About Multigrid Convergence of Some Length Estimators*

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Abstract. An interesting property for curve length digital estimators is the convergence toward the continuous length and the associate convergence speed when the digitization step h tends to 0. On the one hand, it has been proved that the local estimators do not verify this convergence. On the other hand, DSS and MLP based estimators have been proved to converge but only under some convexity and smoothness or polygonal assumptions. In this frame, a new estimator class, the so called *semi-local estimators*, has been introduced by Daurat *et al.* in [4]. For this class, the pattern size depends on the resolution but not on the digitized function. The semi-local estimator convergence has been proved for functions of class $\mathcal{C}^2$ with an optimal convergence speed that is a $\mathcal{O}(h^{\frac{1}{2}})$ without convexity assumption (here, optimal means with the best estimation parameter setting). A semi-local estimator subclass, that we call *sparse estimators*, is exhibited here. The sparse estimators are proved to have the same convergence speed as the semi-local estimators under the weaker assumptions. Besides, if the continuous function that is digitized is concave, the sparse estimators are proved to have an optimal convergence speed in h. Furthermore, assuming a sequence of functions $G_h \colon h\mathbb{Z} \to h\mathbb{Z}$ discretizing a given Euclidean function as h tends to 0, sparse length estimation computational complexity in the optimal setting is a $\mathcal{O}(h^{-\frac{1}{2}})$.

1 Introduction

The ability to perform the measurement of geometric features on digital representations of continuous objects is an important goal in a world becoming more and more digital. We focus in this paper on one classical digital problem: the length estimation. The problem is to estimate the length of a continuous curve S knowing a digitization of S. As information is lost during the digitization step, there is no reliable estimation without *a priori* knowledge and it is difficult to evaluate the estimator performances. In order to refine the evaluation of the estimators, a property, so called *convergence property* is desirable: the estimation convergence toward the true length of the curve S when the grid step h tends to 0. This property can be viewed as a robustness to digitization grid change.

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The local estimators based on a fixed pattern size do not satisfy the convergence property [13]. The adaptive estimators based on the Maximal Digital Straight Segment (MDSS) or the Minimum Length Polygon (MLP) satisfy the convergence property under assumptions of convexity, 4-connectivity for closed simple curves (also called Jordan curves) [3]. The semi-local estimators, introduced by Daurat et al [4] for function graphs, verifies the convergence property under smoothness assumption but without convexity hypothesis. We present here a subclass of the semi-local estimators, the *sparse estimators* that only need information on a small part of the function values and keep the convergence property.

The paper is organized as follows. In Section 2, some necessary notations and conventions are recalled, then existing estimators and their convergence properties are detailed. In Section 3, the sparse estimators are defined, their convergence properties are given in the general case and then in the concave cases (we make a distinction between the concavity of the continuous function and the concavity of the piecewise affine function related to the discretization). Due to the lack of place, no formal proofs can be provided in the present article. Nevertheless a series of lemmas outlines them and an experiment exemplifies the lemmas. The reader that want to dive more deeper in the proofs can find them in [15]. Section 4 concludes the article and gives directions for future works. In Appendix A two counterexamples about the concavity are exhibited. Appendix B presents a minimal error on the sparse estimation of the length of a segment of parabola.

2 Background

2.1 Digitization Model

In this work, we have restricted ourselves to the digitizations of function graphs. So, let us consider a continuous function $g : [a, b] \to \mathbb{R}$ ($a < b$), its graph $\mathcal{C}(g) = \{(x, g(x)) \mid x \in [a, b]\}$ and a positive real number r, the *resolution*. We assume to have an orthogonal grid in the Euclidean space $\mathbb{R}^2$ whose set of grid points is $h\mathbb{Z}^2$ where $h = 1/r$ is the *grid spacing*. We use the following notations: $\lfloor x \rfloor_h$ is the greatest multiple of h less than or equal to x, $\{x\}_h = x - \lfloor x \rfloor_h$. Finally, for any function f defined on an interval, $L(f)$ denotes the length of $\mathcal{C}(f)$, the graph of f ($L(f) \in [0, +\infty]$).

The common methods to model the digitization of the graph $\mathcal{C}(g)$ at the resolution r are closely related to each others. In this paper, we assume an *object boundary quantization* (OBQ). This method associates to the graph $\mathcal{C}(g)$ the *h-digitization* set $\mathcal{D}^{\mathrm{O}}(g, h) = \{(kh, \lfloor g(kh) \rfloor_h) \mid k \in \mathbb{Z}\}$. The set $\mathcal{D}^{\mathrm{O}}(g, h)$ contains the uppermost grid points which lie in the hypograph of g, hence it can be understood as a part of the boundary of a solid object. Provided the slope of g is limited by 1 in modulus, $\mathcal{D}^{\mathrm{O}}(g, h)$ is an 8-connected digital curve. Observe that if g is a function of class C^1 such that the set $\{x \in [a, b] \mid |g'(x)| = 1\}$ is finite, then by symmetries on the graph $\mathcal{C}(g)$, it is possible to come down to the case where $|g'| \leq 1$. So, we assume that g is a Lipschitz function which Lipschitz constant 1. Hence, the set $\mathcal{D}^{\mathrm{O}}(g, h)$ is 8-connected for any h and the curve $\mathcal{C}(g)$

is rectifiable ($L(g) < +\infty$). Moreover, the h-digitization set $\mathcal{D}^{\mathrm{O}}(g,h)$ can be described by its first point and its *Freeman code* [9], $\mathcal{F}(g,h)$, with the alphabet $\{0,1,7\}$. For any word $\omega \in \{0,1,7\}^k$ ($k \in \mathbb{N}$), we set $\|\omega\| = \sqrt{k^2 + j^2}$ where j is the number of letters 1 minus the number of letters 7 in the word ω.

2.2 Local Estimators

Local length estimators (see [10] for a short review) are based on parallel computations of the length of fixed size segments of a digital curve. For instance, an 8-connected curve can be split into 1-step segments. For each segment, the computation return 1 whenever the segment is parallel to the axes (Freeman's code is even) and $\sqrt{2}$ when the segment is diagonal (Freeman's code is odd). Then all the results are added to give the curve length estimation.

This kind of local computation is the oldest way to estimate the length of a curve and has been widely used in image analysis. Nevertheless, it has not the convergence property. In [13], the authors introduce a general definition of local length estimation with sliding segments and prove that such computations cannot give a convergent estimator for straight lines whose slope is small (less than the inverse of the size of the sliding segment). In [17], a similar definition of local length estimation is given with disjoint segments. Again, it is shown that the estimator failed to converge for straight lines (with irrational slopes). This behavior is experimentally confirmed in [3] on a test set of five closed curves. Moreover, the non-convergence is established in [5,18] for almost all parabolas.

2.3 Adaptive Estimators: MDSS and MLP

Adaptive length estimators gather estimators relying on a segmentation of the discrete curve that depends on each point of the curve: a move on a point can change the whole segmentation. Unlike local estimators, it is possible to prove the convergence property of adaptive length estimators under some assumptions. Adaptive length estimators include two families of length estimators, namely the Maximal Digital Straight Segment (MDSS) based length estimators and the Minimal Length Polygon (MLP) based length estimators.

Definition and properties of MDSS can be found in [12,7,3]. Efficient algorithms have been developed for segmenting curves or function graphs into MDSS and to compute their characteristics in a linear time [12,8,7]. The decomposition in MDSS is not unique and depends on the start-point of the segmentation and on the curve travel direction. The convergence property of MDSS estimators has been proved for convex polygons whose MDSS polygonal approximation[1] is also convex [11, Th. 13 and the proof]: given a convex polygon $\mathcal{C}$ and a grid spacing h (below some threshold), the error between the estimated length $L_{\mathrm{est}}(\mathcal{C},h)$ and the true length of the polygon $L(\mathcal{C})$ is such that

[1] Though the digitization of a convex set is digitally convex, it does not mean that a polygonal curve related to a convex polygonal curve via a MDSS segmentation process is also convex.

$$|L(S) - L_{\mathrm{est}}(S,h)| \leq (2+\sqrt{2})\pi h. \qquad (1)$$

Empirical MDSS multigrid convergence has also been tested in [3,6] on smooth nonconvex planar curves. The obtained convergence speed is a O(h) as in the convex polygonal case. Nevertheless it has not been proved under these assumptions. Another way to obtain an estimation of the length of a curve using MDSS is to take the slopes of the MDSSs to estimate the tangent directions and then to compute the length by numerical integration [2,3,14]. The estimation is unique and has been proved to be multigrid convergent for smooth curves (of class C^2 with bounded curvature in [2], of class C^3 with strictly positive curvature in [14]). The convergence speed is a $\mathcal{O}(h^{\frac{1}{3}})$ [14] and thus, worse than (1).

Let $\mathcal{C}$ be a simple closed curve lying in-between two polygonal curves γ_1 and γ_2. Then there is a unique polygon, the MLP, whose length is minimal between γ_1 and γ_2. The length of the MLP can be used to estimate the length of the curve $\mathcal{C}$. At least two MLP based length estimators have been described and proved to be multigrid convergent for convex, smooth or polygonal, simple closed curves, the SB-MLP proposed by Sloboda *et al.* [16] and the AS-MLP, introduced by Asano *et al.* [1]. For both of them, and for a given grid spacing h, the error between the estimated length $L_{\mathrm{est}}(\mathcal{C},h)$ and the true length of the curve $L(\mathcal{C})$ is a $\mathcal{O}(h)$:

$$|L(\mathcal{C}) - L_{\mathrm{est}}(\mathcal{C},h)| \leq Ah$$

where $A = 8$ for SB-MLP and $A \approx 5.844$ for AS-MLP.

On the one hand, as estimators described in this section are adaptive, the convergence theorems are difficult to establish and rely on strong hypotheses. On the other hand, the study of the MDSS in [6] shows that the MDSS size tends to 0 and their discrete length tends toward infinity as the grid step tends to 0. Thereby, one could ask whether combining a local estimation with an increasing window size as the resolution grows would give a convergent estimator under more general assumptions and/or with simpler proofs of convergence. The following sections explore this question.

2.4 Semi-local Length Estimators

The notion of semi-local estimator appears in [4]. At a given resolution, a semi-local estimator resembles a local estimator: it can be implemented via a parallel computation, each processor handling a fixed size segment of the curve. Nevertheless, in the framework of semi-local estimation, the processors must be aware of the resolution from which the size of the segments depends.

More formally, let $g : [a,b] \to \mathbb{R}$ be a 1-Lipschitz function[2]. Hence, at any resolution, the Freeman's code describing the discretization of g belongs to the set $\mathcal{P} = \bigcup_{n\in\mathbb{N}}\{0,1,7\}^n$.

[2] In [4], the hypothesis on g is not clear. On the one hand, the code $\mathcal{F}(g,h)$ is supposed to have $\{0,1\}$ as alphabet. On the other hand, [4, Prop 1] does not retain any hypothesis on g but its class of differentiability. Indeed, in the proof, the derivative of g needs not be positive nor limited by 1.

A *semi-local estimator* is a pair (H, p) where

- $H :]0, \infty[\to \mathbb{N}^*$ gives the relative size of the segments given a grid spacing h and
- $p : \mathcal{P} \to [0, \infty[$ gives the estimated feature (here, the length) associated to a (finite) Freeman's code.

At a given grid spacing h, the Freeman's code describing the digitization of the curve $\mathcal{C}(g)$ is segmented in N_h codes ω_i of length $H(h)$ and a rest $\omega_* \in \{0, 1, 7\}^j$, $j < H(h)$. Then, the length of the curve $\mathcal{C}(g)$ is estimated by

$$L^{\mathrm{SL}}(g, h) = h \sum_{i=0}^{N_h - 1} p(\omega_i).$$

In [4], the authors give a proof of convergence for functions of class C^2.

Theorem 1 ([4, Prop. 1]). *Let (H, p) be a semi-local estimator such that:*

1. $\lim_{h \to 0} hH(h) = 0$,
2. $\lim_{h \to 0} H(h) = +\infty$,
3. $\max \left\{ p(\omega) - \|\omega\| \mid \omega \in \{0, 1, 7\}^k \right\} = \mathrm{o}(k)$ *as* $k \to +\infty$.

Then, for any function $g \in C^2([a, b])$, the estimation $L^{\mathrm{SL}}(g, h)$ converges toward the length of the curve $\mathcal{C}(g)$. Furthermore, if the term $\mathrm{o}(k)$ in the third hypothesis is a constant and $H(h) = \Theta(h^{-\frac{1}{2}})$, then $L(g) - L^{\mathrm{SL}}(g, h) = \mathcal{O}(h^{\frac{1}{2}})$.

$H(h)$ stands for the size of a Freeman's code ω while $hH(h)$ is the real length of the computation step. In the above theorem, the first hypothesis states that the real length $hH(h)$ tends to 0. If instead of diminishing the grid spacing, we keep it constant while doing a magnification of the curve with a factor $1/h$, the second hypothesis states that the size $H(h)$ of a code tends to infinity. Finally, and informally speaking, the last hypothesis states that the function p applied to a code ω must return a value close to the diameter[3] of the subset of $\mathcal{D}^{\mathrm{O}}(g, h)$ associated to ω.

3 Sparse Estimators

In this section, we introduce a new notion, derived from semi-local estimators. Yet, on the contrary to semi-local estimators, we discard the information given by the codes ω_i but their extremities. It is as if we had two resolutions, one for the space (the abscissas), one for the calculus (the ordinates).

We have noted earlier that the hypotheses about semi-local estimators in [4] are ambiguous. May be for the same reasons than Daurat *et al.*, we are tempted to do so. Indeed, in all of our proofs, we do not need the "1" in the 1-Lipschitz hypothesis. But from a practical point of view, k-Lipschitz function for $k > 1$

[3] The maximal Euclidean distance between two points of the subset.

may give non 8-connected digitization and it does not make a lot of sense to measure the length of a set of disconnected points (though we could define a discrete curve as the curve, in the usual mathematical sense, of a function from $\mathbb{Z}$ to $\mathbb{Z}$). Hence, in the following definition, as in the statement of our theorems, we assume a 1-Lipschitz function while we intentionally forget the "1" in the statements of the lemmas.

3.1 Definition

Definition 1. *Let* $H\colon]0,+\infty[\to \mathbb{N}^*$ *such that* $\lim_{h\to 0} H(h) = +\infty$ *and* $\lim_{h\to 0} hH(h) = 0$. *We say that* H *is* sparsity function. *Let* $g : [a,b] \to \mathbb{R}$ *be a 1-Lipschitz function. The* H-sparse estimator *of the length of the curve* $\mathcal{C}(g)$ *is defined by*

$$L^{\mathrm{Sp}}(g,h) = h\sum_{i=0}^{N_h} \|\omega_i\|$$

where $\omega_i \in \{0,1,7\}^{H(h)}$ *for* $i \neq N_h$, $\omega_{N_h} \in \{0,1,7\}^j$ *with* $j \in (0, H(h)]$ *and the concatenation of the words* ω_i *equals* $\mathcal{F}(g,h)$.

An Illustration is given Figure 1.

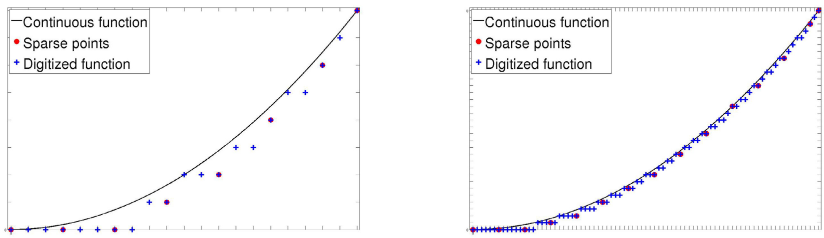


Fig. 1. Sparse estimation at two resolutions

3.2 Convergence

In this section, we establish that the sparse length estimators are convergent for 1-Lipschitz functions. Moreover, Theorem 2 gives a bound on the error at grid spacing h for functions of class C^2.

Let $m = hH(h)$ and A, B be resp. the minimum and the maximum of the integer interval $\{k \in \mathbb{N} \mid kh \in [a,b]\}$. The proof of Theorem 2 relies on two lemmas. The first one evaluates the difference between the length of the curve $\mathcal{C}(g)$ and the length of the curve of the piecewise affine function g_m defined on $[Ah, Bh]$ by $g_m(Ah + km) = g(Ah + km)$ ($k \in \mathbb{N}$) and $g_m(Bh) = g(Bh)$. The second lemma evaluates the difference between $\mathrm{L}(g_m)$ and the length of the piecewise affine function g_m^h defined on $[Ah, Bh]$ by $g_m^h(Ah + km) = \lfloor g_m(Ah + km)\rfloor_h = \lfloor g(Ah + km)\rfloor_h$ ($k \in \mathbb{N}$) and $g_m^h(Bh) = \lfloor g(Bh)\rfloor_h$. Figure 2 shows the three functions g, g_m, g_m^h on an interval $[Ah + km, Ah + (k+1)m]$.

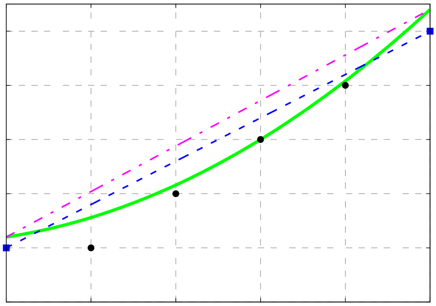


Fig. 2. The two parts of the estimation error: the curve g (in green, solid) to its chord g_m (in magenta, dotted-dashed) then the curve chord to the chord g_m^h (in blue, dashed) of the digitized curve $\mathcal{D}^{\mathrm{O}}(g,h)$ (black points)

Lemma 1. *Let g be a Lipschitz function.*

– *For any sparsity function H, we have*

$$\lim_{h\to 0} L(g_m) = L(g).$$

– *If furthermore g is of class C^2, we have for any h*

$$|L(g_m) - L(g)| \leq m\frac{b-a}{2}\|\varphi'\|_\infty + 2h\|\varphi\|_\infty \tag{2}$$

where the function φ is defined on $\mathbb{R}$ by $\varphi(t) = \sqrt{1+g'(t)^2}$.

Lemma 1 can be seen as an adaptation of a classical result on the approximation of a curve by its chords, the difficulty comes from the 1-Lipschitz hypothesis and the fixed sparsity step $H(h)$.

Lemma 2. *Let f_1 and f_2 be two piecewise affine functions defined on $[c,d] \subset \mathbb{R}$ with a common subdivision having p steps. Suppose that $f_1 \leq f_2$ and $\|f_1 - f_2\|_\infty \leq e$ for some $e \in \mathbb{R}$. Then*

$$|L(f_1) - L(f_2)| \leq p\,e.$$

Theorem 2 relies on Lemma 1 and Lemma 2 which is applied to the piecewise affine functions g_m and g_m^h, taking $e = h$.

Theorem 2. *Let H be a sparsity function and $g : [a,b] \to \mathbb{R}$ a 1-Lipschitz function. Then, the estimator L^{Sp} converge toward the length of the curve $\mathcal{C}(g)$. Furthermore, if g is of class C^2, we have*

$$L(g) - L^{\mathrm{Sp}}(g,h) = \mathcal{O}(hH(h)) + \mathcal{O}\left(\frac{1}{H(h)}\right). \tag{3}$$

Formula 3 shows two opposite trends for the determination of the sparsity step $H(h)$: $\mathcal{O}(hH(h))$ – the discretization error – corresponds to the curve sampling error and tends to reduce the step $H(h)$ while $\mathcal{O}\left(\frac{1}{H(h)}\right)$ – the quantization error – corresponds to the error due to the quantization of the sample points and tends to extend the step. The optimal convergence speed in $h^{\frac{1}{2}}$ is then obtained taking $H(h) = \Theta(h^{-\frac{1}{2}})$. Thus, only one in about $h^{-\frac{1}{2}}$ value is needed to make a sparse estimation (which justifies the adjective sparse). Then, the complexity in the optimal case is a $\mathcal{O}(r^{\frac{1}{2}})$.

3.3 Concave Functions

In this section, we assume besides that the function g is differentiable and concave on $[a, b]$. Under these hypotheses, we can improve the bound on the convergence speed of the estimated length toward the true length of the curve $\mathcal{C}(g)$. The functions g_m and g_m^h are those defined in Section 3.2. Lemmas 3 and 4 are improvements of Lemmas 1 and 2 for concave curves. Figure 3 shows some experiments that illustrate the convergence rate obtained with Theorem 3.

Lemma 3. *If g is of class C^2 and $g'' \leq 0$ on $[a, b]$, then*

$$|L(g) - L(g_m)| \leq \frac{(b-a)\|g''\|_\infty^2}{8} m^2 + 2h\|\varphi\|_\infty \tag{4}$$

where the function φ is defined on $\mathbb{R}$ by $\varphi(t) = \sqrt{1 + g'(t)^2}$.

The right part of Inequality (4) contains two terms as in Inequality (2) but only the first term has been improved with m becoming m^2. The second term standing for the error on the edges remains the same.

Lemma 4.

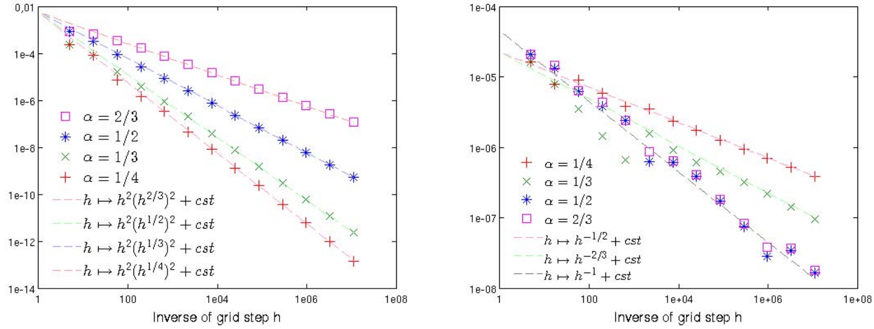
$$\left|L(g_m) - L(g_m^h)\right| \leq (b-a)\frac{1}{H(h)^2} + h\|g'\|_\infty .$$

From Lemma 4 and Lemma 3, we derive the following bound on the speed of convergence when the function g is concave.

Theorem 3. *Let H be a sparsity function and $g : [a, b] \to \mathbb{R}$ a concave 1-Lipschitz function of class C^2. Then, we have*

$$L(g) - L^{\mathrm{Sp}}(g, h) = \mathcal{O}(h^2 H(h)^2) + \mathcal{O}\left(\frac{1}{H(h)^2}\right).$$

Concavity allows squarring each term (compared to Theorem 2), which does not change the optimal size for $H(h)$ but improves the optimal convergence speed up to h.



(a) $|L(g) - L(g_m)|$ (b) $\left|L(g_m) - L(g_m^h)\right|$

Fig. 3. Experimental convergence rates. We have computed the length of the curve $y = \ln(x)$, $x \in [1, 2]$, using the sparse estimators defined by $H(h) = \lfloor h^{-\alpha} \rfloor$ where $\alpha \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, for the resolutions defined by $r = \lfloor 1.5^{1+3n} \rfloor$, $n \in [0, 13]$. (a) Discretization error (the errors on the left and the right bounds of the interval have been withdrew). We observe the convergence in $\mathcal{O}(h^2 H(h)^2)$ which appears in Theorem 3. (b) Quantization error. For $\alpha \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, we observe the convergence is a $\mathcal{O}(1/H(h)^2)$, which appears in Theorem 3. For $\alpha = \frac{2}{3}$, the condition *(iii)* of Prop. 1 is satisfied and thus the piecewise affine function g_m^h is concave. Hence, we can observe that the convergence is a $\mathcal{O}(h)$ as deduced from Lemma 5.

3.4 Strong Concavity

When the function g is concave, the piecewise affine function g_m is clearly also concave. Nevertheless, the second piecewise function g_m^h is not necessary concave even on the sub-domain $J_h = [Ah, Ah + N_0 m]$ where N_0 is the greatest integer such that $J_h \subseteq [a, b]$. Indeed, we exhibit in Appendix A a function g that is concave and for which the function g_m^h is nonconcave for any h below some threshold. This section gives some sufficient conditions for g_m^h to be also concave and studies the consequences on the convergence speed of such an assumption.

Proposition 1. *Let H be a sparsity function and $g : [a, b] \to \mathbb{R}$ a concave function of class C^2. If one of the following condition holds, then there exists $h_0 > 0$ such that, for any $h < h_0$, the piecewise affine function g_m^h is concave on J_h.*

(i) $H(h) = h^{-\frac{1}{2}}$ *and* $\max(g'') < -1$.
(ii) $H(h) = h^{-\frac{1}{2}}$ *and* $g(x) = ax^2 + bx + c$ *where* $a \leq -\frac{1}{2}$.
(iii) $hH(h)^2 \to +\infty$ *as* $h \to 0$ *and* $\max(g'') < 0$.

The following lemma is an improvement of Lemma 4 for two concave piecewise affine functions.

Lemma 5. *Let f_1 and f_2 be two concave piecewise affine functions with the same monotonicity defined on $[c, d] \subset \mathbb{R}$ such that $f_1 \leq f_2$ and $\|f_1 - f_2\|_\infty \leq e$ for some $e \in \mathbb{R}$. Then*

$$|L(f_1) - L(f_2)| \leq B\, e.$$

where B is a constant related to the slope of f_1 and f_2 at c.

Corollary 1. *Let H be a sparsity function and $g : [a, b] \to \mathbb{R}$ a concave function of class C^2. If, for some $h_0 > 0$, the function g_m^h is concave on J_h for any $h < h_0$, then we have*

$$L(g) - L^{\mathrm{Sp}}(g, h) = \mathcal{O}(h^2 H(h)^2) + \mathcal{O}(h).$$

From Corollary 1, it follows that, to speed up the convergence, we shall take the smallest sparsity step $H(h)$ provided the hypothesis about the concavity is satisfied. According to Proposition 1, this should lead us to choose the function H such that H dominates $h^{-\frac{1}{2}}$ as $h \to 0$. For instance, we can take $H(h) = h^{-\frac{1}{2}-\varepsilon}$ where $\varepsilon > 0$ and $\varepsilon \approx 0$. Then, the convergence speed is $h^{1-2\varepsilon}$. Note that h is a minimal error bound that cannot be improved in the general case since for the function g defined by $g(x) = (\frac{19}{48})^2 - x^2$, $x \in [\frac{1}{16}, \frac{19}{48}]$, we have shown that $L(g) - L^{\mathrm{Sp}}(g, h) \geq 0.06h$ (see Appendix B).

4 Conclusion

In this article, we have studied some convergence properties of a class of semi-local length estimators in the concave and the general cases. These estimators need few information about the curve: the proportion of points of the curve used for the computation tends to 0 as the resolution tends toward infinity. That is why we propose to call them sparse estimators. In a future work, we plan to extend our estimators to the nD Euclidean space to compute k-volumes, $k < n$. We have also to study how the material presented in this article behave with Jordan curves obtained as boundary of solid objects through various discretization schemes. Furthermore, the definition of the sparse estimators relies on Jordan's definition for curve length. It would be interesting to keep the main idea from these estimators while relying on the more general definition of Minkowski (as in [2]). This could be more realistic in the framework of multigrid convergence, since physic objects cannot be considered as smooth (nor convex, etc.) at any resolution. Another extension of this work is to check whether the proofs of convergence obtained for sparse estimators can help to obtain new proofs for the convergence of adaptative length estimators as the MDSS. This could lead to the definition of a larger class of geometric feature estimators including sparse estimators and MDSS. Eventually, there is a need to find how to estimate the resolution of a given curve.

A Strong Concavity: Counterexamples

In this appendix, we show that a piecewise affine function can be concave and its digitization, beyond some resolution, never concave (that is, the piecewise affine function g_m^h defined in Sec. 3.2 is not concave for grid spacing h below some threshold). The first counterexample uses a local estimator and the second one uses a sparse estimator. Both counterexamples rely on the following theorem proved in [18] (in fact, an extended version of the theorem is needed for the second counterexample). This theorem asserts that, given a function $x \mapsto ax^2 + bx + c$, the distribution in $[0, h]$ of the values of the expression $\{a(kh)^2 + b(kh) + c\}_h$, $k \in \mathbb{N}$, which are the errors resulting from the quantization, tends toward the equidistribution.

Theorem 4 ([18, Lemma 2 and Prop. 3]). *Let $a, b \in \mathbb{R}$, $a < b$. Let $g : [a, b] \to \mathbb{R}$ be a polynomial function of degre 2 with derivative in $[0, 1]$. Then, for all $[u, v] \subseteq [0, 1]$,*

$$\lim_{h \to 0} \frac{\mathrm{card}\{x \in h\mathbb{N} \cap [a, b] \mid \{g(x)\}_h \in [hu, hv]\}}{\mathrm{card}(h\mathbb{N} \cap [a, b])} = v - u.$$

For the first counterexample, we digitize the parabola associated to the function $g(x) = 2x - x^2$, $x \in [0, 1]$ and we split this parabola into segments of size $5h$. Thanks to Theorem 4, we prove that, for each grid spacing h below some threshold, we can choose an integer p for which the fractional part $\{g_m^h(ph)\}_h$ is such that the finite difference $g_m^h((p + 5)h) - g_m^h(ph)$ is less than or equal to the grid spacing h while the finite difference $g_m^h((p + 10)h) - g_m^h((p + 5)h)$ is greater than or equal to twice the grid spacing h. Thus, the function g_m^h is not concave on $[0, 1]$.

For the second counterexample, we discretize the parabola $y = g(x) = \frac{1}{50}(2x - x^2)$, $x \in [0, 1]$ and we use segments of size $H(h) = \left\lfloor h^{-\frac{1}{2}} \right\rfloor$. Again, we have shown that there exists $h_0 > 0$ such that for any h, $0 < h < h_0$, there exists an integer p for which the slope of g_m^h on $[ph, ph + H(h)h]$ is greater than its slope on $[ph + H(h)h, ph + 2H(h)h]$.

B Inferior Bound for the Method Error in the Concave Case

We give an inferior bound on the difference between the true length $L(g)$ of the parabola $y = g(x) = (\frac{19}{48})^2 - x^2$ for $x \in [\frac{1}{16}, \frac{19}{48}]$ and the length $L^{\mathrm{Sp}}(g, h)$, obtained with the sparse estimator defined by the sparsity function $H(h) = \left\lfloor h^{-\frac{1}{2}} \right\rfloor$. Let g_m and g_m^h be the piecewise affine functions defined in Section 3.2. Then the lengths of their curves satisfy $L(g_m^h) + 0.06h \leq L(g_m) \leq L(g)$ for any $h = (12(8p + 1))^{-2}$ where $p \in \mathbb{N}$. Moreover, the bounds of the interval $[\frac{1}{16}, \frac{19}{48}]$ are multiples of h. Hence, there is no error due to the bounds. Eventually, for any $p \in \mathbb{N}$ and $h = (12(8p + 1))^{-2}$, we get $L(g) - L^{\mathrm{Sp}}(g, h) \geq 0.06h$.

References

1. Asano, T., Kawamura, Y., Klette, R., Obokata, K.: Minimum-length polygons in approximation sausages. In: Arcelli, C., Cordella, L.P., Sanniti di Baja, G. (eds.) IWVF 2001. LNCS, vol. 2059, pp. 103–112. Springer, Heidelberg (2001)

2. Coeurjolly, D.: Algorithmique et géométrie discrète pour la caractérisation des courbes et des surfaces. Ph.D. thesis, Université Lyon 2 (2002)
3. Coeurjolly, D., Klette, R.: A comparative evaluation of length estimators of digital curves. IEEE Trans. Pattern Anal. Mach. Intell. 26(2), 252–257 (2004)
4. Daurat, A., Tajine, M., Zouaoui, M.: Les estimateurs semi-locaux de périmètre. Tech. rep., LSIIT CNRS, UMR 7005, Université de Strasbourg (2011), http://hal.inria.fr/hal-00576881
5. Daurat, A., Tajine, M., Zouaoui, M.: Patterns in discretized parabolas and length estimation. In: Brlek, S., Reutenauer, C., Provençal, X. (eds.) DGCI 2009. LNCS, vol. 5810, pp. 373–384. Springer, Heidelberg (2009)
6. De Vieilleville, F., Lachaud, J.O., Feschet, F.: Convex digital polygons, maximal digital straight segments and convergence of discrete geometric estimators. Journal of Mathematical Imaging and Vision 27(2), 139–156 (2007)
7. Debled-Rennesson, I., Reveillès, J.P.: A linear algorithm for segmentation of digital curves. International Journal of Pattern Recognition and Artificial Intelligence 09(04), 635–662 (1995)
8. Dorst, L., Smeulders, A.W.: Discrete straight line segments: parameters, primitives and properties. In: Vision Geometry: Proceedings of an AMS Special Session Held October 20-21, 1989,[at Hoboken, New Jersey], vol. 119, pp. 45–62 (1991)
9. Freeman, H.: On the encoding of arbitrary geometric configurations. IRE Transactions on Electronic Computers EC-10(2), 260–268 (1961)
10. Klette, R., Rosenfeld, A.: Digital Geometry. Morgan Kaufmann (2004)
11. Klette, R., Žunić, J.: Multigrid convergence of calculated features in image analysis. Journal of Mathematical Imaging and Vision 13(3), 173–191 (2000)
12. Kovalevsky, V.: New definition and fast recognition of digital straight segments and arcs. In: Proceedings of the 10th International Conference on Pattern Recognition, vol. ii, pp. 31–34 (1990)
13. Kulkarni, S.R., Mitter, S.K., Richardson, T.J., Tsitsiklis, J.N.: Local versus nonlocal computation of length of digitized curves. IEEE Trans. Pattern Anal. Mach. Intell. 16(7), 711–718 (1994)
14. Lachaud, J.O.: Espaces non-euclidiens et analyse d' image: modèles déformables riemanniens et discrets, topologie et géométrie discrète. Habilitation á diriger des recherches, Université de Bordeaux 1 (Décembre 2006)
15. Mazo, L., Baudrier, E.: About multigrid convergence of some length estimators (extended version). Tech. rep., ICube, University of Strasbourg, CNRS (2014), http://hal.inria.fr/hal-00990694
16. Sloboda, F., Zatko, B., Stoer, J.: On approximation of planar one-dimensional continua. In: Advances in Digital and Computational Geometry, pp. 113–160 (1998)
17. Tajine, M., Daurat, A.: On local definitions of length of digital curves. In: Nyström, I., Sanniti di Baja, G., Svensson, S. (eds.) DGCI 2003. LNCS, vol. 2886, pp. 114–123. Springer, Heidelberg (2003)
18. Tajine, M., Daurat, A.: Patterns for multigrid equidistributed functions: Application to general parabolas and length estimation. Theoretical Computer Science 412(36), 4824–4840 (2011)