Freeman Digitization and Tangent Word Based Estimators

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Abstract. This paper deals with the digitization of smooth or regular curves (beyond algebraic, analytic or locally convex ones). The first part explains why the Freeman square box quantization is not well-defined for such curves, and discuss possible workarounds to deal with them. In the second part, we prove that first-order differential estimators (tangent, normal, length) based on tangent words are multi-grid convergent, for any (C^1) regular curve, without assuming any form of convexity.

Keywords: Freeman digitization, symbolic coding, cutting sequence, smooth curve, tangent word, tangent estimation, multigrid convergence.

1 Freeman Square Box Quantization ...

In his survey paper [5], Freeman defines the *square box quantization* of tracings as follows: given a continuous curve $\gamma : [0, 1] \to \mathbb{R}^2$ and a (square) grid G, we associate the (ordered) sequence of pixels that are intersected by γ.

There is a slight ambiguity when the curve crosses the grid since an edge belongs to two pixels and a vertex belongs to four pixels. Freeman solves this ambiguity by breaking the rotational symmetry and defines half-open pixels in a way that the pixels form a partition of the plane $\mathbb{R}^2$. The pixel (m, n) is defined as the set $\{(x, y) \in \mathbb{R}^2 \mid (m - 1/2)h < x \leq (m + 1/2)h$ and $(n - 1/2)h < y \leq (n+1/2)h\}$ (here, h denotes the mesh of the grid, m and n are integers, Freeman does not consider any non-integer shift and places $(0, 0)$ at the center of some pixel). A vertex now belongs to its lower left pixel, a horizontal edge belongs to its bottom pixel and a vertical edge belongs to its left pixel (see Figure 1).

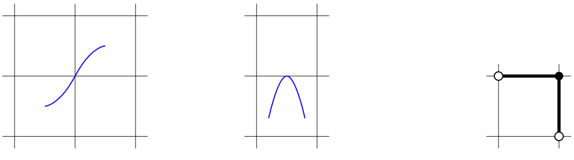

Fig. 1. Two ambiguous discretizations of curves, solved by redefining pixel shape

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Then, noticing that two consecutive pixels in the sequence are "essentially" 1-connected (they share an edge, unless the curves goes to the bottom left vertex of a pixel), Freeman codes the sequence of pixels by a word on the alphabet $\{0, 1, 2, 3\}$, depending on whether a pixel is located right, above, left, or below the previous one. We denote by $F(\gamma, G)$ the Freeman *chain code* of the square box quantization of the curve γ through the grid G (see Figure 2).

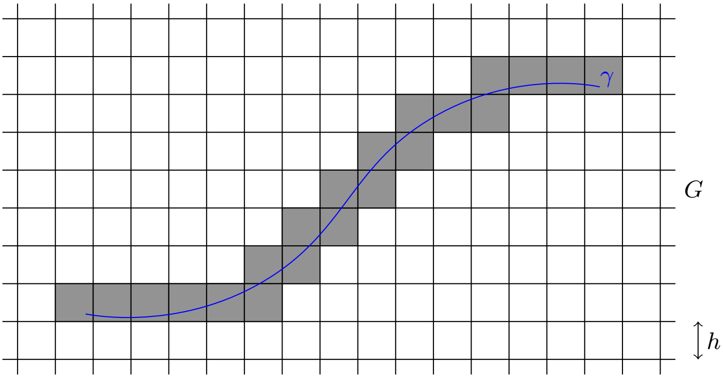

Fig. 2. $F(\gamma, G) = 00000101010101001000$

The hypothesis about the regularity of the curve is very weak (the curve is only assumed to be continuous), and allows a lot of pathological constructions such as plane filling Peano curves that will prevent Freeman quantization to be well defined.

There is a huge literature about the estimation of differential operators applied to a smooth curve through the knowledge of its Freeman discretizations at arbitrary small scales. Most of the proven methods require, in addition to some level of regularity (γ should be C^2 or C^3), the curve to be strictly convex. The aim of this paper, in the sequel of [11], is to understand Freeman discretization of smooth or regular curves, beyond convex of analytic ones.

Let I be the unit interval $[0, 1]$. A curve $\gamma : I \to \mathbb{R}^2$ is said to be *regular* if it is C^1 and if $||\gamma'(t)|| > 0$ for any $t \in I$. It is said to be *smooth* if it is moreover of class C^∞.

2 ... Is Not Well Defined for Smooth Curves

Another problem appears here, which is not addressed in Freeman's survey: the "sequence" of pixels may not be well defined. Let us construct a typical example of bad behaviour. The map $s = \begin{pmatrix} \mathbb{R} \to & \mathbb{R} & \\ x \mapsto & 0 & \text{if } x \leq 0 \\ x \mapsto & e^{-1/x^2} & \text{otherwise} \end{pmatrix}$ is smooth and positive on positive numbers. Hence, the bell map defined by $b(x) = s(x - 1/3)s(2/3 - x)$ is smooth and positive on $(1/3, 2/3)$. Now let us define recursively a sequence of maps b_n by $b_0 = b$ and

$$b_{n+1} = \begin{pmatrix} \mathbb{R} \to & \mathbb{R} & \\ x \mapsto & b_n(3x) & \text{if } 0 < x < 1/3 \\ x \mapsto & b_n(3(x - 2/3)) & \text{if } 2/3 < x < 1 \\ x \mapsto & 0 & \text{otherwise} \end{pmatrix}$$

Let (a_n) be a sequence of positive numbers such that, for any n and any $k \leq n$, $||a_n b_n^{(k)}||_\infty \leq 2^{-n}$. Since the metric space $C^\infty(I, \mathbb{R})$ endowed with the distance defined by $d(f, g) = \sum_{n\geq 0} 2^{-\min(1, \sup_{t\in I} |f^{(n)}(t) - g^{(n)}(t)|)}$ is complete, the map $\phi = \sum_{n\geq 0} a_n b_n$ is well defined and smooth. Its zeroes are located on the Cantor set (see Figure 3).

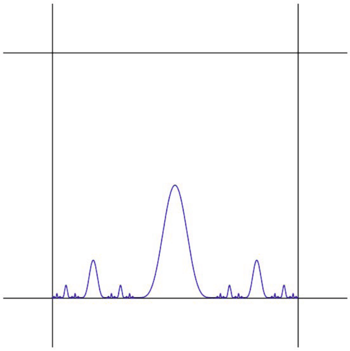

Fig. 3. The curve defined as the graph of ϕ

The Freeman digitization of the smooth curve defined by $\gamma(t) = (t, \phi(t))$ is not well defined with respect to the unit grid (or any grid containing the horizontal axis): since the Cantor set has uncountably many connected components, we created an uncountable sequence of pixels ! But this construction is sensitive to the choice that Freeman did along the edges. For example, replacing $\gamma = (x, y)$ by $(x, -y)$ leads to a single pixel, and it can be opposed that such intersections with the horizontal edge are irrelevant and could be considered as trivial.

To deal with such an objection, let us assume that vertices and edges do not belong to any pixel, that is, a pixel is selected only when the curve pass through its interior. Unfortunately, this is not sufficient to solve our problem: it is possible to construct an oscillating smooth curve that intersects the interior of the pixels infinitely many times. For this purpose, let us define the smooth map $\psi = \sum_{n\geq 0} (-1)^n a_n b_n$ (see Figure 4).

The Freeman digitization of the smooth curve defined by $\eta(t) = (t, \psi(t))$ is not well defined with respect to the unit grid (or any grid containing the horizontal

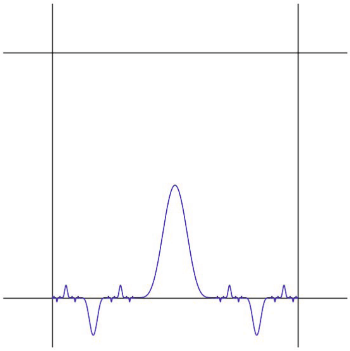

Fig. 4. The curve defined as the graph of ψ

axis): η passes through the interior of different pixels infinitely many times, and no two of them are consecutive (letting the chain coding from the sequence of pixels hard to define)!

Hence, we have a serious problem with the coding of smooth curves that cannot be fixed with a convention on choosing how to code a curve that crosses the vertex of the grid or turns around along an edge. Let us discuss some possible workarounds, leading to different research directions.

3 Some Workarounds

3.1 Restrict

A first possibility to deal with such situations is to forbid them. For example, in [6], the authors impose that the curve "passes only once between two neighboring nodes of the grid". Such a condition impacts both the curve and the grid.

Another possibility is to restrict to a class of curves that can not present such oscillations. This is the case for analytic curves. A curve $\gamma = (x, y) : [0, 1] \to \mathbb{R}^2$ is said to be *analytic* if on a neighbourhood of any time $t_0 \in [0, 1]$, its coordinates $x(t)$ and $y(t)$ can be written as $\sum_{n \geq 0} a_n (t - t_0)^n$. If such an analytic curve meets an (say) horizontal edge infinitely many times, then $x(t)$ takes the same value for infinitely many $t \in [0, 1]$: such an accumulation forces $x(t)$ to be constant and the curve is a horizontal straight line.

A similar representation of curves by words was introduced by dynamicists and are called *cutting sequences.* They have been introduced by Hadamard [7], and are used in the symbolic coding of geodesics in hyperbolic or (piecewise) euclidean spaces. It should be noticed that the problems we encountered are avoided for a similar reason: geodesics are locally straight (while their long-range behaviour may be intricate).

3.2 Extend

Conversely, we can face the problem and extend Freeman chain code to generalized sequences indexed by linear orders (instead of sequences on finite linear orders (words)).

Let us describe the possible orders appearing in a generalized Freeman chain code. Such orders are countable. Indeed, since we count intersections with interiors of pixels and since the curve is assumed to be continuous, the set of times t that γ spends in a pixel has non-empty interior. Since the real line is separable, there are only countably many non-trivial pixel intersections in the sequence. There are no other obstructions on the order type of the generalized sequence of pixels.

Theorem 1. *Any countable linear order can be obtained from the Freeman square box quantization of a smooth curve.*

Proof. The complement of the Cantor set in I (around which we built the map ψ) is a disjoint union of open intervals. This countable set (which we denote by $\mathcal{I}$) inherits from the linear order of I (given two distinct intervals $A = (a, b)$ and $B = (c, d)$ in $\mathcal{I}$, either $a < b < c < d$ or $c < d < a < b$). Since it is dense (for any $A < B$ in $\mathcal{I}$, there exists a $C \in \mathcal{I}$ such that $A < C < B$), and has no maximum nor minimum, it is (order) isomorphic to the chain $\mathbb{Q}$ of rationals. The chain of rationals has the following universal property [2]: any countable linear order is isomorphic to a subset of the chain of the rationals.

Given a countable linear order $\mathcal{L}$, we can see it as a subset of $\mathcal{I}$. Unfortunately, it is not sufficient to keep only the bells that are defined on the elements of $\mathcal{L}$. Indeed, some consecutive elements may appear in $\mathcal{L}$, and the related bells may not have opposite signs (see Figure 5).

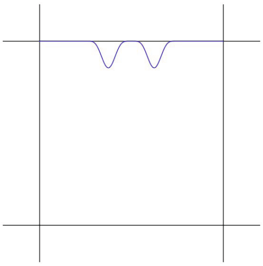

Fig. 5. Two consecutive bells in the same direction select only one pixel

So, we have to ensure local sign alternation for consecutive elements of $\mathcal{L}$. Let $\sim$ be the binary relation defined on $\mathcal{L}$ by $A \sim B$ if, and only if, the set $\{X \in \mathcal{L} \mid A \leq X \leq B \text{ or } B \leq X \leq A\}$ is finite. This defines an equivalence relation whose classes are either (order) isomorphic to a finite linear order $\{0, \ldots, n-1\}$, to the chain $\mathbb{N}$ of non-negative integers, to the chain $\mathbb{Z}\setminus\mathbb{N}$ of negative integers, or to the chain $\mathbb{Z}$ of integers, depending on the existence of a minimum or a maximum. Two consecutive elements of $\mathcal{L}$ belong to the same class.

For each class $\mathcal{C} \subseteq \mathcal{L}$, we can define an oscillating smooth map $\chi_{\mathcal{C}} : I \to \mathbb{R}$ whose graph has a Freeman sequence that is order isomorphic to $\mathcal{C}$. Let us do it for the most complex case where $\mathcal{C}$ is isomorphic to $\mathbb{Z}$. The sequence $b_n = (1+n/(1+|n|))/2$ is an increasing sequence from $\mathbb{Z}$ to I. If $(a_n)_{n\in\mathbb{Z}}$ is a sufficiently fast decreasing sequence of positive real numbers, the map defined by $\chi_{\mathcal{C}}(x) = \sum_{n\in\mathbb{Z}}(-1)^n a_n s(x - b_n)s(b_{n+1} - x)$ is convenient (see Figure 6).

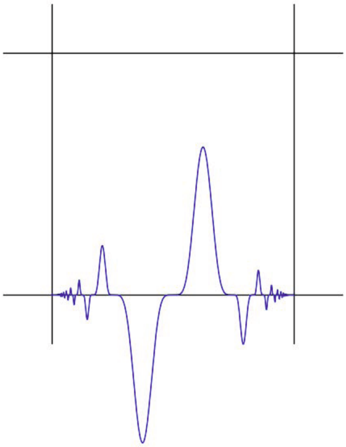

Fig. 6. An oscillating smooth map whose Freeman sequence is order-isomorphic to $\mathbb{Z}$

Now, since equivalent classes form intervals in $\mathcal{L}$, the quotient $\mathcal{J} = \mathcal{L}/\sim$ inherits a countable linear order from the one of $\mathcal{L}$. We can see it as a subset of $\mathcal{I}$. For each element $\mathcal{C}$ of $\mathcal{J}$, which is identified with an open interval (a, b) of I, we can define the smooth map $\lambda_{\mathcal{C}}$ by $\lambda_{\mathcal{C}}(x) = \chi_{\mathcal{C}}((x - a)/(b - a))$, which vanishes out of (a, b), and whose Freeman coding is isomorphic to $\mathcal{C}$.

Again, we can sum the family of maps $(\lambda_{\mathcal{C}})_{\mathcal{C}\in\mathcal{J}}$ in a way that it converges in $C^\infty(I, \mathbb{R})$. We constructed a smooth map $I \to \mathbb{R}$ whose graph is a curve whose Freeman sequence is order isomorphic to $\mathcal{L}$. □

3.3 Blur

The mesh of the grid somehow corresponds to the scale of precision of the optical device. But the constructions above play with the sharpness of the interpixel edges, as if the optical device is infinitely precise there. A possible workaround is to consider that the optical device does not have an infinite precision between consecutive cells. This corresponds to thicken the width of both edges and vertices between pixels in the mathematical model (see Figure 7). Hence, a regular curve can not oscillate between two plain pixels anymore. The blurred edge belongs to both pixels nondeterministically: if a regular curve oscillates between a pixel and a blurred edge, the device may detect it or not, as if the edge belongs to this plain pixel. If the curve passes from a plain pixel to another plain pixel, or if the curve threads its way through blurred edges, the device must detect it. Among all possible outputs, one is a finite word.

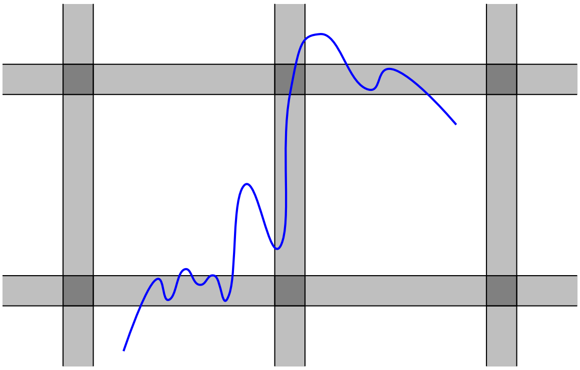

Fig. 7. Thicken the interpixel vertices and edges (but keep the long-range information)

Multigrid convergence (corresponding to using better and better optical devices) will therefore have to deal with two parameters (corresponding to the sources of imprecision) : the mesh of the grid and the width of the blurred interpixel zone. Their ratio may not be constant along the increase of precision, and depending on the speed in which the width becomes small compared to the mesh of the grid, some smoothed fractal details (as defined before) may appear or not.

3.4 Look Almost Everywhere

The set of grids of given mesh $h > 0$, which are translate to each other, can be identified with the torus $\mathbb{R}^2/h\mathbb{Z}^2$ of area h^2 (corresponding to possible shifts of the grid). It therefore inherits a natural finite Lebesgue measure.

The aim of this workaround is to prove that bad phenomena are very rare with respect to this measure.

Theorem 2. *Let γ be a regular curve and h be a positive mesh. For almost every grid G of mesh h, γ does not cross any vertex of G and intersects the edges of G transversally. For such generic grids G, the Freeman chain code $F(\gamma, G)$ is a well defined finite word.*

Proof. Since the image of a regular curve has Lebesgue measure zero in $\mathbb{R}^2$, and since the set of grid vertices is countable, we have that for almost every grid G with mesh h, the curve γ does not hit a vertex of G.

Sard's lemma asserts that given a C^k map $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ (for $k \geq \max(1, m-n+1)$), the image of critical points $f(\{t \in A \mid \mathrm{rank}(\mathrm{Diff}_t(f)) < n\})$ has (Lebesgue) measure zero in $\mathbb{R}^n$. Let us apply this lemma to the first coordinate $x : I \to \mathbb{R}$ of γ. The set $\{X \in \mathbb{R} \mid (\exists t \in I)(X = x(t) \text{ and } x'(t) = 0)\}$ has Lebesgue measure zero. By countable union, the set $\{\kappa \in [0, h] \mid (\exists k \in \mathbb{Z})(\exists t \in I)(x(t) = k + \kappa \text{ and } x'(t) = 0)\}$ also has measure zero. The same holds for the second coordinate y, hence the set $\{(\kappa, \lambda) \in [0, h]^2 \mid ((\exists k \in \mathbb{Z})(\exists t \in I)(x(t) = k + \kappa \text{ and } x'(t) = 0)) \text{ or } (\exists t \in I)(y(t) = l + \lambda \text{ and } y'(t) = 0))$ has measure zero. This set corresponds to the set of grids that are not transversally intersected by γ.

Now, let G be a generic grid. At each time t when γ intersects G, the intersection is transverse: there is an open interval O containing t such that for any $t' \neq t$ in O, $\gamma(t') \notin G$. Moreover, $\gamma(O \cap [0, t))$ and $\gamma(O \cap (t, 1])$ are included in two distinct adjacent connected components of $\mathbb{R}^2 \setminus G$ (pixel interiors). Since the set $\gamma^{-1}(G)$ is a closed subset of the compact interval I and is made of isolated points, it is finite. Hence, there is a finite set of times $\leq 0 < t_0 < \cdots < t_n \leq 1$ such that for each i, $\gamma(t_i)$ is on the grid and $\gamma((t_{i-1}, t_i))$ and $\gamma((t_i, t_{i+1}))$ are included in two distinct edge-adjacent pixel interiors. □

That said, note that the set of grids that are not transversally intersected by γ may not be countable (an antiderivative f of ϕ admits uncountably many singular values, hence the curve defined by $\gamma(t) = (t, f(t))$ intersects uncountably many horizontal lines non-transversally).

With this workaround, by looking modulo almost everywhere, the arbitrary choices made by Freeman in the boundaries of the pixels become irrelevant. Hence, we get a more symmetric and intrinsic discretization scheme that does not have to specify non-canonical choices (since those happen only on a set of zero measure).

This workaround has another big advantage: the action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ that appears in the continued fractions algorithm is central in the study of discrete straight segments (and tangent words), and inherent to the lattice structure of the grid. It sends lines with rational slopes to the vertical and horizontal axes. So, applying such a map to a well-coded curve that oscillates around a rational slope may lead to an ill-coded one. By looking almost everywhere, we avoid such a situation since the set of rational slopes is countable.

This workaround was the one we used in [11] to define tangent words and we will stick to that framework for the remaining of the paper.

4 First-Order Differential Operators via Tangent Words

Given a regular curve, the limit object we get while zooming into a point is a straight line. Freeman chain codes of straight segments are known to be exactly the *balanced* words. Hence, it seems natural to decompose a discretized curve into maximal balanced words in order to approach the tangents of the real curve, a strategy which has been widely studied. But this is not what the multigrid discretization scheme does: it discretizes a curve, at various scales. Tangent words

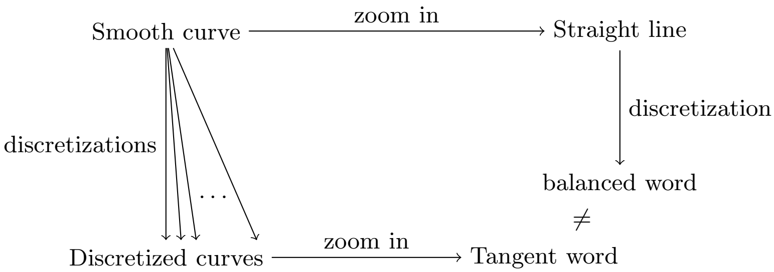

Fig. 8. Not even a noncommutative diagram

are the finite words that appear in the coding of a smooth or regular curve at arbitrary small scale.

More formally, a finite word u is said to be *tangent* if there exists a regular curve $\gamma : [0,1] \to \mathbb{R}^2$ and a sequence of grids (G_n) whose meshes converge to zero, such that u is a factor of the Freeman chain code $F(\gamma, G_n)$ of the curve γ through the grid G_n for any integer n.

Theorem 3. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular curve. For any sequence (G_n) of grids whose meshes converge to zero, the minimal length of a maximal tangent word in $F(\gamma, G_n)$ goes to ∞ with n.*

Proof. Assume by contradiction that there exists a sequence of grids (G_n) whose mesh converges to zero such that there exists an integer L such that, for any N, there exists $n \geq N$ and a maximal tangent word $w(n)$ in $W(n) = F(\gamma, G_n)$ whose length is less than L. Up to taking a subsequence (there are finitely many words of length $\leq L$), we can assume that there exists a finite word w that is a maximal tangent word of $W(n)$, for any n: $w = W(n)_{i_n \dots j_n}$ for some indices i_n, j_n and neither $W(n)_{i_n - 1 \dots j_n}$ nor $w = W(n)_{i_n \dots j_n + 1}$ is tangent (when defined). Since γ is regular, it is not trivial, hence it has positive length. In particular, the length $|W(n)|$ of $W(n)$ goes to infinity with n. Hence, for n large enough, $i_n > 0$ or $j_n < |W(n)| - 1$. Up to symmetry and up to taking a subsequence, we can assume that $j_n < |W(n)| - 1$ for all n. Now, the letter $W(n)_{j_n + 1}$ takes at most four values: one of them, which we call a appears infinitely often: wa is therefore a tangent word, that extends w in $W(n)$ for infinitely many n, which contradicts the maximality of w. □

The simplicity of this result (a pigeonhole argument) should not be surprising: the definition of tangents words contains its adaptation to the behaviour of regular curves. What is lucky is that tangent words have a simple combinatorial characterization and can be recognized in linear time [11], so they are ready to serve as drop-in replacement where balanced words are not optimal.

Now, this feature allows us to easily ensure multigrid convergence of first-order differential operators by replacing decomposition of the curve into maximal balanced words by a decomposition into maximal tangent words. Due to a lack of space, the proofs of the following results are postponed in the appendix.

4.1 Length Estimation

Given a regular curve γ, and a sequence of grids (G_n) whose meshes h_n go to zero, we can estimate its length as follows: for each n, greedily decompose $F(\gamma, G_n)$ into maximal (to the left) tangent words $F(\gamma, G_n) = w_0^n \dots w_{k_n-1}^n$, and compute the length of the associated polygonal line:

$$l_n = h_n \sum_{i=0}^{k_n-1} \sqrt{(|w_i^n|_0 - |w_i^n|_2)^2 + (|w_i^n|_1 - |w_i^n|_3)^2}$$

The sequence l_n converges to the length of γ when n goes to ∞.

4.2 Tangent Estimation

Let us first recall the definition of the directions of a tangent word u (see the definition of slope in [8]). If u is a tangent word, there exists a regular curve $\gamma : I \to \mathbb{R}^2$ and a sequence (G_n) of grids whose meshes go to 0 and such that for all n, u is a factor of $F(\gamma, G_n)$. In particular, for any integer n, there exist two sequences (t_n^1) and (t_n^2) in I such that u is the Freeman code of $\gamma|_{]t_n^1, t_n^2[}$ with respect to the grid G_n. Up to taking a subsequence (the segment I is compact), we can assume that (t_n^1) and (t_n^2) both converge to some $t \in I$: we say that the direction of $\gamma'(t)$ is *a direction* of u. The *set of directions* of u corresponds to all possible choices on the curve γ, the sequence of grids (G_n) and sequences of times (t_n^1) and (t_n^2). The set of directions of a tangent words can be computed from the continued fraction algorithm introduced to recognize it. The set of directions of a balanced word is a non-trivial interval, while the set of directions of a non-balanced tangent words is a singleton.

Now, we can obtain a tangent (or normal) estimator as follows: given a pixel of a discretization of a regular curve γ, we can choose any maximal tangent word that contains this pixel and take any of its directions as an estimation.

As for length estimation, this leads to uniformly multigrid convergent estimators. Convex combinations of estimations (corresponding to λ-MST [10]) also work. While existing convergence results for balanced word based estimators require the curve to be C^3 and piecewise strictly convex, the use of maximal tangent words in place of balanced words only require the curve to be C^1 (with the same algorithms).

4.3 Maximal Symmetric Tangent Words

Feschet and Tougne [4] defined a tangent estimator based on maximal symmetric balanced words around a given pixel (symmetric in the sense that the position of the pixel is in the middle of the word). Since, even under very strong hypotheses, the length of such words does not converges to ∞ around some pixels, [9] proved that it is not a multigrid convergent tangent estimator.

But, the same argument as in Theorem 3 works for maximal symmetric tangent words: any word w can be extended simultaneously in both directions as awb for $(a, b) \in \{0, 1, 2, 3\}^2$ in at most 16 ways. Since 16 is a finite number, the same pigeonhole argument applies. Hence, apart from the ends of the word (a case which can be dealt with by considering closed curves and circular words, or by simply looking away from the ends), the length of the smallest maximal tangent word which is symmetric around any letter converges to ∞ when the mesh of the grid goes to zero, letting the estimator to be uniformly convergent.

5 Why Does Convexity Matter for Maximal Segment Based Estimators?

A more detailed description of the tangent convex words was provided in [8]. In particular, we should notice that any tangent convex word is the concatenation of two balanced words, which should not impact the estimations so much. In particular, while the smallest length of maximal balanced words is not the same as for maximal tangent words, their average lengths are similar.

However, this is not the case for general tangent words, whose minimal factorization into balanced words can have arbitrary many factors, and even linear with respect to the length of the tangent word (think of the tangent words $(0011)^n$).

This difference between (piecewise) convex curves and more general smooth curves is highlighted by this counting argument [12]: the set of tangent convex words of length n has size $\Theta(n^3)$, which is the same as for balanced words, whereas the set of smooth tangent words of length n has exponential size.

Those facts tend to indicate that small oscillations are the main reason for maximal segment-based estimators not to work for general smooth curves.

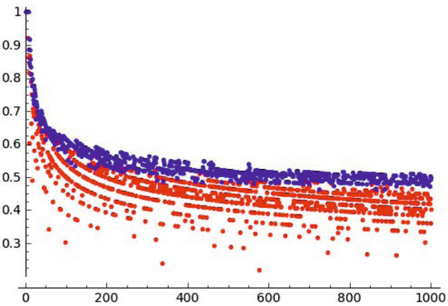

Fig. 9. Smallest maximal balanced (red) and tangent (blue) word in the coding of circles (abscissa=inverse of the mesh, ordinate = length of the word (in a loglog scale))

6 Conclusion

We tried to explain that the problems arising in the digitization of regular curves are related to soften fractal oscillations. We saw that the use of tangent words in place of balanced words has the effect of smoothing those irregularities, especially around points whose curvature vanishes. They are very natural objects, since their definition is adapted to the framework of regular (or smooth) curves, and qualitative convergence properties come for free. Actually, quantitative results on the speed of convergence can also be proven: *if* $\gamma : [0,1] \to \mathbb{R}^2$ *is any regular* C^2 *curve and* (G_n) *is a sequence of grids whose meshes* h_n *converge to zero, the minimal length of a maximal tangent word in* $F(\gamma, G_n)$ *belongs to* $\Omega(h_n^{-\frac{1}{3}})$, a result which requires much weaker hypotheses than in [1] and [3].

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7 Appendix: Postponed Proofs

7.1 Length Estimation (Section 4.1)

If w is a finite word over the alphabet $\{0, 1, 2, 3\}$, let us denote $||w|| = \sqrt{(|w|_0 - |w|_2)^2 + (|w|_1 - |w|_3)^2}$.

Since the curve γ is of class C^1, it is rectifiable and its length can be approximated as follows: if $(\{0 = t_0^n \leq t_1^n \leq \cdots \leq t_{k_n-1}^n \leq t_{k_n}^n = 1\})_{n\in\mathbb{N}}$ is a sequence of subdivisions of the unit interval such that $\lim_{n\to\infty} \max_{0\leq i\leq k_n-1} |t_{i+1}^n - t_i^n| = 0$, then the length of γ is given by

$$l(\gamma) = \lim_{n\to\infty} \sum_{i=0}^{k_n-1} ||\gamma(t_{i+1}^n) - \gamma(t_i^n)||$$

Given a grid G_n and a decomposition of $F(\gamma, G_n)$ into maximal tangent words $w_0^n \ldots w_{k_n-1}^n$, each word w_i^n corresponds to a sequence $s_i^n = (p_0^{i,n}, \ldots, p_{|w_i^n|}^{i,n})$ of $|w_i^n| + 1$ pixels of G_n, where the last pixel of s_i^n is the first pixel of s_{i+1}^n.

The value

$$l_n = h_n \sum_{i=0}^{k_n-1} ||w_i^n||$$

corresponds to the length of the polygonal line $(x_0^n, \ldots, x_{k_n}^n)$, where x_i^n is the center of the pixel $p_i^n := p_{|w_{i-1}^n|}^{i-1,n} = p_0^{i,n}$.

We can pick, for any i, a time $t_i^n \in I$ where $\gamma(t_i^n)$ pass through the pixel p_i^n. Since for any i, $||x_i^n - \gamma(t_i^n)|| \leq h_n$, we have

$$\left| \sum_{i=0}^{k_n-1} ||\gamma(t_{i+1}^n) - \gamma(t_i^n)|| - h_n \sum_{i=0}^{k_n-1} ||w_i^n|| \right| \leq 2h_n k_n$$

Theorem 3 asserts that the minimum length of the w_i^n (for $0 < i < k_n - 1$) goes to infinity, hence $2h_n k_n$ goes to zero (the error we made in the extremities of the segments become negligible with respect to their length).

We are almost done, except that $\max_{0\leq i\leq k_n-1} |t_{i+1}^n - t_i^n|$ does not necessarily converge to zero when n goes to infinity (some w_i^n can be very long compared to $1/h_n$). To achieve this, let us decompose the long maximal tangent words into smaller ones ($F(\gamma, G_n) = \bar{w}_0^n \ldots \bar{w}_{\bar{k}_n-1}^n$) in a way that their minimal length (except on the boundaries) still goes to infinity, but such that the maximal value of $h_n|\bar{w}_i^n|$ goes to zero. Hence, constructing the $\bar{t}_i^n$ accordingly, we now have

$$l(\gamma) = \lim_{n\to\infty} \sum_{i=0}^{\bar{k}_n-1} ||\gamma(\bar{t}_{i+1}^n) - \gamma(\bar{t}_i^n)|| = \lim_{n\to\infty} h_n \sum_{i=0}^{\bar{k}_n-1} ||\bar{w}_i^n||$$

Now, tangent words enjoy some balance property (they stay close to a segment): if w is a tangent word that is written as a concatenation of words $w = w_0 \ldots w_{l-1}$, then $\left| ||w|| - \sum_{i=0}^{l-1} ||w_i|| \right| \leq 4l$. This concludes the proof since it implies $\lim_{n\to\infty} \left| h_n \sum_{i=0}^{\bar{k}_n-1} ||\bar{w}_i^n|| - h_n \sum_{i=0}^{k_n-1} ||w_i^n|| \right| = 0$. □

7.2 Tangent Estimation (Section 4.2)

The structure of this proof is similar as above. Let t be a point in I and (G_n) be a sequence of grids whose meshes go to zero. For any n, let u_n be a maximal tangent factor of $F(\gamma, G_n)$ *containing* $\gamma(t)$, in the sense that there exists an interval $I_{u_n} \subseteq I$ containing t such that $F(\gamma\restriction_{I_{u_n}}, G_n) = u_n$.

The first (resp. last) letter of u_n corresponds to an edge e_n (resp. e'_n) of G_n. The set $D(u_n)$ of directions of u_n is included in the set of directions of vectors $b-a$ for $(a,b) \in e_n \times e'_n$, which we denote by $\bar{D}(u_n)$. Since γ crosses e and e', the mean value theorem implies there exists a point t_n in I_{u_n} such that the direction of $\gamma'(t_n)$ belongs to $\bar{D}(u_n)$.

Theorem 3 asserts that the length of u_n goes to infinity, hence the distance between e_n and e'_n goes to infinity and the diameter of $\bar{D}(u_n)$ converges to zero (to be precise, we should deal separately with the special case along the axes where γ oscillates around a set of parallel edges).

Now, as in the previous proof, if the length of u_n grows too fast, it is possible that t_n does not converge to the point t. Again, we can shorten u_n to a word u'_n that still contains t, whose length still goes to infinity, but whose length is negligible with respect to the inverse of the mesh of G_n. This leads to an interval $I_{u'_n}$, a point t'_n in $I_{u'_n}$ and two sets of slopes $D(u'_n)$ and $\bar{D}(u'_n)$ as before, with the additional property that t'_n converges to t. Even if $\bar{D}(u'_n)$ is not necessarily included in $\bar{D}(u_n)$, the word u'_n is also tangent and $D(u'_n)$ is included in $D(u_n)$ (which is included in $\bar{D}(u_n)$). Since the diameter of $\bar{D}(u_n)$ goes to zero, and contains $\gamma'(t'_n)$ which converges to $\gamma'(t)$, any choice of direction in $\bar{D}(u_n)$ converges to $\gamma'(t)$. $\square$