Nonlocal PDEs Morphology on Graph: A Generalized Shock Operators on Graph

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Abstract. This paper presents an adaptation of the shock filter on weighted graphs using the formalism of Partial difference Equations. This adaptation leads to a new morphological operators that alternate the nonlocal dilation and nonlocal erosion type filter on graphs. Furthermore, this adaptation extends the shock filters applications to any data that can be represented by graphs. This paper also presents examples that illustrate our proposed approach.

1 Introduction

More and more contemporary applications operate with a large amount of data which are collected or represented in the form of graphs and networks or functions on graph. Examples are images, surfaces, 3D meshes, social networks. Processing and analyzing these types of data is a major challenge for both image and machine learning communities. Hence, its very important to transfer many mathematical tools which are initially developed on usual Euclidean space and proven to be efficient for many problems and applications dealing with usual image and signal domains to graph and networks.

Historically, the main tools for the study of graphs or networks come from combinatorial and graph theory. But, there is a growing interest to transpose and generalize classical tools used in image processing to graph such as Partial Differential Equation (PDEs) on graphs, see [1,2,3] and references therein for more details.

In this paper, we present an adaptation of shock filter [7] on weighted graphs using the formalism of Partial difference equations (PdEs) [1]. This adaptation leads to a new morphological family of nonlocal operators. Furthermore, this scheme allows to extend the shock filters applications to any data that can be represented by graphs.

Brief Literature on Shock Filters. The shock filter belongs to a PDEs-based filter class. The basic idea is to perform a dilation that operate around maxima, and an erosion that operate around minima. This operation create a shock between the influence zone of the image extrema. Most of the current shock filters are based on the the definition of Kramer and Bruckner in term of minima/maxima neighborhood filters [8], and on the formulation of Osher and Ruddin [9] in terms of Partial differential equations (PDEs).

There are many advantages that shock filters can offer in the image treatments. 1) They create strong discontinuities at the edges of image, and the filtered signal becomes flat within a region 2) they satisfy a maximum-minimum principle so the range of the filtered image and the initial one remains the same 3) they do not increase the L_1 norms of the derivative of a signal 4) They possess inherent stability properties.

Several modification has been proposed for the original shock filter like morphological toggle mappings [12], PDE-based enhancing [10] as well as coherence-enhancing shock filters [11] combining the stability properties of shock filters with the possibility

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of enhancing flow based in the eigenvalues of the second moment matrix or structure tensor. As the main shock filter is composed from combining the erosion and dilation processes, which are sensible to the noise, the shock filter is unable to remove some basic types of noise such as Gaussian noise, “alt and pepper” noise, etc.

Contribution. In this work, we propose a discretization of the classical shock filter on weighted graph using the formalism of PdEs which leads to a new nonlocal morphological class of operators.

The shock filter proposed by Osher and Rudin [9] is written as:

$$\frac{\partial f}{\partial t} = -sign(\Delta f)|\nabla f|, \tag{1}$$

where Δf represents the laplacian and $|\nabla f|$ represents the norm of gradient. The continuous shock filter formulation proposed by Kramer-Bruckner [7,8] is written as:

$$\frac{\partial f}{\partial t} = -sign(f_{\eta\eta})|\nabla f|, \tag{2}$$

where $f_{\eta\eta}$ represents the local second derivative in the gradient direction of the gradient of f.

In this paper, we propose a formulation that unifies and generalize the above shock filter equations on graph. We define the shock filter for each vertex on a weighted graph as:

$$\frac{\partial f(u)}{\partial t} = -sign(\Delta_{w,p} f(u))|\nabla_w f(u)|_p, \tag{3}$$

where $\Delta_{w,p}$ define the p-laplacian, $|.|_p$ define the norme, w define the weight function on a graph, $\nabla_w f$ define the gradient on a given function f on a weighted graph $G = (V, E, w)$. For $p = 2$, our formalism presents an extension of the shock filter equation proposed by Osher and Rudin on weighted and for $p = \infty$ it presents an extension of Kramer-Bruckner equation.

Paper Organization. The rest of this paper is organized as follows. Section 2 presents a general definition of Partial difference Equations on weighted graph. Section 3 presents our definition of the shock filter and a numerical scheme. Section 4 presents some experiments. Finally, Section 5 concludes this paper.

2 Partial Difference Equation on Graphs

In this Section, we present definitions of operators involved in this paper. All these definitions are borrowed from [1].

Notations and Definitions. Let $G = (V, E, w)$ be a weighted graph composed of two finite sets: $V = \{u_1, ...u_n\}$ of n vertices and $E \subset V \times V$ a set of weighted edges. An edge $(u, v) \in E$ connects two adjacent vertices u and v. The weight w_{uv} of an edge (u, v) can be defined by a function $w : V \times V \rightarrow \mathbb{R}^+$ if $(u, v) \in E$, and $w_{uv} = 0$ otherwise. We denote by $N(u)$ the neighborhood of a vertex u, i.e. the subset of vertices that share an edge with u. In this paper, graphs are assumed to be connected, undirected and with no self loops.

Let $f : V \rightarrow \mathbb{R}$ be a discrete real-valued function that assigns a real value $f(u)$ to each vertex $u \in V$. We denote by $\mathcal{H}(V)$ the Hilbert space of such functions defined on V.

Gradient Operators and p-laplacian. The *external* and *internal weighted discrete partial derivative operators* of $f \in \mathcal{H}(V)$ defined on $G = (V, E, w)$ are defined as:

$$\partial_v^{\pm} f(u) = \sqrt{w_{uv}}(f(v) - f(u))^{\pm} \tag{4}$$

with $(x)^+ = \max(0, x)$ and $(0)^- = \min(0, x)$.
The *weighted morphological external and internal gradient* $(\nabla_w^+ f)(u)$ and $(\nabla_w^- f)(u)$ of a function $f \in \mathcal{H}(V)$ at vertex u are :

$$(\nabla_w^{\pm} f)(u) = \Big((\partial_v^{\pm} f)(u)\Big)^T_{(u,v)\in E} \tag{5}$$

The $\mathcal{L}_p$ norms, $1 \leqslant p < \infty$ and the $\mathcal{L}_\infty$ norm of *upwind weighted gradients* are defined at a vertex u as:

$$\begin{aligned} \|\big(\nabla_w^{\pm} f\big)(u)\|_p &= \Big[\sum_{v\in V} w(u,v)^{p/2}\big(f(v) - f(u)\big)^{\pm}\Big]^{\frac{1}{p}} \\ \|\big(\nabla_w^{\pm} f\big)(u)\|_\infty &= \max_{v\in V}\Big(\sqrt{w(u,v)}|\big(f(v) - f(u)\big)^{\pm}|\Big) \end{aligned} \tag{6}$$

These gradients are used in [4] to adapt the well-known Eikonal equation on continuous domains defined as:

$$\frac{\partial f}{\partial t}(x,t) = F(x)\|(\nabla f)(x,t)\|_p, F(x) \in \mathbb{R}, \tag{7}$$

to the discrete following equation on graph:

$$\frac{\partial f}{\partial t}(u,t) = F^+(u)\|(\nabla_w^+ f)(u)\|_p - F^-(u)\|(\nabla_w^- f)(u)\|_p. \tag{8}$$

This equation summarizes the dilation and erosion processes. When $F > 0$, then the external gradient is used and this equation corresponds to a dilation. When $F < 0$, this equation corresponds to an erosion.

The non-local 2-Laplace operator of $f \in \mathcal{H}(V)$ at a vertex $u \in V$ is be defined by [5]:

$$(\varDelta_{w,2} f)(u) = \frac{1}{\delta_w(u)} \sum_{v\sim u}\big(w(u,v)f(v)\big) - f(u). \tag{9}$$

The infinity Laplacian operator is related to the PDE infinity Laplacian equation and defined as:

$$(\varDelta_{w,\infty} f)(u) = \tfrac{1}{2}\left[\|(\nabla_w^+ f)(u)\|_\infty - \|(\nabla_w^- f)(u)\|_\infty\right]. \tag{10}$$

We define the nonlocal dilation (NLD) and nonlocal erosion (NLE) operators in the case where $p = \infty$ and $p = 2$ respectively as following:

$$\begin{aligned} NLD_\infty\big(f\big)(u) &= f(u) + \|(\nabla_w^+ f)(u)\|_\infty. \\ NLE_\infty\big(f\big)(u) &= f(u) - \|(\nabla_w^- f)(u)\|_\infty. \\ NLD_2\big(f\big)(u) &= f(u) + \Delta t\|(\nabla_w^+ f)(u)\|_2. \\ NLE_2\big(f\big)(u) &= f(u) - \Delta t\|(\nabla_w^- f)(u)\|_2. \end{aligned} \tag{11}$$

The non-local mean filter NLM [6] is defined as:

$$NLM\big(f\big)(u) = \frac{\sum_{v \sim u} w(u,v) f(v)}{\sum_{v \sim u} w(u,v)}. \tag{12}$$

3 A New Family of Shock Filters

In this section, we present our new proposition of the new family of shock operators by adapting the classical shock filter on weighted graphs.

3.1 Extension of Shock Filter Formulation

The continuous shock filter equations (1) (2) can be expressed as:

$$\frac{\partial f}{\partial t} = k|\nabla f|, \tag{13}$$

where $k = -sign(\Delta_p)$ with $p = 2$ or $p = \infty$.

The discretization on weighted graph of equation (13) can be written as:

$$\frac{\partial f}{\partial t} = k^+ \|\nabla_w^+ f\|_p - k^- \|\nabla_w^- f\|_p. \tag{14}$$

Using a time discretization with the conventional notations, the solution of equation (14) for any vertex u can be written as:

$$\frac{f^{n+1}(u) - f^n(u)}{\Delta t} = k^+ \|\nabla_w^+ f^n(u)\|_p - k^- \|\nabla_w^- f^n(u)\|_p. \tag{15}$$

Equation (15) can be rewritten as:

$$f^{n+1}(u) = f^n(u) + \Delta t\, k^+ \|\nabla_w^+ f^n(u)\|_p - \Delta t\, k^- \|\nabla_w^- f^n(u)\|_p \tag{16}$$

Particular Case. In the case where $p = 2$, equation (16) can be interpreted as:

$$f^{n+1}(u) = \begin{cases} f^n(u) + \Delta t\ \|\nabla_w^+ f^n(u)\|_2 & \text{if} \quad k > 0 \\ f^n(u) - \Delta t\ \|\nabla_w^- f^n(u)\|_2 & \text{if} \quad k < 0 \end{cases} \tag{17}$$

with $k = -sign((\Delta_{w,2} f^n)(u))$.

By replacing NLD_2 and NLE_2 by their definitions in equation (17), we define the nonlocal shock filters operator (NLS_2) as an iterative filter as:

$$NLS_2\big(f^n\big)(u) = \begin{cases} NLD_2\big(f^n\big)(u) & \text{if} \quad k > 0 \\ NLE_2\big(f^n\big)(u) & \text{if} \quad k < 0 \\ f^n(u) & \text{if} \quad k = 0 \end{cases} \tag{18}$$

In the case where we have a regular grid graph $G = (V, E, w)$ a grid graph with 4-neighbors and with a weight function $w(u, v) = 1$. For $f : v \subset \mathbb{Z}^2 \to \mathbb{R}$, equation (19) can be interpreted as:

$$NLS_2(f^n)(u) = \begin{cases} \sqrt{\sum\limits_{v \sim u} \left(min\left(f^n\left(v\right) - f^n\left(u\right), 0\right)^2\right)} & \text{if } NLM(f^n)(u) < f^n(u). \\ \\ \sqrt{\sum\limits_{v \sim u} \left(max\left(f^n\left(v\right) - f^n\left(u\right), 0\right)^2\right)} & \text{if } NLM(f^n)(u) > f^n(u). \end{cases} \tag{19}$$

The above formulation (19) corresponds exactly to the discretization proposed by Osher-Ruddin [9] of the shock filter of a gray scale image with shaped diamond structuring element (4-connectivity).

Particular Case. In the case where $p = \infty$, the shock filters formulation (16) can be expressed as:

$$\begin{aligned} NLS_\infty(f^n)(u) = {} & \Delta t\; k^+\; NLD_\infty(f^n)(u) + \\ & \Delta t\; k^-\; NLE_\infty(f^n)(u) + \\ & (1 - \Delta t) f^n(u), \end{aligned} \tag{20}$$

with $k = -sign((\Delta_{w,\infty} f^n)(u))$. Then, equation (20) can be interpreted when $\Delta t = 1$ as:

$$NLS_\infty(f^n)(u) = \begin{cases} NLE_\infty(f^n)(u) & \text{if} \quad \alpha < \beta \\ NLD_\infty(f^n)(u) & \text{if} \quad \alpha > \beta \\ f^n(u) & \text{otherwise,} \end{cases} \tag{21}$$

with α represents $NLD_\infty(f^n)(u) - f^n(u)$ and β represents $f^n(u) - NLE_\infty(f^n)(u)$.

The above formulation corresponds exactly to the discretization proposed by kramer-Brukner [8] of the shock filter when we have an unweighted grid graph with $4-$neighbors.

Our proposed formulation (3) of the shock filter combines a large family of shock filter that depends on the norm p, w parameters and the graph topology.

4 Experimentation

In this Section, we illustrate the behavior of the shock filter equation presented in this paper. The experiments provided are not here to solve a particular application but to illustrate the potentialities of our proposal.

Figure 1 presents our shock filter effects on a blurred Data matrix image with a local and nonlocal graph structure. One can see that the nonlocal graph structure conserve better the image details. For the local graph structure, we constructed a grid graph where each pixel is connected by an edge to its 4 adjacent pixels. Then, the weighted function $(w(u,v) = 1))$. For the nonlocal graph structure, we constructed a k-nn graph where $k = 20$ and the weight function $w(u,v) = e^{\left(-d(f^0(u),f^0(v))/\sigma^2\right)}$ with σ data depends. Figure 2 and figure 3 present the shock filter effect on textured images using a nonlocal graph structure. We used the same structure of the nonlocal graph as the figure 1.

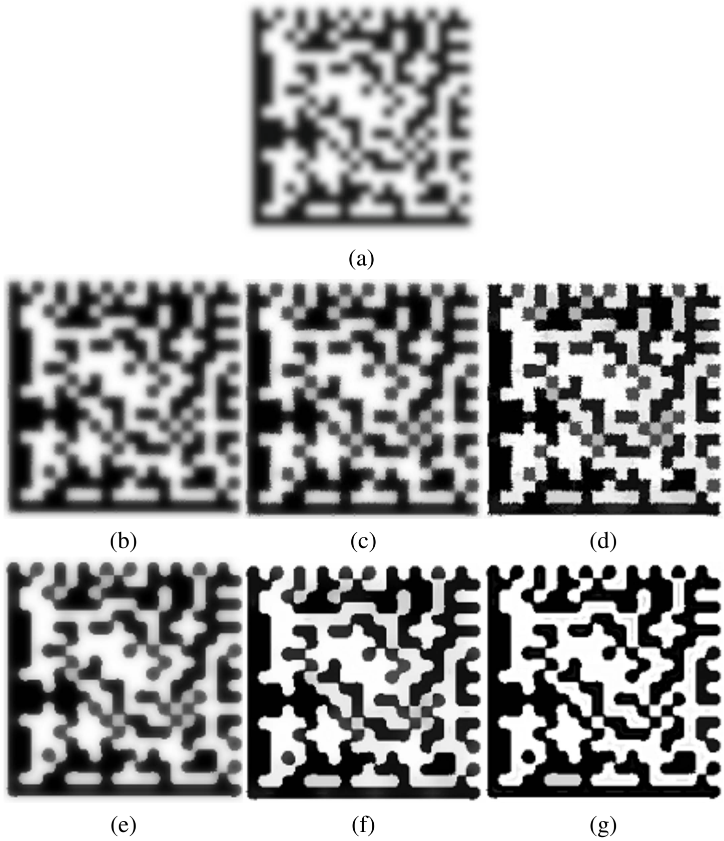
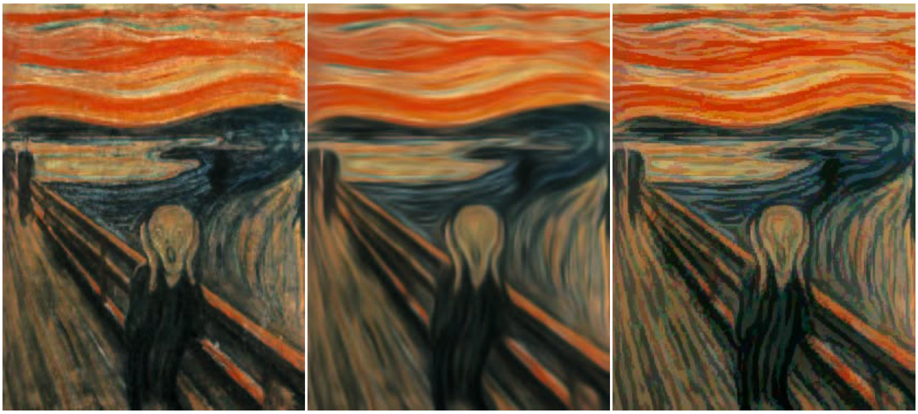


Fig. 1. Shock filter on Data matrix image. (a) presents the blurred image, (b,c,d) present the results within 5,10 and 15 iterations on a local graph structure and (e,f,g) present the results within 5,10 and 15 iterations on a nonlocal graph structure

(a) (b) (c)

Fig. 2. Shock filter on colored images. (a) presents the initial images, (b) presents the blurred images, (c) presents the resulting images using our approach



(a) (b) (c)

Fig. 3. Shock filter on colored images. (a) presents the initial images, (b) presents the blurred images, (c) presents the resulting images using our approach

5 Conclusion

In this paper, we proposed a new class of the shock filter based on Partial difference Equations on weighted graphs. This class complete the morphological nonlocal operators on graph. Furthermore, We have shown that our formalism simplify and unify some numerical schemes used in the morphological mathematics approach defined by Partial Differential Equations. Finally, we have shown that our approach produces robust results.

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