Chapter 19
Beam-Cavity Interaction*

The proper operation of the rf-system in a particle accelerator depends more than any other component on the detailed interaction with the particle beam. This results from the observation that a particle beam can induce fields in the accelerating cavities of significant magnitude compared to the generator produced voltages and we may therefore not neglect the presence of the particle beam. This phenomenon is called beam loading and can place severe restrictions on the beam current that can be accelerated. In this section, main features of such interaction and stability conditions for most efficient and stable particle acceleration will be discussed.

19.1 Coupling Between rf-Field and Particles

In our discussions about particle acceleration we have tacitly assumed that particles would gain energy from the fields in accelerating cavities merely by meeting the synchronicity conditions. This is true for a weak particle beam which has no significant effect on the fields within the cavity. As we try, however, to accelerate an intense beam, the actual accelerating fields become modified by the presence of considerable electrical particle beam currents. This beam loading can ultimately limit the maximum beam intensity.

The phenomenon of beam loading will be defined and characterized in this section leading to conditions and parameters to assure positive energy flow from the rf-power source to the beam. Fundamental consideration to this discussion are the principles of energy conservation and linear superposition of fields which allow us to study field components from one source independent of fields generated by other sources. Specifically, we may treat beam induced fields separately from fields generated by rf-power sources.

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19.1.1 Network Modelling of an Accelerating Cavity

The electrical excitation of a rf-cavity can be accurately described by an oscillator as discussed in Sect. 18.2.4 and we will use therefore characteristic parameters and terminology of externally driven, damped oscillators in our further discussions of rf-systems. Electrically, an accelerating cavity can be represented by a parallel resonant circuit (Fig. 19.1) which is driven by an external rf-current source I_g from a generator and the particle beam I_b.

The amount of rf-power available from the generator in the accelerating cavity depends greatly on the relative impedance of cavity and generator. Both have to be matched to assure optimum power transfer. To derive conditions for that we define the internal impedance of the current source or rf-generator in terms of the cavity shunt impedance R_s of an empty cavity as defined in (18.74)

$$R_g = \frac{R_s}{\beta}\,, \tag{19.1}$$

where β is the coupling coefficient still to be defined. This coefficient depends on the actual hardware of the coupling arrangement for the rf-power from the generator at the entrance to the cavity and quantifies the generator impedance as seen from the cavity in units of the cavity shunt impedance R_s (Fig. 19.1). Since this coupling coefficient depends on the hardware, we need to specify the desired operating condition to determine the proper adjustment of the coupling during assembly. This adjustment is done by either rotating a loop coupler with respect to the cavity axis or adjustment of the aperture in case of capacitive coupling through a hole.

The inductance L and capacitance C form a parallel resonant circuit with the resonant frequency

$$\omega_r = \frac{1}{\sqrt{LC}}\,. \tag{19.2}$$

The rf-power available at the cavity from the generator is

$$P_g = \tfrac{1}{2} Y_L V_g^2\,, \tag{19.3}$$

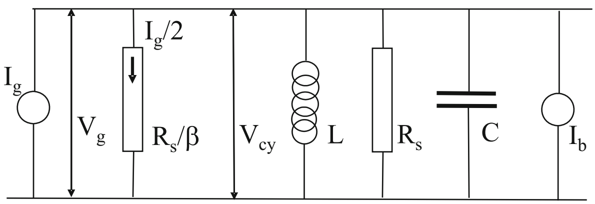

Fig. 19.1 Network model for an rf generator and an accelerating cavity

where Y_L is the loaded cavity admittance including energy transfer to the beam and V_g is the generator voltage. Unless otherwise noted, the voltages, currents and power used in this section are the amplitudes of otherwise oscillating quantities. At resonance where all reactive power vanishes we use the generator current I_g and network admittance $Y = Y_\mathrm{g} + Y_\mathrm{L}$ to replace the generator voltage

$$V_\mathrm{g} = \frac{I_\mathrm{g}}{Y} = \frac{I_\mathrm{g}}{Y_\mathrm{g} + Y_\mathrm{L}}$$

and get after insertion into (19.3) the generator power in the form

$$P_\mathrm{g} = \frac{1}{2} \frac{Y_\mathrm{L}}{\left(Y_\mathrm{g} + Y_\mathrm{L}\right)^2} I_\mathrm{g}^2 \,. \tag{19.4}$$

Noting that the generator power has a maximum, which can be determined from $\partial P_\mathrm{g}/\partial Y_\mathrm{L} = 0$, we obtain the well-known result that the rf-power transfer from the generator becomes a maximum if the load is matched to the internal impedance of the generator by adjusting

$$Y_\mathrm{g} = Y_\mathrm{L} \qquad \text{or} \qquad R_\mathrm{L} = \frac{R_\mathrm{s}}{\beta} \,, \tag{19.5}$$

replacing the admittances by the respective impedances. The maximum available rf-power at the cavity is therefore with $Y_\mathrm{g} = \beta/R_\mathrm{s}$

$$P_\mathrm{g} = \frac{1}{8} \frac{R_\mathrm{s}}{\beta} I_\mathrm{g}^2 \,. \tag{19.6}$$

To calculate the quality factor for a cavity, we note the stored energy is $W = \frac{1}{2} C V^2$ and the energy loss rate $P_\mathrm{cy} = \frac{1}{2} V^2/R$. Using the definition (18.80) the unloaded quality factor becomes with $R = R_\mathrm{s}$ at resonance

$$Q_0 = \omega_\mathrm{r} C R_\mathrm{s} \,. \tag{19.7}$$

The admittance for the total circuit as seen by the beam is that of cavity plus generator or

$$\frac{1}{R_\mathrm{b}} = \frac{\beta}{R_\mathrm{s}} + \frac{1}{R_\mathrm{s}} = \frac{1+\beta}{R_\mathrm{s}} . \tag{19.8}$$

From this and (19.7) we get the loaded quality factor

$$Q = \omega_\mathrm{r} C R_\mathrm{b} = \frac{Q_0}{1+\beta} \,. \tag{19.9}$$

Off resonance the generator voltage and current are no more in phase. The phase difference can be derived from the complex impedance of the network, which is the same seen from the generator as well as seen from the beam

$$\frac{1}{Z} = \frac{1}{R_\mathrm{b}} + \mathrm{i}\omega C + \frac{1}{\mathrm{i}\omega L} . \tag{19.10}$$

The complex impedance becomes with (19.2), (19.9)

$$\frac{1}{Z} = \frac{1}{R_\mathrm{b}} \left(1 + \mathrm{i}Q\frac{\omega^2 - \omega_\mathrm{r}^2}{\omega\,\omega_\mathrm{r}}\right) \tag{19.11}$$

and with $I_\mathrm{g} = V_\mathrm{g}/Z$ the generator current is

$$I_\mathrm{g} = \frac{V_\mathrm{g}}{R_\mathrm{b}} \left(1 + \mathrm{i}Q\frac{\omega^2 - \omega_\mathrm{r}^2}{\omega\,\omega_\mathrm{r}}\right) = \frac{V_\mathrm{g}}{R_\mathrm{b}} \left(1 - \mathrm{i}\tan\Psi\right) . \tag{19.12}$$

Close to resonance the tuning angle Ψ becomes from (19.12) with $\omega \approx \omega_\mathrm{r}$

$$\tan\Psi \approx -Q\frac{\omega^2 - \omega_\mathrm{r}^2}{\omega\,\omega_\mathrm{r}} \approx -2Q\frac{\omega - \omega_\mathrm{r}}{\omega_\mathrm{r}} \tag{19.13}$$

in agreement with (18.64) except for a phase shift of $-90°$, which was introduced here to be consistent with our definition of the synchronous phase ψ_s. The variation of the tuning angle is shown in Fig. 19.2 as a function of the generator frequency. From (19.12), the generator voltage at the cavity is finally

$$V_\mathrm{g} = \frac{I_\mathrm{g} R_\mathrm{b}}{1 - \mathrm{i}\tan\Psi} = I_\mathrm{g} R_\mathrm{b} \cos\Psi\, \mathrm{e}^{\mathrm{i}\Psi} . \tag{19.14}$$

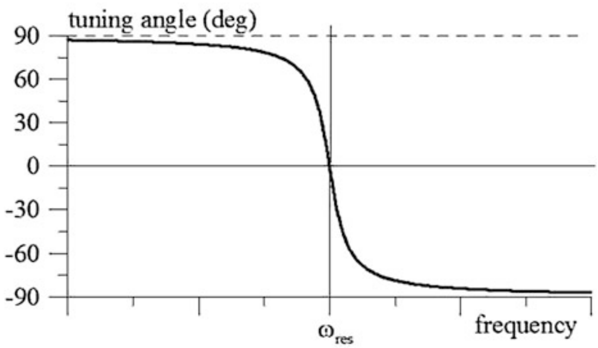

Fig. 19.2 Tuning angle ψ as a function of the generator frequency

At frequencies below the resonance frequency the tuning angle is positive and therefore the generator current lags the voltage by the phase Ψ. This case is also called inductive detuning since the impedance looks mainly inductive. Conversely, the detuning is called capacitive detuning because the impedance looks mostly capacitive for frequencies above resonance frequency.

A bunched particle beam passing through a cavity acts as a current just like the generator current and therefore the same relationships with respect to beam induced voltages exist. In case of capacitive detuning, for example, the beam induced voltage V_b lags in phase behind the beam current I_b.

The effective accelerating voltage in the cavity is a composition of the generator voltage, the induced voltage, and the phase relationships between themselves and relative to the particle beam. To assure a stable beam, the resulting cavity voltage must meet the requirements of particle acceleration to compensate, for example, lost energy into synchrotron radiation. We determine the conditions for that by deriving first the generator voltage V_{gr} at resonance and without beam loading while voltage and current are in phase. From Fig. 19.1 we get

$$V_{gr} = \frac{I_g}{Y_g + Y_L} = \frac{I_g}{\frac{1}{R_s} + \frac{\beta}{R_s}} = \frac{R_s I_g}{1+\beta} \tag{19.15}$$

and with (19.6) the generator voltage at resonance becomes

$$V_{gr} = \frac{2\sqrt{2\beta}}{1+\beta}\sqrt{R_s P_g} \,. \tag{19.16}$$

The generator voltage at the cavity is therefore with (19.6)

$$V_g = V_{gr}\cos\Psi\, e^{i\Psi} \,. \tag{19.17}$$

This is the cavity voltage seen by a negligibly small beam and can be adjusted to meet beam stability requirements by varying the tuning angle Ψ and rf-power P_g.

19.2 Beam Loading and Rf-System

For more substantial beam currents the effect of beam loading must be included to obtain the effective cavity voltage. Similar to the derivation of the generator voltage in a cavity, we may derive the induced voltage from the beam current passing through that cavity. Since there is no fundamental difference between generator and beam current, the induced voltage is in analogy to (19.17)

$$V_b = -V_{br}\cos\Psi\, e^{i\Psi} \,, \tag{19.18}$$

where the negative sign indicates that the induced voltage is decelerating the beam. The particle distribution in the beam occurs in bunches and the beam current therefore can be expressed by a Fourier series. Here we are only interested in the harmonic I_h of the beam current and find for bunches short compared to the rf-wavelength

$$I_\mathrm{h} = 2I_\mathrm{b} \tag{19.19}$$

where I_b is the average beam current and h the harmonic number. The approximation for short bunches with $\ell \ll \lambda_{\mathrm{rf}\text{-}}$ holds as long as $\sin k_{\mathrm{rf}\text{-}}\ell \approx k_{\mathrm{rf}\text{-}}\ell$ with $k_{\mathrm{rf}\text{-}} = 2\pi/\lambda_{\mathrm{rf}\text{-}}$. For longer bunches the factor 2 becomes a more complicated formfactor as can be derived from an appropriate Fourier expansion. At the resonance frequency $\omega_\mathrm{r} = h\omega_0$, the beam induced voltage in the cavity is then with (19.8) from (19.15)

$$V_\mathrm{br} = \frac{R_\mathrm{s} I_\mathrm{h}}{1+\beta} = \frac{2R_\mathrm{s} I_\mathrm{b}}{1+\beta}\,. \tag{19.20}$$

The resulting cavity voltage is the superposition of both voltages, the generator and the induced voltage. This superposition, including appropriate phase factors, is often represented in a phasor diagram. In such a diagram a complex quantity $\tilde{z}$ is represented by a vector of length $|\tilde{z}|$ with the horizontal and vertical components being the real and imaginary part of $\tilde{z}$, respectively. The phase of this vector increases counter clockwise and is given by $\tan\varphi = \mathrm{Im}(\tilde{z})\,/\mathrm{Re}(\tilde{z})$. In an application to rf-parameters we represent voltages and currents by vectors with a length equal to the magnitude of voltage or current and a counter clockwise rotation of the vector by the phase angle φ.

The particle beam current can be chosen as the reference being parallel to the real axis and we obtain from the quantities derived so far the phasor diagram as shown in Fig. 19.3. First we determine the relationships between individual vectors and phases and then the correct adjustments of variable rf-parameters. In Fig. 19.3 the generator current is assumed to have the still to be determined phase ϑ with respect to the beam current while the generator voltage and beam induced voltage lag by the phase Ψ behind the beam current. The resulting cavity voltage $\tilde{\boldsymbol{V}}_\mathrm{cy}$ is the phasor addition of both voltages $\tilde{\boldsymbol{V}}_\mathrm{g} + \tilde{\boldsymbol{V}}_\mathrm{b}$ as shown in Fig. 19.3.

The adjustment of the rf-system must now be performed in such a way as to provide the desired gain in kinetic energy $U_0 = e\,\hat{V}_\mathrm{cy}\sin\psi_\mathrm{s}$ where $\hat{V}_\mathrm{cy}$ is the maximum value of the cavity voltage and ψ_s the synchronous phase. To maximize the energy flow from the generator to the cavity the load must be matched such that it appears to the generator purely resistive. This is achieved by adjusting the phase ψ_g to get the cavity voltage V_cy and generator current I_g in phase which occurs for

$$\psi_\mathrm{g} = \tfrac{1}{2}\pi - \psi_\mathrm{s} \tag{19.21}$$

as shown in Fig. 19.4. Obviously, this is only true for a specific value of the beam current. General operation will deviate from this value and therefore we often match

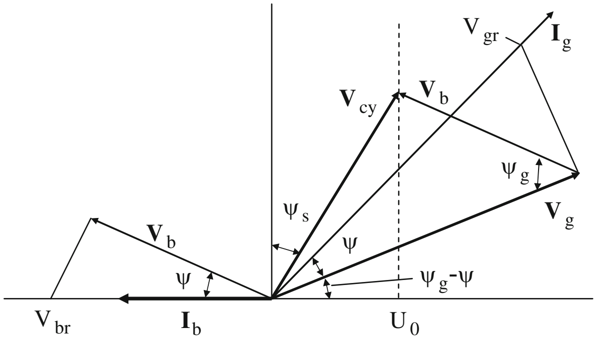

Fig. 19.3 Phasor diagram for an accelerating cavity and arbitrary tuning angle

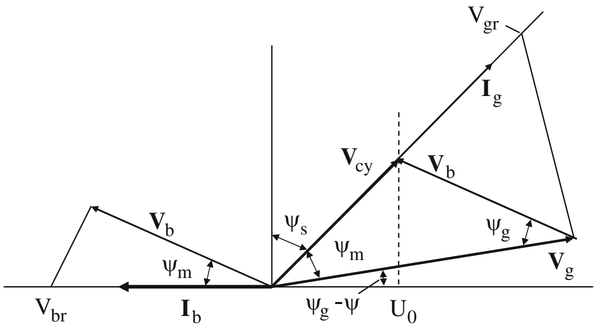

Fig. 19.4 Phasor diagram with optimum tuning angle

to the maximum desired beam current. For lower currents, where the energy transfer is not optimum anymore, some loss of efficiency is acceptable.

The tuning angle adjustment for optimum matching can be derived from Fig. 19.4 and applying the law of sines we have with (19.17)

$$\frac{V_b}{V_{cy}} = \frac{V_{br}\cos\Psi_m}{V_{cy}} = \frac{\sin\Psi_m}{\sin\psi_g} = \frac{\sin\Psi_m}{\cos\psi_s}\,. \tag{19.22}$$

The optimum tuning angle is from (19.22)

$$\tan\Psi_m = \frac{V_{br}}{V_{cy}}\cos\psi_s\,. \tag{19.23}$$

This tuning is effected by a shift in the resonant frequency of the cavity with respect to the generator frequency by, for example, moving a tuner in or out. From (19.13) we get with (19.9), (19.20), (19.23) for the frequency shift or frequency tuning

$$\delta\omega = \omega - \omega_{\mathrm{r}} = -\frac{\omega_{\mathrm{r}}}{2Q_0}\frac{P_{\mathrm{b}}}{P_{\mathrm{cy}}}\cot\psi_{\mathrm{s}}\,, \tag{19.24}$$

where the cavity power is defined by

$$P_{\mathrm{cy}} = \frac{V_{\mathrm{cy}}^2}{2R_{\mathrm{s}}} \tag{19.25}$$

and the beam power by

$$P_{\mathrm{b}} = I_{\mathrm{b}} V_{\mathrm{cy}} \sin\psi_{\mathrm{s}}\,. \tag{19.26}$$

To determine the required generator power the components of the cavity voltage vector can be expressed by other quantities and we get from Fig. 19.4

$$V_{\mathrm{cy}}\sin\psi_{\mathrm{s}} = V_{\mathrm{gr}}\cos\Psi_{\mathrm{m}}\cos\left(\psi_{\mathrm{g}} - \Psi_{\mathrm{m}}\right) - V_{\mathrm{br}}\cos^2\Psi_{\mathrm{m}} \tag{19.27}$$

and

$$V_{\mathrm{cy}}\cos\psi_{\mathrm{s}} = V_{\mathrm{gr}}\cos\Psi_{\mathrm{m}}\sin\left(\psi_{\mathrm{g}} - \Psi_{\mathrm{m}}\right) + V_{\mathrm{br}}\cos\Psi_{\mathrm{m}}\sin\Psi_{\mathrm{m}}\,. \tag{19.28}$$

Combining both equations to eliminate the phase $\left(\psi_{\mathrm{g}} - \Psi_{\mathrm{m}}\right)$, we get

$$V_{\mathrm{gr}}^2 = \left(V_{\mathrm{cy}}\frac{\sin\psi_{\mathrm{s}}}{\cos\Psi_{\mathrm{m}}} + V_{\mathrm{br}}\cos\Psi_{\mathrm{m}}\right)^2 + \left(V_{\mathrm{cy}}\frac{\cos\psi_{\mathrm{s}}}{\cos\Psi_{\mathrm{m}}} - V_{\mathrm{br}}\sin\Psi_{\mathrm{m}}\right)^2 \tag{19.29}$$

and with (19.16), (19.20) the required generator power for the condition of optimum matching is

$$P_{\mathrm{g}} = \frac{V_{\mathrm{cy}}^2}{2R_{\mathrm{s}}}\frac{(1+\beta)^2}{4\beta}\left[\left(\frac{\sin\psi_{\mathrm{s}}}{\cos\Psi_{\mathrm{m}}} + \frac{2R_{\mathrm{s}}I_{\mathrm{b}}}{V_{\mathrm{cy}}(1+\beta)}\cos\Psi_{\mathrm{m}}\right)^2 + \left(\frac{\cos\psi_{\mathrm{s}}}{\cos\Psi_{\mathrm{m}}} - \frac{2R_{\mathrm{s}}I_{\mathrm{b}}}{V_{\mathrm{cy}}(1+\beta)}\sin\Psi_{\mathrm{m}}\right)^2\right]. \tag{19.30}$$

This expression can be greatly simplified with (19.23) to become

$$P_{\mathrm{g,opt}} = \frac{(1+\beta)^2}{8\beta R_{\mathrm{s}}}\left(V_{\mathrm{cy}} + \frac{2R_{\mathrm{s}}I_{\mathrm{b}}}{1+\beta}\sin\psi_{\mathrm{s}}\right)^2. \tag{19.31}$$

Equation (19.31) represents a combination of beam current through I_{b}, rf–generator power P_{g}, coupling coefficient β, and shunt impedance R_{s} to sustain a cavity voltage V_{cy}. Specifically, considering that the rf-power P_{g} and coupling coefficient β is fixed by the hardware installed a maximum supportable beam current can be derived as a function of the desired or required cavity voltage. Solving for the cavity voltage, (19.31) becomes after some manipulation

$$V_{\mathrm{cy}} = \frac{\sqrt{2\beta R_{\mathrm{s}}}}{1+\beta}\left(\sqrt{P_{\mathrm{g,opt}}} + \sqrt{P_{\mathrm{g,opt}} - \frac{1+\beta}{\beta}P_{\mathrm{b}}}\right). \tag{19.32}$$

This expression exhibits a limit for the beam current above which the second square root becomes imaginary. The condition for real solutions requires that

$$P_{\mathrm{b}} \le \frac{\beta}{1+\beta}P_{\mathrm{g,opt}} \tag{19.33}$$

leading to a limit of the maximum sustainable beam current of

$$I_{\mathrm{b}} \le \frac{\beta}{1+\beta}\frac{P_{\mathrm{g}}}{V_{\mathrm{cy}}\sin\psi_{\mathrm{s}}}. \tag{19.34}$$

Inspection of (19.31) shows that the required generator power can be further minimized by adjusting for the optimum coupling coefficient β. Optimum coupling can be derived from $\partial P_{\mathrm{g}}/\partial\beta = 0$ with the solution

$$\beta_{\mathrm{opt}} = 1 + \frac{2R_{\mathrm{s}}I_{\mathrm{b}}}{V_{\mathrm{cy}}}\sin\psi_{\mathrm{s}} = 1 + \frac{P_{\mathrm{b}}}{P_{\mathrm{cy}}}. \tag{19.35}$$

The minimum generator power required to produce an accelerating voltage $V_{\mathrm{cy}}\sin\psi_{\mathrm{s}}$ is therefore from (19.31) with (19.35)

$$P_{\mathrm{g,min}} = \frac{V_{\mathrm{cy}}^2}{2R_{\mathrm{s}}}\beta_{\mathrm{opt}} = \beta_{\mathrm{opt}}P_{\mathrm{cy}} \tag{19.36}$$

and the optimum tuning angle from (19.23)

$$\tan\Psi_{\mathrm{opt}} = \frac{\beta_{\mathrm{opt}}-1}{\beta_{\mathrm{opt}}+1}\cot\psi_{\mathrm{s}}. \tag{19.37}$$

In this operating condition all rf-power from the generator is absorbed by the beam loaded cavity and no power reflection occurs. The maximum beam power is therefore $P_{\mathrm{b}} = P_{\mathrm{g}} - P_{\mathrm{cy}}$ and the maximum beam current

$$I_{\mathrm{b}} \le \frac{P_{\mathrm{g}}}{V_{\mathrm{cy}}\sin\psi_{\mathrm{s}}} - \frac{V_{\mathrm{cy}}}{2R_{\mathrm{s}}\sin\psi_{\mathrm{s}}}. \tag{19.38}$$

Conditions have been derived assuring most efficient power transfer to the beam by proper adjustment of the cavity power input coupler to obtain the optimum coupling coefficient. Of course this coupling coefficient is optimum only for a specific beam current which in most cases is chosen to be the maximum desired beam current.

We are now in a position to determine the total rf-power flow. From conservation of energy we have

$$P_{\mathrm{g}} = P_{\mathrm{cy}} + P_{\mathrm{b}} + P_{\mathrm{r}}\,, \tag{19.39}$$

where P_{r} is the reflected power which vanishes for the case of optimum coupling.

19.3 Higher-Order Mode Losses in an Rf-Cavity

The importance of beam loading for accurate adjustments of the rf-system has been discussed qualitatively but not yet quantitatively. In this paragraph, quantitative expressions will be derived for beam loading. Accelerating cavities constitute an impedance to a particle current and a bunch of particles with charge q passing through a cavity induces electromagnetic fields into a broad frequency spectrum limited at the high frequency end by the bunch length. The magnitude of the excited frequencies in the cavity depends on the frequency dependence of the cavity impedance, which is a function of the particular cavity design and need not be known for this discussion. Fields induced within a cavity are called modes, oscillating at different frequencies with the lowest mode being the fundamental resonant frequency of the cavity. Although cavities are designed primarily for one resonant frequency, many higher-order modes or HOM'scan be excited at higher frequencies. Such modes occur above the fundamental frequency first at distinct well-separated frequencies with increasing spectral densities at higher frequencies.

For a moment we consider here only the fundamental frequency and deal with higher-order modes later. Fields induced by the total bunch charge act back on individual particles modifying the overall accelerating voltage seen by the particle. To quantify this we use the fundamental theorem of beam loading formulated by Wilson [1] which states that each particle within a bunch sees one half of the induced field while passing through the cavity.

We prove this theorem by conducting a Gedanken experiment proposed by Wilson. Consider a bunch of particles with charge q passing through a lossless cavity inducing a voltage $V_{\mathrm{i}1}$ in the fundamental mode. This induced field is opposed to the accelerating field since it describes a loss of energy. While the bunch passes through the cavity this field increases from zero reaching a maximum value at the moment the particle bunch leaves the cavity. Each particle will have interacted with this field and the energy loss corresponds to a fraction f of the induced voltage $V_{\mathrm{i},h}$, where the index h indicates that we consider only the fundamental mode. The total energy lost by the bunch of charge q is

$$\Delta E_1 = -q_1 f\, V_{\mathrm{i},h} \,. \tag{19.40}$$

This energy appears as field energy proportional to the square of the voltage

$$W_1 = c_1 V_{\mathrm{i},h}^2 \,, \tag{19.41}$$

where c_1 is a constant.

Now consider another bunch with the same charge $q_2 = q_1 = q$ following behind the first bunch at a distance corresponding to half an oscillation period at the fundamental cavity frequency. In addition to its own induced voltage this second bunch will see the field from the first bunch, now being accelerating, and will therefore gain an energy

$$\Delta E_2 = q_1 V_{\mathrm{i},h} - q_2 f\, V_{\mathrm{i},h} = q V_{\mathrm{i},h}\,(1 - f\,) \,. \tag{19.42}$$

After passage of the second charge, the cavity returns to the original state before the first charge arrived because the field from the first charge having changed sign exactly cancels the induced field from the second charge. The cavity has been assumed lossless and energy conservation requires therefore that $\Delta E_1 + \Delta E_2 = 0$ or $-qf\, V_{\mathrm{i},h} + qV_{\mathrm{i},h}\,(1 - f\,) = 0$ from which we get

$$f = \tfrac{1}{2} \tag{19.43}$$

proving the statement of the fundamental theorem of beam loading. The energy loss of a bunch of charge q due to its own induced field is therefore

$$\Delta E_1 = -\tfrac{1}{2} q V_{\mathrm{i},h} \,. \tag{19.44}$$

This theorem will be used to determine the energy transfer from cavity fields to a particle beam. To calculate the induced voltages in rf-cavities, or in arbitrarily shaped vacuum chambers providing some impedance for the particle beam can become very complicated. For cylindrically symmetric cavities the induced voltages can be calculated numerically with programs like SUPERFISH [2], URMEL[3] or MAFIA [3].

For a more practical approach Wilson [1] introduced a loss parameter k which can be determined either by electronic measurements or by numerical calculations. This loss parameter for the fundamental mode loss of a bunch with charge q is defined by

$$\Delta E_h = k_h q^2 \tag{19.45}$$

and together with (19.44) we get the induced voltage

$$V_{\mathrm{i},h} = -2k_h q \tag{19.46}$$

or after elimination of the charge

$$\Delta E_h = \frac{V_{i,h}^2}{4k_h}\,, \tag{19.47}$$

where the index h indicates that the parameter should be taken at the fundamental frequency. The loss parameter can be expressed in terms of cavity parameters. From the definition of the cavity quality factor (18.80) and cavity losses from (18.77) we get

$$\frac{2R_{cy}}{Q} = \frac{V^2}{\omega W}\,, \tag{19.48}$$

where ω is the frequency and W the stored field energy in the cavity. Applying this to the induced field, we note that ΔE_h is equal to the field energy W_h and combining (19.47), (19.48) the loss parameter to the fundamental mode in a cavity with shunt impedance R_h and quality factor Q_h is

$$k_h = \frac{\omega_h}{4}\frac{R_h}{Q_h}\,. \tag{19.49}$$

The excitation of higher-order mode fields by the passing particle bunch leads to additional energy losses which are conveniently expressed in units of the energy loss to the fundamental mode

$$\Delta E_{hom} = (r_{hom} - 1)\,\Delta E_h\,, \tag{19.50}$$

where r_{hom} is the ratio of the total energy losses into all cavity modes to the loss into the fundamental mode only. The induced higher order field energy in the cavity is therefore

$$W_{hom} = (r_{hom} - 1)\,W_h\,. \tag{19.51}$$

Again we may define a loss parameter k_n for an arbitrary nth-mode and get analogous to (19.49)

$$k_n = \frac{\omega_n}{4}\frac{2R_n}{Q_n}\,, \tag{19.52}$$

where R_n and Q_n are the shunt impedance and quality factor for the nth-mode or frequency ω_n, respectively. The total loss parameter due to all modes is by linear superposition

$$k = \sum_n k_n\,. \tag{19.53}$$

The task to determine the induced voltages has been reduced to the determination of the loss parameters for individual modes or if this is not possible or desirable we may use just the overall loss parameter k as may be determined experimentally. This is particularly convenient for cases where it is difficult to calculate the mode losses but much easier to measure the overall losses by electronic measurements.

The higher-order mode losses will become important for discussion of beam stability since these fields will act back on subsequent particles and bunches thus creating a coupling between different parts of one bunch or different bunches.

19.3.1 Efficiency of Energy Transfer from Cavity to Beam

Higher-order mode losses affect the efficiency by which energy is transferred to the particle beam. Specifically, since the higher-order mode losses depend on the beam current we must expect some limitation in the current capability of the accelerator.

With these preparations we have now all information to calculate the transfer of energy from the cavity to the particle beam. Just before the arrival of a particle bunch let the cavity voltage as generated by the rf-power source be

$$V_{\mathrm{cy}} = -V_{\mathrm{g}}\,\mathrm{e}^{\mathrm{i}\Psi_{\mathrm{g}}}, \tag{19.54}$$

where V_{g} is the generator voltage and Ψ_{g} the generator voltage phase with respect to the particle beam. To combine the generator voltage with the induced voltage we use phasor diagrams in the complex plane.

The generator voltage is shown in Fig. 19.5 as a vector rotated by the angle Ψ_{g} from the real axis representing the cavity state just before the beam passes. The beam induced voltage is parallel and opposite to the real axis. Both vectors add up to the voltage V just after the beam has left the cavity.

The difference of the fundamental field energy before and after passage of the particle bunch is equal to the energy transferred to the passing particle bunch minus

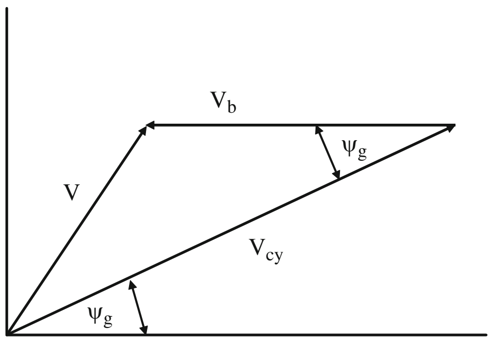

Fig. 19.5 Phasor diagram for cavity voltages with beam loading

the higher-order field energy and is from the phasor diagram

$$\Delta E_{\mathrm{hom}} = \alpha\left(V_{\mathrm{cy}}^2 - V^2\right) - W_{\mathrm{hom}} = \alpha\left(2V_{\mathrm{cy}}V_{\mathrm{b}}\cos\Psi_{\mathrm{g}} - V_{\mathrm{b}}^2\right) - W_{\mathrm{hom}}\,, \tag{19.55}$$

where α is the proportionality factor between the energy gain ΔE and the square of the voltage defined by $\alpha = \Delta E/V^2$. With (19.45), (19.46), we get from (19.55) the net energy transfer to a particle bunch [4]

$$\Delta E_{\mathrm{hom}} = \alpha\left(2V_{\mathrm{cy}}V_{\mathrm{b}}\cos\Psi_{\mathrm{g}} - r_{\mathrm{hom}}V_{\mathrm{b}}^2\right)\,. \tag{19.56}$$

The energy stored in the cavity before arrival of the beam is $W_{\mathrm{cy}} = \alpha V_{\mathrm{cy}}^2$ and the energy transfer efficiency to the beam becomes

$$\eta = \frac{\Delta E_h}{W_{\mathrm{cy}}} = 2\frac{V_{\mathrm{b}}}{V_{\mathrm{cy}}}\cos\Psi_{\mathrm{g}} - r_{\mathrm{hom}}\frac{V_{\mathrm{b}}^2}{V_{\mathrm{cy}}^2}\,. \tag{19.57}$$

It is obvious from (19.57) that energy transfer is not guaranteed by the synchronicity condition or the power of the generator alone. Specifically, the second term in (19.57) becomes dominant for a large beam intensity and the efficiency may even become negative indicating reversed energy transfer from the beam to cavity fields. The energy transfer efficiency has a maximum for $V_{\mathrm{b}} = \frac{\cos\Psi_{\mathrm{g}}}{r_{\mathrm{hom}}}V_{\mathrm{cy}}$ and is

$$\eta_{\max} = \frac{\cos^2\Psi_{\mathrm{g}}}{r_{\mathrm{hom}}}\,, \tag{19.58}$$

a result first derived by Keil et al. [5] and is therefore frequently called the Keil-Schnell-Zotter criterion. The maximum energy transfer efficiency is limited by the phase of the generator voltage and the higher-order mode losses in the cavity.

19.4 Beam Loading

Only one passage of a bunch through a cavity has been considered in the previous section. In circular accelerators, however, particle bunches pass periodically through the accelerating cavities and we have to consider the cumulative build up of induced fields. Whenever a particle bunch is traversing a cavity the induced voltage from this passage is added to those still present from previous bunch traversals. For simplicity, we assume a number of equidistant bunches along the circumference of the ring, where adjacent bunches are separated by an integer number m_{b} of the fundamental rf-wavelength. The induced voltage decays exponentially by a factor $\mathrm{e}^{-\rho}$ between

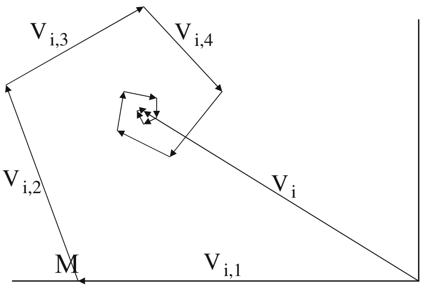

Fig. 19.6 Phasor diagram for the superposition of induced voltages in an accelerating cavity

two consecutive bunches with

$$\rho = \frac{t_b}{t_d}\,, \tag{19.59}$$

where t_b is the time between bunches and t_d the cavity voltage decay time for the fundamental mode. The phase of the induced voltage varies between the passage of two consecutive bunches by

$$\varphi = \omega_h t_b - 2\pi m_b\,. \tag{19.60}$$

At the time a bunch passes through the cavity the total induced voltages are then the superposition of all fields induced by previous bunches

$$V_i = V_{i,h}\left(1 + e^{-\rho}e^{i\varphi} + e^{-2\rho}e^{i2\varphi} + \cdots\right) \tag{19.61}$$

shown in Fig. 19.6 as the superposition of all induced voltages in form of a phasor diagram together with the resultant induced voltage V_i. The sum (19.61) can be evaluated to be

$$V_i = V_{i,h}\frac{1}{1 - e^{-\rho}e^{i\varphi}}\,, \tag{19.62}$$

which is the total induced voltage in the cavity just after the last bunch passes; however, the voltage seen by this last bunch is only half of the induced voltage and the total voltage V_b acting back on the bunch is therefore

$$V_b = V_{i,h}\left(\frac{1}{1 - e^{-\rho}e^{i\varphi}} - \frac{1}{2}\right). \tag{19.63}$$

The voltage $V_{i,h}$ can be expressed in more practical units. Considering the damping time (18.62) for fields in a cavity we note that two damping times exist, one for the empty unloaded cavity t_{d0} and a shorter damping time t_d when there is also a beam present. For the unloaded damping time we have from (18.62)

$$t_{d0} = \frac{2Q_{0h}}{\omega_h}\,, \tag{19.64}$$

where Q_{0h} is the unloaded quality factor. From (19.45), (19.47) we get with $q = I_0 t_b$, where I_0 is the average beam current,

$$V_{i,h} = \frac{\omega_h}{2Q_{0h}} R_h I_0 t_b$$

and with (19.64)

$$V_{i,h} = R_h I_0 \frac{t_b}{t_{d0}}\,. \tag{19.65}$$

Introducing the coupling coefficient β, we get from (19.9), (19.64)

$$t_{d0} = (1+\beta)\, t_b\,. \tag{19.66}$$

In analogy to (19.59) we define

$$\rho_0 = \frac{\rho}{1+\beta} = \frac{t_b}{t_{d0}} \tag{19.67}$$

and (19.65) becomes

$$V_{i,h} = \rho_0 R_h I_0\,. \tag{19.68}$$

We are finally in a position to calculate from (19.63), (19.68) the total beam induced cavity voltage V_b in the fundamental mode for circular accelerators.

19.5 Phase Oscillation and Stability

In the course of discussing phase oscillations we found it necessary to select carefully the synchronous phase depending on the particle energy being below or above the transition energy. Particularly, we found that phase stability for particles above transition energy requires the rf-voltage to decrease with increasing phase. From the derivative of (19.27) with respect to ψ_s we find with (19.21) and since $V_{gr} \cos \Psi > 0$ the condition for phase stability $\sin(\psi_g - \Psi_m) < 0$ or

$$\tfrac{1}{2}\pi < \psi_s + \Psi_m < \tfrac{3}{2}\pi\,. \tag{19.69}$$

From (19.28) and (19.69) we get

$$-V_{\mathrm{cy}}\,|\cos\psi_{\mathrm{s}}| - V_{\mathrm{br}}\sin\Psi_{\mathrm{m}}\cos\Psi_{\mathrm{m}} < 0$$

or with (19.23)

$$V_{\mathrm{br}}\sin\psi_{\mathrm{s}} < V_{\mathrm{cy}}\,, \tag{19.70}$$

which is Robinson's phase-stability criterion or the Robinson condition [6] for the tuning angle of the accelerator cavity. The maximum current that can be accelerated in a circular accelerator with stable phase oscillations is limited by the effective cavity voltage. In terms of rf-power, (19.70) is with (19.20) equivalent to

$$P_{\mathrm{b}} \le (1+\beta)\,P_{\mathrm{cy}} \tag{19.71}$$

and the stability condition for the coupling coefficient is from (19.35)

$$\beta > \beta_{\mathrm{opt}} - 2\,. \tag{19.72}$$

The stability condition is always met for rf-cavities with optimum coupling $\beta = \beta_{\mathrm{opt}}$.

19.5.1 Robinson Damping

Correct tuning of the rf-system is a necessary but not a sufficient condition for stable phase oscillations. In Chap. 12 we found the occurrence of damping or anti-damping due to forces that depend on the energy of the particle. Such a case occurs in the interaction of bunched particle beams with accelerating cavities or vacuum chamber components which act like narrow band resonant cavities. The revolution time of a particle bunch depends on the average energy of particles within a bunch and the Fourier spectrum of the bunch current being made up of harmonics of the revolution frequency is therefore energy dependent. On the other hand by virtue of the frequency dependence of the cavity impedance, the energy loss of a bunch in the cavity due to beam loading depends on the revolution frequency. We have therefore an energy dependent loss mechanism which can lead to damping or worse anti-damping of coherent phase oscillation and we will therefore investigate this phenomenon in more detail. Robinson [6] studied first the dynamics of this effect generally referred to as Robinson damping or Robinson instability.

Above transition energy the revolution frequency is lower for higher bunch energies compared to the reference energy and vice versa. To obtain damping of coherent phase oscillations, we would therefore tune the cavity such that the bunch would loose more energy in the cavity while at higher energies (lower frequency) during the course of coherent synchrotron oscillation and loose less energy at

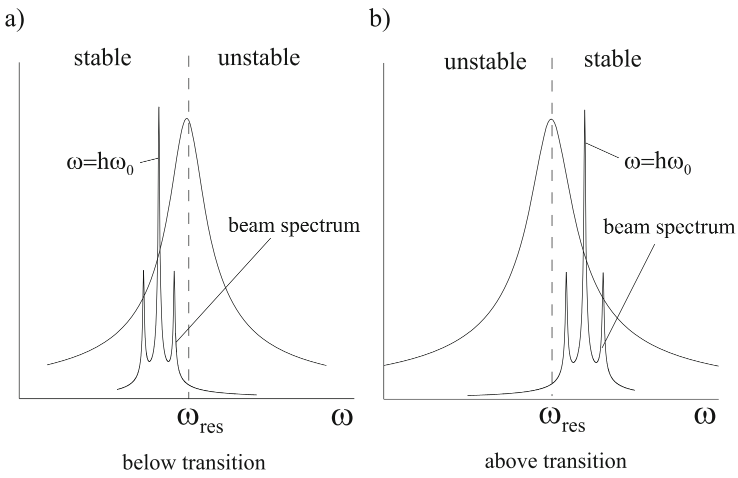

Fig. 19.7 Cavity tuning for positive Robinson damping below and above transition energy. (**a**) Below transition. (**b**) Above transition

lower energies (higher frequency). In this situation, the impedance of the cavity should decrease with increasing frequency for damping to occur as demonstrated in Fig. 19.7.

Here the resonance curve or impedance spectrum is shown for the case of a resonant frequency above the beam frequency $h\omega_0$ in Fig. 19.7a and below the beam frequency in Fig 19.7b. Consistent with the arguments made above we would expect damping in case of Fig. 19.7b for a beam above transition and anti-damping in case of Fig. 19.7a. Adjusting the resonance frequency of the cavity to a value below the beam frequency $h\omega_0$where $\omega_0$is the revolution frequency, is called capacitive detuning. Conversely, we would tune the cavity resonance frequency above the beam frequency ($\omega_\mathrm{r} > h\omega_0$) or inductively detune the cavity for damping below transition energy (Fig. 19.7a).

In a more formal way we fold the beam-current spectrum with the impedance spectrum of the cavity and derive scaling laws for the damping as well as the shift in synchrotron frequency. During phase oscillations the revolution frequency is modulated and as a consequence the beam spectrum includes in addition to the fundamental frequency two side bands or satellites. The beam-current spectrum is composed of a series of harmonics of the revolution frequency up to frequencies with wavelength of the order of the bunchlength

$$I(t) = I_\mathrm{b} + \sum_{n>0} I_\mathrm{n} \cos\left(n\omega_0 t - \varphi\right) , \tag{19.73}$$

where I_b is the average circulating beam current and φ a phase shift with respect to the reference particle. The Fourier coefficient for bunches short compared to the wavelength of the harmonic is given by

$$I_\mathrm{n} = 2I_\mathrm{b}\,. \tag{19.74}$$

Here we restrict the discussion to the interaction between beam and cavity at the fundamental cavity frequency and the only harmonic of interest in the beam spectrum is therefore the hth harmonic

$$I_h(t) = 2I_\mathrm{b}\cos\left(h\omega_0 t - \varphi\right)\,. \tag{19.75}$$

By virtue of coherent synchrotron oscillations the phase oscillates for each particle in a bunch like

$$\varphi(t) = \varphi_0 \sin \Omega_\mathrm{s} t\,, \tag{19.76}$$

where φ_0 is the maximum amplitude and Ω_s the synchrotron oscillation frequency of the phase oscillation. We insert this into (19.75) and get after expanding the trigonometric functions for small oscillation amplitudes $\varphi_0 \ll 1$

$$I_h(t) = 2I_\mathrm{b}\cos\left(h\omega_0 t\right) - I_\mathrm{b}\varphi_0\left[\cos\left(h\omega_0 t + \Omega_\mathrm{s} t\right) - \cos\left(h\omega_0 t - \Omega_\mathrm{s} t\right)\right]. \tag{19.77}$$

This expression exhibits clearly sidebands or satellites in the beam spectrum at $h\omega_0 \pm \Omega_\mathrm{s}$. Folding the expression for the beam current with the cavity impedance defines the energy loss of the particle bunch while passing through the cavity. The cavity impedance is a complex quantity which was derived in (19.11) and its real part is shown together with the beam spectrum in Fig. 19.8. The induced voltage in the cavity by a beam $I_h(t) = I_h \cos h\omega_0 t$ is

$$V_h = -ZI_h(t) = -Z_\mathrm{r} I_h \cos\left(h\omega_0 t\right) - Z_\mathrm{i} I_h \sin\left(h\omega_0 t\right)\,, \tag{19.78}$$

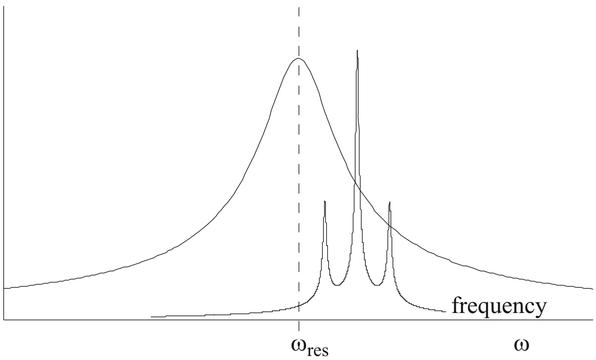

Fig. 19.8 Cavity impedance and beam spectrum in the vicinity of the fundamental rf frequency $\omega_\mathrm{rf} = h\omega_0$

where we have split the impedance in its real and imaginary part and have expressed the imaginary part of the induced voltage by a $-\pi/2$ phase shift. Applying (19.78) to all components of the beam current (19.77) we get the induced voltage in the cavity

$$\begin{aligned} V_h = &-Z_\mathrm{r}^0 2I_\mathrm{b}\cos h\omega_0 t - Z_\mathrm{i}^0 2I_\mathrm{b}\sin h\omega_0 t \\ &+ Z_\mathrm{r}^+ I_\mathrm{b}\varphi_0\cos h\omega_0 t\,\cos\Omega_\mathrm{s}t - Z_\mathrm{r}^+ I_\mathrm{b}\varphi_0\sin h\omega_0 t\,\sin\Omega_\mathrm{s}t \\ &+ Z_\mathrm{i}^+ I_\mathrm{b}\varphi_0\sin h\omega_0 t\,\cos\Omega_\mathrm{s}t + Z_\mathrm{i}^+ I_\mathrm{b}\varphi_0\cos h\omega_0 t\,\sin\Omega_\mathrm{s}t \\ &- Z_\mathrm{r}^- I_\mathrm{b}\varphi_0\cos h\omega_0 t\,\cos\Omega_\mathrm{s}t - Z_\mathrm{r}^- I_\mathrm{b}\varphi_0\sin h\omega_0 t\,\sin\Omega_\mathrm{s}t \\ &- Z_\mathrm{i}^- I_\mathrm{b}\varphi_0\sin h\omega_0 t\,\cos\Omega_\mathrm{s}t + Z_\mathrm{i}^- I_\mathrm{b}\varphi_0\cos h\omega_0 t\,\sin\Omega_\mathrm{s}t\,, \end{aligned} \tag{19.79}$$

where Z^0, Z^+ and Z^- are the real r and imaginary i cavity impedances at the frequencies $h\omega_0$, $h\omega_0 + \Omega_\mathrm{s}$, $h\omega_0 - \Omega_\mathrm{s}$ respectively. We make use of the expression for the phase oscillation (19.76) and its derivative

$$\dot{\varphi}(t) = \varphi_0\Omega_\mathrm{s}\cos\Omega_\mathrm{s}t, \tag{19.80}$$

multiply the induced voltage spectrum (19.79) by the current spectrum (19.77) and get after averaging over fast oscillating terms at frequency $h\omega_0$

$$\langle V_h I_h\rangle = -2I_\mathrm{b}^2\left\{Z_\mathrm{r}^0 - \left[Z_\mathrm{i}^0 - \tfrac{1}{2}\left(Z_\mathrm{i}^+ + Z_\mathrm{i}^-\right)\right]\varphi - \frac{Z_\mathrm{r}^+ - Z_\mathrm{r}^-}{2\Omega_\mathrm{s}}\dot{\varphi}\right\}. \tag{19.81}$$

This is the rate of energy loss of the particle bunch into the impedance of the cavity. Dividing by the total circulating charge $T_0 I_\mathrm{b}$ we get the rate of relative energy loss per unit charge

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\frac{\langle eV_h I_h\rangle}{T_0 I_\mathrm{b} E_0} = +\frac{\ddot{\varphi}}{\beta c k_h\,|\eta_\mathrm{c}|}, \tag{19.82}$$

where T_0 is the revolution time and I_b the average beam current.

We made use of the relation between the energy deviation from the ideal energy and the rate of change of the phase (9.17) on the r.h.s. of the equation. From (19.81), (19.82) and making use of the definition of the synchrotron frequency in (9.32) $\Omega_{\mathrm{s}0}^2 = \frac{ck_\mathrm{h}|\eta_\mathrm{c}|}{E_0 T_0} eV_\mathrm{cy}\,|\cos\psi_\mathrm{s}|$, we get a differential equation of the form

$$\ddot{\varphi} + 2\alpha_R\dot{\varphi} + \Delta\Omega^2\varphi = 0 \tag{19.83}$$

with a Robinson damping decrement

$$\alpha_R = -\frac{\beta\Omega_{\mathrm{s}0}}{2V_\mathrm{cy}\,|\cos\psi_\mathrm{s}|}\left(Z_\mathrm{r}^+ - Z_\mathrm{r}^-\right)I_\mathrm{b}\,, \tag{19.84}$$

and a shift in synchrotron oscillation frequency

$$\Delta\Omega^2 = -\frac{2\beta\Omega_{s0}^2}{V_{cy}\,|\cos\psi_s|}\left[Z_i^0 - \tfrac{1}{2}\left(Z_i^+ + Z_i^-\right)\right] I_b\,. \tag{19.85}$$

The unperturbed phase equation (9.26) is

$$\ddot{\varphi} + 2\alpha_{s0}\dot{\varphi} + \Omega_{s0}^2\,\varphi = 0 \tag{19.86}$$

and combining both, we derive a modification of both the damping and oscillation frequency. The combined damping decrement is

$$\alpha_s = \alpha_{s0} - \frac{\beta\Omega_{s0}}{V_{cy}\,|\cos\psi_s|}\left(Z_r^+ - Z_r^-\right) I_b > 0 \tag{19.87}$$

where α_{s0} is the radiation damping in electron accelerators. The total damping decrement must be positive for beam stability. The interaction of the beam with the accelerating cavity above transition is stable for all values of the beam current if $Z_r^+ < Z_r^-$ or if the cavity resonant frequency is capacitively detuned. Due to the imaginary part of the impedance the interaction of beam and cavity leads to a synchrotron oscillation frequency shift given by

$$\Omega_s^2 = \Omega_{s0}^2 - \frac{2\beta\Omega_{s0}^2}{V_{cy}\,|\cos\psi_s|}\left[Z_i^0 - \tfrac{1}{2}\left(Z_i^+ + Z_i^-\right)\right] I_b. \tag{19.88}$$

This frequency shift has two components, the incoherent frequency shift due to the impedance Z_i^0 at the fundamental beam frequency $h\omega_0$ and a frequency shift for coherent bunch-phase oscillations due to the imaginary part of the cavity impedances. For small frequency shifts $\Delta\Omega_s = \Omega_s - \Omega_{s0}$, (19.88) can be linearized for

$$\frac{\Delta\Omega_s}{\Omega_{s0}} = -\frac{I_b\beta}{V_{cy}\,|\cos\psi_s|}\left[Z_i^0 - \tfrac{1}{2}(Z_i^+ + Z_i^-)\right]. \tag{19.89}$$

The cavity impedance is from (19.10)

$$Z = R_s\,\frac{1 - \mathrm{i}Q_0\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q_0^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}. \tag{19.90}$$

From the imaginary part of the cavity impedance and capacitive detuning we conclude that above transition energy, the incoherent synchrotron tune shift is positive

$$\Delta\Omega_{s,\text{incoh}} > 0 \tag{19.91}$$

while the coherent synchrotron tune shift is negative

$$\Delta\Omega_{s,coh} < 0. \tag{19.92}$$

This conclusion may in special circumstances be significantly different due to other passive cavities in the accelerator. The shift in the synchrotron tune is proportional to the beam current and can be used as a diagnostic tool to determine the cavity impedance or its deviation from the ideal model (19.90).

In the preceding discussion it was assumed that only resonant cavities contribute to Robinson damping. This is correct to the extend that other cavity like structures of the vacuum enclosure in a circular accelerator have a low quality factor Q for the whole spectrum or at least at multiples of the revolution frequency and therefore do not contribute significantly to this effect through a persistent energy loss over many turns. Later we will see that such low-Q structures in the vacuum chamber may lead to other types of beam instability.

19.5.2 Potential Well Distortion

The synchrotron frequency is determined by the slope of the rf-voltage at the synchronous phase. In the last subsection the effect of beam loading at the cavity fundamental frequency was discussed demonstrating the need to include the induced voltages in the calculation of the synchrotron oscillation frequency. These induced voltages cause a perturbation of the potential well and as a consequence a change in the bunch length. In this subsection we will therefore also include higher-order interaction of the beam with its environment.

It is not possible to derive a general expression for the impedance of all components of a vacuum chamber in a circular accelerator. However, measurements [7] have shown that the impedance spectrum of circular accelerator vacuum chambers, while excluding accelerating cavities, has the form similar to that of the SPEAR storage ring shown in Fig. 19.9.

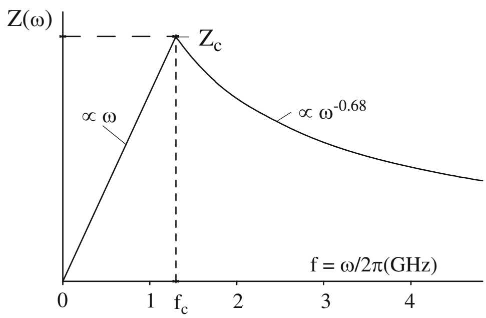

Fig. 19.9 SPEAR impedance spectrum [7]

Up to the transition frequency f_t, which is determined by vacuum chamber dimensions, the impedance is predominantly inductive and becomes capacitive above the transition frequency. We are looking here only for fields with wavelength longer than the bunch length which may distort the rf-voltage waveform such as to change the slope for the whole bunch. Later we will consider shorter wavelength which give rise to perturbations within the bunch. Because the bunch length is generally of the order of vacuum chamber dimensions we only need to consider the impedance spectrum below transition frequency which is predominantly inductive. To preserve generality, however, we assume a more general but still purely imaginary impedance defined by

$$Z(\omega)_\| = \mathrm{i}\omega\mathcal{Z}_\| . \tag{19.93}$$

Studying the modification of a finite bunch length due to potential-well distortions we use for mathematical simplicity a parabolic particle distribution [8] in phase (Fig. 19.10) normalized to the bunch current $\int_{-\varphi_\ell}^{\varphi_\ell} I(\varphi)\,\mathrm{d}\varphi = I_\mathrm{b}$

$$I(\varphi) = \frac{3I_\mathrm{b}}{4\varphi_\ell}\left(1 - \frac{\varphi^2}{\varphi_\ell^2}\right), \tag{19.94}$$

where $2\varphi_\ell$ is the bunch length expressed in terms of a phase with respect to the fundamental rf-wavelength. The combined induced voltage in the whole vacuum chamber is

$$V_\mathcal{Z} = \mathcal{Z}_\| \frac{\mathrm{d}I}{\mathrm{d}t} = h\omega_0\mathcal{Z}_\| \frac{\mathrm{d}I}{\mathrm{d}\varphi} = h\mathrm{Im}\left(Z_\|/n\right)\frac{\mathrm{d}I}{\mathrm{d}\varphi}, \tag{19.95}$$

where we have introduced the normalized impedance

$$\frac{Z_\|}{n} = \mathrm{i}\omega_0\mathcal{Z}_\| , \tag{19.96}$$

which is the longitudinal impedance divided by the frequency in units of the revolution frequency or by the mode number $n = \omega/\omega_0$. Inserting (19.94)

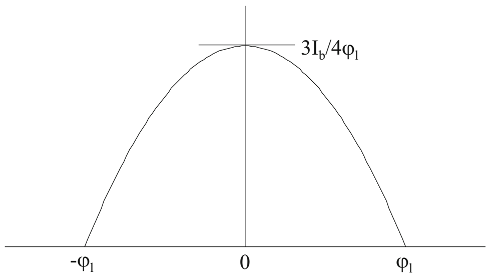

Fig. 19.10 Current distribution for potential-well distortion

into (19.95) we get the induced voltage

$$V_{\mathcal{Z}} = -\frac{3hI_{\mathrm{b}}\,\mathrm{Im}\left(Z_{\|}/n\right)}{2\varphi_{\ell}^{3}}\varphi\,, \tag{19.97}$$

which must be added to the rf-voltage $V_{\text{rf-}} = V_{\text{cy}} \sin\left(\psi_{\text{s}} + \varphi\right)$. Forming an effective voltage we get

$$V_{\text{eff}} = V_{\text{cy}}\cos\psi_{\text{s}}\left(1 - \frac{3hI_{\mathrm{b}}\,\mathrm{Im}\left(Z_{\|}/n\right)}{2\varphi_{\ell}^{3}V_{\text{cy}}\cos\psi_{\text{s}}}\right)\varphi + V_{\text{cy}}\sin\psi_{\text{s}}\,. \tag{19.98}$$

This modification of the effective cavity voltage leads to an incoherent shift of the synchrotron oscillation frequency

$$\frac{\Omega_{\mathrm{s}}^{2}}{\Omega_{\mathrm{s0}}^{2}} = 1 - \frac{3\eta_{\mathrm{c}}eI_{\mathrm{b}}}{4\pi\varphi_{\ell}^{3}E\nu_{\mathrm{s}}^{2}}\mathrm{Im}\left(Z_{\|}/n\right)\,, \tag{19.99}$$

where we used the definition of the synchrotron tune $\nu_{\mathrm{s}}^{2} = \frac{\eta_{\mathrm{c}}eV_{\mathrm{cy}}\cos\psi_{\mathrm{s}}}{2\pi hE}$.

Above transition energy $\eta_{\mathrm{c}}\cos\psi_{\mathrm{s}} > 0$ and therefore the frequency shift is positive for $\mathrm{Im}(Z_{\|}/n) < 0$ and negative for $\mathrm{Im}\{Z_{\|}/n\} > 0$. We note specifically that the shift depends strongly on the bunch length and increases with decreasing bunch length, a phenomenon we observe in all higher-order mode interactions.

Note that this shift of the synchrotron oscillation frequency does not appear for coherent oscillations since the induced voltage also moves with the bunch oscillation. The bunch center actually sees always the unaltered rf-field and oscillates according to the slope of the unperturbed rf-voltage. The coherent synchrotron oscillation frequency therefore need not be the same as the incoherent frequency. This has some ramification for the experimental determination of the synchrotron oscillation frequency.

The shift in incoherent synchrotron oscillation frequency reflects also a change in the equilibrium bunch length which is different for proton or ion beams compared to an electron bunch. The energy spread of radiating electron beams is determined only by quantum fluctuations due to the emission of synchrotron radiation and is independent of rf-fields. The electron bunch length scales therefore inversely proportional to the synchrotron oscillation frequency and we get with $\Omega_{\mathrm{s}}/\Omega_{\mathrm{s0}} = \sigma_{\ell 0}/\sigma_{\ell}$ from (19.99) after solving for $\sigma_{\ell}/\sigma_{\ell 0}$

$$\frac{\sigma_{\ell}^{3}}{\sigma_{\ell 0}^{3}} - \frac{\sigma_{\ell}}{\sigma_{\ell 0}} - \frac{8\eta_{\mathrm{c}}eI_{\mathrm{b}}}{9\pi^{2}\sqrt{2\pi}\sigma_{\ell 0}^{3}E\nu_{\mathrm{s}}^{2}}\mathrm{Im}\left(\frac{Z_{\|}}{n}\right) = 0\,, \tag{19.100}$$

where we replaced the parabolic current distribution by a Gaussian distribution with equal total bunch current and equal intensity in the bunch center by setting $\varphi_{\ell} = 3\sqrt{2\pi}/4h\sigma_{\ell}/\overline{R}$ and where $\sigma_{\ell 0}$ is the unperturbed bunch length.

Non-radiating particles, in contrast, must obey Liouville's theorem and the longitudinal beam emittance $\ell\,\Delta p$ will not change due to potential-well distortions. For proton or ion bunches we employ the same derivation for the bunch lengthening but note that the bunch length scales with the energy spread in such a way that the product of bunch length ℓ and momentum spread Δp remains constant. Therefore $\ell \propto 1/\sqrt{\Omega_{\rm s}}$ and the perturbed bunch length is from (19.99) with $\ell = (\overline{R}/h)\,\varphi_\ell$

$$\frac{\ell^4}{\ell_0^4} - \frac{3\eta_{\rm c} e I_{\rm b}}{4\pi E\,\nu_{\rm s}^2}\frac{\overline{R}^3}{\ell_0^3}\mathrm{Im}\left(\frac{Z_\|}{n}\right)\frac{\ell}{\ell_0} - 1 = 0\,. \tag{19.101}$$

Of course, along with this perturbation of the proton or ion bunch length goes an opposite perturbation of the energy spread.

Problems

19.1 (S). Consider an electron storage ring to be used as a damping ring for a linear collider. The energy is $E = 1.21$ GeV, circumference $C = 35.27$ m, bending radius $\rho = 2.037$ m, momentum compaction factor $\alpha_{\rm c} = 0.01841$, rf harmonic number $h = 84$, cavity shunt impedance of $R_{\rm cy} = 8.4\ \mathrm{M\Omega}$. An intense bunch of $N_{\rm e} = 5 \times 10^{10}$ particles is injected in a single pulse and is stored for only a few msec to damp to a small beam emittance. Specify and optimize a suitable rf-system and calculate the required rf-cavity power, cavity voltage, coupling factor first while ignoring beam loading and then with beam loading. Assume a quantum lifetime of 1 h.

19.2 (S). Show that for bunches short compared to the rf-wavelength the harmonic amplitudes are $I_{\rm h} = 2I_{\rm b}$.

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