Chapter 3
Circular Accelerators

Parallel with the development of electrostatic and linear rf accelerators the potential of circular accelerators was recognized and a number of ideas for such accelerators have been developed over the years. Technical limitations for linear accelerators encountered in the early 1920s to produce high-power rf waves stimulated the search for alternative accelerating methods or ideas for accelerators that would use whatever little rf fields could be produced as efficiently as possible.

Interest in circular accelerators quickly moved up to the forefront of accelerator design and during the 1930s made it possible to accelerate charged particles to many million electron volts. Only the invention of the rf klystron by the Varian brothers at Stanford in 1937 gave the development of linear accelerators the necessary boost to reach par with circular accelerators again. Since then both types of accelerators have been developed further and neither type has yet outperformed the other. In fact, both types have very specific advantages and disadvantages and it is mainly the application that dictates the use of one or the other.

Circular accelerators are based on the use of magnetic fields to guide the charged particles along a closed orbit. The acceleration in all circular accelerators but the betatron is effected in one or few accelerating cavities which are traversed by the particle beam many times during their orbiting motion. This greatly simplifies the rf system compared to the large number of energy sources and accelerating sections required in a linear accelerator. While this approach seemed at first like the perfect solution to produce high energy particle beams, its progress soon became limited for the acceleration of electrons by copious production of synchrotron radiation.

The simplicity of circular accelerators and the absence of significant synchrotron radiation for protons and heavier particles like ions has made circular accelerators the most successful and affordable principle to reach the highest possible proton energies for fundamental research in high energy physics. Protons are being accelerated into the TeV range in the Large Hadron Collider (LHC) at CERN in Geneva, Switzerland [1].

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For electrons the principle of circular accelerators has reached a technical and economic limit at about 28 GeV [2] due to synchrotron radiation losses, which make it increasingly harder to accelerate electrons to higher energies [3]. Further progress in the attempt to reach higher electron energies is being pursued through the principle of linear colliders [4, 5], where synchrotron radiation is avoided.

Many applications for accelerated particle beams, however, exist at significantly lower energies and a multitude of well developed principles of particle acceleration are available to satisfy those needs. We will discuss only the basic principles behind most of these low- and medium-energy accelerators in this text and concentrate in more detail on the beam physics in synchrotrons and storage rings. Well documented literature exists for smaller accelerators and the interested reader is referred to the bibliography at the end of this text.

3.1 Betatron

The first "circular electron accelerator" has been invented and developed a hundred years ago in the form of an electrical current transformer. Here we find the electrons in the wire of a secondary coil accelerated by an electro motive force generated by a time varying magnetic flux through the area enclosed by the secondary coil. This idea was picked up independently by several researchers [6, 7]. Wideroe finally recognized the importance of a fixed orbit radius and formulated the Wideroe $\frac{1}{2}$-condition, which is a necessary although not sufficient condition for the successful operation of a beam transformer or betatron as it was later called, because it functions optimally only for the acceleration of beta rays or electrons [8].

The betatron makes use of the transformer principle, where the secondary coil is replaced by an electron beam circulating in a closed doughnut shaped vacuum chamber. A time-varying magnetic field is enclosed by the electron orbit and the electrons gain an energy in each turn which is equal to the electro-motive force generated by the varying magnetic field. The principle arrangement of the basic components of a betatron are shown in Fig. 3.1.

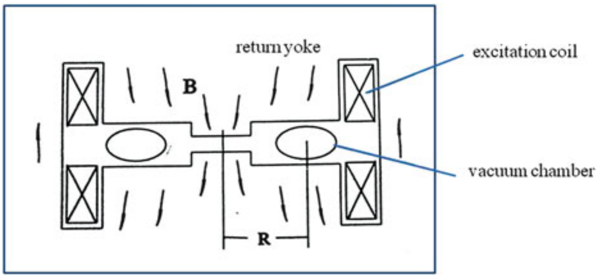

Fig. 3.1 The principle of acceleration in a betatron (schematic)

The accelerating field is determined by integrating Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{3.1}$$

and utilizing Stokes's theorem, we obtain the energy gain per turn

$$\oint \mathbf{E}\mathrm{d}\mathbf{s} = -\frac{\partial \Phi}{\partial t}\,, \tag{3.2}$$

where Φ is the magnetic flux enclosed by the integration path, which is identical to the design orbit of the beam. The particles follow a circular path under the influence of the Lorentz force in a uniform magnetic field. We use a cylindrical coordinate system (r, φ, y), where the particles move with the coordinate φ clockwise along the orbit. From (1.74) we get for the particle momentum

$$cp = \gamma cmv = ecrB_\perp. \tag{3.3}$$

The accelerating force is equal to the rate of change of the particle momentum and can be obtained from the time derivative of (3.3). This force must be proportional to the azimuthal electric field component E_φ on the orbit

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -e\left(\frac{\mathrm{d}r}{\mathrm{d}t}B_\perp + r\frac{\mathrm{d}B_\perp}{\mathrm{d}t}\right) = eE_\varphi. \tag{3.4}$$

Following Wideroe's requirement for a constant orbit $\mathrm{d}r/\mathrm{d}t = 0$ allows the containment of the particle beam in a doughnut shaped vacuum chamber surrounding the magnetic field. The induced electric field has only an angular component E_φ since we have assumed that the magnetic field enclosed by the circular beam is uniform or at least rotationally symmetric. While noting that for a positive rate of change for the magnetic field the induced azimuthal electric field is negative, the left hand side of (3.2) then becomes simply

$$\oint \mathbf{E}\mathrm{d}\mathbf{s} = -\int E_\varphi R\,\mathrm{d}\varphi = -2\pi R E_\varphi\,. \tag{3.5}$$

On the other hand, we have from (3.4)

$$eE_\varphi = -eR\frac{\mathrm{d}B_\perp(R)}{\mathrm{d}t} \tag{3.6}$$

and using (3.5), (3.6) in (3.2) we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\pi R^2\frac{\mathrm{d}B_\perp(R)}{\mathrm{d}t}. \tag{3.7}$$

Noting that the complete magnetic flux enclosed by the particle orbit can also be expressed by an average field enclosed by the particle orbit, we have $\Phi = \pi R^2 \overline{B}(R)$, where $\overline{B}(R)$ is the average magnetic induction within the orbit of radius R. The rate of change of the magnetic flux becomes

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \pi R^2 \frac{\mathrm{d}\overline{B}_\perp(R)}{\mathrm{d}t} \tag{3.8}$$

and comparing this with (3.7) we obtain the Wideroe $\frac{1}{2}$-condition

$$B_\perp(R) = \tfrac{1}{2}\overline{B}_\perp(R), \tag{3.9}$$

which requires for orbit stability that the field at the orbit be half the average flux density through the orbit. This condition must be met in order to obtain orbital-beam stability in a betatron accelerator. By adjusting the total magnetic flux through the particle orbit such that the average magnetic field within the orbit circle is twice the field strength at the orbit, we are in a position to accelerate particles on a circle with a constant radius R within a doughnut shaped vacuum chamber.

The basic components of a betatron, shown in Fig. 3.1, have rotational symmetry. In the center of the magnet, we recognize two magnetic gaps of different aperture. One gap at R provides the bending field for the particles along the orbit. The other gap in the midplane of the central return yoke is adjustable and is being used to tune the magnet such as to meet the Wideroe $\frac{1}{2}$-condition. The magnetic field is generally excited by a resonance circuit cycling at the ac frequency of the main electricity supply. In this configuration the magnet coils serve as the inductance and are connected in parallel with a capacitor bank tuned to the ac frequency of 50 or 60 Hz.

The rate of momentum gain is derived by integration of (3.4) with respect to time and we find that the change in momentum is proportional to the change in the magnetic field

$$\triangle p = R \int \frac{\mathrm{d}B_\perp}{\mathrm{d}t}\mathrm{d}t = eR\Delta B_\perp\,. \tag{3.10}$$

The particle momentum depends only on the momentary magnetic field and not on the rate of change of the field. For slowly varying magnetic fields the electric field is smaller but the particles will make up the reduced acceleration by travelling around the orbit more often. While the magnet cycling rate does not affect the particle energy it certainly determines the available flux of accelerated particles per unit time. The maximum particle momentum is determined only by the orbit radius and the maximum magnetic field at the orbit during the acceleration cycle

$$cp_{\max} = ecRB_{\max}(R). \tag{3.11}$$

The betatron principle works for any charged particle and for all energies since the stability condition (3.9) does not depend on particle parameters. In praxis, however, we find that the betatron principle is unsuitable to the acceleration of heavy particles like protons. The magnetic fields in a betatron as well as the size of the betatron magnet set practical limits to the maximum momentum achievable. Donald Kerst built the largest betatron ever constructed with an orbit radius of $R = 1.23$ m, a maximum magnetic field at the orbit of 8.1 kG and a total magnet weight of 350 tons reaching the maximum expected particle momentum of 300 MeV/c at 60 Hz.

For experimental applications we are interested in the kinetic energy of the accelerated particles. In case of electrons the rest mass is small compared to the maximum momentum of $cp = 300\,\mathrm{MeV}$ and therefore the kinetic electron energy from this betatron is

$$E_{\mathrm{kin}} \approx cp = 300\,\mathrm{MeV}\,. \tag{3.12}$$

In contrast to this result, we find the achievable kinetic energy for a proton to be much smaller

$$E_{\mathrm{kin}} \approx \frac{1}{2}\frac{(cp)^2}{m_{\mathrm{p}}c^2} = 48\,\mathrm{MeV}\,, \tag{3.13}$$

because of the large mass of protons.

The betatron produces a pulse of accelerated particles once per ac cycle. To gain the maximum energy, the ac field is biased by a dc current and acceleration occurs from the minimum ac field to the maximum ac field. At the maximum field the beam can be ejected for applications.

Different, more efficient accelerating methods have been developed for protons, and betatrons are therefore used exclusively for the acceleration of electrons as indicated by it's name. Most betatrons are designed for modest energies of up to 45 MeV and are used to produce electron and hard x-ray beams for medical applications or in technical applications to, for example, examine the integrity of full penetration welding seams in heavy steel containers.

3.2 Weak Focusing

The Wideroe $\frac{1}{2}$-condition is a necessary condition to obtain a stable particle orbit at a fixed radius R. This stability condition, however, is not sufficient for particles to survive the accelerating process. Any particle starting out with, for example, a slight vertical slope would, during the acceleration process, follow a continuously spiraling path until it hits the top or bottom wall of the vacuum chamber and gets lost. Constructing and testing the first, although unsuccessful, beam transformer, Wideroe recognized [8] the need for beam focusing, a need which has become a fundamental part of all future particle accelerator designs. First theories on beam stability and focusing have been pursued by Walton [9] and later by Steenbeck, who formulated a stability condition for weak focusing and applied it to the design of the first successful construction and operation of a betatron in 1935 at the Siemens-Schuckert Company in Berlin reaching an energy of 1.9 MeV [10] although at a very low intensity measurable only with a Geiger counter. The focusing problems in a betatron were finally solved in a detailed orbit analysis by Kerst and Serber [11].

To derive the beam stability condition we note that (1.74) is true only at the ideal orbit $r = R$. For any other orbit radius r the restoring force is

$$F_x = \frac{\gamma m v^2}{r} - evB_y\,. \tag{3.14}$$

Here we use a cartesian coordinate system which moves with the particle along the orbit with $\mathbf{x}$ pointing in the radial and $\mathbf{y}$ in the axial direction.

In a uniform magnetic field the restoring force would be zero for any orbit. To include focusing we assume that the magnetic field at the orbit includes a gradient such that for a small deviation x from the ideal orbit, $r = R + x = R\,(1 + x/R)$, the magnetic guide field becomes

$$B_y = B_{0y} + \frac{\partial B_y}{\partial x}x = B_{0y}\left(1 + \frac{R}{B_{0y}}\frac{\partial B_y}{\partial x}\frac{x}{R}\right). \tag{3.15}$$

After insertion of (3.15) into (3.14) the restoring force is

$$F_x \approx \frac{\gamma m v^2}{R}\left(1 - \frac{x}{R}\right) - evB_{0y}\left(1 - n\frac{x}{R}\right), \tag{3.16}$$

where we assumed $x \ll R$ and defined the field index

$$n = -\frac{R}{B_{0y}}\frac{\partial B_y}{\partial x}. \tag{3.17}$$

With (1.74) we get for the horizontal restoring force

$$F_x = -\frac{\gamma m v^2}{R}\frac{x}{R}(1-n). \tag{3.18}$$

The equation of motion under the influence of the restoring force in the deflecting or horizontal plane is with $F_x = \gamma m\ddot{x}$

$$\ddot{x} + \omega_x^2 x = 0, \tag{3.19}$$

which has the exact form of a harmonic oscillator with the frequency

$$\omega_x = \frac{v}{R}\sqrt{1-n} = \omega_0\sqrt{1-n}, \tag{3.20}$$

where ω_0 is the orbital revolution frequency. The particle performs oscillations about the ideal or reference orbit with the amplitude $x(z)$ and the frequency ω_x. Because this focusing feature was discovered in connection with the development of the betatron we refer to this particle motion as betatron oscillations with the betatron frequency ω_x. From (3.20) we note a stability criterion, which requires that the field index not exceed unity to prevent the betatron oscillation amplitude to grow exponentially,

$$n < 1\,. \tag{3.21}$$

The particle beam stability discussion is complete only if we also can show that there is stability in the vertical plane. A vertical restoring force requires a finite horizontal field component B_x and the equation of motion becomes

$$\gamma m\ddot{y} = evB_x. \tag{3.22}$$

Maxwell's curl equation $\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$ can be integrated and the horizontal field component is with (3.15), (3.17)

$$B_x = \int \frac{\partial B_y}{\partial x}\,\mathrm{d}y = -\int n\frac{B_{0y}}{R}\mathrm{d}y = -n\frac{B_{0y}}{R}y. \tag{3.23}$$

Insertion of (3.23) and (1.74) into (3.22) results in the equation of motion for the vertical plane in the form of

$$\ddot{y} + \omega_y^2 y = 0, \tag{3.24}$$

where the vertical betatron oscillation frequency is

$$\omega_y = \omega_0\sqrt{n}. \tag{3.25}$$

Particles perform stable betatron oscillations about the horizontal mid plane with the vertical betatron frequency ω_y as long as the field index is positive

$$n > 0\,. \tag{3.26}$$

In summary, we have found that a field gradient in the magnetic guide field can provide beam stability in both the horizontal and vertical plane provided that the field index meets the criterion

$$0 < n < 1\,, \tag{3.27}$$

which has been first formulated and applied by Steenbeck [10] and is therefore also called Steenbeck's stability criterion.

A closer look at (3.20) shows that the field index actually provides defocusing in the horizontal plane. The reason why we get focusing in both planes is that there is a strong natural focusing from the sector type magnet which is larger than the defocusing from the field gradient. This focusing is of geometric nature and relates to the length of the orbit. A particle travelling, for example, parallel to and outside the ideal orbit is deflected more because it follows a longer path in the uniform magnetic field than a particle following the ideal orbit leading to effective focusing toward the ideal orbit. Conversely, a particle traveling parallel to and inside the ideal orbit is deflected less and therefore again is deflected toward the ideal orbit.

The stability condition (3.21) actually stipulates that the defocusing from the field index in the horizontal plane be less than the focusing of the sector magnet allowing to choose the sign of the field index such that it provides focusing in the vertical plane. Basically the field gradient provides a means to distribute the strong sector magnet focusing. This method of beam focusing is known as weak focusing in contrast to the principle of strong focusing, which will be discussed extensively in the remainder of this text.

3.3 Adiabatic Damping

During the discussion of transverse focusing we have neglected the effect of acceleration. To include the effect of acceleration into our discussion on beam dynamics, we use as an example the Lorentz force equation for the vertical motion. The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{y}) = ev_z B_x\,, \tag{3.28}$$

where we used the fields $\mathbf{B} = (B_x, B_y, 0)$ in a cartesian coordinate system (x, y, z). Evaluating the differentiation, we get the equation of motion at the equilibrium orbit

$$\gamma m\ddot{y} + \dot{\gamma} m\dot{y} = e\omega_0 R B_x. \tag{3.29}$$

Inserting (3.23) into (3.29) results in the equation of motion in the vertical plane under the influence of accelerating electrical and focusing magnetic fields

$$\ddot{y} + \frac{\dot{E}}{E}\dot{y} + n\omega_0^2 y = 0, \tag{3.30}$$

where $\dot{E}$ is the energy gain per unit time. This is the differential equation of a damped harmonic oscillator with the solution

$$y = y_0 \mathrm{e}^{-\alpha_y t} \cos\omega_y t, \tag{3.31}$$

where $\omega_y \approx \omega_0 \sqrt{n}$ and the damping decrement

$$\alpha_y = \frac{1}{2}\frac{\dot{E}}{E}. \tag{3.32}$$

For technically feasible acceleration $\dot{E}$ the damping time $\tau_y = \alpha_y^{-1}$ is very long compared to the oscillation period and we therefore may consider for the moment the damping as a constant. The envelope $y_{\max} = y_0 e^{-\alpha_y t}$ of the oscillation (3.31) decays like

$$\mathrm{d}y_{\max} = -\frac{1}{2}\frac{\dot{E}}{E} y_{\max} \mathrm{d}t\,, \tag{3.33}$$

which after integration becomes

$$\frac{y_{\max}}{y_{0,\max}} = \sqrt{\frac{E_0}{E}}. \tag{3.34}$$

The betatron oscillation amplitude is reduced as the particle energy increases. This type of damping is called adiabatic damping. Similarly, the slope y' as well as the horizontal oscillation parameters experience the same effect of adiabatic damping during acceleration. This damping is due to the fact that the longitudinal particle momentum is increasing during acceleration while the transverse momentum is not. For a particle beam we define a beam emittance in both planes by the product $\epsilon_u = u_{\max}\, u'_{\max}$, where u stands for x or y. Due to adiabatic damping this beam emittance is reduced inversely proportional to the energy like

$$\epsilon \sim \frac{1}{E}. \tag{3.35}$$

No specific use has been made of the principle of betatron acceleration to derive the effect of adiabatic damping. We therefore expect this effect to be generally valid for any kind of particle acceleration.

The development of the betatron was important for accelerator physics for several reasons. It demonstrated the need for particle focusing, the phenomenon of adiabatic damping and stimulated Schwinger to formulate the theory of synchrotron radiation [3]. He realized that the maximum achievable electron energy in a betatron must be limited by the energy loss due to synchrotron radiation. Postponing a more detailed discussion of synchrotron radiation to Chap. 24, we note that the instantaneous synchrotron radiation power is given by $P_\gamma \propto E^2 F_\perp^2$, where $F_\perp$ is the transverse force on the particle and the energy loss per turn to synchrotron radiation in a circular electron accelerator is given by (24.41). The energy loss increases rapidly with the fourth power of energy and can lead quickly to an energy limitation of the accelerator when the energy loss per turn becomes equal to the energy gain.

3.4 Acceleration by rf Fields

Most types of circular particle accelerators utilize compact accelerating cavities, which are excited by rf sources. Particles traverse this cavity periodically and gain energy from the electromagnetic fields in each passage. The bending magnet field serves only as a beam guidance system to allow the repeated passage of the particle beam through the cavity. Technically, this type of accelerator seem to be very different from the principle of the betatron. Fundamentally, however, there is no difference. We still rely on the transformer principle, which in the case of the betatron looks very much like the familiar transformers at low frequencies, while accelerating cavities are transformer realizations for very high frequencies. Electric fields are generated in both cases by time varying magnetic fields.

Since the cavity fields are oscillating, acceleration is not possible at all times and for multiple accelerations we must meet specific synchronicity condition between the motion of particles and the field oscillation. The time it takes the particles to travel along a full orbit must be an integer multiple of the oscillation period for the radio frequency field. This synchronization depends on the particle velocity, path length, magnetic fields employed, and on the rf frequency. Specific control of one or more of these parameters defines the different types of particle accelerators to be discussed in the following subsections.

3.4.1 Microtron

The schematic configuration of a microtron [12] is shown in Fig. 3.2. Particles emerging from a source pass through the accelerating cavity and follow then a circular orbit in a uniform magnetic field B_y leading back to the accelerating cavity. After each acceleration the particles follow a circle with a bigger radius till they reach the boundary of the magnet.

The bending radius of the orbit can be derived from the Lorentz force equation (1.74)

$$\frac{1}{\rho} = \frac{ecB}{cp} = \frac{ecB_y}{\gamma\beta mc^2}, \tag{3.36}$$

and the revolution time for a particle traveling with velocity v is

$$\tau = \frac{2\pi\rho}{v} = \frac{2\pi mc}{e}\frac{\gamma}{B_y}. \tag{3.37}$$

The revolution time is therefore proportional to the particle energy and inversely proportional to the magnetic field. It is interesting to note that for subrelativistic particles, where $\gamma \approx 1$, the revolution time is constant even though the particle

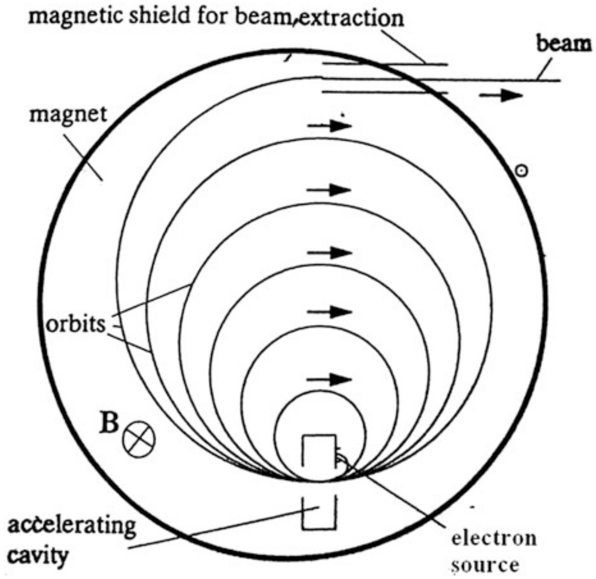

Fig. 3.2 The principle of a microtron accelerator (schematic)

momentum increases. The longer path length for the higher particle momentum is compensated by the higher velocity. As particles reach relativistic energies, however, this synchronism starts to fail. To still achieve continued acceleration, specific conditions must be met.

A particle having completed the nth turn passes through the cavity and it's energy is increased because of acceleration. The change in the revolution time during the $(n+1)$st turn compared to the nth turn is proportional to the energy increase $\Delta\gamma$. The increase in the revolution time must be an integer multiple of the radio frequency period. Assuming that the revolution time along the first innermost circle, when the particle energy is still $\gamma \approx 1$, is equal to one rf period we conclude that synchronism is preserved for all turns if

$$\Delta\gamma = 1 \tag{3.38}$$

or integer multiples. In order to make a microtron functional the energy gain from the accelerating cavity in each passage must be

$$\begin{aligned} \Delta E_{\mathrm{e}} &= 511\ \mathrm{keV} && \text{for electrons and} \\ \Delta E_{\mathrm{p}} &= 938\ \mathrm{MeV} && \text{for protons.} \end{aligned} \tag{3.39}$$

While it is possible to meet the condition for electrons it is technically impossible at this time to achieve accelerating voltages of almost 1 GV in an accelerating cavity

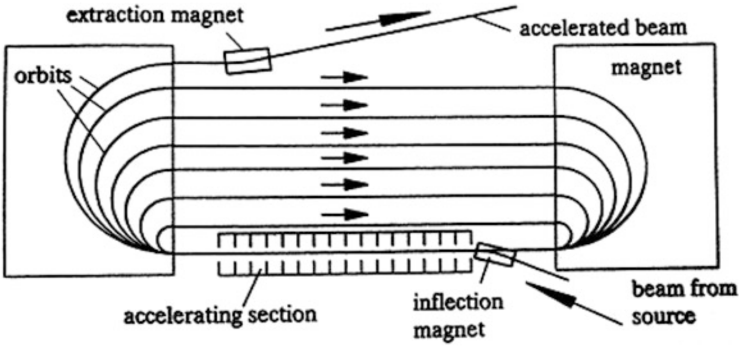

Fig. 3.3 Race track microtron [13] (schematic)

of reasonable length. The principle of microtrons is therefore specifically suited for the acceleration of electrons.

The size of the magnet imposes a practical limit to the maximum particle energy. A single magnet scales like the third power of the bending radius and therefore the weight of the magnet also scales like the third power of the maximum particle energy. Single magnet microtrons are generally used only to accelerate electrons to energies up to about 25–30 MeV.

To alleviate the technical and economic limitations as well as to improve control of the synchronicity condition, the concept of a race track microtron has been developed [13, 14]. In this type of microtron the magnet is split in the middle and normal to the orbit plane and pulled apart as shown in Fig. 3.3. The space opened up provides space for a short linear accelerator which allows the acceleration of electrons by several units in γ thus reducing the number of orbits necessary to reach the desired energy. The magnets are flat and scale primarily only like the square of the bending radius.

3.4.2 Cyclotron

The synchronicity condition of a microtron proved to be too severe for the successful acceleration of heavier particles like protons. In drawing this conclusion, however, we have ignored the trivial synchronicity condition $\Delta\gamma = 0$. This condition demands that the particle energy be nonrelativistic which limits the maximum achievable energy to values much less than the rest energy. This limitation is of no interest for electrons since electro-static accelerators would provide much higher energies than that. For protons, however, energies much less than the rest energy of 938 MeV are of great interest. This was recognized in 1930 by Lawrence and Edlefsen [15] in the process of inventing the principle of the cyclotron and the first such device was built by Lawrence and Livingston in 1932 [16].

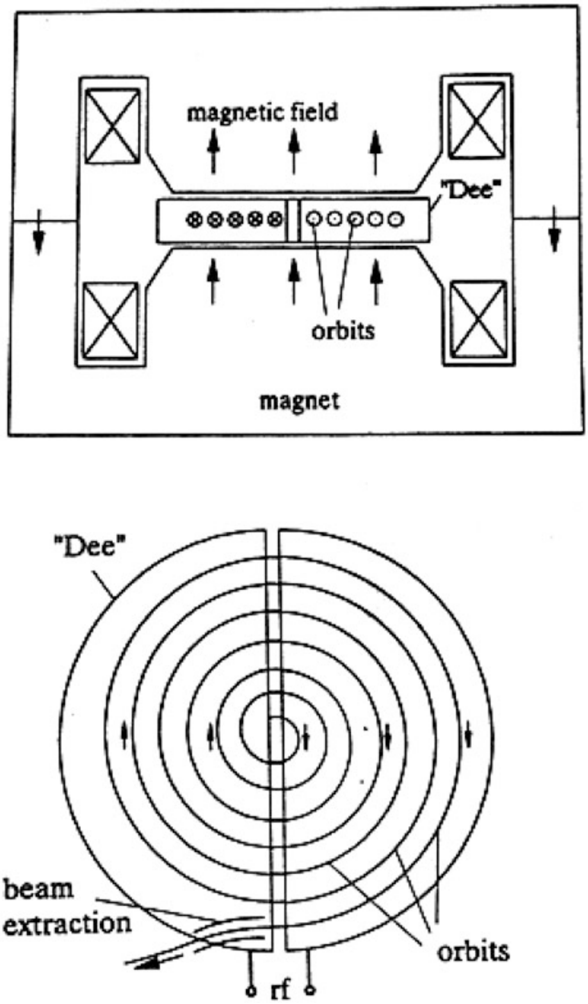

Fig. 3.4 Principle of a cyclotron [16] (schematic). Vertical (*top*) and horizontal (*bottom*) mid plane cross section

The cyclotron principle employs a uniform magnetic field and an rf cavity that extends over the whole aperture of the magnet as shown in Fig. 3.4. The accelerating cavity has basically the form of a pill box cut in two halves, where the accelerating fields are generated between those halves and are placed between the poles of the magnet. Because of the form of the half pill boxes, these cavities are often called the *D's* of a cyclotron. The particle orbits occur mostly in the field free interior of the *D*'s and traverse the accelerating gaps between the two *D*'s twice per revolution. Due to the increasing energy, the particle trajectories spiral to larger and larger radii. The travel time within the *D*'s is adjusted by the choice of the magnetic field such that it is equal to half the radio frequency period. The principle of the cyclotron is basically the application of the Wideroe linac in a coiled up version to save space and rf equipment. Fundamentally, however, the use of field free tubes or D's with increasing path length between accelerating gaps is the same.The revolution time in a cyclotron is given by (3.37) where now $\gamma = 1$ and the acceleration of ions with a charge multiplicity Z is allowed. Keeping the magnetic field constant, we have a constant revolution frequency and may therefore apply a constant radio frequency

$$f_{\mathrm{rf}} = \frac{ZeB_y}{2\pi mc\gamma} h = \mathrm{const.} \tag{3.40}$$

The principle of the cyclotron is limited to nonrelativistic particles. Protons are sufficiently nonrelativistic up to kinetic energies of about 20–25 MeV or about 2.5 % of the rest energy. As the particles become relativistic the revolution frequency becomes smaller and the particles get out of synchronism with the radio frequency f_{rf}.

The radio frequency depends on the charge multiplicity Z of the particles to be accelerated and on the magnetic field B_y. Evaluating (3.40), the following frequencies are required for different types of particles

$$\begin{aligned} f_{\mathrm{rf}}\,[\mathrm{MHz}] &= 1.53 \cdot B_y[\mathrm{kG}] \qquad \text{for protons,} \\ &= 0.76 \cdot B_y[\mathrm{kG}] \qquad \text{for deuterons,} \\ &= 0.76 \cdot B_y[\mathrm{kG}] \qquad \text{for He}^{++}. \end{aligned} \tag{3.41}$$

A closer inspection of the synchronicity condition, however, reveals that these are only the lowest permissible rf frequencies. Any odd integer multiple of the frequencies (3.41) would be acceptable too.

As long as particles do not reach relativistic energies, the maximum achievable kinetic energy E_{kin} depends on the type of the particle, the magnetic field B, and the maximum orbit radius R possible in the cyclotron and is given by

$$E_{\mathrm{kin}} = \tfrac{1}{2}mv^2 = \frac{(cp)^2}{2\,mc^2} = \frac{Z^2e^2B_y^2R^2}{2\,mc^2}. \tag{3.42}$$

Examples of numerical relations are

$$\begin{aligned} E_{\mathrm{kin}}[\mathrm{MeV}] &= 0.48B_y^2[\mathrm{kG}^2]R^2[\mathrm{m}^2] \qquad \text{for protons,} \\ &= 0.24B_y^2[\mathrm{kG}^2]R^2[\mathrm{m}^2] \qquad \text{for deuterons,} \\ &= 0.48B_y^2[\mathrm{kG}^2]R^2[\mathrm{m}^2] \qquad \text{for He}^{++}. \end{aligned} \tag{3.43}$$

The particle flux reflects the time structure of the radio frequency field. For a continuous radio frequency field the particle flux is also "continuous" with micro bunches at distances equal to the oscillation period of the accelerating field. For a pulsed rf system obviously the particle flux reflects this macropulse structure on top of the micropulses.

3.4.3 Synchro-Cyclotron

The limitation to nonrelativistic energies of the cyclotron principle is due only to the assumption that the radio frequency be constant. This mode of operation for rf systems is desirable and most efficient but is not a fundamental limitation. Technical means are available to vary the radio frequency in an accelerating cavity.

As the technology for acceleration to higher and higher energies advances, the need for particle beam focusing becomes increasingly important. In the transverse plane this is achieved by the weak focusing discussed earlier. In the longitudinal phase space stability criteria have not been discussed yet. Veksler [17] and McMillan [18] discovered and formulated independently the principle of phase focusing, which is a fundamental focusing property for high energy particle accelerators based on accelerating microwave frequency fields and was successfully tested only one year later [19]. We will discuss this principle of phase focusing in great detail in Chap. 9.

Both the capability of varying the rf frequency and the principle of phase focusing is employed in the synchro-cyclotron. In this version of the cyclotron, the microwave frequency varies as the relativistic factor γ deviates from unity. Instead of (3.40) we have for the revolution frequency or microwave frequency

$$f_{\mathrm{rf}} = \frac{ZeB_y}{2\pi\gamma mc} h, \tag{3.44}$$

where h is an integer called the harmonic number. Since $B = \text{const}$ the radio frequency must be adjusted like

$$f_{\mathrm{rf}} \sim 1/\gamma(t) \tag{3.45}$$

to keep synchronism. The momentary particle energy $\gamma(t)$ can be derived from the equation of motion $1/\rho = ZeB/(cp)$ which we solve for the kinetic energy

$$\sqrt{E_{\mathrm{kin}}(E_{\mathrm{kin}} + 2\,mc^2)} = eZB_y\rho. \tag{3.46}$$

The largest accelerator ever built based on this principle is the 184 inch synchro cyclotron at the Lawrence Berkley Laboratory (LBL) in 1946 [20]. The magnet weight was 4,300 tons, produces a maximum magnetic field of 15 kG and has a maximum orbit radius of 2.337 m. From (3.46) we conclude that the maximum kinetic proton energy should be $E_{\text{kin,max}} = 471\,\text{MeV}$ while 350 MeV have been achieved. The discrepancy is mostly due to the fact that the maximum field of 15 kG does not extend out to the maximum orbit due to focusing requirements. The principle of the synchro cyclotron allows the acceleration of particles to rather large energies during many turns within the cyclotron magnet. This long path requires the addition of weak focusing as discussed in connection with the betatron principle to obtain a significant particle flux at the end of the acceleration period. From the discussion in Sect. 3.2 we know that efficient focusing in both planes requires the vertical magnetic field component to drop off with increasing radius. For equal focusing in both planes the field index should be $n = \frac{1}{2}$ and the magnetic field scales therefore like

$$B_y(\rho) \sim \frac{1}{\sqrt{\rho}}. \tag{3.47}$$

The magnetic field is significantly lower at large radii compared to the center of the magnet.

This magnetic field dependence on the radial position leads also to a modification of the frequency tracking condition (3.45). Since both the magnetic field and the particle energy change, synchronism is preserved only if the rf frequency is modulated like

$$f_{\text{rf}} \sim \frac{B_y[\rho(t)]}{\gamma(t)}. \tag{3.48}$$

Because of the need for frequency modulation, the particle flux has a pulsed macro structure equal to the cycling time of the rf modulation. A detailed analysis of the accelerator physics issues of a synchro-cyclotron can be found in [21].

3.4.4 Isochron Cyclotron

The frequency modulation in a synchro cyclotron is technically complicated and must be different for different particle species. A significant breakthrough occurred in this respect when Thomas [22] realized that the radial dependence of the magnetic field could be modified in such a way as to match the particle energy γ. The condition (3.48) becomes in this case

$$f_{\text{rf}} \sim \frac{B_y[\rho(t)]}{\gamma(t)} = \text{const.} \tag{3.49}$$

To reconcile (3.49) with the focusing requirement, strong azimuthal variations of the magnetic fields are introduced

$$\frac{\partial B_y(\rho,\varphi)}{\partial \varphi} \neq 0. \tag{3.50}$$

In essence, the principle of weak focusing is replaced by strong focusing, to be discussed later, with focusing forces established along the particle trajectory while meeting the synchronicity condition only on average in each turn such that

$$\frac{1}{2\pi}\oint B_y[\rho(t),\varphi]\mathrm{d}\varphi \sim \gamma(t)\,. \tag{3.51}$$

Isochron cyclotrons produce a continuous beam of micro bunches at the rf frequency and are used frequently for acceleration of protons and ions for cancer therapy.

The development of circular accelerators has finally made a full circle. Starting from the use of a constant radio frequency field to accelerate particles we found the need for frequency modulation to meet the synchronicity condition for particles through the relativistic transition regime. Application of sophisticated magnetic focusing schemes, which are now known as strong focusing, finally allowed to revert back to the most efficient way of particle acceleration with constant fixed radio frequency fields.

3.4.5 Synchrotron

The maximum particle energy is limited to a few hundred MeV as long as one stays with the basic cyclotron principle because the volume and therefore cost for the magnet becomes prohibitively large. Higher energies can be achieved and afforded if the orbit radius R is kept constant. In this case the center of the magnet is not needed anymore and much smaller magnets can be employed along the constant particle orbit. Equation (3.36) is still applicable but we keep now the orbit radius constant and have the design condition

$$\frac{1}{R} = \frac{ecB_y}{cp} = \text{const}\,. \tag{3.52}$$

This condition can be met for all particle energies by ramping the magnet fields proportional to the particle momentum. Particles are injected at low momentum and are then accelerated while the bending magnet fields are increased to keep the particles on a constant radius while they gain energy. The particle beam from such a synchrotron is pulsed with a repetition rate determined by the magnetic field cycling. The synchronicity condition

$$f_{\mathrm{rf}} = \frac{Z\,ecB_y}{2\pi\gamma mc^2}h, \tag{3.53}$$

is still valid, but because the magnetic field is varied proportional to the particle momentum we expect the frequency to require adjustment as long as there is sufficient difference between particle energy and momentum.

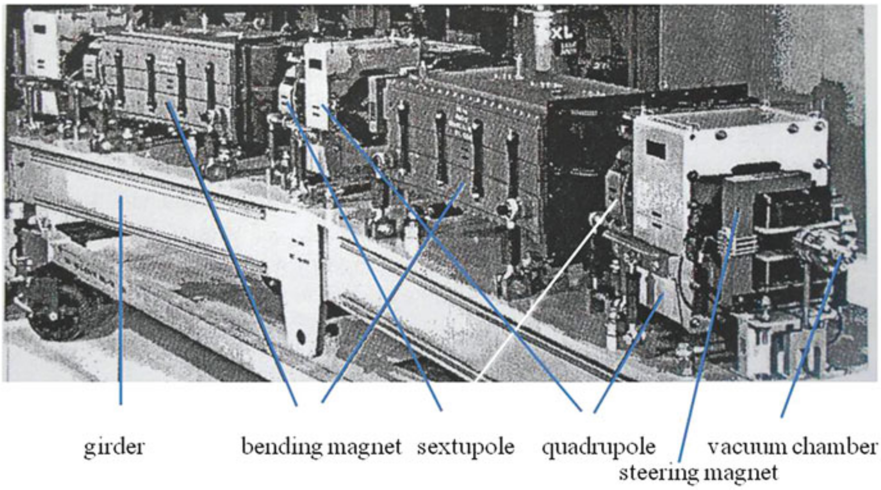

Fig. 3.5 Magnet arrangement in a synchrotron [23]

For highly relativistic particles a solution for particle acceleration has been found which does not require a prohibitively large magnet and where the radio frequency fields can be of constant frequency for optimal efficiency. This is the case for electron synchrotrons with an initial energy of at least a few tens of MeV. For this reason electrons are generally injected into a synchrotron at energies of more than about 10–20 MeV from a linear accelerator or a microtron. Figure 3.5 shows an example of a synchrotron used for the injection of electrons into a storage ring [23].

For heavier particles, however, we are back to the need for some modest frequency modulation during the early phases of acceleration. From (3.53) we expect the revolution frequency to vary like

$$f_{\text{rev}}(t) = \frac{Z\,ecB_y}{2\pi cp}\beta(t) \propto \beta(t) \tag{3.54}$$

To preserve the synchronicity condition, the radio frequency must be an integer multiple of the revolution frequency and must be modulated in proportion of the varying revolution frequency. The ratio of the rf frequency to revolution frequency is called the harmonic numberdefined by

$$f_{\text{rf}} = hf_{\text{rev}}. \tag{3.55}$$

The maximum energy in a synchrotron is determined by the ring radius R, and the maximum magnetic field B_y, and is

$$cp_{\max}(\text{GeV}) = \sqrt{E_{\text{kin}}(E_{\text{kin}} + 2mc^2)} = C_p B_y\,[\text{T}]\,R\,[\text{m}], \tag{3.56}$$

where

$$C_p = 0.2997926 \frac{\text{GeV}}{\text{Tm}}. \tag{3.57}$$

Early synchrotrons have been constructed with weak focusing bending magnets which in addition to the dipole field component also included a field gradient consistent with a field index meeting the focusing condition (3.27). More detailed information about early weak focusing synchrotrons can be obtained from [21, 24].

With the discovery of the principle of strong focusing by Christofilos [25] and independently by Courant et. al. [26] in 1952 much more efficient synchrotrons could be designed. The apertures in the magnets could be reduced by up to an order of magnitude thus allowing the design of high-field magnets at greatly reduced overall magnet size and cost. The physics of strong focusing will be discussed in great detail in subsequent chapters.

Synchrotrons are the workhorse in particle acceleration and are applied for electron acceleration as well as proton and ion acceleration to the highest energies. In more modern proton accelerators superconducting magnets are employed to reach energies in excess of 1 TeV.

3.4.6 Storage Ring

Although not an accelerator in the conventional sense, a particle storage ring can be considered as a synchrotron frozen in time. While the basic functioning of a storage ring is that of a synchrotron, particle beams are generally not accelerated but only stored to orbit for long times of several hours. Bruno Touschek and R. Wideroe invented this principle in 1941 while working (not completely by their own free will) at the Hamburg University for application in high energy physics to bring two counter rotating beams of particles and antiparticles into collision and study high energy elementary particle processes. A newer and more copious application of the storage ring principle arose from the dedicated production of synchrotron radiation for basic and applied research and technology. The principles, details and functioning of synchrotrons and storage rings will occupy most of our discussions in this text.

3.4.7 Summary of Characteristic Parameters

It is interesting to summarize the basic principles for the different particle accelerators discussed. All are based on two relations, the equation of motion (3.14) and the synchronicity condition (3.37). Depending on which parameter in these two relations we want to keep constant or let vary, different acceleration principles apply

Table 3.1 Parameter properties for different acceleration principles

Principle	Energy	Velocity	Orbit	Field	Frequency	Particle flux
	γ	v	ρ	B	f_{rf}	
Microtron	var.	c	$\sim p$	const.	const.	const.[a]
Cyclotron	1	var.	$\sim v$	const.	const.	const.[a]
Synchro-cyclotron	var.	var.	$\sim p$	$B(\rho)$	$\sim \frac{B(\rho)}{\gamma(t)}$	pulsed
Isochron cyclotron	var.	var.	$\rho\,(p)$	$B(\rho,\varphi)$	const.	const.[a]
Proton synchrotron	var.	var.	R	$\sim p(t)$	$\sim v\,(t)$	pulsed
Electron synchrotron	var.	c	R	$\sim p(t)$	const.	pulsed

[a] continuous beam, although rf-modulated

with varying advantages and disadvantages. In Table 3.1 these parameters and their disposition are compiled for the acceleration principles discussed above.

There are two particle parameters and three technical device parameters which define the mode of accelerator operation. Interestingly enough, most of the discussed acceleration methods have their proper application and are used either as stand alone accelerators for research and technology or are part of an acceleration chain for high energy particle accelerators.

For example, it makes no sense to construct a proton synchrotron, where the protons must be injected directly from the source at very low energies. The proper way is to first accelerate the protons with electro-static fields, for example in a Cockcroft-Walton accelerator followed by a medium energy linear accelerator (e.g. an Alvarez structure) to reach a high enough energy of a few hundred MeV for efficient injection into a synchrotron. Similarly, we accelerate an electron beam first in a linac or microtron before injection into a synchrotron or storage ring.

Problems

3.1 (S). Consider the Kerst betatron cycling at 60 Hz. Electrons are injected at 50 keV kinetic energy into this betatron. Calculate the magnetic field on the orbit at injection and the energy gain per turn for the first turn and at a time when the electron has gained 20 MeV. Discuss the reason for the difference in the energy gain per turn (use linear dependence of field, $B \approx B_0 \omega t$).

3.2 (S). What is the total excitation current in each of the two coils for a betatron with an orbit radius of $R = 0.4$ m, a maximum electron momentum of $cp = 42$ MeV and a gap of $g = 10$ cm between the poles (hint: apply Ampère's law).

3.3 (S). Calculate the frequency variation required to accelerate protons or deuterons in a synchro cyclotron from a kinetic energy of $E_{\mathrm{kin}\,0} = 100$ keV to an end energy of $E_{\mathrm{kin}} = 600$ MeV. Keep the magnetic field constant and ignore weak focussing. Derive formula for the rf frequency as a function of the kinetic energy and generate a graph of $f_{\mathrm{rf}}/f_{\mathrm{rf},0}$ vs. E_{kin} from just a few points. How big would the frequency swing be for electrons ?

3.4 (S). Calculate the electron beam current in the Kerst betatron that would produce a total synchrotron radiation power of 1 Watt at $cp = 300\,\mathrm{MeV}$.

3.5 (S). Try to "design" a microtron for a maximum electron energy of $E = 25\,\mathrm{MeV}$ at a magnetic field of $B = 2140\,\mathrm{G}$. (a) What is the diameter of the last circular trajectory at 25 MeV ? (b) Sketch a cross section of the magnet with excitation coils. Magnet poles must extend radially at least by 1.5 gap heights beyond the maximum orbit to obtain good field quality. Use a total coil cross section of $5\,\mathrm{cm}^2$. Choose your own gap height. (c) What is the electrical power required to operate each coil assuming copper and a copper fill factor of 75 %. That means 75 % of the coil cross section is copper and the rest is for insulation and cooling. Do you think your coil needs water cooling? (for simplicity assume the coil length to be equal to the length of the last trajectory plus 10 %.) (d) How does the electrical power requirement change if you change the number of turns in the coil thereby changing the electrical current. Keep in either case 75 % fill factor.

3.6 (S). Consider the Fermilab 400 GeV/c synchrotron which has a circumference of 6,000 m. Protons are injected at 10 GeV/c momentum. Calculate the frequency swing necessary for synchronicity during the acceleration cycle.

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