

Chapter 4

Signs of Market Orders and Human Dynamics

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Abstract A time series of signs of market orders was found to exhibit long memory. There are several proposed explanations for the origin of this phenomenon. A cogent one is that investors tend to strategically split their large hidden orders into small pieces before execution to prevent the increase in the trading costs. Several mathematical models have been proposed under this explanation.

In this paper, taking the bursty nature of the human activity patterns into account, we present a new mathematical model of order signs that have a long memory property. In addition, the power law exponent of distribution of a time interval between order executions is supposed to depend on the size of hidden order. More precisely, we introduce a discrete time stochastic process for polymer model, and show it's scaled process converges to a superposition of a Brownian motion and countably infinite number of fractional Brownian motions with Hurst exponents greater than one-half.

4.1 Introduction

Empirical studies [2, 6, 8, 11] on high frequency financial data of stock markets that employ the continuous double auction method have revealed a time series of signs of market orders has long memory property. In contrast, a time series of stock returns is known to have short memory property. A time series of *order signs* is defined by changing transactions at the best ask price into +1 and transactions at the best bid price into -1. The auto-correlation function of the order signs decays as a power law of the lag and the exponent of the decay is less than 1, which is equivalent to a *Hurst exponent* of the time series is greater than one-half.

In this paper, we propose a new mathematical model which takes account of origin of the long memory in order signs. As a first step, we define a discrete time stochastic process of cumulative order signs in accordance with some explanation

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for the origin of the phenomenon. Subsequently, we verify increments of the process has the long memory property. In general, there are three ways to verify a discrete time process has some property. The first one runs computer simulations. The second calculates the distribution of the discrete time process directly. And the third, which we use in this paper, is to show that the scaled discrete time process converges to a continuous process which has that property.

There are various explanations for the origin of the long memory property of order signs [3]. A cogent one, which was proposed by Lillo et al. [9], is that investors tend to strategically split their large *hidden orders* into small pieces before execution to prevent the increase in the trading costs. Empirical findings partially support this explanation. A long memory phenomenon is found in a time series of order signs of transactions initiated by a single *member* of the stock market [3, 8]. Investors enter their orders into the market through one of its members.

Assuming the *size* of hidden orders distributes as a power law, Lillo et al. [9] considered a discrete time mathematical model with this explanation. Under an additional technical assumption that the number of hidden orders is fixed, they showed rigorously the model has a long memory property. However, this technical assumption does not seem natural.

Taking account of the bursty nature of human dynamics [1], Kuroda et al. [7] proposed another theoretical model with this explanation. They assumed that a time interval between order executions distributes as a power law, and that the power law exponent does not depend on the size of hidden order. Under an additional technical assumption that the size of hidden order is bounded above, they showed the scaled discrete time process converges to a superposition of a Brownian motion and a finite number of *fractional Brownian motions* with Hurst exponents greater than one-half. Moreover, the number of hidden orders is not fixed in their model, and it randomly varies. Although, the maximum Hurst exponent of obtained process depends on the largest hidden order.

The Hurst exponent of order signs expected by the theory of splitting large hidden order is smaller than the value of the empirical study [3]. About stocks with high liquidity, the fluctuation of Hurst exponents of order signs is small by a stock and a period [6]. These findings suggest that there might be some other cause about the long memory of order signs. We can pay attention not only to large hidden orders but also to small hidden orders. Vázquez et al. found two universality class in human dynamics [13]. On the other hand, Zhou et al. observed that in an online movie rating site, a power law exponent of the time interval between user's postings depends on user's activity [15].

In this paper, we propose a new mathematical model with an explanation for the origin of long memory of order signs that investors split their hidden order of any size into small pieces before execution. We assumed that the power law exponent of distribution of a time interval between order executions depends on the size of hidden order. We showed the scaled discrete time process converges to a superposition of a Brownian motion and countably infinite number of fractional Brownian motions with Hurst exponents greater than one-half. We note that the number of hidden orders randomly varies, that Hurst exponents are not bounded

above and that the maximum Hurst exponent of obtained process depends on hidden order of medium size.

4.2 Model

In this section, we introduce a probability space (Ω_n, P_n) , where n is a natural number. In the next section, we will define a discrete time stochastic process in time interval $\Lambda_n = \{1, 2, \dots, n\}$ which describes *cumulative order signs* on the probability space. And we will show the increment of the process has a long memory property. We note that in this paper we only study the order sign and do not consider the stock price.

All essential assumptions our model requires for the market is as follows:

- Investors tend to split their hidden orders into small pieces before execution.
- The distribution of a time interval between order executions obeys a power law.
- The power law exponent of the inter-event distribution depends on the size of the hidden order.

A hidden order of one investor in a stock market and execution times of its small pieces is denoted by \mathbf{p} . Namely, \mathbf{p} has two quantities: the order sign $s(\mathbf{p}) = s$ and the set of times of executions $b(\mathbf{p}) = \{u_1, \dots, u_m\}$, where $m \geq 1$ is the number of small pieces of the hidden order split by the investor. We call \mathbf{p} a *polymer* using the terminology of a mathematical method called the *cluster expansion*, which we will use to prove our main theorem. A method of the cluster expansion is developed in the study of the statistical physics and is applied for instance to convergence theorems of the phase separation line of the two dimensional Ising model [4, 12]. Since the cluster expansion is defined in an abstract setting [5], it can be applied to a financial model [7, 14].

It is known that a time series of trading volume in a stock market exhibits long memory [10]. However, we do not consider the memory of the trading volume in this paper; we suppose the volume of each piece is 1 just for the sake of simplicity, and we emphasize it is not technical assumption. Consequently, the number m of small pieces is equivalent to the size (or the total volume) of the hidden order. For any polymer \mathbf{p} , we can also regard m as the amount of *activity* of a investor in time period of her holding the polymer. Meanwhile, investors often do not split their own orders and submit it in a stock market at once. This situation is also included in our model as $m = 1$. Although the model proposed by Kuroda et al. [7] assumed that m is bounded above, that is, the maximum value of m is finite, our model does not require any restriction on the maximum value of m . More precisely, m has an upper bound $\log \log n$, and n tends to infinity in our main theorem.

The order sign $s(\mathbf{p})$ is assigned to $+1$ or -1 according to whether the hidden order is a buy order or a sell order. For any polymer \mathbf{p} , its order sign $s(\mathbf{p})$ is a single value. Obviously, different valued order signs are possibly assigned to different polymers possessed by one investor. In our model, the distribution of order signs

is symmetry:

$$P_n(s(\mathbf{p}) = +1) = P_n(s(\mathbf{p}) = -1) = \frac{1}{2}. \quad (4.1)$$

Each element of the set of times of executions $b(\mathbf{p}) = \{u_1, \dots, u_m\}$ is an integer. Their magnitude relation is given as $u_1 < u_2 < \dots < u_m$, that is, the first piece of the hidden order is executed at u_1 , and the last one is executed at u_m . For any distinct two polymers \mathbf{p}_1 and \mathbf{p}_2 , their execution times do not overlap:

$$b(\mathbf{p}_1) \cap b(\mathbf{p}_2) = \emptyset. \quad (4.2)$$

Since we will observe the discrete time stochastic process of cumulative order signs in time interval Λ_n , it is enough to consider only polymers \mathbf{p} which satisfy

$$b(\mathbf{p}) \cap \Lambda_n \neq \emptyset. \quad (4.3)$$

Taking the bursty nature of the human activity patterns into account, we assume that the distribution of a time interval between order executions obeys a power law and that its exponents $\alpha(m)$ depends on the size (or the activity) of the polymer:

$$P_n(u_i, u_{i-1} \in b(\mathbf{p}), \mathbf{p} \text{ is a polymer of size } m) \propto (u_i - u_{i-1})^{-\alpha(m)}. \quad (4.4)$$

According to an empirical study on human dynamics [15], power law exponents of the inter-event times are increasing in a parameter of activity. Hence, it suggests that the exponent $\alpha(m)$ is increasing in m . Our model does not require that the exponent is increasing, though it requires some condition on the exponent.

In the following, we define our model using the mathematical terminology. Let n be a natural number and $\Lambda_n = \{1, 2, \dots, n\}$ be an observation time of a discrete time process. We describe a hidden order and its execution times of small pieces by a polymer.

Let $\mathcal{P}_{n,1}$ be the set of all polymers corresponds to hidden order of size 1:

$$\mathcal{P}_{n,1} = \{\mathbf{p} = (s, u); s \in \{+1, -1\}, u \in \Lambda_n\}. \quad (4.5)$$

For each polymer $\mathbf{p} = (s, u) \in \mathcal{P}_{n,1}$, we denote the order sign by $s(\mathbf{p}) = s$, the time of execution by $b(\mathbf{p}) = \{u\}$ and the size of hidden order by $|\mathbf{p}| = 1$.

For any m , ($2 \leq m \leq \log \log n$), we define the set of all polymers corresponds to hidden order of size m by

$$\begin{aligned} \mathcal{P}_{n,m} = \{ & \mathbf{p} = (s, u_1, \dots, u_m); s \in \{+1, -1\}, \{u_1, \dots, u_m\} \cap \Lambda_n \neq \emptyset, \\ & 1 \leq u_i - u_{i-1} \leq n \quad (i = 2, \dots, m)\}. \end{aligned}$$

For each polymer $\mathbf{p} = (s, u_1, \dots, u_m) \in \mathcal{P}_{n,m}$, we denote the order sign by $s(\mathbf{p}) = s$, the set of times of executions by $b(\mathbf{p}) = \{u_1, \dots, u_m\}$ and the size of hidden order by $|\mathbf{p}| = m$. The set of all polymers is denoted by

$$\mathcal{P}_n = \bigcup_{m=1}^{\log \log n} \mathcal{P}_{n,m}. \tag{4.6}$$

The configuration space is denoted by

$$\Omega_n = \bigcup_{k=1}^n \{ \omega = \{ \mathbf{p}_1, \dots, \mathbf{p}_k \} \subset \mathcal{P}_n; b(\mathbf{p}_i) \cap b(\mathbf{p}_j) = \emptyset, (1 \leq i < j \leq k) \}. \tag{4.7}$$

Example 1 We consider the case that $n = 10$ and $\omega = \{ \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \} \in \Omega_{10}$. Each polymer has order signs and the set of execution times as follows:

Polymer	Order sign	The set of execution times	Size
\mathbf{p}_1	$s(\mathbf{p}_1) = -1$	$b(\mathbf{p}_1) = \{-1, 3\}$	$m = 2$
\mathbf{p}_2	$s(\mathbf{p}_2) = +1$	$b(\mathbf{p}_2) = \{2, 5, 6\}$	$m = 3$
\mathbf{p}_3	$s(\mathbf{p}_3) = -1$	$b(\mathbf{p}_3) = \{4, 9, 11, 13\}$	$m = 4$
\mathbf{p}_4	$s(\mathbf{p}_4) = +1$	$b(\mathbf{p}_4) = \{8\}$	$m = 1$

We note that the sets of execution times intersect with Λ_{10} and do not intersect each other (see Fig. 4.1).

The power law exponent $\alpha(m)$ of distribution of a time interval between order executions depends on the size m of hidden order, and satisfies

$$1 - \frac{1}{m-1} < \alpha(m) < 1 - \frac{1}{m} \left(\frac{3}{4} \right)^m. \tag{4.8}$$

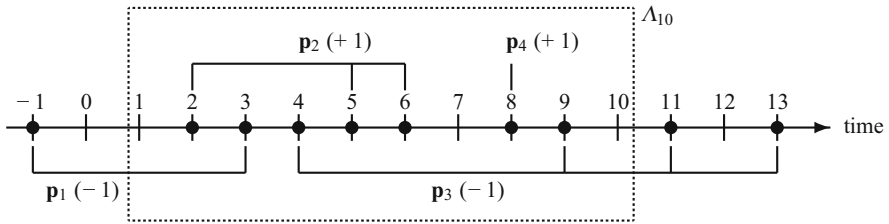


Fig. 4.1 Configuration of Example 1. The configuration consists of four polymers \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 and \mathbf{p}_4 . Numbers +1 or -1 in parentheses are order signs of polymers. *Black circles* are times of executions. A discrete time stochastic process of cumulative order signs will be defined in time interval Λ_{10}

A probability intensity function of a polymer $\mathbf{p} \in \mathcal{P}_n$ is given by

$$\varphi(\mathbf{p}) = \begin{cases} d(n, 1) & (\mathbf{p} = (s, u) \in \mathcal{P}_{n,1}) \\ d(n, m) \prod_{i=2}^m (u_i - u_{i-1})^{-\alpha(m)} & (\mathbf{p} = (s, u_1, \dots, u_m) \in \mathcal{P}_{n,m}, \\ & 2 \leq m \leq \log \log n), \end{cases} \quad (4.9)$$

where scale factors $d(n, 1)$ and $d(n, m)$ are given by

$$d(n, 1) = c (\log n)^{-4},$$

$$d(n, m) = d(n, 1) \left\{ \frac{c \cdot (1 - \alpha(m))}{e \cdot n^{1-\alpha(m)}} \right\}^{m-1},$$

and c ($0 < c < 1$) is a constant. We define a probability measure on Ω_n by

$$P_n(\omega) = \frac{1}{\mathcal{E}_n} \prod_{\mathbf{p} \in \omega} \varphi(\mathbf{p}), \quad (\omega \in \Omega_n),$$

where $\mathcal{E}_n = \sum_{\omega \in \Omega_n} \prod_{\mathbf{p} \in \omega} \varphi(\mathbf{p})$ is a normalization constant.

4.3 Main Theorem

We define a discrete time stochastic process of cumulative order signs by

$$S_u(\omega) = \sum_{\mathbf{p} \in \omega} s(\mathbf{p}) \sum_{v=1}^u 1_{\{v \in b(\mathbf{p})\}}, \quad (u \in \Lambda_n, \omega \in \Omega_n). \quad (4.10)$$

We note that the increment $S_u(\omega) - S_{u-1}(\omega)$ of the process is order signs. In order to verify that the increment exhibits long memory, we show that the scaled process of the discrete time process converges to a continuous time stochastic process, the increment of which has a long memory property.

Example 2 Let $n = 10$ and the configuration $\omega = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ be the same one given in Example 1. The discrete time stochastic process is as follows:

Time u	1	2	3	4	5	6	7	8	9	10
Order sign		+1	-1	-1	+1	+1		+1	-1	
$S_u(\omega)$	0	+1	0	-1	0	+1	+1	+2	+1	+1

A scaled process of $S_u(\omega)$ is given by

$$X_t^{(n)}(\omega) = \frac{1}{c(n)} \sum_{\mathbf{p} \in \omega} S_{[nt]}(\mathbf{p}), \quad (0 \leq t \leq 1, \omega \in \Omega_n), \quad (4.11)$$

where $c(n) = \sqrt{n \cdot d(n, 1)} = \sqrt{cn}^{1/2} (\log n)^{-2}$ is a scale function, and $[nt]$ indicates the greatest integer less than or equal to nt .

Theorem 1 *The distribution of $X_t^{(n)}$ weakly converges to the distribution of*

$$X_t = \sqrt{c_1} B_t + \sum_{m=2}^{\infty} \sum_{\ell=1}^{m-1} \sqrt{c_2(m, \ell)} B_t^{H_{m, \ell}}, \quad (0 \leq t \leq 1), \quad (4.12)$$

where

$$c_1 = 2 \sum_{m=1}^{\infty} m \left(\frac{c}{e}\right)^{m-1},$$

$$c_2(m, \ell) = \frac{4(1 - \alpha(m))^{\ell-1} (m - \ell) B_{\ell}(\alpha(m))}{\ell \{(1 - \alpha(m)) \ell + 1\}} \left(\frac{c}{e}\right)^{m-1},$$

$$B_{\ell}(\alpha(m)) = \frac{\Gamma(1 - \alpha(m))^{\ell}}{\Gamma(\ell(1 - \alpha(m)))},$$

B_t is a standard Brownian motion, $B_t^{H_{m, \ell}}$ is a fractional Brownian motion with Hurst exponent

$$H_{m, \ell} = \frac{1}{2} \{(1 - \alpha(m)) \ell + 1\} \quad (4.13)$$

and $\{B_t, B_t^{H_{m, \ell}}; m \geq 2, 1 \leq \ell \leq m - 1\}$ are independent.

Remark 1 For any $m \geq 2$ and $1 \leq \ell \leq m - 1$, it follows from the condition

$$1 - \frac{1}{m-1} < \alpha(m) \quad (4.14)$$

that $H_{m, \ell} < 1$. And it follows from the condition

$$\alpha(m) < 1 - \frac{1}{m} \left(\frac{3}{4}\right)^m \quad (4.15)$$

that

$$\sum_{m=2}^{\infty} \sum_{\ell=1}^{m-1} c_2(m, \ell) < \infty. \quad (4.16)$$

Remark 2 Let us consider a continuous time stochastic process Z_t of a superposition of two independent (fractional) Brownian motions B_t^H and $B_t^{\tilde{H}}$ with Hurst exponents $\frac{1}{2} \leq H < \tilde{H} < 1$:

$$Z_t = aB_t^H + \tilde{a}B_t^{\tilde{H}} \quad (4.17)$$

where a, \tilde{a} are constants. We note that when $H = \frac{1}{2}$, since $B_t^{1/2}$ is a Brownian motion, the process Z_t is a superposition of a Brownian motion and a fractional Brownian motion. We define an increment of the process by $\Delta Z_t = Z_t - Z_{t-1}$. Since $E[\Delta Z_t] = 0$ and $\text{Var}(\Delta Z_t) = a^2 + \tilde{a}^2$, the auto-correlation function of the increment is

$$\begin{aligned} \rho_Z(\tau) &= \frac{E[\Delta Z_t \Delta Z_{t+\tau}]}{a^2 + \tilde{a}^2} = \frac{a^2}{a^2 + \tilde{a}^2} E[\Delta B_t^H \Delta B_{t+\tau}^H] + \frac{\tilde{a}^2}{a^2 + \tilde{a}^2} E[\Delta B_t^{\tilde{H}} \Delta B_{t+\tau}^{\tilde{H}}] \\ &\sim \frac{\tilde{a}^2}{a^2 + \tilde{a}^2} \tilde{H}(2\tilde{H} - 1) \tau^{2\tilde{H}-2} \quad (\tau \rightarrow \infty). \end{aligned} \quad (4.18)$$

Hence, we see that the Hurst exponent of Z_t is $\tilde{H} = \max\{H, \tilde{H}\}$. In a similar way, it can be verified that the Hurst exponent of the process X_t in Theorem 1 is

$$\begin{aligned} H_{\max} &= \max\{H_{m,\ell} ; m \geq 2, 1 \leq \ell \leq m-1\} \\ &= \max_{m \geq 2} \frac{1}{2} \{(1 - \alpha(m))(m-1) + 1\}. \end{aligned} \quad (4.19)$$

Remark 3 In the model of Kuroda et al. [7], since they assume that the exponent of inter-event time distribution is a constant $\alpha(m) = \alpha$, they need to put a limitation on the size of hidden order: $m \leq m_{\max}$ where m_{\max} is a positive number. Then, they derive finite number of fractional Brownian motions. And the maximum Hurst exponents H_{\max} is attained by the largest size m_{\max} of hidden orders.

In our model, we set no limitation on the size m of hidden orders. As a result, we derive countably infinite number of fractional Brownian motions. On the other hand in the empirical study the size of the hidden order has some limitation, and the upper bound of the size possibly depends on markets or stocks. A finite number of fractional Brownian motions appears in the case.

If exponents $\alpha(m)$ is increasing in size m [15], then

$$\frac{1}{2} \{(1 - \alpha(m))(m-1) + 1\} \quad (4.20)$$

is not monotone function in m . Hence the maximum Hurst exponents in (4.19) is attained by middle size m^* of hidden orders.

4.4 Outline of the Proof of Theorem 1

In order to prove the main theorem, we show a convergence of a finite dimensional distribution, and show the tightness. In this section, we give an outline of the proof of Theorem 1. The detail of the proof is complicated. The interested reader is referred to Kuroda et al. [7].

For any $0 < t_1 < \dots < t_k \leq 1$ and any $\mathbf{z} = (z_1, \dots, z_k) \in \mathbf{R}^k$, we define

$$Y^{(n)}(\omega) = Y_{\mathbf{t}, \mathbf{z}}^{(n)}(\omega) = \prod_{\mathbf{p} \in \omega} \sum_{i=1}^k z_i X_{t_i}^{(n)}(\mathbf{p}), \quad (\omega \in \Omega_n). \quad (4.21)$$

Its characteristic function is denoted by

$$\phi_{\mathbf{t}}^{(n)}(\mathbf{z}) = E_n \left[e^{\sqrt{-1} Y^{(n)}} \right] \quad (4.22)$$

Using the method of the cluster expansion [5, 12], we have

$$\log \phi_{\mathbf{t}}^{(n)}(\mathbf{z}) = \sum_{A \in \mathcal{A}_n} \left(e^{\sqrt{-1} Y^{(n)}(A)} - 1 \right) \varphi(A) \frac{\alpha^T(A)}{A!} \quad (4.23)$$

where $\mathcal{A}_n = \{A : \mathcal{P}_n \rightarrow \{0, 1, 2, \dots\}\}$, and for any $A \in \mathcal{A}_n$, $A! = \prod_{\mathbf{p} \in \mathcal{P}_n} A(\mathbf{p})!$,

$$\begin{aligned} Y^{(n)}(A) &= \sum_{\mathbf{p} \in \mathcal{P}_n} Y^{(n)}(\mathbf{p}) A(\mathbf{p}) = \sum_{\mathbf{p} \in \mathcal{P}_n} \sum_{i=1}^k z_i X_{t_i}^{(n)}(\mathbf{p}) A(\mathbf{p}), \\ \varphi(A) &= \prod_{\mathbf{p} \in \mathcal{P}_n} \varphi(\mathbf{p})^{A(\mathbf{p})}, \\ \alpha(A) &= \begin{cases} 1 & A! = 1 \text{ and } \text{supp}(A) \in \Omega_n \\ 0 & \text{o.w.}, \end{cases} \end{aligned}$$

$\text{supp}(A) = \{\mathbf{p}\}$ and $\alpha^T(A) = \text{Log } \alpha(A)$. Applying the Taylor's expansion, we obtain

$$\log \phi_{\mathbf{t}}^{(n)}(\mathbf{z}) = \sqrt{-1} I_1(n) - \frac{1}{2} \left\{ I_2(n) + \hat{I}_2(n) \right\} - \frac{\sqrt{-1}}{3!} \left\{ I_3(n) + \hat{I}_3(n) \right\} \quad (4.24)$$

where

$$I_1(n) = \sum_{A \in \mathcal{A}_n} Y^{(n)}(A) \varphi(A) \frac{\alpha^T(A)}{A!},$$

$$\begin{aligned}
I_2(n) &= \sum_{\mathbf{p} \in \mathcal{P}_n} \{Y^{(n)}(\mathbf{p})\}^2 \varphi(\mathbf{p}), & \hat{I}_2(n) &= \sum_{A \in \mathcal{A}_n, |A| \geq 2} \{Y^{(n)}(A)\}^2 \varphi(A) \frac{\alpha^T(A)}{A!}, \\
I_3(n) &= \sum_{\mathbf{p} \in \mathcal{P}_n} \{Y^{(n)}(\mathbf{p})\}^3 e^{\sqrt{-1}\theta Y^{(n)}(\mathbf{p})} \varphi(\mathbf{p}), \\
\hat{I}_3(n) &= \sum_{A \in \mathcal{A}_n, |A| \geq 2} \{Y^{(n)}(A)\}^3 e^{\sqrt{-1}\theta Y^{(n)}(A)} \varphi(A) \frac{\alpha^T(A)}{A!}
\end{aligned}$$

for some $\theta \in (0, 1)$ and $|A| = \sum_{\mathbf{p} \in \mathcal{P}_n} A(\mathbf{p})$ for any $A \in \mathcal{A}_n$. From the symmetry property of the model, we have

$$I_1(n) = 0. \quad (4.25)$$

It is easy to see that

$$\lim_{n \rightarrow \infty} I_3(n) = 0. \quad (4.26)$$

It follows from the theory of Kotecký and Preiss [5] and the Cauchy formula that

$$\lim_{n \rightarrow \infty} \hat{I}_2(n) = 0, \quad \lim_{n \rightarrow \infty} \hat{I}_3(n) = 0. \quad (4.27)$$

It can be seen that

$$\begin{aligned}
\lim_{n \rightarrow \infty} I_2(n) &= c_1 \sum_{i=1}^k \sum_{j=1}^k z_i z_j \min\{t_i, t_j\} \\
&\quad + \sum_{i=1}^k \sum_{j=1}^k z_i z_j \sum_{m=2}^{\infty} \sum_{\ell=1}^{m-1} c_2(m, \ell) \frac{1}{2} \left\{ t_i^{H_{m,\ell}} + t_j^{H_{m,\ell}} - |t_i - t_j|^{H_{m,\ell}} \right\}.
\end{aligned}$$

Hence we have shown a convergence of the finite dimensional distribution.

Using Pfister's lemma (Lemma 3.5 in [12]), it can be shown that there are a positive constant $c_3 > 0$ and a positive number $n_0 \in \mathbf{N}$ such that for any $n \geq n_0$ and any $0 \leq r \leq s \leq t \leq 1$,

$$E_n \left[\left(X_s^{(n)} - X_r^{(n)} \right)^2 \left(X_t^{(n)} - X_s^{(n)} \right)^2 \right] \leq c_3 (t - r)^2. \quad (4.28)$$

Hence we have shown the tightness condition. Therefore we complete the proof of Theorem 1.

4.5 Conclusion

Using a method of the cluster expansion developed in the study of the statistical physics, we introduced a new mathematical model with the explanation for the origin of long memory of order signs that investors split their hidden order of any size into small pieces before execution. The power law exponent of distribution of a time interval between order executions was supposed to depend on the size of hidden order. The limit process of the scaled discrete time process was found to be a superposition of a Brownian motion and countably infinite number of fractional Brownian motions with Hurst exponents greater than one-half. Namely, increments of the limit process have a long memory property. The maximum Hurst exponent of obtained process was described as

$$H_{\max} = \max_{m \geq 2} \frac{1}{2} \{ (1 - \alpha(m)) (m - 1) + 1 \}. \quad (4.29)$$

The power law exponent $\alpha(m)$ of distribution of a time interval between order executions was supposed to be increasing, cf. [15]. Thus, investors having a hidden order of medium size m^* , which attains the maximum in (4.29), have an influence on the Hurst exponent of order signs. It should be noted that in the empirical study, the power law exponent of the auto-correlation function $\rho(\ell)$ of order signs is determined by middle region of lag ℓ . Hence, investors who have an influence on the Hurst exponent of order signs depends on size m of their hidden order, the power law exponent $\alpha(m)$ and the distribution of investors.

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