On Dummett's "Proof-Theoretic Justifications of Logical Laws"

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Abstract This paper deals with Michael Dummett's attempts at a proof-theoretic justification of the laws of (intuitionistic) logic, pointing to several critical problems inherent in this approach. It discusses in particular the role played by "boundary rules" in Dummett's semantics. For a revised approach based on schematic validity it is shown that the rules of intuitionistic logic can indeed be justified, but it is argued that a schematic conception of validity is problematic for Dummett's philosophy of logic.

Keywords Proof-theoretic justification · Logical laws · Dummett

Can logical laws be justified? Of course, the question can be answered, trivially, in the affirmative: a logical law can be justified by deriving it from other logical laws. But the question is meant to ask something deeper, something like: can *the* logical laws be justified, all of them. Or, at least, can the logical laws be justified on the basis of some small fragment of them, a fragment deductively weaker than the whole?

To this question it seems plausible that the answer is negative. Early analytic philosophers might have argued that since the logical laws provide the canons of justification, it does not even make sense to seek to justify them. (This view is, I take it, near to the surface, if not completely explicit, in Frege. It is the cornerstone of Carnap's thought, when he takes the specification of a linguistic framework—including all the logical laws—as a precondition for any rational inquiry or debate at all.) This philosophical view is supported by, or mirrored in, an obvious technical point: any justification would involve a deductive argument; this argument would use logical laws, so that the justification would presuppose what it is supposed to justify. Thus it would be circular, and not a justification at all. This is well illustrated

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This paper was written in 1998. In 1999, at the first Tübingen conference on Proof-Theoretic Semantics, it was presented as a manuscript, with a talk by Michael Dummett devoted to it. It has been widely circulating since, and has been intensively discussed by many authors. It is here published in its original form.

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by soundness proofs for deductive systems: ordinarily, in showing soundness of a particular axiom or rule, one uses logical reasoning that is the direct analogue in the metalanguage of that very axiom or rule.

Nonetheless, as Michael Dummett has long urged (see, e.g., [1]), a negative answer might be too quick.

It might be proposed, for example, that it is the meaning of our words that have, as upshots, the acceptability of the logical laws; might not an account of those meanings therefore be able to play the role of supporting, or even fully justifying logical laws? To put a finer point on it, the suggestion is that logical laws are true by dint of the meanings of the words in them—specifically the meanings of the logical particles; and hence one might be able to find justifications of those laws simply by unfolding what the meanings of the logical particles are. The hope is that this might be done *without* invoking the full panoply of logical laws that use those particles, so as to obtain noncircular justifications.

In an odd sense, the idea goes back to Wittgenstein's discovery of truth-functional analysis: for the validity of the truth-functional laws follows at once from the stipulation of the truth-functions that the connectives represent. (I say "in an odd sense", since for Wittgenstein the logical laws have no content, and it is surely odd to speak of justifying something without content: what is there to justify?) But it should be noted that a strong assumption underlies Wittgenstein's procedure, namely his notion of propositions as bipolar—possibly true, possibly false, and determinately either one or the other. That is a highly suspect assumption, at least to those like Dummett who wish to question classical two-valued logic. So perhaps the question should be rephrased as: can we find noncircular justifications of logical laws by unfolding the meanings of the logical particles, without making strong meaning-theoretic assumptions?

Gerhard Gentzen's work in proof theory in the 1930s proved to be suggestive in this regard. Gentzen had developed logical systems in which the role of each connective was isolated, so that each basic inference rule was "about" one and only one connective. Indeed, he showed that two sorts of rules for each connective suffice. One sort allows for the introduction of the connective, and one for its elimination. In the context of a system for natural deduction (rather than in Gentzen's sequent calculus), the rule of $\wedge$-introduction is that which licenses the inference of $A \wedge B$ from premises A and B; the rules of $\wedge$-elimination license the inference of A from $A \wedge B$ and of B from $A \wedge B$. The rule of $\rightarrow$-introduction is the rule of discharge of premises: if B has been deduced from premises including A, then we may infer $A \rightarrow B$ while striking A from the list of premises. The rule of $\rightarrow$-elimination is just modus ponens, licensing the inference of B from A and $A \rightarrow B$. Gentzen suggested that introduction rules have much the same status as definitions: they fix the meaning of the connectives they introduce, at least in part. That is, an introduction rule for a connective gives the conditions under which a statement with that connective as its main connective can be inferred. Those conditions can be thought of as simply stipulated, and once stipulated, as constitutive of the meaning of the connective.

With respect to the project of justifying logical laws on the basis of the meaning of the logical particles, if we accept this view of introduction rules then clearly those rules stand in no further need of justification. As Dummett puts it, they are "self-justifying". The question then is whether such self-justifying rules can be used to endow further logical rules with justification, in particular, rules beyond those that amount to iterated use of introduction rules.

In Chaps. 11–13 of *The Logical Basis of Metaphysics* [3], Michael Dummett formulates a method for providing what he argues are just such justifications. The introduction rules for the connectives are taken as furnishing the *canonical means* of establishing sentences whose main connectives are one of those the rules introduce. Dummett's method then seeks to show, of an inference, that any canonical argument for the premises of the inference can be transformed into a canonical argument for the conclusion. Dummett's claim is that if this can be shown, the inference is justified.

The clearest illustrative case is an inference by an elimination rule, say, an inference from $F \wedge G$ to G. A canonical argument for the premise $F \wedge G$ would end in an application of the rule of $\wedge$-introduction, that is, would end in an inference of $F \wedge G$ from F and G. But then the argument already contains a canonical argument for G. Thus, the inference is justified, since we can transform the given argument for $F \wedge G$ into a canonical argument for G simply by extracting the subargument for G. The basic idea here stems from Gentzen's [6] technique of normalization of proofs, which he devised to prove his cut-elimination theorem. Dummett's use of the technique as a justificatory procedure is inspired by a similar proposal of Dag Prawitz from the early 1970s (especially in [7]), although there are differences in formulation and in scope.

This method of justifying logical laws is important to Dummett for several reasons. First, it provides a sense in which logical inferences *can* be justified, in a way that is clearly noncircular, and so stills the doubt I mentioned at the start as to whether any such program could make sense. Moreover, although the method presupposes the self-justifying nature of introduction rules, and so relies on a view of the meaning-endowing nature of those rules, the method need not invoke a full-fledged and comprehensive theory of meaning, as Dummett's better-known arguments criticizing classical logic and supporting intuitionistic logic do. Since we seem to be no closer to obtaining a comprehensive Dummettian theory of meaning for natural language than we were when Dummett formulated his meaning-theoretic program 25 years ago, this avoidance of invoking such a theory makes the method more credible and presumably less open to controversy.

As it turns out, or so Dummett asserts, the method provides justification for intuitionistic logic but not for classical logic, at least not for the classical laws about negation. Thus it gives important support to his position that intuitionistic logic is preferable to classical. Indeed, it exhibits a virtue of intuitionistic logic—justifiability on the basis of laws that merely express the meaning of the connectives—that classical logic fails to have: "[Intuitionistic logic's] logical constants can be understood, and its logical laws acknowledged, without appeal to any semantic theory and with only a very general meaning-theoretical background." [3, p. 300] The failure of this method for the laws of classical negation thus allows an invidious distinction to be made.

In this paper I investigate Dummett's method, as it applies to sentential logic.[1] I shall show that, even in this restricted domain, Dummett's method won't do: it provides "justifications" for obviously invalid inferences. I shall consider how to repair the damage, and analyze the question of whether the repair restores confidence in the philosophical framework underlying Dummett's claim that his method does indeed justify. The results are, I think, suggestive of some overlooked, and possibly deep, difficulties in Dummett's overarching project of marrying intuitionism and a verificationistic theory of meaning.

1 Analysis of the Method

In order to make the method precise, we must define the notion of a canonical argument, for, to repeat, the idea is that an inference is justified if any canonical argument for its premises can be transformed into a canonical argument for its conclusion. The definition should make plausible the following: if a logically complex proposition is provable at all, then it could in principle be proved by a canonical argument. For only if that condition is met will Dummett's procedure have any plausible claim to justificatory force. In line with the underlying idea, it might be tempting to define a canonical argument as one composed only of introduction rules. This does not work, however, because of the nature of the introduction rule for the conditional, to wit: $F \to G$ may be inferred from a subsidiary argument from premise F to conclusion G, discharging the premise F. Since F itself may be logically complex, the argument from F to G cannot be restricted to those that use introduction rules only, or else many elementary logical truths will not be obtainable by canonical arguments, for example, $A \wedge B \to B$. That is, a canonical argument will end in $\to$-introduction:

$$\frac{\begin{array}{c}[A \wedge B] \\ \\ B\end{array}}{A \wedge B \to B}$$

But if we are constrained to using only introduction rules, we will not be able to fill in the middle part. Hence the subsidiary arguments, the ones starting from premises that will eventually be discharged, cannot be constrained to contain only introduction rules. All that can be required of such subsidiary arguments is that they themselves be already recognized as justified. The result is a definition, by simultaneous induction on the complexity of the statements in the arguments, of the notion of "valid canonical argument" along with the notion of "valid argument":

[1]Dummett actually proposes the method for full first-order logic. Moreover, he aims at definitions that could apply to arbitrary new connectives, as well as our familiar ones.

> A *valid canonical argument* is a deduction whose premises are all atomic sentences and that uses only introduction rules except when auxiliary premises are introduced; at any point when such are introduced, the subargument from the point of the introduction of the first new premise to the last step before the discharge of the last new premise must be a valid argument.
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> A *valid argument* is an inference I such that any valid canonical argument (i.e., any valid canonical argument with any atomic premises) for the premises of I can be transformed into a valid canonical argument, with the same atomic premises, for the conclusion of I.

We have simplified matters slightly by omitting what Dummett calls "boundary rules", which allow the inference of one atomic sentence from others.[2] For example, these may be empirical laws, connecting the primitive notions of the vocabulary. Dummett allows the employment of such rules in valid canonical arguments. For the moment we take there to be no such rules, since the mathematics is clearer without them. In the next section, we shall allow boundary rules and investigate their impact.

The validity of an argument depends only on its premises and conclusion, and not on any intervening steps. Hence the second definition is framed as applying to inferences, rather than deductions. The simultaneous induction works because discharge of premises increases logical complexity. Thus, whether a deduction with conclusion F is a valid canonical argument depends on the validity of arguments whose premises and conclusion are of strictly lesser logical complexity than F.

These definitions are far from transparent. Applying them involves tracking through the tree structures of deductions in natural deduction systems. Most importantly, the definitions do not readily yield any general information about the range of inferences that are valid or not.

However, the definitions can be greatly clarified if we focus not on the proof-theoretic layout but rather on the relation that holds between a set α of atomic sentences and a formula F when there is a valid canonical argument with conclusion F and premises among the atomic sentences in α. Let us use "$\alpha \Vdash F$" for this relation. Using this notation we may frame the definition of "valid" thus: an inference from premises $F_1, \ldots, F_n$ to conclusion G is valid iff, for all sets α, if $\alpha \Vdash F_i$ for each i, then $\alpha \Vdash G$. (It may seem that this reformulation ignores a constructivity requirement, implicit in the phrase "we can transform" of the original definition. However, since we are dealing with sentential logic only, all notions are decidable and all quantifiers in the metalanguage are constructively evaluable.)

We can now investigate the relation $\alpha \Vdash F$, by looking at how its behaviour for logically complex F depends on its behaviour on the constituents of F. A valid canonical argument for $F \wedge G$ is just a valid canonical argument for F and a valid canonical argument for G, put together by means of a final inference to $F \wedge G$, using the rule of $\wedge$-introduction. A valid canonical argument for $F \vee G$ is either a valid canonical argument for F followed by one application of $\vee$-introduction or else a

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[2] These amount to the definitions given by Dummett in [3, p. 261], simplified by the absence of boundary rules and (more importantly) of the need to deal with free variables.

valid canonical argument for G followed by one application of $\vee$-introduction. These observations immediately yield:

$$\alpha \Vdash F \wedge G \text{ iff } \alpha \Vdash F \text{ and } \alpha \Vdash G \tag{1}$$

$$\alpha \Vdash F \vee G \text{ iff } \alpha \Vdash F \text{ or } \alpha \Vdash G \tag{2}$$

A valid canonical argument for $F \rightarrow G$ with atomic premises in α is a valid inference I to G from premises F and members of α, followed by an application of $\rightarrow$-introduction, discharging the premise F and yielding $F \rightarrow G$. The inference I will be valid provided that every valid canonical argument for F whose premises may include members of α and possibly some other atomic sentences can be transformed into a valid canonical argument for G whose premises are either in α or are among those others. This yields the condition:

$$\alpha \Vdash F \rightarrow G \text{ iff } \forall\beta(\text{if } \alpha \subseteq \beta \text{ and } \beta \Vdash F, \text{ then } \beta \Vdash G). \tag{3}$$

(1)–(3) show that the relation $\Vdash$ is, in fact, a familiar one from the semantics of intuitionistic logic, since they are nothing other than rules for the treatment of the connectives in the usual Kripke model semantics, when we take the sets α of atomic sentences as the nodes (worlds) of the model, and the relation $\alpha \subseteq \beta$ as the relation of extension. Thus the proof-theoretic trappings of Dummett's presentation conceal a notion whose structure is the same as the standard model-theoretic or semantic one.

One connective remains to be considered, namely, negation. As Dummett notes, the only way to treat negation that is consonant with his general procedure is to take $\neg F$ as an abbreviation for $(F \rightarrow \bot)$, where $\bot$ is a sentential constant governed by the following introduction rule: from premises that are all the atomic sentences, it may be inferred. Dummett allows there to be infinitely many atomic sentences; in fact, this treatment of negation fares poorly if there are not. For if $A_1, \ldots, A_n$ exhaust the atomic sentences, then the introduction rule just mentioned yields the validity of inferring $\neg(A_1 \wedge \ldots \wedge A_n)$ with no premises. Thus on logical grounds alone we would be able to infer that not every atomic statement is true, and this is surely an unacceptable result.

If there are infinitely many atomic sentences, then this treatment of negation can most easily be incorporated into our forcing relation by requiring that the domain of sets of atomic sentences α that we consider is always finite. Then the stipulation above becomes:

$$\alpha \Vdash \bot \text{ for no } \alpha. \tag{4}$$

The resulting rule for negation is then: $\alpha \Vdash \neg F$ iff $\forall\beta(\text{if } \alpha \subseteq \beta \text{ then not } \beta \Vdash F)$. This is just the standard rule for the treatment of negation in the semantics of intuitionistic logic.

The characterization of the forcing relation will be complete once we give the clause governing atomic sentences themselves. Since we are at the moment allowing no boundary rules, we have:

$$\text{for any atomic sentence } A, \alpha \Vdash A \text{ iff } A \in \alpha. \tag{5}$$

As we've just seen, Dummett's notion of valid canonical model yields a relation $\Vdash$ that obeys just the usual semantic rules for models of intuitionism, as given by (1)–(4). However, there is a key difference between $\Vdash$ as used in Dummett's method and the ordinary model-theory of intuitionistic logic. In the latter, the validity of an inference would mean that at each node (world) in *every* Kripke model, if the premises are true then the conclusion is true. Dummett's method, in contrast, amounts to considering only one particular structure, the Kripke model in which every finite set of atomic sentences is a distinct node, and for every finite set of atomic sentences there is exactly one node at which all and only those sentences are true, namely, the node that is the set of those sentences. This restriction to one particular structure yields anomalous results.

Counterexample 1 *If F does not contain $\perp$, then the inference from no premise to $\neg\neg F$ is valid.*

Proof It is easily shown by induction on the construction of F that if F does not contain $\perp$ then for every α there exists β with $\alpha \subseteq \beta$ and $\beta \Vdash F$. But then for no γ do we have $\gamma \Vdash \neg F$. Hence, for every α, $\alpha \Vdash \neg\neg F$. □

By the way, since we have shown that, for F that do not contain $\perp$, $\gamma \Vdash \neg F$ for no F, we also have the conclusion that, for such F and any G, the inference from no premises to $\neg F \to G$ is valid.

Counterexample 2 *Let F be a sentence not containing $\perp$ and G a sentence having no atomic sentences in common with F. Then the inference from premise $F \to G$ to conclusion G is valid.*

Proof Suppose $\alpha \Vdash F \to G$; we must show $\alpha \Vdash G$. By (3), for any β with $\alpha \subseteq \beta$, if $\beta \Vdash F$ then $\beta \Vdash G$. Moreover, as noted in the previous proof, there exists a β such that $\alpha \subseteq \beta$ and $\beta \Vdash F$.

The following is easily shown by induction on the construction of sentences: for any sentence H and any sets γ and δ, if $A \in \gamma$ iff $A \in \delta$ for all atomic sentences A that occur in H, then $\gamma \Vdash H$ iff $\delta \Vdash H$.

Thus, if β' is the subset of β containing just those atomic sentences either in α or occurring in F, we have $A \in \beta$ iff $A \in \beta'$ for all A that occur in F. Since $\beta \Vdash F$, it follows that $\beta' \Vdash F$. Hence $\beta' \Vdash G$. Since F and G have no atomic sentence in common, no atomic sentence occurring in G is in $\beta' - \alpha$. Thus $A \in \beta'$ iff $A \in \alpha$ for all A that occur in G. Hence $\alpha \Vdash G$. □

Thus there are many inferences that turn out valid under Dummett's definition, and yet are logically valid in no plausible sense. The counterexamples show that such inferences exist even in the fragment of the language that does not contain $\perp$, and so does not contain negation. As a particularly vivid case, we have the validity of the inference from $A \to B$ to B whenever A and B are distinct atomic sentences! We must conclude that Dummett's method has no justificatory force whatsoever.

2 Boundary Rules

To see how the trouble arises in terms of canonical arguments, rather than the relation $\Vdash$, it is helpful to consider the case of the inference from $A \to B$ to B, where A and B are distinct atomic sentences. If there were to be a valid canonical argument for $A \to B$, it would have to enable us to transform any valid canonical argument for A into one for B. Since B is atomic, the only valid canonical argument for B is the one-step argument of taking B as a premise. Hence a valid canonical argument for $A \to B$ must have B as an (undischarged) premise; and so it will be transformable into a valid canonical argument for B. The problem, in short, is that there is no way of getting from A to B, except by taking B as premise.

Here, it might be thought, is where Dummett's boundary rules can play a role, since boundary rules license inferences from atomic formulas to atomic formulas. However, three considerations—one technical and two philosophical—show that the problems in the method cannot be avoided by boundary rules as Dummett envisages them.

First, if the counterexamples are to be avoided, there are going to have to be an inordinate number of boundary rules. To forestall the validity of the inference from $A \to B$ to B, there must be a rule allowing the inference of B from A (and possibly other premises not including B) for *any* pair (A, B) of distinct atomic sentences. To forestall the validity of the inference from no premise to $\neg\neg A$, there must be a rule allowing the inference of $\bot$ from A (again, possibly with other premises). Rules that avoid some anomalies may engender others. For example, if $\bot$ can be inferred by boundary rules from premises A and B, and from premises A and C, but not from A and any other premises, then although the inference from no premise to $\neg\neg A$ is no longer valid, the inference from $\neg A$ to $B \vee C$ is. It appears, then, that it is unreasonable to expect that boundary rules will avoid the difficulty.

(By the way, it is not clear that a rule allowing the inference of $\bot$ from atomic premises should count as a boundary rule at all. Dummett characterizes boundary rules as "rules governing ... non-logical expressions." Allowing $\bot$ as a conclusion violates this description. After all, a rule allowing the inference of $\bot$ from premises A and B is just a rule allowing the inference of $\neg B$ from A, and of $\neg A$ from B. This significantly weakens the claim that $\bot$ is given meaning only by its introduction rule; indeed, it seems to me to weaken the contrast Dummett makes between intuitionistic negation and classical, saying of the latter "there is no way of attaining an understanding of the classical negation operator if one does not have it already" [3, p. 299] Nonetheless, if we are to block the anomalies given by Counterexample 1, we must allow boundary rules with conclusion $\bot$.)

Alongside the technical difficulties there are philosophical ones. To use boundary rules in the manner envisioned makes the validity of inferences dependent on which boundary rules there are, and hence, in particular, on empirical claims about the connections of different empirical basic sentences. This is not consistent with the claim that the validity of the logical inferences comes only from the meaning of the logical connectives (as based on the introduction rules).

Finally, even if the latter difficulty is set aside, there is another disturbing consequence, namely, that it becomes impossible to put forth a link between atomic sentences as a supposition, and draw consequences from it. For either the link is taken as a boundary rule, and hence becomes part of the logical framework, usable in any argument anywhere and playing a role in the criterion of validity; or else there is no link, in which case having $A \to B$ as a supposition yields B as a valid conclusion, and therefore we can infer from the conditional everything that is yielded by its consequent alone. The irony here is that we have landed in a position akin to Frege's odd-sounding view that "Only true thoughts can be premises of inferences." [5, p. 335][3]

The true nature of the difficulty should be apparent, by now. The intuitionist reading of $F \to G$ is, roughly, "from any demonstration of F we can obtain a demonstration of G." In Brouwer and the early intuitionistic tradition, the notion of demonstration here is taken to be open-ended, identified not with any particular formal system, indeed, not with the entirety of means of demonstration we currently have at our disposal, but as anything that we might come to accept as a demonstration. In later studies, particularly those inspired by Kreisel's work of the 1950s, the generality in talking of "any demonstration" is expressed by speaking of the intuitionist $\to$ as being "impredicative": $F \to G$ implicitly quantifies over all demonstrations, including those that may contain the very demonstration of $F \to G$. Dummett, in contrast, wants to read "any demonstration" here as meaning "any valid canonical argument", where this notion is defined in an inductive and hence purely predicative way. It is this restriction that gives rise to the difficulties above, both in the case without boundary rules, and the peculiarities of trying to use a fixed set of boundary rules to block those difficulties.

It is I think far more natural to use the notion of boundary rule in a way not envisaged by Dummett, and in fact inconsistent with Dummett's aim. The definition of "valid" can be revised so that what counts as a valid inference is one that was valid in the old sense given *any* assumption of boundary rules.[4] This revision avoids both of the philosophical difficulties just canvassed. It does not restrict allowable arguments to a fixed set of accepted ones, but rather allows any collection of possible arguments from atomic sentences to atomic sentences. Since all sets of boundary rules are considered, there is no need for empirical input to determine which boundary rules should be adopted.

Technically, the consideration of all sets of boundary rules amounts to the consideration of different model-theoretic structures. There are two equivalent ways of formulating this. Given a set S of boundary rules, the relation $\Vdash$-relative-to-S, or $\Vdash_S$ as we shall write it, can be defined by appropriate changes in clauses (4) and (5), keeping clauses (1)–(3) as is. Alternatively, (1)–(5) can be kept as is, and the domain of sets altered to contain all and only sets α that are closed under all the boundary rules in S and do not contain $\perp$. For our purposes, the latter procedure is more convenient. For any set α of atomic sentences, let $cl_S(\alpha)$ be the closure of α

[3] For Dummett's appraisal of this view, see [4, p. 313].

[4] This is the idea in the work of Prawitz [7, p. 236].

under the rules in S, that is, the smallest set β such that $\alpha \subseteq \beta$ and if S contains a rule "infer B from $A_1, \ldots, A_n$" and $A_1, \ldots, A_n$ are in β, then B is in β.

It is easy to show that every inference that is valid in the revised sense is classically valid. Suppose the inference from premise F to conclusion G is valid in the revised sense. Let T be a (classical) truth-assignment to the atomic sentences in F and G under which F comes out true; we must show that G also comes out true under T. Let S be the set of boundary rules containing "from no premise infer A" for every atomic sentence A to which T assigns truth, and "from A infer $\perp$" for every other atomic sentence A. Obviously, there is only one set α that is closed under S and does not contain $\perp$, namely, the set of atomic sentences assigned truth by T. But then $\Vdash_S$ behaves classically on the connectives, so that $\alpha \Vdash_S F$. Since the inference is valid in the revised sense, $\alpha \Vdash_S G$. Hence G is true under T.

From this we can surmise that there will be no counterexamples of the alarming sort encountered above. However, validity in the revised sense still does not coincide with intuitionistic validity.

Counterexample 3 *Let A be an atomic sentence, and G and H any sentences. Then the inference from premise $A \rightarrow (G \vee H)$ to conclusion $(A \rightarrow G) \vee (A \rightarrow H)$ is valid in the revised sense.*

Proof Let S be a set of boundary rules, and suppose α is an S-closed set not containing $\perp$ such that $\alpha \Vdash_S A \rightarrow (G \vee H)$. Let β be the S-closure of $\alpha \cup \{A\}$. If $\perp \in \beta$, then $\alpha \Vdash_S A \rightarrow F$ for every F, so $\alpha \Vdash_S (A \rightarrow G) \vee (A \rightarrow H)$; hence we may suppose $\perp \notin \beta$. If $\beta \Vdash_S G$, then $\alpha \Vdash_S A \rightarrow G$, for if γ is any S-closed extension of α with $\gamma \Vdash_S A$ then $\beta \subseteq \gamma$, so that $\gamma \Vdash_S G$; similarly if $\beta \Vdash_S H$ then $\alpha \Vdash_S A \rightarrow H$; in either case $\alpha \Vdash_S (A \rightarrow G) \vee (A \rightarrow H)$. But if neither, then β is an S-closed extension of α such that $\beta \Vdash_S A$ while not $\beta \Vdash_S G \vee H$, which contradicts the hypothesis that $\alpha \Vdash_S A \rightarrow (G \vee H)$. □

In the usual model-theory of intuitionistic logic, say via Kripke trees, one obtains a model of $A \rightarrow (G \vee H)$ that is not a model of $(A \rightarrow G) \vee (A \rightarrow H)$ by having two nodes v_1 and v_2, one of which models A and G but not H, the other models A and H but not G. For this it is essential that there be no u with $u \leq v_1$ and $u \leq v_2$ that models A; for if the root of the tree is to model $A \rightarrow (G \vee H)$ any such u would have to model $G \vee H$, and thus have to model G or model H, but every node above u would also model G or every node would also model H, thus defeating the example. The problem is that, using $\Vdash$ and boundary rules, these strictures cannot be met. For example, suppose G and H are also atomic. Using boundary rules one can insure that there is a closed set containing A and G and a distinct one containing A and H, but then there will also be a closed set containing A that is a subset of each of those, and in order to insure that $A \rightarrow (G \vee H)$ holds, that subset will have to contain either G or H.

It may be helpful to translate the situation back into Dummett's proof-theoretic language. Again suppose G and H, as well as A, are atomic. The counterexample shows that any valid canonical argument for $A \rightarrow (G \vee H)$ can be transformed into one for $(A \rightarrow G) \vee (A \rightarrow H)$. Suppose, then, there is a valid canonical argument from premises α to conclusion $A \to (G \vee H)$. This is just to say that the inference from α and A to $G \vee H$ is valid, which in turn means that every valid canonical argument for α and A can be transformed into one for $G \vee H$. Since there is a valid canonical argument from α and A to α and A, there must be one from α and A to $G \vee H$. The last step of this must be an application of $\vee$-introduction. Hence there is either a valid canonical argument from α and A to G or one from α and A to H, and so there is a valid canonical argument from α to $(A \to G) \vee (A \to H)$. The idea is that there is only one way to demonstrate A, so to speak.

3 Schematic Inferences

The counterexamples I have presented are not *schematic* inferences, that is, inferences that rely only on the forms of the premises and conclusion. The inferences that I showed to be valid-by-Dummett's lights (although not valid in any ordinary sense) were further constrained, e.g., in Counterexample 2 the formulas could have no atomic constituent in common, and in Counterexample 3 the antecedent had to be atomic.

The question naturally arises as to how Dummett's definitions fare on schematic inferences. Let us call an inference *schematically* valid in Dummett's original sense iff the inference and all its instances are valid in Dummett's original sense. (An instance is simply any inference obtained from the original one by replacing atomic sentences with other sentences.)

Now any inference that is schematically valid is classically valid, since if F does not imply G in the classical sense, a truth-assignment T that makes F true and G false can be mimicked by the substitution instances of F and G in which sentence letters assigned truth by T are replaced with "$p \to p$" and those assigned falsity are replaced with "$\perp$". The resulting instances F^* and G^* are such that $\emptyset \Vdash F^*$ but not $\emptyset \Vdash G^*$ (since forcing will then just amount to two-valued truth-computation).

However, schematic validity outstrips intuitionistic logic.

Counterexample 4[5] *The inference from no premise to $\neg F \vee \neg\neg F$ is schematically valid (that is, for any F the inference from no premise to $\neg F \vee \neg\neg F$ is valid in Dummett's original sense).*

Proof If not $\alpha \Vdash \neg F$, then there exists β such that $\alpha \subseteq \beta$ and $\beta \Vdash F$. But then, for any γ such that $\alpha \subseteq \gamma$, there exists δ such that $\gamma \subseteq \delta$ and $\delta \Vdash F$, namely, $\delta = \gamma \cup \beta$. Thus, for any γ such that $\alpha \subseteq \gamma$, not $\gamma \Vdash \neg F$. That is, $\alpha \Vdash \neg\neg F$. □

However, we can obtain a positive result if we combine the notion of schematic inference with that of validity-in-the-revised-sense, that is, validity given any collection of boundary rules. That is, it is possible to prove the following:

[5] I owe this counterexample to Philip Kremer.

Theorem *If every instance of the inference from F to G is valid in the revised sense, then the inference from F to G is intuitionistically valid.*

For the proof, see the Appendix.

4 Assessment

Dummett can't take too much comfort in this positive result. Dummett is careful to point out, in framing his method, that the inferences treated are actual inferences, involving particular meaningful sentences, the atomic components of which are actual atomic sentences, not schematic parts [3, p. 254]. That is, he treats the language as fully interpreted. On the view he is propounding, an inference is justified by its validity; the justification of a schematic inference (an inference rule) can lie only in the fact that each of its instances is justified. There is simply no room, on his view, for a position to the effect that validity does not justify an inference unless all inferences like it in being subsumable under a particular rule are also justified.

Let us return to Counterexample 3. Intuitionistically, the inference from $A \rightarrow (G \vee H)$ to $(A \rightarrow G) \vee (A \rightarrow H)$ is incorrect, because the former means "any demonstration of A can be transformed either into one for G or into one for H" whereas the latter means "any demonstration of A can be transformed into one for G or any demonstration of A can be transformed into one for H." This inference turns out to be valid, in the revised sense, because—so to speak—the method treats an atomic formula as its own proof. That is, the criterion of the identity of a proof is just the atomic formulas it has as premises or are implied by its premises. Thus distinct ways of proving an atomic A will not register as distinct. Now this view of proofs arises because the whole set-up envisages all proofs as, ultimately from atomic formulas. And this is the upshot of the set-up's being part of, or the beginnings of, a verificationist meaning-theory.

I believe the basic views that lead to Dummett's difficulties are well exhibited in the following remark (he is speaking here only of mathematics, but presumably he would maintain the same for a language with empirical vocabulary as well):

> If the intuitionistic explanations of the logical constants and, more generally, of the meanings of mathematical statements are to be considered as constituting a coherent theory of meaning for the language of mathematics, then the notion of proof which is appealed to must be such that we can fully grasp the concept of a proof of any constituent of a given sentence in advance of grasping that of a proof of that sentence. It cannot, therefore, be identified with the notion of the sort of proof that we may, at some future time, come to consider valid ... [2, p. 402]

This remark expresses a fundamental view of Dummett's; and from it we can see three sources of the problems with his program for "proof-theoretic justification". Of course, most generally, his underlying concern to meld intuitionistic logic with theory of meaning impels him to differ with the Brouwerian tradition of the open-endedness of the notion of demonstration, and as we saw that was key to the anomalies exemplified by Counterexamples 1 and 2. But the remark also expresses Dummett's commitment to molecularity: that what it is to prove a sentence is explained in terms of what it is to prove each constituent of the sentence. That, of course, is a denial of the impredicative nature of intuitionistic conditionals; and it signals his commitment to just the view of the proofs of atomic sentences as, if you like, logically unanalyzable, that engenders Counterexample 3.

The difference between Dummett and the treatments of intuitionism standard in mathematical logic on these two points has not been sufficiently explored. In classical truth-functional semantics, atomic sentences are the basic building blocks, and it is clear why. Dummett takes atomic sentences to be the basic building blocks of a proof-theoretic semantics—and on both the "basic" aspect and the "building block" aspect he differs from classical intuitionism. It is not clear why one should believe this, except perhaps for the conflation of the notion of verification and a mathematical notion of proof.

The third factor expressed in Dummett's remark is that there will be no definite meaning ascribed to a sentence unless it is fixed what a demonstration is for that sentence. That will presuppose that not just the logical rules but also the boundary rules are fixed. This tells us that, from Dummett's viewpoint, the revised notion of validity is not acceptable. For, in considering all possible boundary rules, it should be clear, the revised notion of validity treats the atomic components of sentences in abstraction from their actual content. It takes them to be schematic, in that their connections to one another (and hence also to complex sentences) are varied at will, but this is precisely what Dummett's insistence that he seeks to justify actual inferences, not schematic ones, would rule out.[6]

Finally, I suppose the following line might be taken. The claim that Dummett's method provides justifications of logical laws might be abandoned or weakened, while still it be pressed that the method does show *something*. That, under the revised notion of validity, the inference rules that yield valid inferences under all substitutions are not the classical ones but precisely the intuitionistic ones, surely supports the ascription of *some* advantageous status to intuitionistic logic. But here we should note at once that the method—in taking the introductory rules as definitory of the connectives—identifies the sense of $F \to G$ as "G can be validly inferred from F", and then goes on to define the latter as "every valid canonical argument for F can be transformed into a valid canonical argument for G". Thus, built into the method at the start is the intuitionistic construal of the conditional. As pointed out above, this is just what leads to condition (3) on the $\Vdash$-relation, and that in turn is the characteristic of the model theory of intuitionism. If (3) were to be replaced by $\alpha \Vdash F \to G$ iff (if $\alpha \Vdash F$ then $\alpha \Vdash G$), then what we obtain will be a classical

[6] Nor is Dummett's insistence ill-placed. The justificatory force of his method rests on what he calls the "fundamental assumption", that a logically complex sentence, if demonstrated, could have been demonstrated by a (valid) canonical argument. For this reason, Dummett spends an entire chapter of [3] investigating the exact sense and the plausibility of the fundamental assumption. Clearly, for the fundamental assumption to make sense at all requires that the sentences about which it speaks have content. If they are merely schemata, it is unclear what the assumption could mean, unless it is to be true by fiat.

notion of validity. In short, it should occasion no surprise, that the (revised) method yields intuitionistic validity, because the method is based on, or presupposes, an intuitionistic reading of the conditional. (Although attention is usually focused on negation as that which marks the difference between classical and intuitionistic, a case can be made that it's the conditional. A classical conditional combined with the definition of $\neg F$ as $F \to \bot$ would still yield the classical laws of negation.) And for this reason, that the method yields the intuitionistic inferences once it is applied schematically does not signal any greater virtue of intuitionistic logic, at least as framable from neutral ground. What is odd, and perhaps even undermining of the claims to virtue of intuitionistic logic, is that even when the intuitionistic reading of the conditional is built into the project at the start, it still takes lots of fussing here and jiggling there to get the method to yield just the intuitionistic laws.

Appendix

Theorem *Let F and G be sentential formulas such that the inference from F to G is not intuitionistically valid. Then there are instances F^* and G^* of F and G, a set S of boundary rules, and a set α of atomic sentences such that $\alpha \Vdash_S F^*$ but not $\alpha \Vdash_S G^*$.*

Proof Let Σ be the set of atomic sentences occurring in F or in G. Since the inference from F to G is not intuitionistically valid, there is a Kripke tree $(W, \leq, I)$ with root w such that $(W, \leq, I) \vDash_w F$ but not $(W, \leq, I) \vDash_w G$. Here $(W, \leq)$ is a tree (we take the root as being at the bottom), and I is a mapping from W to subsets of Σ: for each $u \in W$, $I(u)$ is the set of atomic sentences true at u. I is subject to the constraint that if $u \leq v$ then $I(u) \subseteq I(v)$. We wish to obtain sets of atomic sentences that "mimic" $(W, \leq, I)$. For this we shall need additional atomic sentences for there may be distinct nodes in W making the same atomic sentences of Σ true, but to these nodes we want to have correspond distinct sets of atomic sentences. For each u in W let u^* be a distinct atomic sentence not in Σ, and for each u in W let $\varphi(u) = \{v^* \mid v \in W \text{ and } v \leq u\}$. $\varphi(u)$ will be the set of atomic sentences corresponding to the node u. We now so formulate boundary rules that the only S-closed sets that do not contain $\bot$ are precisely the sets $\varphi(u)$ for $u \in W$. In fact, let S be the following set of boundary rules:

"Infer $\bot$ from $A_1, \ldots, A_n$ whenever $\{A_1, \ldots, A_n\} \subseteq \varphi(u)$ for no $u \in W$;"
"Infer v^* from u^* whenever $u, v \in W$ and $v < u$."

Now, for each $p \in \Sigma$, let $D(p)$ be the disjunction of all atomic sentences u^* such that p is in $I(u)$. Finally, for any sentence H constructed from members of Σ, let H^* be obtained from H by replacing each atomic sentence p with $D(p)$.

It is a routine matter to show by induction on the construction of formulas that, for any H and any $u \in W$, $(W, \leq, I) \vDash_u H$ iff $\varphi(u) \Vdash_S H^*$. It then follows from the supposition that $\varphi(u) \Vdash_S F^*$ but not $\varphi(u) \Vdash_S G^*$, so that the inference from F^* to G^* is not valid, in the revised sense. □

Author's Postscript, January 2015

From 1999 to 2007 I presented this paper at various universities and conferences. Often audience members raised stimulating points, particularly about my suggestions in Sect. 4, which I hoped to address in an expanded version, but I never managed to do so to my satisfaction. However, the basic issues still seem to me to be well-framed in the original version that is printed here.

The last presentation I gave was in September 2007 at the Oxford philosophy of mathematics seminar led by Daniel Isaacson. I was delighted that Michael Dummett was able to attend, despite infirmities of age. (Sir Michael and I had been on warm terms since his spring semester 1976 residence at Harvard, when he delivered the William James Lectures from which *The Logical Basis of Metaphysics* [3] evolved, and I was in my first year on the Harvard faculty.) It was particularly pleasing that at the 2007 seminar one of the younger Oxford philosophers, Ofra Magidor, raised the same objection that Sir Michael had framed nine years earlier in a letter to me, namely that Counterexamples 1 and 2 (from Sect. 1) are really not worrisome at all. In his 1998 letter he wrote, "I do not accept that your counter-examples are genuinely such." His point and Magidor's was that if G and F have no atomic sentences in common, and F has no occurrences of $\bot$, then the only reason to assert $F \to G$ would be that one had independent reason for thinking that F were false or G were true, and since the former is ruled out by F not containing $\bot$, it isn't at all surprising that we can infer G.

However this point seems to me mistaken: for if the rule (infer G from $F \to G$, when F does not contain $\bot$ and G and F contain no atomic parts in common) is justified, it is justified in application not just to assertions but also to *suppositions*. On the ordinary understanding of what it is to suppose $F \to G$, this is simply untenable. So if the rule is to be accepted, it would have to be argued that the ordinary understanding of supposition is incorrect, and in fact to suppose $F \to G$ is to suppose something like "every canonical argument for F can be transformed into a canonical argument for G". But this is clearly wrong if F and G are empirical. (It might be maintained for mathematical F and G, but in this case it would again appear that a bias in favor of intuitionistic logic were being built in at the ground level.)

Even though I was criticizing his position, Sir Michael clearly enjoyed my presentation at the seminar, no doubt because he thought the issues needed more discussion. Despite his infirmities, he maintained his famously cheerful humour as well as his robust sense of what philosophy could aspire to do. I dedicate this publication to his memory.

References

1. Dummett, M.: The Justification of Deduction. Oxford University Press, Oxford (1974). Reprinted in M. Dummett, Truth and Other Enigmas, Duckworth, London (1978), 290–318
2. Dummett, M.: Elements of Intuitionism. Oxford University Press, Oxford (1977)
3. Dummett, M.: The Logical Basis of Metaphysics. Harvard University Press, Cambridge (1991)
4. Dummett, M.: Frege: Philosophy of Language, 2nd edn. Harvard University Press, Cambridge (1993)
5. Frege, G.: Collected Papers on Mathematics, Logic and Philosophy. Basil Blackwell, New York. Translated by M. Black, V. Dudman, P. Geach, H. Kaal, E.-H.W. Kluge, B. McGuinness & R.H. Stoothoff(1984)
6. Gentzen, G.: Untersuchungen über das logische Schließen. Mathematische Zeitschrift 39, 176–210, 405–431 (1934/35). Translated in M.E. Szabo (ed.), The Collected Papers of Gerhard Gentian, 68–131. North-Holland, Amsterdam (1969)
7. Prawitz, D.: Towards a foundation of a general proof theory. In: Suppes, P., Henkin, L., et al. (eds.) Logic, Methodology, and Philosophy of Science IV, 225–250. North-Holland, Amsterdam (1973)