Surface Matching and Registration by Landmark Curve-Driven Canonical Quasiconformal Mapping

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Abstract. This work presents a novel surface matching and registration method based on the landmark curve-driven canonical surface quasiconformal mapping, where an open genus zero surface decorated with landmark curves is mapped to a canonical domain with horizontal or vertical straight segments and the local shapes are preserved as much as possible. The key idea of the canonical mapping is to minimize the harmonic energy with the landmark curve straightening constraints and generate a quasi-holomorphic 1-form which is zero in one parameter along landmark and results in a quasiconformal mapping. The mapping exists and is unique and intrinsic to surface and landmark geometry. The novel shape representation provides a conformal invariant shape signature. We use it as Teichmüller coordinates to construct a subspace of the conventional Teichmüller space which considers geometry feature details and therefore increases the discriminative ability for matching. *Furthermore*, we present a novel and efficient registration method for surfaces with landmark curve constraints by computing an optimal mapping over the canonical domains with straight segments, where the curve constraints become linear forms. Due to the linearity of 1-form and harmonic map, the algorithms are easy to compute, efficient and practical. Experiments on human face and brain surfaces demonstrate the efficiency and efficacy and the potential for broader shape analysis applications.

1 Introduction

In computer vision, efficient shape representations for surfaces are highly desired to effectively deal with the shape analysis problems, such as shape indexing, matching, recognition, classification, and registration [6,8,9,16]. *Canonical* surface mappings such as conformal mappings provide shape representations with good properties, which are global and intrinsic and have the guarantee of existence and uniqueness.

In this work, we compute a special category of quasiconformal mappings for surfaces decorated landmark curves, whose angle distortion (quasiconformality) is implied by the landmark curve straightening constraints. Such mappings are *canonical* and *intrinsic*. They give a novel type of shape representation, which encodes the landmark curves' geometry and their relation to background surface context, and provides a global and intrinsic shape signature to classify surfaces in shape space. Specially, the shape signature is invariant if the surface encounters a conformal transformation (conformal invariant). Using this as the Teichmüller coordinates, we construct a landmark-driven Teichmüller space, which is a subspace of the conventional Teichmüller space, constrained by the

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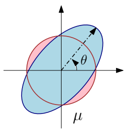
landmark curves. We then apply this canonical shape representation for matching and registration purposes for the case of *landmark curve decorated surfaces*.

1.1 Motivation

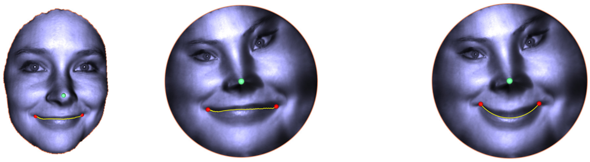
In practice, features on surface are preferred for surface matching and registration purposes. Such features usually include feature points, landmark curves, or regions of interest. For example, in medical applications, *anatomical* landmarks are used in computer-aided diagnosis and tumor or abnormality detection, such as sulci and gyri curves in brain mapping and facial symmetry curves in adolescent idiopathic scoliosis (AIS) and autism diagnosis. These landmarks may be manually labeled by doctors or automatically extracted. Conformal mapping is computed for *pure surfaces* without any interior constraints. Surfaces to be registered can be first mapped to 2D canonical domains conformally and then a mapping over them is built with feature constraints [7]. One strategy to handle landmark curve constraints is to slice surface open along them, and map them to boundaries of canonical domain by hyperbolic metric [18], but it is highly nonlinear.

In general, surface mapping will introduce angle and/or area distortions inevitably. If angle distortion is reduced to the limit (zero), then the mapping is *conformal* (C). If the angle distortion is bounded, then the mapping is *quasiconformal* (QC). Geometrically, conformal mapping maps infinitesimal circles to circles, while quasiconformal mapping maps infinitesimal ellipses to circles. The distortion from ellipse to circle is encoded into Beltrami coefficient, denoted as μ, which is complex-valued. A conformal mapping has zero μ everywhere. A quasiconformal mapping corresponds to a μ; and a μ determines a quasiconformal mapping uniquely up to a Möbius transformation.



In practice, it is hard to prescribe μ for the desired mapping; however, it is easy to set target canonical shapes for landmark curves (straight lines or circular arcs/loops). Therefore, in this work, we use landmark straightening constraints to adapt surface conformal structure, such that the resulting quasiconformal mapping preserves the local shapes as much as possible (see Fig. 1). This mapping is intrinsic to surface and landmark geometry and reveals the characteristics of landmark curves.

This landmark-driven canonical form will pave a novel way for efficient and effective matching and registration of surfaces decorated with landmark curve cases. It provides a global shape signature by combining the conformal module of the background domain and the configuration of the canonically mapped landmark curves, and used to construct a Teichmüller space which considers more dimensional information (surface features) besides the surface itself. *For example*, conformal mapping cannot differentiate topological disk surfaces, because they share the same conformal structure. If we consider the landmark curve constraints, then the topological disks can be compared using the locations and sizes of the straightened landmark curves on the canonical domain. *Moreover*, the canonical quasiconformal mapping provides an approach to introducing the landmark curve constraints to registration process in a linear way, which is efficient. It deals with landmark curves with surface together, without changing topology, and is linear and easy to compute. It is fundamental and will foster a broad range of real applications with landmark curve constraints in both engineering and medicine.



(a) 3D surface (b) conformal map (c) canonical quasiconformal map

Fig. 1. Surface mapping for a human facial surface. The mouth landmark l_m is employed

1.2 Related Works

In the past decade, a lot of research [5,17] focuses on conformal mapping methods, including the least square conformal maps [11], differential forms [3], discrete curvature flows [2], and so on. According to surface uniformization theorem [4], any arbitrary surface can be conformally mapped to one of three canonical spaces, the unit sphere, the Euclidean plane or the hyperbolic disk, which has been carried out [7].

As a general mapping, quasiconformal mapping has been arousing more and more attention recently. The surface conformal mapping framework can be generalized to compute surface quasiconformal mappings by an auxiliary metric[22] or holomorphic Beltrami flow [12], with a given Beltrami coefficient μ. Recently, extremal quasiconformal map with a unique extremal μ becomes an active topic [19]. In our q.c. mapping, we don't have μ as input; μ is induced by the landmark constraints intrinsically.

Furthermore, feature landmarks, usually feature *points*, are applied to adapt the conformal mapping to be quasiconformal mapping such that the points are aligned to the prescribed targets and used for surface registration [22]. Sparse landmarks were also introduced in Kurtek et al.'s approach [10], which computes the registration and deformation process simultaneously for genus-0 surfaces. In this work, we focus on landmark *curves*, their mapping positions are not prescribed but computed automatically, thus the mapping is intrinsic to surface and landmark curve geometry. Recent work [18] treats landmark curves as surface boundaries based on nonlinear hyperbolic harmonic map.

Recently, Teichmüller space, which studies conformal equivalence class of surfaces, has been studied for shape indexing, dynamics analysis, and morphology analysis. Different Teichmüller coordinates were introduced. Classical geodesic length spectrum for high genus surfaces [8] and conformal module for genus zero surfaces with boundaries [23] describe surface conformal structure directly. Conformal welding signatures for 2D shapes by Sharon and Mumford [16,13] and for 3D shapes [24] describe correlation among non-intersecting *contour(s)* on surface through conformal structures of surface components surrounded by contours. All above are created on canonical conformal mappings. Instead, our proposed landmark-driven canonical quasiconformal mapping gives a novel Teichmüller coordinates, which describes correlation among *open curve(s)* and *non-trivial loop(s)* on surface through one single quasiconformal structure. Here, each landmark has no self-intersection; "horizontal" landmarks may only intersect "vertical" landmarks, and vice versa. Also, this signature is conformal invariant and intrinsic to surface and landmark geometry.

1.3 Approach Overview

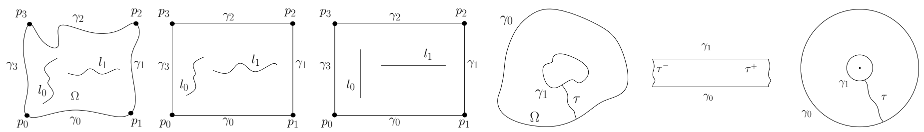
In detail, for a genus zero surface with landmark curves, we map it to a canonical parameter domain $D(u,v)$ such that each landmark curve is mapped to a straight segment parallel to u-axis (horizontally) or v-axis (vertically), while the local shapes are preserved as much as possible. The computational strategy is to incorporate landmark straightening conditions into the computation of conformal mapping based on the holomorphic 1-form method. Mathematically, this problem is formulated as solving a sparse linear system with boundary conditions. According to Hodge decomposition theorem [4], any differential 1-form can be composed of a closed 1-form, an exact 1-form, and a harmonic 1-form. A conformal mapping can be generated by integrating a holomorphic 1-form of the surface. Correspondingly, a quasiconformal mapping can be induced by a harmonic 1-form, plus an exact 1-form which is also closed. We compute a special exact 1-form to constrain the final 1-form to be zero along landmark curves in u or v direction (vertically or horizontally), while minimizing the harmonic energy of the desired 1-form. The resulting 1-form is quasi-holomorphic and its integration generates a rectangular quasiconformal mapping. By an exponential map, the rectangular map can be mapped to a circle domain, where straight segments become concentric arcs. Figure 1 shows the mappings for a human facial surface (a small puncture at nose tip, which is mapped to the disk center), where the curved mouth landmark curve on conformal map is mapped to a circular (horizontal) arc on the canonical quasiconformal map. The variation from surface conformal structure is driven by landmark straightening constraints.

The result exists and is unique. Due to the linear nature of 1-form, the algorithm has linear time complexity, and is efficient and practical. The induced intrinsic shape signature, Teichüller coordinates, is then applied to construct a Teichmüller space surface matching. The L^2 norm between Teichmüller coordinates defines the similarity metric of two surfaces. Using the canonical mapping, the surfaces with landmark curve constraints can be registered over the canonical domains by linear harmonic map, where the landmark constraints between straight segments are converted to linear forms.

1.4 Contributions and Novelties

This work presents a *novel* method for surface matching and registration based on the *novel* canonical surface quasiconformal mapping for landmark curve decorated surfaces. To our best knowledge, this is the *first* work to conquer landmark curves on surface cases in the way of canonical quasiconformal map. The details are as follows:

1. To present a *novel* canonical surface quasiconformal mapping based on holomorphic 1-form, where the quasiconformality is driven by landmark curve straightening constraints intrinsically. Besides that, the mapping is coherent to surface uniformization theorem; surfaces are mapped to rectangle or circle domain and the local shapes are preserved as much as possible. We call this technique *Quasiconformal Straightening (QCS)*. The method is linear, stable and easy to compute.
2. To obtain a *novel* shape representation for landmark curve decorated surfaces, which generates an intrinsic, unique and global shape signature, called *QCS Signature*, as shape index for matching. We employ this conformal invariant as Teichmüller coordinates to construct a subspace of the conventional Teichmüller space,



domain conformal quasiconformal domain rectangular map circle domain

(a) quadrilateral domain (b) doubly-connected domain

Fig. 2. Illustrations of canonical surface mappings

which is aware of geometry of landmark curves and increases the discriminative ability for real applications, such as to differentiate topological disks in Teichmüller space. The surface distance is given by the L^2 norm between the signatures.

3. To present a *novel* landmark curve constrained surface registration method using the proposed canonical quasiconformal mappings. The landmark curve decorated surfaces become straight segment decorated 2D domains; then an optimal mapping is built over them. By taking the linear advantage of straight segments, the curves can be easily aligned. Similarly, the method is linear, robust and efficient in practice.

Experiments on a diverse set of 3D human facial and brain surfaces were performed to demonstrate the efficiency and efficacy of the proposed framework for surface matching and registration. The proposed landmark-driven framework is fundamental and practical for general shape analysis purposes in various engineering and medical fields especially for those anatomical geometric data.

2 Theoretical Background

2.1 Quasiconformal Mapping

Consider a complex-valued function $\phi : \mathbb{C} \to \mathbb{C}$ mapping the z-plane to the w-plane, $z = x + iy$, $w = u + iv$. Suppose ϕ is differentiable. The *complex partial derivative* is defined as $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The *Beltrami equation* for ϕ is given by

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z)\frac{\partial \phi}{\partial z}, \tag{1}$$

where μ is called the *Beltrami coefficient*, which is a complex-valued function. If $\mu = 0$, then ϕ satisfies the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and is called a *holomorphic function*, which preserves angles. The resulting mapping is a *conformal mapping*. Otherwise, if $0 < \|\mu\|_\infty < 1$, where $\|\cdot\|_\infty$ denotes the L^∞ norm, then ϕ is a *quasiconformal mapping* with bounded angle distortion.

2.2 Holomorphic 1-form for Conformal Mapping

We use general genus zero surfaces with boundaries (embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$) to illustrate the computational method of conformal mappings based on Hodge theory [4,7].

Quadrilateral Domain. For a *quadrilateral domain* Ω with four boundary segment components, $\partial\Omega = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$, i.e., with four boundary corners, p_0, p_1, p_2, p_3, the holomorphic 1-form $\omega = (\tau_1, {}^*\tau_1)$ is computed by two exact harmonic 1-forms, $\tau_1 = df_1, \tau_2 = df_2$, such that

$$\begin{cases} \Delta f_1 = 0 \\ f_1|_{\gamma_3} = 0 \\ f_1|_{\gamma_1} = 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_0 \cup \gamma_2} = 0 \end{cases} \text{and} \begin{cases} \Delta f_2 = 0 \\ f_2|_{\gamma_0} = 0 \\ f_2|_{\gamma_2} = 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1 \cup \gamma_3} = 0 \end{cases}. \tag{2}$$

The conjugate ${}^*\tau_1 = c\tau_2$, where c is a scalar function. The integration of ω gives a rectangular conformal map, where γ_0, γ_2 are mapped to horizontal boundaries of the rectangle while γ_1, γ_3 are mapped to vertical ones (see Fig. 2(a)).

Doubly-Connected Domain. If a compact domain Ω has only two boundary components, $\partial\Omega = \gamma_0 - \gamma_1$, then it is called a *doubly-connected domain*. The whole domain is mapped to an annulus, where two boundaries are mapped to concentric circle boundaries. The holomorphic 1-form τ_1 or ${}^*\tau_1$ is orthogonal to both boundaries, $\tau_1 = \omega_1 + c^*\omega_1$, where ω_1 corresponds to γ_1, such that $\int_{\gamma_j} \omega_i = \delta_i^j$, where δ_i^j is the Kronecker symbol. The integration of such a holomorphic 1-form from base point p generates the rectangular map, where the domain is sliced open by a curve τ. Then by the exponential map, the circular map is generated (see Fig. 2(b)).

Simply-Connected Domain. Suppose Ω is a compact domain on the complex plane $\mathbb{C}$. If Ω has a single boundary component, then it is called a *simply-connected domain*. By a puncture at an interior point, the domain becomes a doubly-connected one and then the above computation can be applied. The whole domain is mapped to a unit disk. Such kind of mappings differ by Möbius transformations.

Multiply-Connected Domain. Suppose Ω has multiple boundary components, $\partial\Omega = \gamma_0 - \gamma_1 - \gamma_2 \cdots \gamma_n$, where γ_0 represents the exterior boundary component and $\gamma_i, i = 1..n$ represent the interior ones, then Ω is called a *multiply-connected domain*. It can be mapped to a unit disk with circular holes, called *circle domain*, where one boundary is mapped to the exterior unit circle, and others are mapped to inner circles. The computation is to iteratively perform the basic operation of mapping a doubly-connected domain to a canonical annulus based on Koebe's iteration method [7].

The existence and uniqueness of conformal and quasiconformal mappings for multiply-connected domains are guaranteed by the generalized Riemann mapping theorem [4] and the generalized measurable Riemann mapping theorem [1].

Theorem 1 (Generalized Measurable Riemann Mapping [1]). *Suppose $\Omega \subset \mathbb{C}$ is a multiply-connected domain. Suppose $\mu : \Omega \to \mathbb{C}$ is a measurable complex function, such that $\|\mu\|_\infty < 1$. There exists a quasiconformal mapping $\phi : \Omega \to D$ whose Beltrami coefficient is μ, where D is a circle domain. Such kind of quasiconformal mappings differ by Möbius transformations.*

2.3 Conformal Module

Let S be a topological surface and all the possible Riemannian metrics on S be $G = \{\mathbf{g}\}$. Two metrics $\mathbf{g}_1$, $\mathbf{g}_2$ are *conformally equivalent*, $\mathbf{g}_1 \sim \mathbf{g}_2$, if there exists a function $\lambda : S \to \mathbb{R}$, such that $\mathbf{g_1} = e^{2\lambda}\mathbf{g}_2$.

All domains can be classified by conformal equivalence relation. Each class shares the same *conformal invariant*, called *conformal module*, which defines a unique and global shape signature. According to Reimann mapping theorem [4], every simply-connected domain is conformally equivalent to the open unit disk and such kind of mappings differ by Möbius transformations. Therefore,

Theorem 2. *All simply-connected domains are conformally equivalent.*

The conformal module for a rectangle domain is defined as the ratio of the height over the width. For a circle domain it is represented as the centers and radii of inner circles. By a Möbius normalization mapping one inner circle to be concentric, the topological annulus only requires 1 parameter in its conformal module. In general case, there are $n > 1$ inner circles, the conformal module requires $3n-3$ parameters. All conformal equivalence classes form a $3n-3$ Riemannian manifold, the so-called *Teichmüller space*. The conformal module can be treated as the *Teichmüller coordinates*.

3 Algorithm for Canonical Surface Quasiconformal Mapping

The main goal of our algorithm is to compute the canonical quasiconformal mapping with landmark straightening constraints for genus zero surfaces with boundaries. In practice, the surfaces are approximated by triangular meshes embedded in $\mathbb{R}^3$, denoted as $M = (V,E,F)$, where V,E,F are the sets of vertices, edges, and faces, respectively.

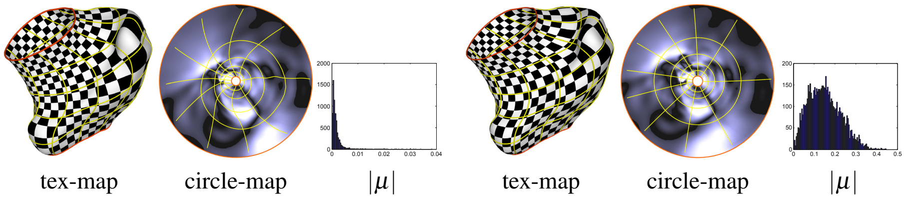
Assume the desire mapping is $f : (M,L) \to (D,\ell)$, surface mesh M is mapped to a planar parameter domain D, and D has the local coordinates (u,v). Here, we use the quadrilateral case for discussion. The computational pipeline is as follows:

Step 1: Prepare Landmark Curves. We use $L = \{l_k, k = 1..m\}$ to denote the set of m interior landmark curves on M. Assume $L = L^H \cup L^V$, where L^H, L^V are to be mapped to horizontal and vertical straight segments, respectively. Each landmark is represented as a chain of vertices, $l_k = [v_1, v_2, \ldots, v_{n_k}]$, where n_k is the number of the vertices on l_k.

Step 2: Compute Quasi-holomorphic 1-form. The quasi-holomorphic 1-form to be computed will be $\omega = \tau_1 + \sqrt{-1}{}^*\tau_1$, where $\tau_1 = df_1$, ${}^*\tau_1 = \lambda\tau_2$, and $\tau_2 = df_2$:
1. Compute the harmonic functions f_1, f_2: Combining the straightening constraint conditions into Eqn. (2), we have

$$\begin{cases} \Delta f_1 = 0 \\ f_1|_{\gamma_3} = 0 \\ f_1|_{\gamma_1} = 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_0 \cup \gamma_2} = 0 \\ f_1|_{l_k^V} = s_k \end{cases} \text{and} \quad \begin{cases} \Delta f_2 = 0 \\ f_2|_{\gamma_0} = 0 \\ f_2|_{\gamma_2} = 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1 \cup \gamma_3} = 0 \\ f_2|_{l_k^H} = t_k \end{cases}, \tag{3}$$

where $\mathbf{n}$ is the normal vector to the boundary, s_k, t_k are unknown variables, computed automatically for each landmark curve. Two types of straightening constraints are: 1)



(a) conformal map (inner radius $r_1 = 0.057$) (b) quasiconformal map (inner radius $r_1 = 0.068$)

Fig. 3. Landmark-driven quasiconformal mapping for a doubly-connected domain. It contains four (4) horizontal landmarks (loops) and ten (10) vertical landmarks. Checker-board texture mappings and histograms of Beltrami coefficients (by $|\mu|$) demonstrate the quasiconformality.

Horizontal: for $l_k \in L^H$, $v(v_i) = s_k, i = 1 \dots n_k$; and 2) *Vertical*: $l_k \in L^V$, $u(v_i) = t_k, i = 1 \dots n_k$. The Laplace-Beltrami operator Δ is approximated by the cotangent weight w_{ij}, $\Delta f(v_i) = \sum_{[v_i,v_j] \in E} w_{ij}(f(v_j) - f(v_i))$. For edge $[v_i, v_j]$, suppose two adjacent faces are $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$. Then its *weight* is defined as $w_{ij} = \cot \theta_k^{ij} + \cot \theta_l^{ij}$, $[v_i, v_j] \notin \partial M$; or $w_{ij} = \cot \theta_k^{ij}$, $[v_i, v_j] \in \partial M$, where θ_k^{ij} is the corner angle at v_k in $[v_i, v_j, v_k]$.
2. Compute harmonic 1-forms by gradient computation: $\tau_1 = \nabla f_1, \tau_2 = \nabla f_2$.
3. Compute conjugate 1-form of τ_1 *by Hodge star operator:* $^\star\tau_1 = \lambda \tau_2$ (λ is a scalar), by minimizing the energy $E(\lambda) = \sum_{[v_i,v_j,v_k] \in F} |\nabla f_2 - \lambda \mathbf{n} \times \nabla f_1|^2 A_{ijk}$, where A_{ijk} is the area of face $[v_i, v_j, v_k]$, and $\mathbf{n}$ is the normal vector to the face.
4. Compute $\omega = \tau_1 + \sqrt{-1}\, ^\star\tau_1$.

The problem turns to minimizing the harmonic energy, $E(f) = \sum_{[v_i,v_j]} w_{ij}(f(v_j) - f(v_i))^2$, by considering the landmark constraints in Eqn. (3).

Step 3: Computing Quasiconformal Mapping. We generate the quasiconformal mapping by integrating the obtained quasi-holomorphic 1-form ω over M, $f(q) = \int_{\gamma(p,q) \in M} \omega$, $\forall q \in V$, where $\gamma(p,q)$ is an arbitrary path from the base vertex p to the current vertex q. On the planar domain, $f(p) = (0,0)$, $f(\gamma_i), f(l_k)$ are all straight lines.

The computational algorithms for other genus zero surface cases are similar. They share the same key component of minimizing harmonic energy with landmark straightening constraints. By the exponential map, the rectangular domain is converted to circle domain, where the straight lines are mapped to circular arcs. Figure 3 shows a doubly-connected case. The difference from the conformal map can be evaluated by the angle distortion of checker-board textures and the distributions of Beltrami coefficients (mean of $|\mu|$: 0.15 vs. 0.001). Figure 4 shows the examples for a human facial surface with multiple landmark curves. In theory, the solution exists and is unique [15]. The resulting mapping preserves local shapes as much as possible and is intrinsic to geometry of both surface and landmark curves. The algorithm solves sparse linear systems and have linear complexity. In practice, the conjugate gradient method is applied.

4 Algorithm for Surface Matching

The proposed quasiconformal mapping $\phi : (S, L) \to (D, \ell)$ offers an intrinsic canonical shape representation for surfaces decorated with landmark curves. We employ the positions and the lengths of the canonical-shaped landmarks and the conformal modules of the background domain as the shape signature, called *QCS signature*, which is a conformal invariant. We use it as Teichmüller coordinates to construct the Teichmüller space. The L^2 norm between signatures gives the distance between two decorated surfaces.

Quadrilaterals. A quadrilateral surface with m landmarks is mapped to a rectangle domain with horizontally or vertically straightened landmarks (see Fig. 4). Assume the bottom-left corner of the rectangle domain is set to be the origin $(0,0)$. Then the QCS signature is defined as

$$QCS(S) = \{\frac{x_j^H}{w}, \frac{y_j^H}{h}, \frac{d_j^H}{w}\} \bigcup \{\frac{x_k^V}{w}, \frac{y_k^V}{h}, \frac{d_k^V}{h}\} \bigcup \mathrm{Mod}(D), \tag{4}$$

where h, w denote the height and the width of D, respectively, (x_j^H, y_j^H) represents the left endpoint of $\phi(l_j)$, $l_j \in L^H$, (x_k^V, y_k^V) represents the bottom endpoint of $\phi(l_k)$, $l_k \in L^V$, d_k denotes the length of the segment $\phi(l_k)$, $l_k \in L$, and Mod is the conformal module of D, h/w, as defined in Sect. 2. Then the Teichmüller space is $3m+1$ dimensional.

$(n+1)$**-Connected Domains.** General genus zero $(g=0)$ domains with $n+1$ boundaries and m landmarks are mapped to circle domains (normalized onto unit disk) with radial straight or concentric circular landmarks. Then the QCS signature is defined as

$$QCS(S) = \{r_j, \frac{\theta_j^1}{2\pi}, \frac{\theta_j^2}{2\pi}\} \bigcup \{r_k^1, r_k^2, \frac{\theta_k}{2\pi}\} \bigcup \mathrm{Mod}(D), \tag{5}$$

where $(r_j, \theta_j^1, \theta_j^2)$ denotes the radius and argument angles of the concentric circular arc $\phi(l_j)$, $l_j \in L^H$; (r_k^1, r_k^2, θ_k) denotes the radii and the argument angle of the radial straight segment $\phi(l_k)$, $l_k \in L^V$, and Mod is the conformal module of circle domain D, including the center positions and radii of the inner circles, as defined in Sect. 2. Assume D is normalized by a Möbius transformation. We use $T_{0,n,m}$ to denote the Teichmüller space of open genus zero $(g=0)$ surfaces with n inner boundaries and m landmarks.

- For multiply-connected domains, $\dim T_{0,n>1,m>0} = 3m + (3n-3)$.
- For simply-connected domains, $\dim T_{0,n=0,m>0} = 3m-2$.
- For doubly-connected domains, $\dim T_{0,n=1,m>0} = (3m-1)+1 = 3m$.

For each open landmark curve with two endpoints on boundary in quadrilaterals or non-trivial landmark loop in connected domains, the total dimension decreases by 2.

5 Algorithm for Surface Registration

The main strategy is to map decorated surfaces to canonical planar domains with *canonically shaped* (*straight*) landmark curves using the proposed quasiconformal map and then convert surface registration problems to image registration problems, where the landmark curve constraints become *linear* constraints between images.

$$\begin{array}{ccc} (S_1,L_1) & \xrightarrow{\quad f \quad} & (S_2,L_2) \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ (D_1,\ell_1) & \xrightarrow{\quad h \quad} & (D_2,\ell_2) \end{array}$$

Registration Framework. Suppose (S_k, L_k), $k = 1,2$ are the *source* and *target* surfaces S_k decorated with landmark curves L_k, respectively. In order to compute the *registration* $f : (S_1,L_1) \to (S_2,L_2)$, we first map decorated surfaces to *decorated canonical domains*, $\phi_k : (S_k,L_k) \to (D_k,\ell_k)$, then construct the optimal mapping $h : (D_1,\ell_1) \to (D_2,\ell_2)$, such that straight line ℓ_1 is aligned with straight line ℓ_2 and the harmonic energy of the mapping is minimized. The registration is given by $f = \phi_2^{-1} \circ h \circ \phi_1$.

This novel quasiconformal map-based registration framework works for landmark *curve* constrained surfaces with general deformations, subsuming rigid motion, isometry, and conformal transformation. This framework can be generalized to more general surfaces and handle point-curve mixed constraints. In contrast, existing works using conformal map-based framework [22,21] mainly focus on registration of surfaces decorated with feature *point* constraints. They cannot introduce each landmark curve as a whole; accordingly, a heuristic alternative is to sample the curve to isolated points then apply the point-constrained registration method [14] .

Algorithm with Landmark Curve Constraints. The computation is based on the optimization of constrained harmonic energy to smooth out distortion as much as possible. We generate a harmonic map $h : (D_1,\ell_1) \to (D_2,\ell_2)$, $\nabla h = (h_1,h_2)$, to minimize the energy $E(h) = \int_{D_1} |\nabla \cdot \nabla h|^2 dA$ with the Dirichlet and Neumann boundary conditions on both canonical domain boundaries and horizontal or vertical landmarks. Suppose D_k are rectangles, $\partial D_k = \gamma_0^k + \gamma_1^k + \gamma_2^k + \gamma_3^k$, where $(\gamma_i^1, \gamma_i^2), i = 0..3$ denotes a pair of corresponding boundaries, and $L_k = L_k^H \bigcap L_k^V$. We use (l_1,l_2) and (p_1,p_2) to denote a pair of segments and endpoints to be aligned, respectively, and solve

$$\begin{cases} \Delta h_1 = 0 \\ h_1|_{\gamma_3^1} = \phi_2^u|_{\gamma_3^2} - \phi_1^u|_{\gamma_3^1} \\ h_1|_{\gamma_1^1} = \phi_2^u|_{\gamma_1^2} - \phi_1^u|_{\gamma_1^1} \\ \frac{\partial h_1}{\partial \mathbf{n}}|_{\gamma_0 \bigcup \gamma_2} = 0 \\ h_1|_{l_1^V} = \phi_2^u|_{l_2^V} - \phi_1^u|_{l_1^V} \\ h_1|_{p_1^H} = \phi_2^u|_{p_2^H} - \phi_1^u|_{p_1^H} \end{cases} \text{and} \begin{cases} \Delta h_2 = 0 \\ h_2|_{\gamma_0^1} = \phi_2^v|_{\gamma_0^2} - \phi_1^v|_{\gamma_0^1} \\ h_2|_{\gamma_2^1} = \phi_2^v|_{\gamma_2^2} - \phi_1^v|_{\gamma_2^1} \\ \frac{\partial h_2}{\partial \mathbf{n}}|_{\gamma_1 \bigcup \gamma_3} = 0 \\ h_2|_{l_1^H} = \phi_2^v|_{l_2^H} - \phi_1^v|_{l_1^H} \\ h_2|_{p_1^V} = \phi_2^v|_{p_2^V} - \phi_1^v|_{p_1^V} \end{cases}. \quad (6)$$

Then $h(p) = p + \nabla h(p), p \in D_1$. The desired mapping $f = \phi_2^{-1} \circ h \circ \phi_1$.

The registration accuracy can be evaluated by an energy form $E(f) = \int_{p \in S_1} (H(p) - H(f(p)))^2 + (K(p) - K(f(p)))^2 + (\lambda(p) - \lambda(f(p)))^2$, where H, K, λ denote the mean curvature, the Gauss curvature, and the conformal factor, respectively.

6 Experimental Results

The proposed landmark-driven canonical quasiconformal mapping provides a fundamental approach to surface matching and registration, and have broad applications in vision, graphics, and medical imaging. Here, we develop experiments on human facial and brain surfaces to demonstrate the efficiency and efficacy.

Experimental Settings. We consider a set of anatomical facial landmark curves: 1) the curve along eyebrows l_b; 2) the geodesic curve between inner eye corners l_e; 3)

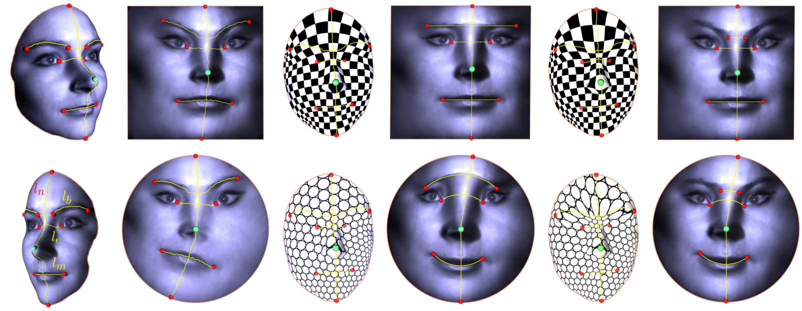


Fig. 4. Landmark-driven canonical quasiconformal mappings for a human facial surface

the geodesic curve between mouth corners l_m; and 4) the symmetry axis curve $l_n = l_{n_1} \cup l_{n_2}$, split into two parts by nose tip, as shown in Fig. 4(a). In canonical mapping, we set $L^H = \{l_b, l_e, l_m\}$ and $L^V = \{l_n\}$ or $\{l_{n_1}, l_{n_2}\}$. For brain surfaces, we consider the anatomical sulci and gyri curves. The algorithms are tested on a desktop with 3.7GHz CPU and 16GB RAM. The whole pipeline is automatic. For mapping a facial surface with $120k$ triangles and a brain surface with $20k$ triangles, the averaged running times are 10 seconds and 3 seconds, respectively. The registration process including two such q.c. maps and another mapping over 2D domains totally costs 3 times the mapping time.

6.1 Feature-Aware Shape Signature

Figure 4 shows the mapping results for a 3D facial surface decorated with multiple landmark curves. *Upper row*: Surface with prescribed four boundary corners, a quadrilateral, is mapped to a rectangle; *Bottom row*: Surface is mapped to a unit disk. *Column*: (a) 3D views of the landmark curve decorated surface; (b) conformal maps; (c) canonical quasiconformal maps with landmark straightening constraints; (d) quasiconformal maps with a constraint change on eyebrow landmark. The angle distortions are illustrated by the texture mapping results. Table 1 gives the numerical results for QCS signatures of the two cases in (c), computed by Eqn. (4).

6.2 Content-Based Surface Matching

As stated in Theorem 2, conformal mappings keep the same structure for the topological disk surfaces and cannot differentiate them in Teichmüller space. The QCS signature is aware of feature details and has the discriminative ability. In our experiments, we employ the QCS signature for shape matching on two categories of human facial scans which have been successfully tested: 1) 5 human facial expression sequences from the same subject (each has 400 frames) (see Fig. 5), and 2) human facial surfaces from different subjects (BU-3DFE database [20], 100 subjects with various expressions) (see Fig. 6). Table 1 gives the QCS signatures computed by Eqn. (5).

In Fig. 5, the expression change is mainly around mouth area, so we introduce one landmark l_m to study the dynamics to conquer the conformal equivalence of topological

Table 1. Shape signatures

Models	Signatures
Fig. 4(c) upper-bottom ($l_b;l_e;l_m;l_n;Mod$)	(0.260,0.849,0.638; 0.433,0.728,0.278; 0.371,0.285,0.398; 0.576; 0.954) (0.373,0.135,0.363; 0.238,0.141,0.346; 0.254,0.617,0.849; 0.475)
Fig. 5 left-right (l_m)	(0.508,0.214), (0.519,0.266), (0.492,0.294), (0.527,0.282)
Fig. 6 left-right ($l_m;l_e$)	(0.398,0.234;0.334,0.149,0.356), (0.415,0.223;0.344,0.145,0.36), (0.384,0.213;0.355,0.143,0.365), (0.426,0.183;0.307,0.147,0.376), (0.427,0.197;0.384,0.14,0.359), (0.463,0.236;0.366,0.156,0.343)

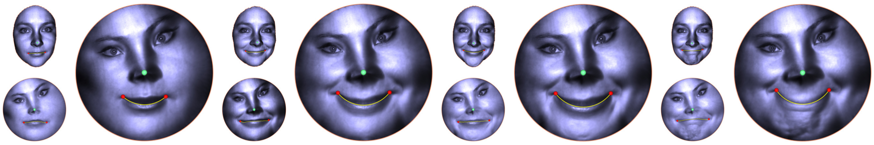


Fig. 5. Surface matching for a deforming facial expression sequence from the same subject. The mouth landmark l_m is employed. The bottom-left small disk shows the conformal mapping result.

disk faces. The L^2 norm geometric distances (similarity of expressions) to the leftmost expression $d(S_k,S_0) = 0, 0.053, 0.071, 0.082, k = 0..3$. In Fig. 6, we use QCS signatures to differentiate the faces from different subjects. The geometric distances (similarity of faces) to the leftmost $d(S_k,S_0) = 0, 0.023, 0.035, 0.067, 0.069, 0.074, k = 0..5$.

The experimental results show that the QCS signature is promising for large-scale content-based shape retrieval applications for 2D images and 3D objects, such as face recognition, brain surface classification, and general geometric search engine.

6.3 Landmark-Curve Constrained Surface Registration

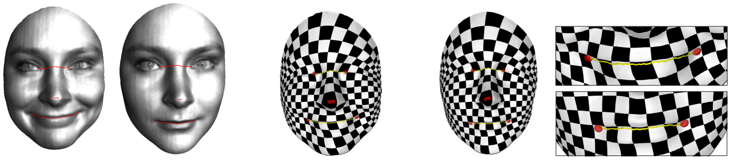
Our registration algorithm has been successfully tested on two categories of 3D nonrigid surfaces: 1) human facial surfaces with different expressions from the same subject (see Fig. 7), and neutral faces from different subjects (see Fig. 8), and 2) human brain surfaces from control group and patient group (see Fig. 9). Each pair has big geometry variance and the deformation is quasiconformal. The geometric registration accuracy is evaluated by the energy form $E(f)$ in Sect. 5. $E(f) = 0.03$ for Fig. 7, $E(f) = 0.05$ for Fig.8, and $E(f) = 0.08$ for Fig. 9. The registration effects can also be visually checked by the consistent texture mapping results.

Our proposed method takes the curve constraints as linear constraints. For better understanding, we performed a comparison test, as shown in Fig. 8(b), where 9 point constraints are sampled to replace curve constraint. The result is not smooth, significantly different from our result (see the close-up view in (b)). The accuracy can be improved by dense sampling, but this will generate more cost. All the results demonstrate the advantages of our method in terms of the accuracy, efficiency and practicality.

6.4 Performance Discussion

Efficiency, Robustness and Generality: The 1-form method solves positive definite sparse linear systems, therefore has linear time complexity. The solution exists and is

Fig. 6. Surface matching by landmark-driven quasiconformal mappings for neutral facial surfaces from different subjects. The inner eye corner and mouth landmarks l_e, l_m are employed.



(a) S_1 (b) S_2 (c) QC-registration (d) C-map (e) close-up view

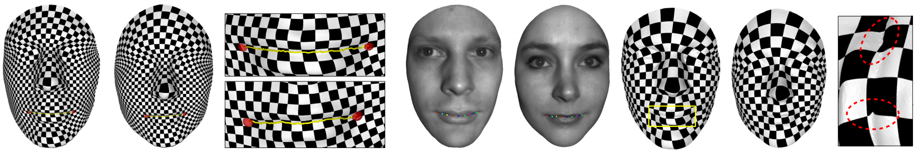
Fig. 7. Facial surface registration between different expressions from the same subject

unique in theory [15]. The 1-form method is stable and robust to handle small geometric and topological noises or variations under different modalities (resolution, quality, smoothness, boundary noise, small holes or handles) and multiple and complicated decorative landmark curves (open and closed); it is easy to implement and fast to compute. The proposed QCS technique can be incorporated into the 1-form computation for general topological surfaces [7]. *In addition*, due to the generality of quasiconformal map, the proposed registration method is general to handle any types and intensities of deformations, including small or significantly large rigid motions, isometries, conformal transformations, and quasiconformal deformations.

Novelty, Comparison and Practicability: To our best knowledge, this is the *first* work to compute landmark curve driven canonical quasiconformal map, and use the canonical shape representation for landmark curve-decorated surface matching and registration.

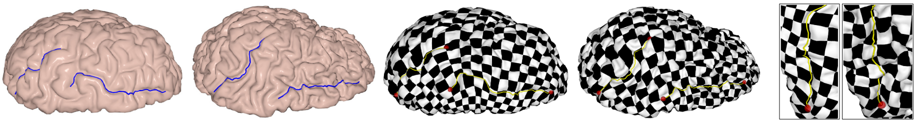
Compared with conformal maps of pure surfaces, the generated intrinsic canonical quasiconformal map studies the nature of curves in surface and the influence to conformal structure, and provides the feature-aware shape signature and global representation. The conformal welding signature [16,13] needs to divide the whole surface to multiple components by closed contours; it cannot deal with open curves. Different from other conformal invariant shape signatures of pure surfaces such as conformal modules [23,8], the QCS signature encodes landmark geometry, therefore is more capable to represent a shape both globally and locally; and especially it can be used to differentiate the topological disk surfaces.

Most existing conformal map based registration methods can only handle point constraints, or treat the landmark curves as isolated points in the mapping process [23,14]. The hyperbolic harmonic map method [18] slices the landmark curves open to be boundaries and uses the hyperbolic metric; the computation is highly nonlinear. Our proposed method has significant difference: it provides linear constraints between straightened curves and has practical advantage due to linearity of 1-forms. Our method deals with surface and landmark curves as a whole without changing topology. Kurtek



(a) curve constraint (b) point constraints

Fig. 8. Facial surface registration for two subjects. (a) lip landmark curve constraint is employed; (b) curve constraint is sampled as point constraints.



(a) S_1 (b) S_2 (c) QC-registration (d) C-map (e) close-up view

Fig. 9. Brain surface registration with convoluted landmark curves

et al.'s approach [10] handles sparse feature points; given two isometric surfaces with different embeddings, our shape metric gives 0, and theirs doesn't.

Flexibility and Potential Impacts: The user can freely select landmark curves and design their straightening styles according to the application needs. In order to arrange the landmarks properly and avoid large distortions, we can first observe their shapes on conformal mapping domain, for example in Fig. 4(b), where the styles can be automatically extracted by comparing the averaged slope of the curve with the u and v-axis. It has potential for artwork design and large-scale shape retrieval in industry.

7 Conclusion

We present the novel surface matching and registration method based on the intrinsic canonical surface quasiconformal mapping, which maps the landmark curves to be the horizontal or vertical straight lines on canonical 2D domain and preserves the local shapes as much as possible. The mapping is unique and intrinsic to surface and landmark curve geometry. It gives the novel conformal invariant shape signature to construct the feature-aware Teichmüller space for surface matching. It is easy to deal with surface registration with straightened landmark curve constraints. All the algorithms are based on 1-form and have linear time complexity and are efficient and practical. Experiments on matching and registering facial and brain scans demonstrate the efficiency and efficacy, which is promising for broader computer vision applications where landmark curves are naturally associated. In future, we will explore more under the proposed framework.

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