Chapter 6
Summary: Decomposition of Second Rank Tensors

Abstract This chapter provides a summary of formulae for the decomposition of a Cartesian second rank tensor into its isotropic, antisymmetric and symmetric traceless parts.

Any second rank tensor $A_{\mu\nu}$ can be decomposed into its isotropic part, associated with a scalar, its antisymmetric part, linked a vector, and its irreducible, symmetric traceless part:

$$A_{\mu\nu} = \frac{1}{3} A_{\lambda\lambda} \delta_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\lambda} c_\lambda + \overline{A_{\mu\nu}} . \tag{6.1}$$

The dual vector $\mathbf{c}$ is linked with the antisymmetric part of the tensor by

$$c_\lambda = \varepsilon_{\lambda\sigma\tau} A_{\sigma\tau} = \varepsilon_{\lambda\sigma\tau} \frac{1}{2} (A_{\sigma\tau} - A_{\tau\sigma}). \tag{6.2}$$

The symmetric traceless second rank tensor, as defined previously, is

$$\overline{A_{\mu\nu}} = \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{3} A_{\lambda\lambda} \delta_{\mu\nu}. \tag{6.3}$$

Similarly, for a dyadic tensor composed of the components of the two vectors $\mathbf{a}$ and $\mathbf{b}$, the relations above give

$$a_\mu b_\nu = \frac{1}{3} (\mathbf{a} \cdot \mathbf{b}) \delta_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\lambda} c_\lambda + \overline{a_\mu b_\nu} . \tag{6.4}$$

The isotropic part involves the scalar product $(\mathbf{a} \cdot \mathbf{b})$ of the two vectors. The antisymmetric part is linked with the cross product of the two vectors, here one has

$$c_\lambda = \varepsilon_{\lambda\sigma\tau} a_\sigma b_\tau = (\mathbf{a} \times \mathbf{b})_\lambda. \tag{6.5}$$

The symmetric traceless part of the dyadic tensor is

$$\overline{a_\mu b_\nu} = \frac{1}{2} (a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{3} a_\lambda b_\lambda \delta_{\mu\nu}. \tag{6.6}$$

S. Hess, *Tensors for Physics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-12787-3_6