrm{Dt} + [1/\gamma]\ \nabla\pi_{2B} = 0,\\
&\mathrm{St}\,\mathrm{Dw}_{0B}/\mathrm{Dt} + [1/\gamma]\ \partial\pi_{2B}/\partial z - B^*\theta_{1B} = 0,\\
&\omega_{1B} = -\theta_{1B},\\
&\partial u_{0B}/\partial x + \partial v_{0B}/\partial y + \partial w_{0B}/\partial z = 0,\\
&\mathrm{St}\,\mathrm{D}\theta_{1B}/\mathrm{Dt} + B^*\{[(\gamma - 1)/\gamma] - \Gamma_{st}(0)\}w_{0B} = 0,
\end{aligned} \tag{2.18f}$$

since in dimensionless form, we obviously have $T_{st}(o) \equiv 1$, but in a *general case* $\Gamma_{st}(0) = \mathrm{d}T_{st}(z_{st})/\mathrm{d}z_{st|z_{st}} = 0$ is obviously *different from zero!*

We note also that the above relation (2.17b) was taken into account. Concerning the validity of the above system of *Boussinesq equations,* from the similarity relation (2.18c), where $B^* = O(1)$, we have

$$B^* = \mathrm{Bo}/\mathrm{M} = O(1) \Rightarrow \mathrm{H_c} \approx [\mathrm{U_c}/g](\mathrm{RT_{st}}(0)/\gamma)^{1/2} \equiv H_B \approx\ 10^3 m, \tag{2.18g}$$

and this leads to a strong restriction on the application of the Boussinesq equations (2.18f) in the whole troposphere ($\mathrm{H_c} = \mathrm{H_{st}} = \mathrm{RT_{st}}(0)/g \approx 10^4$ m)!

The equations (2.18f) derived above are indeed only *shallow Boussinesq equations*. Concerning more particularly the *lee waves downstream of a mountain in the whole troposphere*, one must assume that $\mathrm{Bo} \equiv 1$, $z_{st} \equiv z$. In the 2003 paper <8>, the reader can find a more general derivation of the Boussinesq equations applicable to atmospheric motions

2.6.4 Deep Equations "à la Zeytounian"

If we take into account that, *in the troposphere, the air temperature on average decreases with height at an overall positive rate (of about 6.5 °C/km)*, then it seems to me not too bad a hypothesis to assume that $-\Gamma_{st}(z) = -\mathrm{d}T_{st}/\mathrm{d}z$ *is very close to* $[(\gamma - 1)/\gamma]$ *in the troposphere,* and write

$$\Gamma_{st}(z) = -\,\mathrm{d}T_{st}/\mathrm{d}z = [(\gamma - 1)/\gamma] + \mathrm{M}^2\chi_{st}(z), \tag{2.19a}$$

the function χ_{st} (z) being a known function of z which takes into account a *weak stratification with altitude z* in a *standard troposphere,* and $|\chi_{st}(z)| = O(1)$.

In such a case, with

$$\lim_{\mathrm{M}\to 0}[\text{with t, x, y, z, and St, } \gamma, \text{ Bo, } \chi_{st}(z) \text{ fixed}], \tag{2.19b}$$

instead of the above system (2.18f) *of shallow Boussinesq equations, we derive* the following so-called *deep equation* (*à la Zeytounian*) (see Sect.10.2.2 in [22], cited in "Introduction"):

$$
\begin{aligned}
&\gamma\ T_d(z)\,[\partial u_{od}/\partial x + \partial v_{od}/\partial y + \partial w_{od}] = w_{od},\\
&St\, Dv_{od}/Dt + (1/\gamma)\,T_d(z)\ \nabla\pi_{2d} = 0,\\
&St\, Dw_{od}/Dt + (1/\gamma)[T_d(z)\partial\pi_{2d}/\partial z - \theta_{2d}] = 0,\\
&St\, D\theta_{2d}/Dt - [(\gamma-1)/\gamma]\ St\, D\pi_{2d}/Dt + [1/T_d(z)]\chi_{st}(z)w_{od} = 0,\\
&\pi_{2d} = \theta_{2d} + \omega_{2d},\ T_d(z) = 1 - [(\gamma-1)/\gamma]\,z,
\end{aligned}
\tag{2.19c}
$$

with $D/Dt = \partial/\partial t + u_{od}\partial/\partial x + v_{od}\partial/\partial y + w_{od}\partial/\partial z$, when $M \to 0$ with t, x, y, z, and St, γ, $Bo \equiv 1$, fixed, with:

$$
\begin{aligned}
&(u,\ v,\ w) = (u_{0d},\ v_{0d},\ w_{0d}) + \ldots,\\
&(\omega,\ \theta,\ \pi) = M^2(\omega_{2d},\ \theta_{2d},\ \pi_{2d}) + \ldots .
\end{aligned}
\tag{2.19d}
$$

The above model equations (2.19c) are the true "*anelastic*" non-dissipative equations, the equations of *Ogura* and *Phililips (1962), being rather an ad hoc system of equations!*

These *deep convection equations* (2.19c) *are valid in the whole troposphere, while the shallow Boussinesq equations* (2.18f) *can be considered as a reduced form of these deep convection equations* (2.19c)*, only valid in the vicinity of the ground, in a layer close to a flat ground surface with a thickness of only about* 10^3 m.

In the *2D steady case*, from *each of the above systems of equations*—(2.18f) and (2.19c)—one can derive a *single equation for a stream function*—a very pleasing derivation for a curious and motivated student!

After Chap. 6, in "*Some Concluding Remarks about Part III*", the reader can find the *RAM derivation* of a more complete *anelastic, deep, non-adiabatic, viscous and heat conducting system of equations for dissipative atmospheric thermal convection.*

2.7 Complementary Remarks

In our books published by *Springe*r (Heidelberg) during the years 1974–2014, the reader can find many complementary presentations of very different theoretical subjects relating to fluid dynamics—the reader will find some of these below! But first it should be noted that much of the impetus for research on *Newtonian fluid dynamics* during the past 50 *years* was created by the rapid development of "*asymptotics*" and "*modelling*".

In [10], cited in "Introduction" and in the paper by *Guiraud* [7], cited in Chap. 1 entitled *Going on with asymptotics*, and also in my 2002 book, *FMIA 64* [7], cited in "Introduction" (pp. 4–18 contain a *short summary of* Chaps. 2–12), the reader can find discussion of the following: *Newtonian fluid flow equations and conditions*; *asymptotic analysis and modelling*; *useful limiting forms of the NSF equations*; *the Navier-Fourier model*; *the Euler model*; *boundary layer models*; *models of nonlinear acoustics*; *low-Reynolds number asymptotics; asymptotic modelling of thermal convection and interfacial phenomena; meteo fluid dynamics models; singular coupling and the triple-deck model.*

On the other hand, in the book by *R. E. Meyer <9>,* the curious reader will find an *introduction to mathematical fluid dynamics* laying out the *basic concepts* of a *semi-axiomatic foundation,* but *without abstract nonlinear functional analysis,* in contrast to the two books by *P. L. Lions, published in 1996 and 1998* by Oxford University Press, *where pure mathematics plays the main role*. We note that, *for a mathematics student, such a treatment (à la Meyer) helps to dispel the common impression that the whole subject is built on a quicksand of assorted intuitions. It remains to hope that our RAM approach will lead to more positive feelings!*

Of course, in Meyer's book, the account is not axiomatic; the "postulates" are used to illuminate the subject, not to deduce it, and mathematicians will be even more disappointed by the lack of any attempt to prove an existence theorem, or even to talk seriously about partial differential equations. There would seem to be room, however, for an introduction to what the whole subject is about, before the Navier-Stokes equations are tackled. In fact, many mathematicians will complain that the book contains hardly (!) any mathematics—which is also the case in the present book—and many theoretical fluid dynamicists, that the book is far too abstract—*this indicates a gap which should be filled*!

Note, however, that, in Sect. 38 of Chap. 6 of Meyer, 1971, the reader can find a *short but well argued discussion* concerning the Navier-Stokes equations, referred to in the present book as the *Navier-Stokes-Fourier (NSF) equations*. In particular, in pp. 100–110 of Chap. 4 of Meyer, 1971, the reader will find *the formulation, solution, and discussion of the basic classical Blasius problem of the theory of fluids with small viscosity, considering a steady incompressible flow past a solid flat plate placed edgewise in a uniform stream.* The Blasius problem for viscous fluid flow is considered for the case of a slightly compressible fluid in Sect. 8.5.

Chapter 5 of Meyer, 1971, entitled *Some aspects of rotating fluids*, gives useful complementary info*rmation about* Sect. 2.6 *of the present book, and the singular perturbation example* in his Chap. 4 *seems to me a good simple case, illustrating asymptotics!*

In our two volumes [13] and [14], cited in "Introduction", the reader will find many and varied examples of the application of *asymptotics* to both *non-viscous* and *viscous* fluid dynamics problems. In particular, *in* Chap. 3 of [14], cited in "Introduction", the reader will find some *simple* (but *fundamental*) examples of viscous fluid flows.

For instance, Sect. 3.1 considers the *plane Poiseuille* flow and the *Orr-Sommerfeld* equation, which is solved by a *double-scale asymptotic technique* when

$$ik\text{Re} = 1/\varepsilon >> 1. \tag{2.20a}$$

The unknown stream function ψ(t, x, z), satisfying the perturbed vorticity Navier equation, is written in the following form:

$$\psi(z) = \phi(z) \exp[ik(x - ct))], \tag{2.20b}$$

and ϕ(z) is the solution of the *Orr-Sommerfeld* equation (see p. 58–60 of [14], cited in "Introduction"). With the boundary conditions dϕ/dz = 0 and ϕ = 0 on z = −1 and z = +1, this is an *eigenvalue problem.*

Writing $c = c_r + ic_i$, then if $c_i = 0$, there is sustained oscillation, and the condition $c_i = 0$ leads to a relation between k and Re—a *neutral stability curve* in the (*k*, Re) plane, as shown in Fig. 2.1.

Section 3.2 of [14], cited in "Introduction", is devoted to *steady flow* through an *arbitrary cylinder under pressure,* and three particular cases are considered—the case of a *circular* cylinder, the case of *an annular region* beween *concentric cylinders*, and the case of a cylinder of *arbitrary section.*

Section 3.3 of [14], cited in "Introduction", *investigates* the problem of the *steady-state Couette flow between two concentric circular cylinders of radii* r_1, r_2, rotating about their common axis at angular velocities ω_1, ω_2. This leads to the *classic Taylor problem*, where the *Taylor number for a small* $\varepsilon = d/r_m \ll 1$ is

$$\text{Ta} = (\rho^\circ/\mu^\circ)^2 (\omega_1)^2 d^4 \left[\left(1 - \Omega^2\right)/\varepsilon\right], \tag{2.21a}$$

and the ratio $(1 - \Omega^2)/\varepsilon$, of two small parameters is assumed to have a finite value. In $\varepsilon = d/r_m$, $d = r_2 - r_1$, and $r_m = (½) [r_2 + r_1]$, while $\Omega = \omega_2/\omega_1$. Again we have an eigenvalue problem that yields an eigenvalue relation, and in Fig. 2.2 the neutral

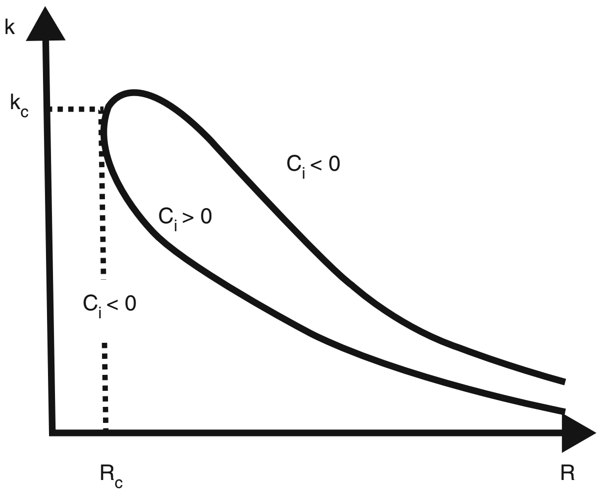

Fig. 2.1 Neutral curve for plane Poiseuille flow. From p. 59 of [14], cited in "Introduction"

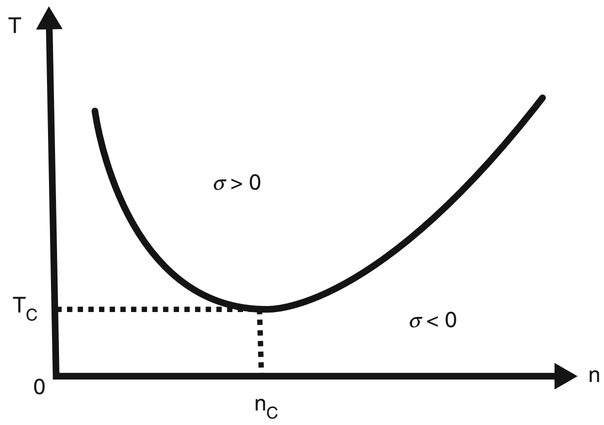

Fig. 2.2 Neutral curve for circular Couette flow. From p. 67 of [14], cited in "Introduction"

stability curve is given by the value $\sigma = 0$. This is the *principle of exchange of stabilities*!

The flow is described by

$$u = w = 0, \quad v = V(r) = Ar + B/r, \quad (1/\rho^\circ)dP(r)/dr = (1/r)\,[V(r)]^2, \tag{2.21b}$$

and, following *Taylor (1923)*, it is assumed that the perturbed flow has the form $\mathbf{u} = (u', V(r) + v', w')$, $p = P(r) + p'$. In Fig. 2.2, n = *md with m the axial wave number.*

The Bénard problem is discussed in pp. 68–70 of Sect. 3.4 of [14], cited in "Introduction", as in Sect. 6.4 *of the present*, *but in the framework of a linear theory, when perturbations relative to a basic equilibrium state are small. In such a case, using the usual linearization, we derive a linear system à la Boussinesq. When we analyze the perturbations into normal mode,* we must then consider, for the *function W(z),*

$$(w', \; T') = [W(z), \; \Theta(z)] \; f(x, \; y) \exp(\sigma \; t), \tag{2.22}$$

where $\sigma = \sigma_r + i\sigma_i$. *In this case, the equation* for *W(z)* with the *boundary conditions determines a so-called self-adjoint eigenvalue problem for Rayleigh number, Ra = Pr Gr, where according to (3.7)* $Gr = \alpha / Fr_d^2$ *is the Grashof number with α as a small dilatation parameter* (see the Sect. 6.4).

When Ra exceeds the critical value Ra_c, instability occurs in the form of convection in (*Bénard*) *cells* forming a polygonal platform. Once again, the case $\sigma = 0$ represents neutral (marginal) stability (see Fig. 2.3).

The *formation of Bénard cells* in a weakly expansible liquid layer is one of the most remarkable examples of a *bifurcation phenomenon. Bifurcations in dissipative (dynamical) systems are investigated in* Chap. 10 (entitled *A Finite-Dimensional*

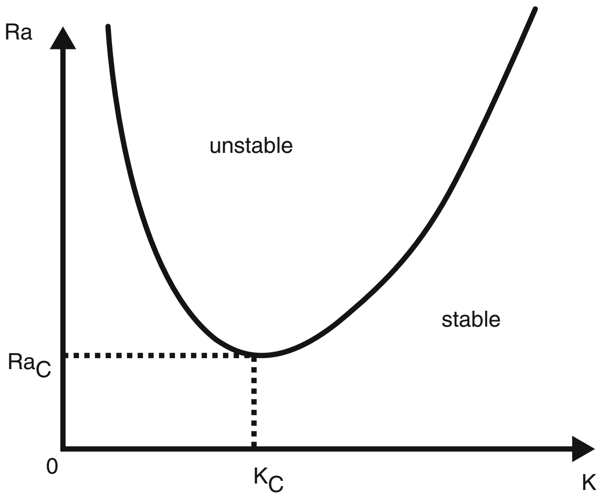

Fig. 2.3 The neutral stability curve for the onset of Bénard thermal convection. From p. 70 of [14], cited in "Introduction"

Dynamical System Approach to Turbulence) of the book [14], cited in "Introduction", devoted to *the theory and applications of viscous fluid flows.*

The linear problem with a *free surface* open to *ambiant passive air* is considered in Sect. 3.5 *of* [14], cited in "Introduction". This problem represents one of the most important cases in which capillary forces come into play. More precisely, the motion induced by *tangential gradients* of variable (only *temperature dependent*) *surface tension:*

$$\sigma(T) = \sigma(T^\circ) - \gamma^\circ\,(T - T^\circ), \tag{2.23a}$$

where T° is the constant temperature of the free surface in the basic equilibrium state

$$\gamma^\circ = -\,(\mathrm{d}\sigma/\mathrm{dT})_{T^\circ} = \mathrm{const}, \tag{2.23b}$$

is customarily called the *Marangoni* effect.

But, according to Zeytounian (see Chap. 7 of [25] cited in "Introduction"), from an *asymptotic* viewpoint, *it is not consistent to take into account the buoyancy effect and the deformation of the free surface simultaneously* in the Bénard thermal convection model problem for a weakly expansible liquid!

In the *Bénard-Marangoni thermocapillary problem, three non-dimensional parameters play a significant role:*

$$\mathrm{Ma} = \gamma^\circ \mathrm{d}^\circ \left[\Delta T^\circ/\rho^\circ \nu^{\circ 2}\right], \tag{2.24a}$$

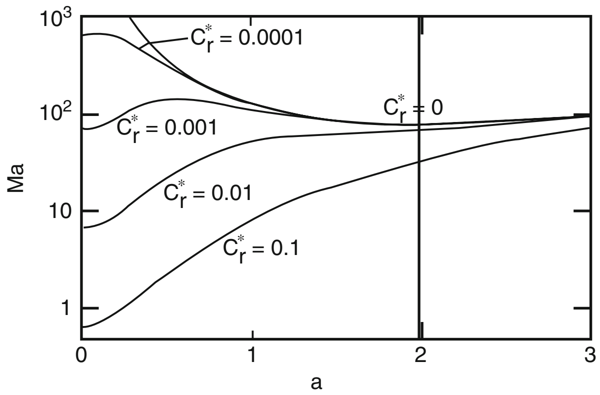

Fig. 2.4 Neutral stability curves for various values of Cr (for Bi = 0) and Bn = 0.1. From p. 74 of [14], cited in "Introduction"

$$\mathrm{We} = \sigma(T^\circ)\left[\mathrm{d}^\circ/\rho^\circ \nu^{\circ 2}\right], \tag{2.24b}$$

$$\mathrm{Bi} = \mathrm{q}^\circ \mathrm{d}^\circ / k^\circ, \tag{2.24c}$$

which are respectively the *Marangoni, Weber, and Biot numbers. Here* ΔT° is a difference of constant temperatures, and q° in the expression for the Biot number is the *thermal surface conductance constant* between a free surface and air.

Figure 2.4 shows the neutral stability curves for various values of the *crispation number,* according to the paper by *Regnier and Lebon (1995). F*or Bi = 0 and *Bond number* Bn = Pr [Cr/Fr$^{\circ 2}$] = $g\rho^\circ[\mathrm{d}^{\circ 2}/\sigma(T^\circ)] = 0.1$, the regions below each of curves represent the stable state in the plan (a, Ma). When Cr = We/Pr → 0, i.e., for a *non-deformable free surface*, and *Froude number* Fr$^{\circ 2}$ → 0, such that Bn ≠ 0, we obtain once again the result due to *Pearson (1958)*, viz., $\mathrm{Ma_c} \approx 80$ and $\mathrm{a_c} \approx 2.0$!

Concerning Sects. 6 and 7 in Chap. 3 of [14], cited in "Introduction", we note the following. *First*, in Sect. 6, viscous flow (ν° is the kinematic viscosity coefficient) due to a disc (in the plane z = 0) rotating at constant angular velocity Ω° is considered for small and large values of $\zeta = [\nu^\circ/\Omega^\circ]^{1/2}$ z. Then, in Sect. 7, the corresponding expansions are "joined" in a rather heuristic manner using a trial and error approach.

Second, in Sect. 7, the *Rayleigh flow problem* is investigated using the one-dimensional unsteady-state NSF equations, caused by an impulsively started flate plate, when

$$\mathrm{Pr} = 1,\ \mathrm{Re} = \mathrm{O}(1),\ \mathrm{M}^2 \to 0,\ \text{and}\ \tau^\circ \to 0, \tag{2.25a}$$

where $\tau^\circ = \Delta T^\circ/T^\circ$, and we assume that, for t > 0 on z = 0, the temperature changes *instantaneously* from T° to T° + ΔT°, with ΔT° > 0. We first consider the *small* M^2 *asymptotic* solution *close* to the flat plane, but *far from* the initial time, which is an *inner* expansion. Then we examine the *small* M^2 *asymptotic* solution far from a flat plate, an *outer* expansion. As a result, it appears necessary to include a term proportional to M^3 in the expansions as a consequence of matching. However, the above procedure yields only an asymptotic solution for the time $t = O(1)$, and is not valid near $t = 0$, because the above asymptotic solutions would imply that an infinite impulse per unit area is required to set the flat plate in motion!

The main problem is to match the solution valid in the initial transient layer with the above asymptotic solution, which is valid for fixed time $t > 0$. In fact, for *small* M^2 *close t*o the initial time, one must first introduce a *short time* adapted to the *initial transient layer. Hence, we write*

$$t^* = t/M^{\beta}, \;\; \beta > o, \tag{2.25b}$$

and with this short time (2.25b), we consider an *initial limiting process*:

$$M \to 0 \text{ with } t^* \text{ and } z \text{ fixed and } Re = 0(1), \tag{2.25c}$$

with an *initial asymptotic expansion. If we assume that, close $t^* = 0$,* the *significant* system of equations is the *classical system of acoustic equations, then in the initial asymptotic expansion (valid close to $t^* = 0$) we have a term* proportional to the Mach number M, and $\beta = 1$ in (2.25b).

In *Some Concluding Remarks for Part I* below, the reader can find further information concerning this unsteady *Rayleigh flow problem.*

As a conclusion of this rather long Sect. 2.7, I wish to quote below some comments by *M.F. Platzer* (from the Dept of Aeronautics and Astronautics at the Naval Postgraduate School, Code *AA/PL*, Monterey CA 93943-5000) in the Book Reviews section of *Appl Mech Rev*, vol 57, No. 3, May 2004. Platzer reviewed my book entitled *Theory* and *Applications of Viscous Fluid Flows* [14], cited in "Introduction":

> It is evident from this brief summary that the author's emphasis is on the mathematical aspects of the viscous flow equations and their various asymptotic limit cases and analytical solution methods.
>
> His choice of topic and flow problems is meant to provide young researchers in fluid mechanics, applied mathematics and theoretical physics with an up-to-date presentation of some key problems in the analysis of viscous fluid flows.
>
> Although the author intentionally limited himself to a select few topics, teachers of advanced viscous flow courses and researchers in this field will welcome this book for its thorough review of current work and the listing of 1156 relevant papers.
>
> In my judgment, it meets the stated objective of bridging the gap between standard undergraduate texts in fluid mechanics and specialized monographs.

The last chapter (Chap. 10) of our *Theory and Applications of Viscous Fluid Flows* [14]cited in "Introduction" presents the *finite-dimensional dynamical systems approach to turbulence* by reviewing the classical *Landau-Hopf*, *Ruelle-Takens-Newhouse*, *Feigenbaum,* and *Pomeau-Maneville transition scenarios to turbulence*. In pp. 414–443 of Sect. 10.4 of [14], cited in "Introduction" the reader will find a collection of *strange attractors (see* Figs. 2.5 and 2.6) for various viscous flow phenomena. These two strange attractors are related to the thermocapillary instabilities in a free-falling vertical 2D film, when the amplitude KS equation (2.26) is considered, where the function H(t, x) is related with the film thickness.

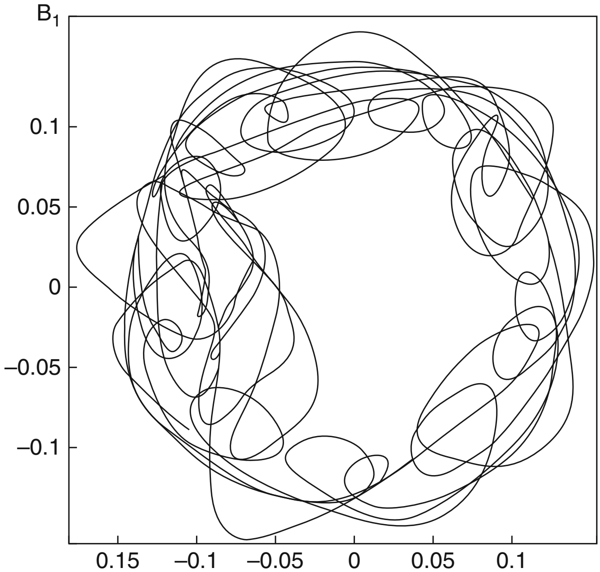

Fig. 2.5 Metastable chaos regime at k = 0. 273. From p. 424 of [14], cited in "Introduction"

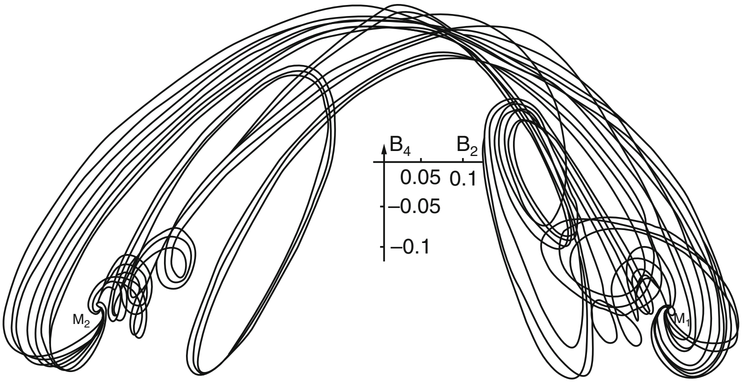

Fig. 2.6 Complex movement is observed in the vicinity of limit cycles. From p. 423 of [14], cited in "Introduction"

$$\partial H/\partial t + 4H\partial H/\partial x + \partial^2 H/\partial x^2 + \partial^4 H/\partial x^4 = 0. \tag{2.26}$$

References

<1> Rayleigh Lord (1916) On convection currents in horizontal layer of fluid when the higher temperature is on the under side. Philos Mag Ser 6 32:529–546 (192)

<2> Coulouvrat F (1992) On the equations of nonlinear acoustics. J Acoustique 321–359, 5

<3> Zeytounian RKh (1991) Meteorological fluid dynamics. LNP m5. Springer, Heidelberg

<4> Zeytounian RKh (1968) Études Hydrodynamiques des Phénomènes Méso -météorologiques. Cours à l' École de la Météorologie(pp. 163). Direction de la Météorologie—Quai Branly à Paris

<5> Zeytounian RKh (1985) Recent advances in asymptotic modelling of tangent atmospheric motions. Int J Eng Sci 23:1239–1288

<6> Zeytounian RKh (1991) Fluid dynamics limits of meterological equations. Adv Mech 14(3):3–86 (Warszawa)

<7> Zeytounian RKh (2004) Weather prediction as a problem in fluid dynamics (On the 100th birthday of I A Kibel). Izv Atmos Ocean Phys 40(5):593–605

<8> Zeytounian RKh (2004) On the foundations of the Boussinesq approximation applicable to atmospheric motions. Izv Atmos Ocean Phys 39:S1–S14

<9> Meyer RE (1971) Introduction to mathematical fluid dynamics. Wiley, New York, NY. Re-edition by Dover Publications