

# Distribution of FRFT Coefficients of Natural Images

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**Abstract.** For the convenience of providing temporal and spectral information by a single variable, fractional Fourier transformation (FRFT) is more and more applied to image processing recently. This paper focuses on the statistical regularity of FRFT coefficients of natural images and proposes that the real and imaginary parts of FRFT coefficients of natural images take on the generalized Gaussian distribution, the coefficient modulus follow the gamma distribution and the coefficient phase angles tend to the uniform distribution, moreover, the real and imaginary parts of coefficient phases similar to the extended beta distribution. These underlying statistics can provide theoretical basis for image processing in FRFT, such as dimensionality reduction, feature extraction, smooth denoising, digital forensics, watermarking, etc.

**Keywords:** Fractional Fourier transformation · Distribution of FRFT coefficients · Generalized Gaussian distribution · Gamma distribution · Uniform distribution · Beta distribution

Fractional Fourier transformation (FRFT), as a kind of generalized Fourier transformation (FT), can be interpreted as a rotation of the signal in the time-frequency plane. Different from wavelet transformation, short-time Fourier transformation, Gabor transformation and other common two-parameters time frequency distributions, FRFT can provide the related local information of the signal in the time domain and the frequency domain simultaneously by a single variable, thus it has a wide application prospect in the field of signal processing [1]. FRFT is also introduced into the image processing, which is a significant branch of signal processing, such as the data compression [2], the image registration [3], the facial expression recognition [4], the image encryption [5], the digital watermarking [6] and so on. For the image processing based on FRFT, it is necessary to learn the related prior knowledge of FRFT coefficients of natural images. However, to the author's knowledge, there has not been relevant research report currently. Thus, this paper will focus on the statistical probability distribution of FRFT coefficients of natural images. As the FRFT phase gaining more and more attention for carrying the important image texture information [4], this paper will explore the statistical distribution of the amplitude and phase parts also with the real and imaginary parts of FRFT coefficients of natural images.

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The following paper is organized as follows: firstly, introduces the FRFT briefly; and then discusses the probability distribution of the FRFT coefficients of natural images, including the real part, the imaginary part, the amplitude part and the phase part; finally, makes conclusions and points out the future research directions.

## 1 Fractional Fourier Transformation

The fractional Fourier transformation (FRFT) of the one-dimensional function  $x(t)$  is defined as the follows:

$$X_p(u) = \int_{-\infty}^{\infty} x(t) \cdot K_p(t, u) dt \tag{1}$$

where the kernel function of FRFT is given as:

$$K_p(t, u) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{\frac{j(t^2+u^2)}{2}\cot\alpha - jtu\csc\alpha} & \alpha \neq n\pi \\ \delta(t - u) & \alpha = 2n\pi \\ \delta(t + u) & \alpha = (2n + 1)\pi \end{cases} \tag{2}$$

where  $\alpha = p\pi/2$  is the order of FRFT and  $p$  is a real number. For  $p$  only existing in the trigonometric function of the definition of FRFT, as (1) shows, the FRFT follows a period of  $p = 4$ . Furthermore,  $k_p(t, u)$  shows symmetry properties in the range of  $p \in (-2, 2]$ , and so, only FRFT with  $p \in [0, 1]$  is generally taken into account [7]. It must be pointed out that the FRFT of signal with  $p = 0$  is just the original signal, while the FRFT of signal with  $p = 1$  is the ordinary FT of signal. Therefore, when  $p$  changes from 0 to 1, the result of FRFT changes from the original signal to the ordinary FT signal smoothly, and the corresponding data domain changes from the time domain to the frequency domain.

The FRFT of two-dimensional function  $x(s, t)$  is defined as the follows:

$$X_{p_1, p_2}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s, t) K_{p_1, p_2}(s, t, u, v) ds dt \tag{3}$$

where the kernel function,  $K_{p_1, p_2}(s, t, u, v)$ , is given as:

$$K_{p_1, p_2}(s, t, u, v) = \frac{1}{2\pi} \sqrt{1 - j \cot \alpha} \sqrt{1 - j \cot \beta} \cdot e^{\frac{j(s^2+u^2)}{2}\cot \alpha - jsu \csc \alpha} \cdot e^{\frac{j(t^2+v^2)}{2}\cot \beta - jsu \csc \beta} \tag{4}$$

where  $\alpha = p_1\pi/2$ ,  $\beta = p_2\pi/2$  are the orders of 2D-FRFT and  $p_1, p_2$  are real numbers. According to the commutative and associative of multiplication,  $K_{p_1, p_2}(s, t, u, v)$  can be decomposed as:

$$\begin{aligned}
 K_{p_1,p_2}(s, t, u, v) &= \left[ \sqrt{\frac{1 - j \cot \alpha}{2\pi}} \cdot e^{\frac{j(s^2+u^2)}{2} \cot \alpha - jsu \csc \alpha} \right] \\
 &\cdot \left[ \sqrt{\frac{1 - j \cot \beta}{2\pi}} \cdot e^{\frac{j(t^2+v^2)}{2} \cot \beta - jsu \csc \beta} \right] = K_{p_1}(s, u) \cdot K_{p_2}(t, v)
 \end{aligned} \tag{5}$$

Namely, the kernel of 2D-FRFT can be decomposed as the multiplication of two kernels of 1-D FRFT. Substituting (4) into (3), there are:

$$\begin{aligned}
 X_{p_1,p_2}(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s, t) K_{p_1,p_2}(s, t, u, v) ds dt \\
 &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(s, t) K_{p_1}(s, u) ds \right] K_{p_2}(t, v) dt
 \end{aligned} \tag{6}$$

Comparing (5) with (1), it can be known that 2D-FRFT can be decomposed into column 1-D FRFT of original data matrix and then row 1-D FRFT of column transformed data matrix.

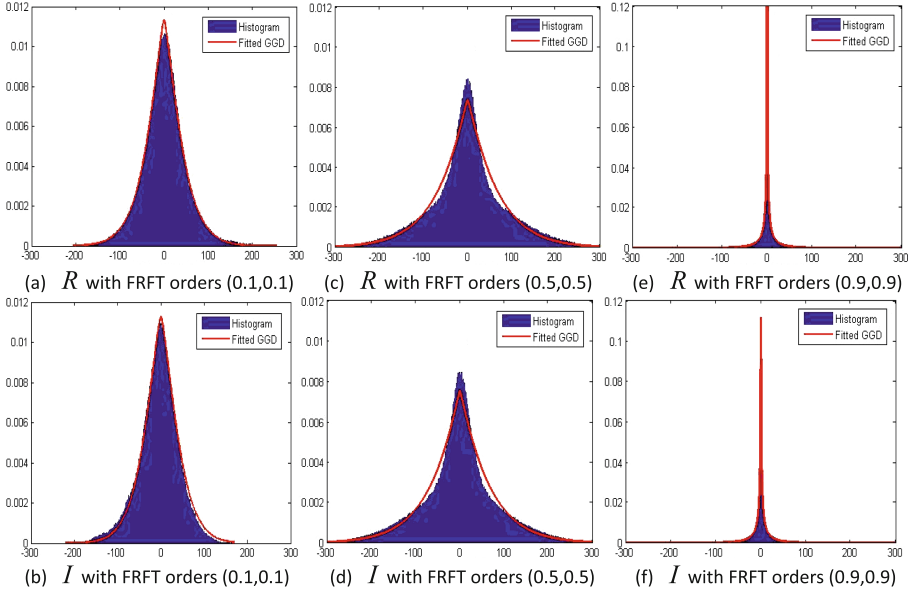
## 2 Statistic Distribution Modeling of the FRFT Coefficients of Natural Images

Through FRFT, the natural image data will become complex number, thus, the statistical analysis of the real and imaginary parts of FRFT coefficients of natural images will all be made in this paper. Furthermore, considering that more and more image processing algorithms pay attention to the FRFT phase for its carrying the importance image texture information [4], we also focus on the statistical analysis of the amplitude and phase parts of FRFT coefficients of natural images.

According to the analysis of the previous section, two FRFT orders  $p_1, p_2 \in [0, 1]$  with the interval  $\Delta = 0.1$  are adopted when analyzing the statistical probability distribution of FRFT coefficients of natural images. The experimental database  $\{I_i\}$  consists of 96 commonly used  $128 \times 128$  gray images, including Lena, Baboon etc. All images in  $\{I_i\}$  are sequentially transformed with FRFT for different orders and the FRFT coefficients with the same orders  $(p_1^i, p_2^i)$ , although from the different images, are stored together forming a data set for the statistical analysis.

### 2.1 Statistical Distribution of the Real and Imaginary Parts

Figure 1 shows the histograms for the real part  $R = \text{real}(X_{p_1,p_2}(u, v))$  and the imaginary part  $R = \text{imag}(X_{p_1,p_2}(u, v))$  of the FRFT coefficients of natural images along with the fitted generalized Gaussian distribution (GGD).



**Fig. 1.** Statistical histograms for the FRFT coefficients of natural images and the fitted GGD curves

The probability density function (pdf) of GGD can be defined as the follows:

$$f_{\text{GGD}}(x|\alpha, \beta) = \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-(|x|/\alpha)^\beta} \tag{7}$$

where  $|\cdot|$  is the modulo operation,  $\Gamma(\cdot)$  is the Gamma function, and the real parameter  $\alpha$  is proportional to the peak width of pdf, while the real parameter  $\beta$  is inversely proportional to the peak rate of decline. Generally,  $\alpha$  is called scale parameter and  $\beta$  is called shape parameter. Moreover, when  $\beta = 2$ , GGD becomes the Gaussian distribution; when  $\beta = 1$ , GGD is the Laplace distribution. Sharifi and Leon-Garcia [8] had given the moment estimation of the two parameters in GGD:

$$\alpha = \frac{\sigma}{\sqrt{\Gamma(3/\beta)/\Gamma(1/\beta)}} \tag{8}$$

$$\beta = F^{-1}(m/\sigma) \tag{9}$$

where  $m = \frac{1}{N} \sum_{i=1}^N |x_i|$  is the mean of the sample modulus,  $\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N x_i^2}$  is the standard deviation of samples, and  $F^{-1}(\cdot)$  is the inverse function of  $F(x) = \Gamma(2/x)/\sqrt{\Gamma(1/x) \cdot \Gamma(3/x)}$ .

Although Fig. 1 seems clear that the GGD can accurately models the statistical distribution of  $R$  and  $I$ , this paper will further provide the quantitative analysis of fitting

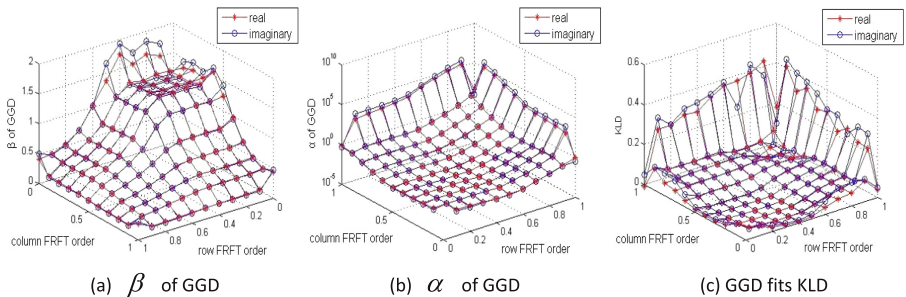
by the Kullback-Leibler divergence (KLD). KLD is a measure of the asymmetry between two discrete distributions with probabilities  $P$  and  $Q$ :

$$KLD(P, Q) = \sum_{i=1}^M P_i \ln \frac{P_i}{Q_i} \tag{10}$$

where  $M$  is the number of the sample subspaces,  $P_i$  and  $Q_i$  represent the probabilities of the sample subspace  $i$  of  $P$  and  $Q$ ,  $\ln(\cdot)$  is the natural logarithm. Equation (9) makes sense only if  $P_i > 0$  and  $Q_i > 0$ , else  $P_i \ln(P_i/Q_i) = 0$ . If  $P = Q$ , then  $KLD(P, Q) = 0$ . Namely, the smaller the KLD, the higher of the similarity between  $P$  and  $Q$ .

The  $\alpha$ ,  $\beta$  and KLD of the GGD fitting curve of the real and imaginary parts of FRFT coefficients with different orders of natural images are shown as the Fig. 2, from which it can be seen that:

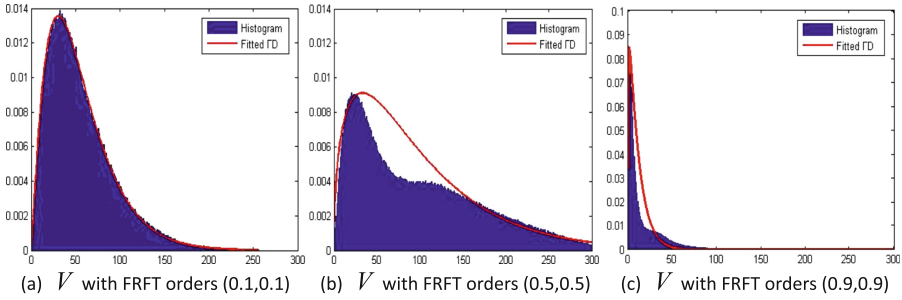
- The real and imaginary parts of the FRFT coefficients of natural images are all similar to the GGD distribution. As the FRFT orders increase, the parameter  $\alpha$  increases, while the shape parameter  $\beta$  decreases. Moreover,  $\alpha$  and  $\beta$  are symmetric about the diagonal of FRFT orders, that is, the real and imaginary parts of FRFT coefficients with the orders  $(i, j)$  and  $(j, i)$  are similar to the same GGD distribution.
- The GGD shape parameters  $\beta$  of the real and imaginary parts of coefficients with different FRFT orders are all less than 2, namely, the statistical probability distributions of the real and imaginary parts of FRFT coefficients of natural images neither the Gaussian nor fixing on the Laplace.
- According to the KLD, when the FRFT order tends to 1, the fitting degree of the GGD curve, which is obtained by Sharifi and Leon-Garcia [8], decreases. This is mainly because the FRFT gradually transforms into the FT when its orders tend to 1. Due to the prominent energy aggregation property of FT, more transform coefficients turn to zeros, all the energy focuses on fewer singular values, and the peak of the statistical probability density curve becomes sharper, which is reflected in the GGD parameters  $\alpha$  reduces to 0, while  $\beta$  increases to infinity, as is shown in Fig. 2. At this time, any slight error of the GGD parameters' moment estimation may greatly influence on the curve fitness. However, according to Fig. 1, a remarkable similarity can be found between the statistical distribution of the real and imaginary parts of FRFT coefficients and the GGDs, even when the FRFT orders tend to 1.



**Fig. 2.** The parameters of the fitted GGD curve with different FRFT orders

### 2.2 Statistical Distribution of the Coefficient Amplitudes

Taking the amplitude of FRFT coefficients of natural images as  $V = |X_{p_1, p_2}(u, v)|$ , it can be discovered that the statistical distribution of  $V$  is similar to the Gamma distribution (GD), as is shown in Fig. 3.



**Fig. 3.** Statistical histograms for the FRFT amplitude of natural images and the fitted GD curves

The pdf of GD is defined as the follows:

$$f_{GD}(x|\alpha, \beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \tag{11}$$

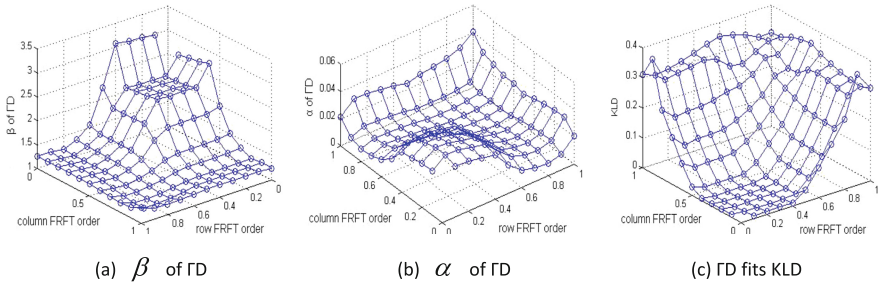
where  $x \geq 0$ , the positive real parameter  $\alpha$  is proportional to the peak width while the positive real parameter  $\beta$  is inversely proportional to the peak rate of decline. Thus,  $\alpha$  is usually called the scale parameter and  $\beta$  is usually called the shape parameter. It is noted that GD becomes exponential distribution when  $\beta = 1$ . The maximum likelihood estimation of two parameters GD is given by Dang and Weerakkody [10]:

$$\alpha = \beta/\bar{x} \tag{12}$$

$$\beta = \frac{1 + \sqrt{1 + 4\log(\bar{x}/\hat{x})/3}}{4\ln(\bar{x}/\hat{x})} \tag{13}$$

where,  $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$  is the arithmetic mean of samples, and  $\hat{x} = (\prod_{i=1}^N x_i)^{1/N}$  is the geometric mean of samples.

As the Fig. 3 shows, when the FRFT orders are small, the statistical distribution of the amplitude of FRFT coefficients of natural images can be well fitted by GD, but as the orders become higher, there are significant differences between them. From the  $\alpha, \beta$  and KLD of the GD fitting curve with difference FRFT orders in Fig. 4, some further conclusions can be drawn as:

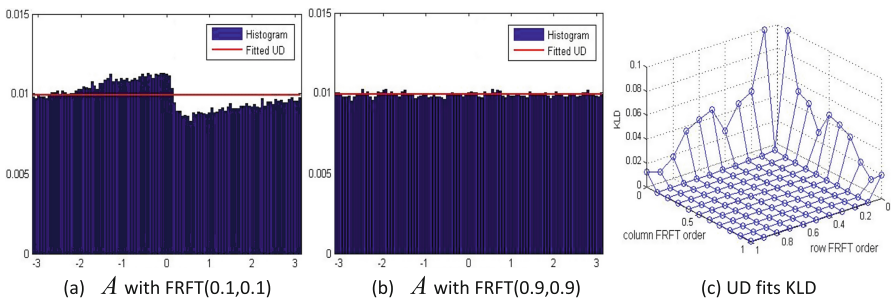


**Fig. 4.** The parameters of the fitted  $\Gamma$ D curve with different FRFT orders

- The probability density of the amplitudes of the FRFT coefficients of natural images is similar to the  $\Gamma$ D and  $\beta > 1$ , which means the amplitudes of the FRFT coefficients of natural images do not follow the exponential distribution.
- With the FRFT orders becoming higher, the  $\beta$  obtained by the maximum likelihood estimation method in Dang and Weerakkody [10] decreases gradually. However, when FRFT orders higher than 0.5,  $\beta < 2$ , and there are significant differences between the statistical probability distribution of the amplitudes of FRFT coefficients of natural images and the  $\Gamma$ D fitting curve, just as Fig. 4(c) shows. Therefore, it's necessary to carry out further research on getting the fitting curve of the FRFT amplitudes with high orders of natural images.

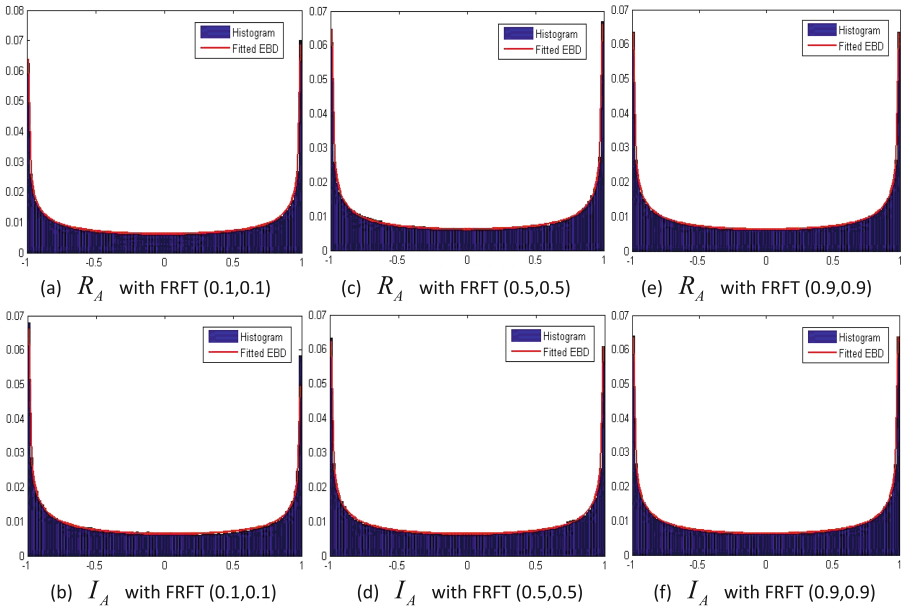
### 2.3 Statistical Distribution of the Coefficient Phase

Taking the phase angles of the FRFT coefficients of natural images as  $A = \arg tg(R/I)$ , the statistical probability distribution of  $A$  gradually tends to the uniform distribution (UD) with the rising of the FRFT orders (Fig. 5).



**Fig. 5.** Statistical histograms for the FRFT phase angles of natural images and its fitted UD curves

For the further study on the statistical probability distribution of the phases of FRFT coefficients of natural images, we takes the phases of FRFT coefficients as  $A' = X_{p_1, p_2}(u, v)/V$ . Then we can find that, the real part  $R_A = real(A')$  and imaginary part  $I_A = imag(A')$  of the complex number  $A'$  are all similar to the beta distribution (BD) with the domain of  $(-1,1)$ , as is shown in Fig. 6.



**Fig. 6.** Statistical histograms for the FRFT phases of natural images and its fitted BD curve.

The traditional BD is a two- parameters probability distribution with the domain of  $(0,1)$ , and its pdf is:

$$f_{BD}(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1} \tag{14}$$

where  $0 < x < 1$ ,  $\alpha$  and  $\beta$  are positive real numbers. According to the definition of traditional BD, the pdf of the extended BD with the domain of  $(-1,1)$  can be obtained as \*appendix for derivation:

$$f'_{BD}(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot 2^{1-(\alpha+\beta)} (1 + x)^{\alpha-1} (1 - x)^{\beta-1} \tag{15}$$

where  $-1 < x < 1$ . And according to the  $f'_{BD}(x|\alpha, \beta)$ , the first moment  $m_1$  and the second moment  $m_2$  of the extended BD can be obtained respectively as:



$$m_1 = E(x) = \int_{-1}^1 x \cdot f_{BD}(x|\alpha, \beta) dx = \frac{\alpha - \beta}{\alpha + \beta} \quad (16)$$

$$\begin{aligned} m_2 = E(x^2) &= \int_{-1}^1 x^2 \cdot f_{BD}(x|\alpha, \beta) dx \\ &= 1 - \frac{4\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)} \end{aligned} \quad (17)$$

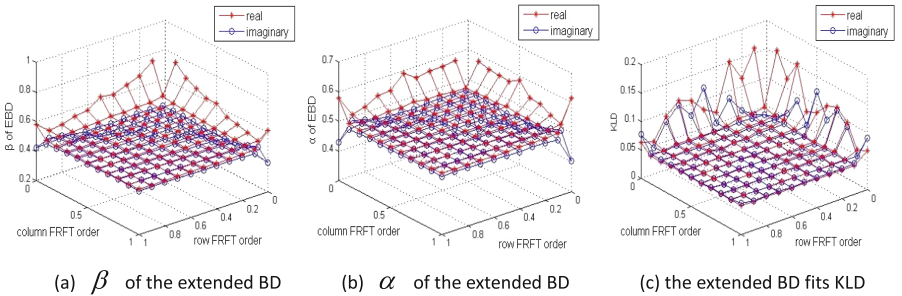
And we can derive the two parameters  $\alpha$  and  $\beta$  of the extended BD from (15) and (16) as:

$$\alpha = \frac{1 + m_1}{1 - m_1} \beta \quad (18)$$

$$\beta = \frac{1 - m_1 - m_2}{2(m_1^2 + m_2)} \quad (19)$$

The  $\alpha\beta$  and KLD of the extended BD fitting curve of the real and imaginary parts of FRFT phases with different orders of natural images are listed in Fig. 7, and some further conclusions can be drawn as:

- The real and imaginary parts of the FRFT phases of natural images are all similar to the extended BD distribution. If the FRFT orders are non-zero, the  $\alpha$  and  $\beta$  have no significant differences as the transformation order changing.
- The parameters  $\alpha$  and  $\beta$  of the extended BD of the real and imaginary parts of FRFT phases of natural images are approximately equal to 0.5, and furthermore, they are symmetrical about 0.5 in a certain sense: the extended BD parameters of the real part ( $\alpha_r, \beta_r$ ) and the extended BD parameters of the imaginary part ( $\alpha_i, \beta_i$ ) with the same FRFT orders satisfy  $\alpha_r - 0.5 \approx 0.5 - \alpha_i$  and  $\beta_r - 0.5 \approx 0.5 - \beta_i$ .
- According to the KLD, if there is zero in the FRFT orders, the fitting degree between the extended BD and the statistical distribution of the real and imaginary parts of FRFT phases of natural images may decrease. This is mainly because the natural images data remains unchanged when the FRFT orders are zero, which means the data is still the original real data and the phase equals zero.



**Fig. 7.** The parameters of the fitted BD curve with different FRFT orders.

### 3 Conclusion

In this paper, we have studied the statistical distribution of FRFT coefficients of natural images and proposed that the real and imaginary parts of FRFT coefficients take on the generalized Gaussian distribution, the FRFT modulus follow the gamma distribution and the FRFT phase angles tend to the uniform distribution, moreover, the real and imaginary parts of FRFT phases meet the extended beta distribution with the definition domain of  $(-1,1)$ . These underlying statistics rules are expected to provide theoretical basis for the image processing in FRFT, such as dimensionality reduction, feature extraction, smooth de-noising, digital forensics and watermarking etc. These will be the key of our future work.

### Appendix

The extended beta distribution with the definition domain of  $(-1,1)$ .

The traditional beta distribution (BD) originated from such a question: sampling  $n$  times for the random variable  $X$  uniformly distributed on  $(0,1)$ , then the distribution of the  $k$ th largest  $x_k$  is BD and its probability density function is as the follows:

$$f_{BD}(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1 \tag{20}$$

where  $\alpha = k$  and  $\beta = n - k + 1$ . And then, we can extended the definition domain of BD to  $(-1,1)$ , which is to find the distribution of the  $k$ th largest  $x_k$  in the  $n$  times sampling of the random variable  $X$  uniformly distributed on  $(-1,1)$ .

Let  $x_k \in (x, x + \Delta x)$ , then its probability is:

$$\begin{aligned} P(x < x_k \leq x + \Delta x) &= C_n^1 \cdot C_n^{k-1} \cdot \left[ \frac{x - (-1)}{2} \right]^{k-1} \cdot \frac{x + \Delta x - x}{2} \cdot \left[ \frac{1 - (x + \Delta x)}{2} \right]^{n-k} \\ &= \frac{n!}{(k-1)!(n-k)!} \cdot 2^{-n} \cdot (x+1)^{k-1} \cdot (1-x)^{n-k} \Delta x + o(\Delta x) \end{aligned} \tag{21}$$

where  $o(\Delta x)$  is the infinitesimal of  $\Delta x$ , then the probability density of  $x_k$  is:

$$f(x_k) = \lim_{\Delta x \rightarrow 0} \frac{P(x < x_k \leq x + \Delta x)}{\Delta x} = \frac{n!}{(k-1)!(n-k)!} \cdot 2^{-n} \cdot (x+1)^{k-1} \cdot (1-x)^{n-k} \tag{22}$$

For the known gamma function  $\Gamma(n) = (n-1)!$ , let  $\alpha = k, \beta = n - k + 1$ , then the above equation can be expressed as:

$$f(x_k) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot 2^{1-(\alpha+\beta)}(1+x)^{\alpha-1}(1-x)^{\beta-1} \quad (23)$$

That is, the pdf of the extended BD defined on  $(-1,1)$  is as the follows:

$$f'_{\text{BD}}(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot 2^{1-(\alpha+\beta)}(1+x)^{\alpha-1}(1-x)^{\beta-1} \quad (24)$$

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