

Vertical Visibility Among Parallel Polygons in Three Dimensions

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Abstract. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ denote a collection of translates of a regular convex k -gon in the plane with the stacking order. The collection \mathcal{C} forms a *visibility clique* if for every $i < j$ the intersection C_i and C_j is not covered by the elements that are stacked between them, i.e., $(C_i \cap C_j) \setminus \bigcup_{i < l < j} C_l \neq \emptyset$.

We show that if \mathcal{C} forms a visibility clique its size is bounded from above by $O(k^4)$ thereby improving the upper bound of 2^{2^k} from the aforementioned paper. We also obtain an upper bound of $2^{2^{\binom{k}{2}+2}}$ on the size of a visibility clique for homothetes of a convex (not necessarily regular) k -gon.

1 Introduction

In a visibility representation of a graph $G = (V, E)$ we identify the vertices of V with sets in the Euclidean space, and the edge set E is defined according to some visibility rule. Investigation of visibility graphs, driven mainly by applications to VLSI wire routing and computer graphics, goes back to the 1980s [12, 14]. This also includes a significant interest in three-dimensional visualizations of graphs [3, 4, 8, 10].

Babilon et al. [1] studied the following three-dimensional visibility representations of complete graphs. The vertices are represented by translates of a regular convex polygon lying in distinct planes parallel to the xy -plane and two translates are joined by an edge if they can *see* each other, which happens if it is possible to connect them by a line segment orthogonal to the xy -plane avoiding all the other translates. They showed that the maximal size $f(k)$ of a clique represented by regular k -gons satisfies $\lfloor \frac{k+1}{2} \rfloor + 2 \leq f(k) \leq 2^{2^k}$ and that $f(3) \geq 14$. Hence, $\lim_{k \rightarrow \infty} f(k) = \infty$. Fekete et al. [8] proved that $f(4) = 7$ thereby showing that $f(k)$ is not monotone in k . Nevertheless, it is plausible that $f(k+2) \geq f(k)$ for every k , and surprisingly enough this is stated as an open problem in [1]. Another interesting open problem from the same paper is to decide if the limit

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$\lim_{k \rightarrow \infty} \frac{f(k)}{k}$ exists. In the present note we improve the above upper bound on $f(k)$ to $O(k^4)$ ¹ and we extend our investigation to families of homothetes of general convex polygons. The main tool to obtain the result is Dilworth Theorem [6], which was also used by Babilon et al. to obtain the doubly exponential bound in [1]. Roughly speaking, our improvement is achieved by applying Dilworth Theorem only once whereas Babilon et al. used its k successive applications.

Fekete et al. [8] observed that a clique of arbitrary size can be represented by translates of a disc. Their construction can be adapted to translates of any convex set whose boundary is partially smooth, or to translates of possibly rotated copies of a convex polygon. The same is true for non-convex shapes, see Fig. 1.

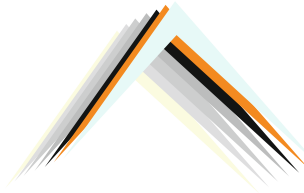


Fig. 1. A visibility clique formed by translates of a non-convex 4-gon.

An analogous question was extensively studied for arbitrary, i.e. not necessarily translates or homothetes of, axis parallel rectangles [3, 8], see also [11]. Bose et al. [3] showed that in this case a clique on 22 vertices can be represented. On the other hand, they showed that a clique of size 57 cannot be represented by rectangles.

For convenience, we restate the problem of Babilon et al. as follows. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ denote a collection of sets in the plane with the *stacking order* given by the indices of the elements in the collection. By a standard perturbation argument, we assume that the boundaries of no three sets in \mathcal{C} pass through a common point. The collection \mathcal{C} forms a *visibility clique* if for every i and j , $i < j$, the intersection C_i and C_j is not covered by the elements that are stacked between them, i.e., $(C_i \cap C_j) \setminus \bigcup_{i < k < j} C_k \neq \emptyset$. Note that reversing the stacking order of \mathcal{C} does not change the property of \mathcal{C} forming a visibility clique. We are interested in the maximum size of \mathcal{C} , if \mathcal{C} is a collection of translates and homothetes, resp., of a convex k -gon. We prove the following.

Theorem 1. *If \mathcal{C} is a collection of translates of a regular convex k -gon forming a visibility clique, the size of \mathcal{C} is bounded from above by $O(k^4)$.*

Theorem 2. *If \mathcal{C} is a collection of homothetes of a convex k -gon forming a visibility clique, the size of \mathcal{C} is bounded from above by $2^{2\binom{k}{2}+2}$.*

The paper is organized as follows. In Sect. 2 we give a proof of Theorem 1. In Sect. 3 we give a proof of Theorem 2. We conclude with open problems in Sect. 4.

¹ After acceptance of the paper the authors became aware of the fact that the upper bound of $O(k^4)$ was previously proven by Štola [13].

2 Proof of Theorem 1

We let $\mathcal{C} = \{C_1, \dots, C_n\}$ denote a collection of translates of a regular convex k -gon C in the plane with the stacking order given by the indices of the elements in the collection. Let \mathbf{c}_i denote the center of gravity of C_i . We assume that \mathcal{C} forms a visibility clique. We label the vertices of C by natural numbers starting in the clockwise fashion from the topmost vertex, which gets label 1. We label in the same way the vertices in the copies of C . The proof is carried out by successively selecting a large and in some sense regular subset of \mathcal{C} . Let W_i be the convex wedge with the apex \mathbf{c}_1 bounded by the rays orthogonal to the sides of C_1 incident to the vertex with label i . The set \mathcal{C} is *homogenous* if for every $1 \leq i \leq k$ all the vertices of C_j 's with label i are contained in W_i . We remark that already in the proof of the following lemma our proof falls apart if C can be arbitrary or only centrally symmetric convex k -gon.

Lemma 1. *If C is a regular k -gon then \mathcal{C} contains a homogenous subset of size at least $\Omega\left(\frac{n}{k^2}\right)$.*

Let $(C_{i_1}, \dots, C_{i_n})$ be the order in which the ray bounding W_i orthogonal to the segment $i[(i - 1) \bmod k]$ of C_1 intersects the boundaries of C_j 's. The set \mathcal{C} forms an *i -staircase* if the order $(C_{i_1}, \dots, C_{i_n})$ is the stacking order. As a direct consequence of Dilworth Theorem or Erdős–Szekeres Lemma [6, 7] we obtain that if \mathcal{C} is homogenous, it contains a subset of size at least $\sqrt{|\mathcal{C}|}$ forming an *i -staircase*.

A graph $G = (\{1, \dots, n\}, E)$ is a *permutation graph* if there exists a permutation π such that $ij \in E$, where $i < j$, iff $\pi(i) > \pi(j)$. Let $G_i = (C', E)$ denote a graph such that C' is a homogenous subset of \mathcal{C} , and two vertices C'_j and C'_k of G_i are joined by an edge if and only if the orders in which the rays bounding W_i intersect the boundaries of C'_j and C'_k are reverse of each other. In other words, the boundaries of C'_j and C'_k intersect inside W_i , see Fig. 2(a). Thus, G_i 's form a family of permutation graphs sharing the vertex set. Note that every pair of boundaries of elements in C' cross exactly twice.

Since for an even k a regular k -gon is centrally symmetric the graphs G_i and $G_{i+k/2 \bmod k}$ are identical. For an odd k , we only have $G_i \subseteq G_{i+[k/2] \bmod k} \cup G_{i+[k/2] \bmod k}$. The notion of the *i -staircase* and homogenous set is motivated by the following simple observation illustrated by Fig. 2(b).

Observation 1. *If C' forms an i -staircase then there do not exist two indices i and j , $i \neq j$, such that both G_i and G_j contain the same clique of size three.*

The following lemma lies at the heart of the proof of Theorem 1.

Lemma 2. *Suppose that C' forms an i -staircase, and that there exists a pair of identical induced subgraphs $G'_i \subseteq G_i$ and $G'_j \subseteq G_j$, where $i \neq j$, containing a matching of size two. Then C' does not form a visibility clique.*

Proof. The lemma can be proved by a simple case analysis as follows. There are basically two cases to consider depending on the stacking order of the elements

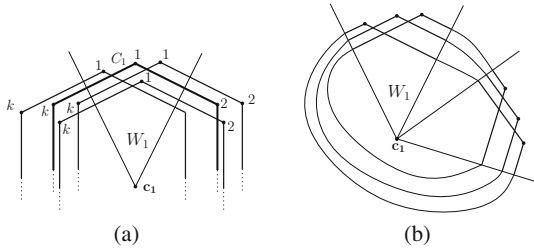


Fig. 2. (a) The wedge W_1 containing all the copies of vertex 1. (b) The 1-staircase giving rise to a clique of size three in G_1 and G_j for some j that cannot appear in a visibility clique.

of \mathcal{C}' supporting the matching M of size two in G'_i . Let u_1, v_1 and u_2, v_2 , respectively, denote the vertices (or elements of \mathcal{C}') of the first and the second edge in M , such that u_1 is the first one in the stacking order. By symmetry and without loss of generality we assume that the ray R bounding W_i orthogonal to the segment $i[(i - 1) \bmod k]$ of C_1 intersects the boundary of u_1 before intersecting the boundaries of u_2, v_1 and v_2 , and the boundary of u_2 before v_2 .

First, we assume that R intersects the boundary of u_2 before the boundary of v_1 . In the light of Observation 1, u_1, v_1 and u_2 look combinatorially like in the Fig. 3(a). Then all the possibilities for the position of v_2 cause that the first and last element in the stacking order do not see each other. Otherwise, R intersects the boundary of v_1 before the boundary of u_2 . In the light of Observation 1, u_1, v_1 and u_2 look combinatorially like in the Fig. 3(b), but then v_2 cannot see u_1 . ■

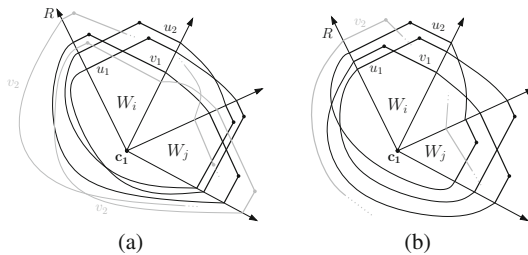


Fig. 3. The case analysis of possible combinatorial configurations of the boundaries of u_1, v_1, u_2 and v_2 , after the first three boundaries were fixed. (a) If R intersects the boundary of u_2 before v_1 the first and the last element in the stacking order cannot see each other. (b) If R intersects the boundary of v_1 before u_2 then u_1 cannot see v_2 .

Finally, we are in a position to prove Theorem 1. We consider two cases depending on whether k is even or odd. First, we treat the case when k is even which is easier.

Thus, let \mathcal{C} be a regular convex k -gon for an even k . By Lemma 1 and Dilworth Theorem we obtain a homogenous subset \mathcal{C}' of \mathcal{C} of size at least $\Omega(\sqrt{\frac{n}{k^2}})$

forming a 1-staircase. Note that for \mathcal{C}' the hypothesis of Lemma 2 is satisfied with $i = 1$ and $j = 1 + k/2$. Since \mathcal{C}' forms a visibility clique, the graph G_1 does not contain a matching of size two. Hence, $G_1 = (\mathcal{C}' = \mathcal{C}_1, E)$ contains a dominating set of vertices \mathcal{C}'_1 of size at most two. Let $\mathcal{C}_2 = \mathcal{C}_1 \setminus \mathcal{C}'_1$. Note that \mathcal{C}_2 forms a 2-staircase and that the hypothesis of Lemma 2 is satisfied with $\mathcal{C}' = \mathcal{C}_2, i = 2$ and $j = 2 + k/2 \pmod k$. Thus, $G_2 = (\mathcal{C}_2, E)$ contains a dominating set of vertices \mathcal{C}'_2 of size at most two. Hence, $\mathcal{C}_3 = \mathcal{C}_2 \setminus \mathcal{C}'_2$ forms a 3-staircase. In general, $\mathcal{C}_i = \mathcal{C}_{i-1} \setminus \mathcal{C}'_{i-1}$ forms an i -staircase and the hypothesis of Lemma 2 is satisfied with $\mathcal{C}' = \mathcal{C}_i, i = i$ and $j = i + k/2 \pmod k$. Note that $|\mathcal{C}_{k/2+1}| \leq 1$. Thus, $|\mathcal{C}'| \leq k + 1$. Consequently, $n = O(k^4)$.

In the case when k is odd we proceed analogously as in the case when k was even except that for \mathcal{C}' as defined above the hypothesis of Lemma 2 might not be satisfied, since we cannot guarantee that G_i and G_j are identical for some $i \neq j$. Nevertheless, since the two tangents between a pair of intersecting translates of a convex k -gon in the plane are parallel we still have $G_i \subseteq G_{i+\lceil \frac{k}{2} \rceil \pmod k} \cup G_{i+\lfloor \frac{k}{2} \rfloor \pmod k}$. The previous property will help us to find a pair of identical induced subgraphs in G_i , and $G_{i+\lceil \frac{k}{2} \rceil \pmod k}$ or $G_{i+\lfloor \frac{k}{2} \rfloor \pmod k}$ to which Lemma 2 can be applied, if G_i contains a matching M of size c , where c is a sufficiently big constant determined later. It will follow that G_i does not contain a matching of size c , and thus, the inductive argument as in the case when k was even applies. (Details will appear in the full version.)

3 Homothetes

The aim of this section is to prove Theorem 2. Let C denote a convex polygon in the plane. Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ denote a finite set of homothetes of C with the stacking order. Unlike as in previous sections, this time we assume that the indices correspond to the order of the centers of gravity of C_i 's from left to right. Let \mathbf{c}_i denote the center of gravity of C_i . Let $x(\mathbf{p})$ and $y(\mathbf{p})$, resp., denote x and y -coordinate of \mathbf{p} . Thus, we assume that $x(\mathbf{c}_1) < x(\mathbf{c}_2) < \dots < x(\mathbf{c}_n)$

Suppose that \mathcal{C} forms a visibility clique. Similarly as in the previous sections we label the vertices of C by natural numbers starting in the clockwise fashion from the topmost vertex, which gets label 1. We label in the same way the vertices in the copies of C . Consider the poset (\mathcal{C}, \subset) and note that it contains no chain of size five. By Dilworth theorem it contains an anti-chain of size at least $\frac{1}{4}|\mathcal{C}|$. Since we are interested only in the order of magnitude of the size of the biggest visibility clique, from now on we assume that no pair of elements in \mathcal{C} is contained one in another.

Every pair of elements in \mathcal{C} has exactly two common tangents, since every pair intersect and no two elements are contained one in another. We color the edges of the clique $G = (\mathcal{C}, \binom{\mathcal{C}}{2})$ as follows. Each edge $C_i C_j, i < j$, is colored by an ordered pair, in which the first component is an unordered pair of vertices of G supporting the common tangents of C_i and C_j , and the second pair is an indicator equal to one if C_i is below C_j in the stacking order, and zero otherwise.

Lemma 3. *The visibility clique G does not contain a monochromatic path of length two of the form $C_i C_j C_k$, $i < j < k$.*

We say that a path $P = C_1 C_2 \dots C_k$ in G is monotone if $x(\mathbf{c}_1) < x(\mathbf{c}_2) < \dots < x(\mathbf{c}_k)$. It was recently shown [9, Theorem 2.1] that if we color the edges of an ordered complete graph on $2^c + 1$ vertices with c colors we obtain a monochromatic monotone path of length two. We remark that this result is tight and generalizes Erdős–Szekeres Lemma [7]. Thus, if G contains more than $2^{2^{\binom{k}{2}+2}}$ vertices it contains a monochromatic path of length two which is a contradiction by Lemma 3.

4 Open Problems

Since we could not improve the lower bound from [1] even in the case of homothetes, we conjecture that the polynomial upper bound in k on the size of the visibility clique holds also for any family of homothetes of an arbitrary convex k -gon. To prove Theorem 2 we used a Ramsey-type theorem [9, Theorem 2.1] for ordered graphs. We wonder if the recent developments in the Ramsey theory for ordered graphs [2, 5] could shed more light on our problem.

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