# Superpatterns and Universal Point Sets

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**Abstract.** An old open problem in graph drawing asks for the size of a *universal point set*, a set of points that can be used as vertices for straight-line drawings of all $n$-vertex planar graphs. We connect this problem to the theory of permutation patterns, where another open problem concerns the size of *superpatterns*, permutations that contain all patterns of a given size. We generalize superpatterns to classes of permutations determined by forbidden patterns, and we construct superpatterns of size $n^2/4+\Theta(n)$ for the 213-avoiding permutations, half the size of known superpatterns for unconstrained permutations. We use our superpatterns to construct universal point sets of size $n^2/4-\Theta(n)$, smaller than the previous bound by a 9/16 factor. We prove that every proper subclass of the 213-avoiding permutations has superpatterns of size $O(n\log^{O(1)} n)$, which we use to prove that the planar graphs of bounded pathwidth have near-linear universal point sets.

## 1 Introduction

Fary's theorem tells us that every planar graph can be drawn with its edges as non-crossing straight line segments. As usually stated, this theorem allows the vertex coordinates of the drawing to be drawn from an uncountable and unbounded set, the set of all points in the plane. It is natural to ask how tightly we can constrain the set of possible vertices. In this direction, the *universal point set problem* asks for a sequence of point sets $U_n \subseteq \mathbf{R}^2$ such that every graph with $n$ vertices can be straight-line embedded with vertices in $U_n$ and such that the size of $U_n$ is as small as possible.

So far the best known upper bounds for this problem have considered sets $U_n$ of a special form: the intersection of the integer lattice with a convex polygon. In 1988 de Fraysseix, Pach and Pollack showed that a triangular set of lattice points within a rectangular grid of $(2n-3)\times(n-1)$ points forms a universal set of size $n^2-O(n)$ [1,2], and in 1990 Schnyder found more compact grid drawings within the lower left triangle of an $(n-1)\times(n-1)$ grid [3], a set of size $n^2/2-O(n)$. Using the method of de Fraysseix et al., Brandenburg found that a triangular subset of a of a $\frac{4}{3}n\times\frac{2}{3}n$ grid, of size $\frac{4}{9}n^2+O(n)$, is universal [4]. Until now his bound has remained the best known.

On the other side, Dolev, Leighton, and Trickey [5] used the *nested triangles graph* to show that rectangular grids that are universal must have size at least $n/3\times n/3$, or with a fixed choice of planar embedding and outer face $2n/3\times 2n/3$. Thus, if we wish to find subquadratic universal point sets we must consider sets not forming a grid. However,

⋆ This research was supported in part by the National Science Foundation under grants 0830403 and 1217322, and by the Office of Naval Research under MURI grant N00014-08-1-1015.

S. Wismath and A. Wolff (Eds.): GD 2013, LNCS 8242, pp. 208–219, 2013.

the known lower bounds that do not make this grid assumption are considerably weaker. In 1988 de Fraysseix, Pach and Pollack proved the first nontrivial lower bounds of $n+\Omega(\sqrt{n})$ for a general universal point set [1]. This was later improved to $1.098n-o(n)$ by Chrobak and Payne [2]. Finally, Kurowski improved the lower bound to $1.235n$ [6], which is still the best known lower bound.[1]

With such a large gap between these lower bounds and Brandenburg's upper bound, obtaining tighter bounds remains an important open problem in graph drawing [8].

Universal point sets have also been considered for subclasses of planar graphs. For instance, every set of $n$ points in general position (no three collinear) is universal for the $n$-vertex outerplanar graphs [9]. Universal point sets of size $O\left(n(\log n/\log\log n)^2\right)$ exist for simply-nested planar graphs (graphs that can be decomposed into nested induced cycles) [10], and planar 3-trees have universal point sets of size $O(n^{5/3})$ [11]. Based in part on the results in this paper, the graphs of line and pseudoline arrangements have been shown to have universal point sets of size $O(n\log n)$ [12].

In this paper we provide a new upper bound on universal point sets for general planar graphs, and improved bounds for certain restricted classes of planar graphs. We approach these problems via a novel connection to a different field of study than graph drawing, the study of patterns in permutations.[2] A permutation $\sigma$ is said to *contain* the pattern $\pi$ (also a permutation) if $\sigma$ has a (not necessarily contiguous) subsequence whose elements are in the same relative order with respect to each other as the elements of $\pi$. The permutations that do not contain any pattern in a given set $F$ of forbidden patterns are said to be $F$*-avoiding*; we define $S_n(F)$ to be the length-$n$ permutations avoiding $F$. Researchers in permutation patterns have defined a *superpattern* to be a permutation that contains all length-$n$ permutations among its patterns, and have studied bounds on the lengths of these patterns [14, 15], culminating in a proof by Miller that there exist superpatterns of length $n^2/2+\Theta(n)$ [16]. We generalize this concept to an $S_n(F)$-superpattern, a permutation that contains all possible patterns in $S_n(F)$; we prove that for certain sets $F$, the $S_n(F)$-superpatterns are much shorter than Miller's bound.

As we show, the existence of small $S_n(213)$-superpatterns leads directly to small universal point sets for arbitrary planar graphs, and the existence of small $S_n(F)$-superpatterns for $F$ containing 213 leads to small universal point sets for subclasses of the planar graphs. Our method constructs a universal set $U$ that has one point for each each element of the superpattern $\sigma$. It uses two different traversals of a depth-first-search tree of a canonically oriented planar graph $G$ to derive a permutation $\mathrm{cperm}(G)$ from $G$, and it uses the universality of $\sigma$ to find $\mathrm{cperm}(G)$ as a pattern in $\sigma$. Then, the positions of the elements of this pattern in $\sigma$ determine the assignment of the corresponding vertices of $G$ to points in $U$, and we prove that this assignment gives a planar embedding of $G$. A similar but simpler reduction uses $S_n$-superpatterns to construct universal point sets for *dominance drawings* of transitively reduced $st$-planar graphs.

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[1] The validity of this result was originally questioned by Mondal [7], but later confirmed.

[2] A different connection between permutation patterns and graph drawing is being pursued independently by Bereg, Holroyd, Nachmanson, and Pupyrev, in connection with bend minimization in bundles of edges that realize specified permutations [13].

Specifically our contributions include proving the existence of:

- superpatterns for 213-avoiding permutations of size $n^2/4+\Theta(n)$;
- universal point sets for planar graphs of size $n^2/4-\Theta(n)$;
- universal point sets for dominance drawings of size $n^2/2+\Theta(n)$;
- superpatterns for every proper subclass of the 213-avoiding permutations of size $O(n\log^{O(1)} n)$;
- universal point sets for graphs of bounded pathwidth of size $O(n\log^{O(1)} n)$; and
- universal point sets for simply-nested planar graphs of size $O(n\log n)$.

In addition, we prove that every superpattern for $\{213,132\}$-avoiding permutations has length $\Omega(n\log n)$, which in turn implies that every superpattern for 213-avoiding permutations has length $\Omega(n\log n)$. It was known that $S_n$-superpatterns must have quadratic length—otherwise they would not have enough length-$n$ subsequences to cover all $n!$ permutations [14]—but such counting arguments cannot provide nonlinear bounds for $S_n(F)$-superpatterns due to the now-proven Stanley–Wilf conjecture that $S_n(F)$ grows singly exponentially [17]. Instead, our proof finds an explicit set of $\{213,132\}$-avoiding permutations whose copies within a superpattern cannot share many elements.

## 2 Preliminaries

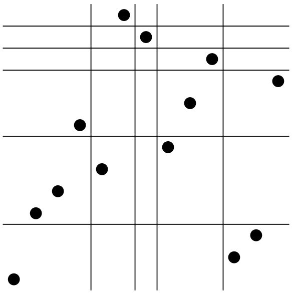
Let $S_n$ denote the set of all *permutations* of the numbers from 1 to $n$. We will normally specify a permutation as a sequence of numbers: for instance, the six permutations in $S_3$ are 123, 132, 213, 231, 312, and 321. If $\pi$ is a permutation, then we write $\pi_i$ for the element in the $i$th position of $\pi$, and $|\pi|$ for the number of elements in $\pi$.

We say that a permutation $\pi$ is a *subpattern* of a permutation $\sigma$ of length $n$ if there exists a sequence of integers $1\le \ell_1<\ell_2<\cdots<\ell_{|\pi|}\le n$ such that $\pi_i<\pi_j$ if and only if $\sigma_{\ell_i}<\sigma_{\ell_j}$. In other words, $\pi$ is a subpattern of $\sigma$ if $\pi$ is order-isomorphic to a subsequence of $\sigma$. We say that a permutation $\sigma$ *avoids* a permutation $\phi$ if $\sigma$ does not contain $\phi$ as a subpattern. A *permutation class* is a set of permutations with the property that all patterns of a permutation in the class also belong to the class; every permutation class may be defined by a (not necessarily finite) set of *forbidden permutations*, the minimal patterns that do not belong to the class. Define $S_n(\phi_1,\ldots,\phi_k)$ to be the set of all length-$n$ permutations that avoid all of the (forbidden) patterns $\phi_1,\ldots,\phi_k$. Given a set of permutations $P\subseteq S_n$, we define a *P-superpattern* to be a permutation $\sigma$ with the property that every $\pi\in P$ is a subpattern of $\sigma$.

One of the most important permutation classes in the study of permutation patterns is the class of *stack-sortable permutations* [18], the permutations that avoid the pattern 231. Knuth's discovery that these are exactly the permutations that can be sorted using a single stack [19] kicked off the study of permutation patterns. The 213-avoiding permutations that form the focus of our research are related to the 231-avoiding permutations by a simple transformation, the replacement of each value $i$ in a permutation by the value $n+1-i$, that does not affect the existence or size of superpatterns.

Given a permutation $\pi$ we define a *column* of $\pi$ to be a maximal ascending run of $\pi$, and we define a *row* of $\pi$ to be a maximal ascending run in $\pi^{-1}$, where a *run* is a contiguous monotone subsequence of the permutation. We define a *block* of $\pi$ to be a set of consecutive integers that appear contiguously (but not necessarily in order) in $\pi$. For instance, $\{3,4,5\}$ forms a block in 14352. (Our definition of rows and columns differs from that of Miller [16]: for our definition, the intersection of a row and column is a block that could contain more than one element, whereas in Miller's definition a row and column intersect in at most one element.)

We define the *chessboard representation* of a permutation $\pi$ to be an $r \times c$ matrix $M = \text{chessboard}(\pi)$, where $r$ is number of rows in $\pi$ and $c$ is the number of columns in $\pi$, such that $M(i,j)$ is the number of points in the intersection of the $i^{\text{th}}$ column and the $j^{\text{th}}$ row of $\pi$. An example of a chessboard representation can be seen in Figure 1. To recover a permutation from its chessboard representation, start with the lowest row and work upwards assigning an ascending subsequence of values to the squares of each row in left to right order within each row. If a square has label $i$, allocate $i$ values for it. Then, after this assignment has been made, traverse each column in left-to-right order, within each column listing in ascending order the values assigned to each square of the column. The sequence of values listed by this column traversal is the desired permutation.



| | | | | |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 2 |

**Fig. 1.** The permutation $\pi = 1\ 4\ 5\ 8\ 6\ 13\ 12\ 7\ 9\ 11\ 2\ 3\ 10$ represented by its scatterplot (the points $(i,\pi_i)$) with lines separating its rows and columns (left), and by chessboard$(\pi)$ (right)

## 3 From Superpatterns to Universal Point Sets

In this section, we show how 213-avoiding superpatterns can be turned into universal point sets for planar graphs. Let $G$ be a planar graph. We assume $G$ is *maximal planar*, meaning that no additional edges can be added to $G$ without breaking its planarity; this is without loss of generality, because a point set that is universal for maximal planar graphs is universal for all planar graphs. Additionally, we assume that $G$ has a fixed plane embedding; for maximal planar graphs, such an embedding is determined by the choice of which of the triangles of $G$ is to be the outer face.

### 3.1 Canonical Representation

As in the grid drawing method of de Fraysseix, Pach and Pollack [1], we use *canonical representations* of planar graphs; these are sequences $v_1, v_2, \ldots v_n$ of the vertices of the given maximal plane graph $G$ with the following properties:

– $v_1v_2v_n$ is the outer triangle of the embedding, in clockwise order $v_1$, $v_n$, $v_2$.
– Each vertex $v_i$ with $i \geq 3$ has two or more earlier neighbors in the sequence, and these neighbors form a contiguous subset of the cyclic ordering of neighbors around $v_i$ in the embedding of $G$.

Given a canonical representation, let $G_k$ be the induced subgraph of $G$ with vertex set $\{v_1, v_2, \ldots v_k\}$. Then $G_k$ is necessarily 2-connected; its induced embedding has as its exterior face a simple cycle $C_k$ containing $v_k$ and the edge $v_1v_2$, and the neighbors of $v_k$ in $G_k$ form an induced path in $C_{k-1}$.

As de Fraysseix, Pach and Pollack proved, every embedded maximal planar graph has at least one canonical representation. For the rest of this section, we will assume that the vertices $v_i$ of the given maximal planar graph $G$ are numbered according to such a representation. The definition of a canonical representation implies that the outer face of $G$ is the triangle $v_1v_2v_n$; we will assume that this triangle is oriented so that $v_1, v_n, v_2$ are in clockwise order.

For each vertex $v_i$ with $i > 2$, let $\text{parent}(v_i)$ be the most clockwise smaller-numbered neighbor of $v_i$. By following a path of edges from vertices to their parents, each vertex can reach $v_1$, so these edges form a tree $\text{ctree}(G)$ having $v_1$ as its root; this same tree may also be obtained by orienting each edge of $G$ from lower to higher numbered vertices, and then performing a depth-first search of the resulting oriented graph that visits the children of each vertex in clockwise order, starting from $v_1$.[3] For each vertex $v_i$ of $G$, let $\text{pre}(v_i)$ be the position of $v_i$ in a pre-order traversal of $\text{ctree}(G)$ that visits the children of each node in clockwise order, and let $\text{post}(v_i)$ be the position of $v$ in a sequence of the nodes of $\text{ctree}(G)$ formed by reversing a post-order clockwise traversal. These two numbers may be used to determine the ancestor-descendant relationships in $\text{ctree}(G)$: a node $v_i$ is an ancestor of a node $v_j$ if and only if both $\text{pre}(v_i) < \text{pre}(v_j)$ and $\text{post}(v_i) < \text{post}(v_j)$ [20].

**Lemma 1.** *Let G be a canonically-represented maximal plane graph, and renumber the vertices of G in order by their values of* $\text{post}(v_i)$. *Then the result is again a canonical representation of the same embedding of G, and for each induced subgraph $G_k$ of this new canonical representation, the clockwise ordering of the vertices along the exterior cycle $C_k$ is in sorted order by the values of* $\text{pre}(v_i)$.

*Proof.* The fact that $\text{post}(v_i)$ gives a canonical representation comes from the fact that it is a reverse postorder traversal of a depth-first search tree. Reverse postorder traversal gives a topological ordering of every directed acyclic graph, from which it follows that every vertex in $G$ has the same set of earlier neighbors when ordered by $\text{post}(v_i)$ as it did in the original ordering.

The statement on the ordering of the vertices of $C_k$ follows by induction, from the fact that $v_k$ has a larger value of $\text{pre}(v_i)$ than its earliest incoming neighbor (its parent in $\text{ctree}(G)$) and a smaller value than all of its other incoming neighbors. □

[3] Although we do not use this fact, $\text{ctree}(G)$ is also part of a Schnyder decomposition of $G$, together with a second tree rooted at $v_2$ connecting each vertex to its most counterclockwise earlier neighbor and a third tree rooted at $v_n$ connecting each vertex to the later vertex whose addition removes it from $C_k$.

Let cperm$(G)$ be the permutation in which, for each vertex $v_i$, the permutation value in position pre$(v_i)$ is post$(v_i)$. That is, cperm$(G)$ is the permutation given by traversing ctree$(G)$ in preorder and listing for each vertex of the traversal the number post$(v_i)$.

**Lemma 2.** *For every canonically-represented maximal planar graph $G$, the permutation $\pi = \mathrm{cperm}(G)$ is 213-avoiding.*

*Proof.* Let $i < j < k$ be an arbitrary triple of indexes in the range from 1 to $n$, corresponding to the vertices $u_i$, $u_j$ and $u_k$. If $\pi_j$ is not the smallest of these three values, then $\pi_i$, $\pi_j$, and $\pi_k$ certainly do not form a 213 permutation pattern. If $\pi_j$ is the smallest of these three values, then, since $\mathrm{pre}(u_i) < \mathrm{pre}(u_j)$ but $\mathrm{post}(u_i) > \mathrm{post}(u_j)$, $u_i$ is not an ancestor or descendant of $u_j$, and $u_j$ is an ancestor of $u_k$. Therefore $u_i$ is also not an ancestor or descendant of $u_k$, from which it follows that $\pi_i > \pi_k$ and the pattern formed by $\pi_i$, $\pi_j$, and $\pi_k$ is 312 rather than 213. Since the choice of indices was arbitrary, no three indices can form a 213 pattern and $\pi$ is 213-avoiding. □

We observe that cperm$(G)$ has some additional structure, as well: its first element is 1, its second element is $n$, and its last element is 2.

### 3.2 Stretching a Permutation

It is natural to represent a permutation $\sigma$ by the points with Cartesian coordinates $(i, \sigma_i)$, but for our purposes we need to stretch this representation in the vertical direction; we use a transformation closely related to one used by Bukh, Matoušek, and Nivasch [21] for weak epsilon-nets, and by Fulek and Tóth [11] for universal point sets for plane 3-trees. Letting $q = |\sigma|$, we define

$$\mathrm{stretch}(\sigma) = \left\{(i, q^{\sigma_i}) \mid 1 \le i \le q\right\}.$$

**Lemma 3.** *Let $\sigma$ be an arbitrary permutation with $|\sigma| = q$, let $p_i$ denote the point in* stretch$(\sigma)$ *corresponding to position $i$ in $\sigma$, let $i$ and $j$ be two indices with $\sigma_i < \sigma_j$, and let $m$ be the absolute value of the slope of line segment $p_i p_j$. Then $q^{\sigma_j - 1} \le m < q^{\sigma_j}$.*

*Proof.* The minimum value of $m$ is obtained when $|i - j| = q - 1$ and $\sigma_i = \sigma_j - 1$, for which $q^{\sigma_j - 1} = m$. The maximum value of $m$ is obtained when $|i - j| = 1$ and $\sigma_i = 1$, for which $m = q^{\sigma_j} - q < q^{\sigma_j}$. □

**Lemma 4.** *Let $\sigma$ be an arbitrary permutation with $|\sigma| = q$, let $p_i$ denote the point in* stretch$(\sigma)$ *corresponding to position $i$ in $\sigma$, and let $i$, $j$, and $k$ be three indices with* $\max\{\sigma_i, \sigma_j\} < \sigma_k$ *and $i < j$. Then the clockwise ordering of the three points $p_i$, $p_j$, and $p_k$ is $p_i$, $p_k$, $p_j$.*

*Proof.* The result follows by using Lemma 3 to compare the slopes of the two line segments $p_i p_j$ and $p_i p_k$. □

**Lemma 5.** *Let $\sigma$ be an arbitrary permutation with $|\sigma| = q$, let $p_i$ denote the point in* stretch$(\sigma)$ *corresponding to position $i$ in $\sigma$, and let $h$, $i$, $j$, and $k$ be four indices with* $\max\{\sigma_h, \sigma_i, \sigma_j\} < \sigma_k$ *and $h < j$. Then line segments $p_h p_j$ and $p_i p_k$ cross if and only if both $h < i < j$ and* $\max\{\sigma_h, \sigma_j\} > \sigma_i$.

*Proof.* A crossing occurs between two line segments if and only if the endpoints of each segment are on opposite sides of the line through the other segment. The endpoints of $p_i p_k$ are on opposite sides of line $p_h p_j$ if and only if the two triangles $p_h p_i p_j$ and $p_h p_k p_j$ have opposite orientations; with the assumption that $\sigma_k$ is the largest of the three values, this is equivalent by Lemma 4 to the condition that $\sigma_i$ is not the second-largest. The endpoints of $p_h p_j$ are on opposite sides of line $p_i p_k$ if and only if the two triangles $p_i p_h p_k$ and $p_i p_j p_k$ have opposite orientations; this is equivalent by Lemma 4 to the condition that $h < i < j$. □

### 3.3 Universal Point Sets

If $\sigma$ is any permutation, we define augment($\sigma$) to be a permutation of length $|\sigma| + 3$, in which the first element is 1, the second element is $|\sigma| + 3$, the last element is 2, and the remaining elements form a pattern of type $\sigma$. It follows from Lemma 2 that, if $\sigma$ is an $S_{n-3}(213)$-superpattern and if $G$ is an arbitrary $n$-vertex maximal plane graph, then cperm$(G)$ is a pattern in augment($\sigma$).

**Theorem 1.** *Let $\sigma$ be an $S_{n-3}(213)$-superpattern, and let $U_n =$ stretch(augment($\sigma$)). Then $U_n$ is a universal point set for planar graphs on n vertices.*

*Proof.* Let $G$ be an arbitrary maximal plane graph, let $v_1, v_2, \ldots v_n$ be a canonical representation of $G$, and let $x_i$ denote a sequence of positions in augment($\sigma$) that form a pattern of type cperm$(G)$, with position $x_i$ in augment($\sigma$) corresponding to position pre$(v_i)$ in cperm$(G)$. Let $q = |\text{augment}(\sigma)|$, and for each $i$, let $y_i = q^j$ where $j$ is the value of augment($\sigma$) at position $x_i$. Embed $G$ by placing vertex $v_i$ at the point $(x_i, y_i) \in U_n$.

Let $v_h$, $v_i$, $v_j$, and $v_k$ be four vertices in $G$ such that $v_h v_j$ and $v_i v_k$ are edges in $G$. We may choose these indices in such a way that post$(v_k)$ is larger than the post values of the other three vertices. If these two edges crossed in the given embedding of $G$, then by Lemma 5 we would necessarily have pre$(v_h) <$ pre$(v_i) <$ pre$(v_j)$, and post$(v_i) <$ max$\{$post$(v_h)$, post$(v_j)\}$. By Lemma 1, $v_i$ would not be on the outside face of the graph induced by the vertices with post values at most max$\{$post$(v_h)$, post$(v_j)\}$, and could not be a neighbor of $v_k$ in the canonical representation given by the post values. This contradiction shows that no crossing is possible, so the embedding is planar. □

## 4 $S_n(213)$-Superpatterns

In this section we construct a $S_n(213)$-superpattern of size $n^2/4 + n + ((-1)^n - 1)/8$. An exhaustive computer search has shown that this size is minimal for $n \leq 6$. We start with a lemma about $S_n(213)$-superpatterns with $n$ rows and $n$ columns (the minimal amount of each), demonstrating their recursive structure.

**Lemma 6.** *If $\sigma$ is a $S_n(213)$-superpattern and has n rows and n columns, then the permutation described by the intersection of columns $n - j + 1$ to $n - i + 1$ and rows i to j of $\sigma$ is a $S_{j-i+1}(213)$-superpattern.*

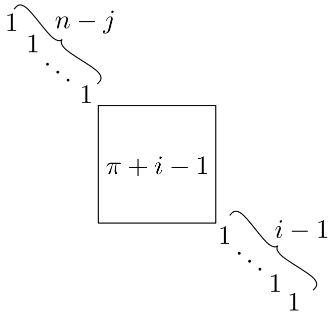



**Fig. 2.** The permutation τ constructed from π in the proof of Lemma 6

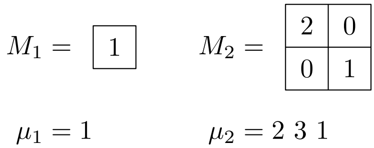



**Fig. 3.** The base case permutations for constructing $\mu_n$ for $n > 2$ and their chessboard representations

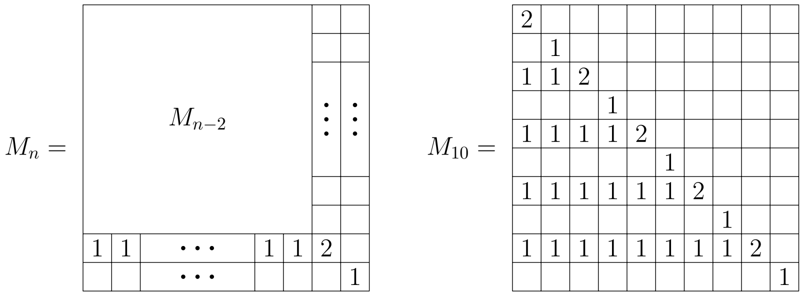



**Fig. 4.** The inductive construction of chessboard$(\mu_n)$ from chessboard$(\mu_{n-2})$. Cells of the matrix containing zero are shown as blank

*Proof.* Let π be an arbitrary 213-avoiding permutation of length $j-i+1$ and consider the $n$-element 213-avoiding permutation

$$\tau = n(n-1)\dots(j+1)(\pi_1+i-1)(\pi_2+i-1)\dots(\pi_{j-i+1}+i-1)(i-1)(i-2)\dots 321.$$

(See Fig. 2.) By the assumption that σ is a superpattern, τ has an embedding into σ. Because there are $n-j$ descents in τ before the first element of the form $\pi_i+i-1$, this embedding cannot place any element $\pi_i+i-1$ into the first $n-j$ columns of σ. Similarly because there are $i$ descents in τ after the last element of the form $\pi_i+i-1$, this embedding cannot place any element $\pi_i+i-1$ into the last $i-1$ columns of σ. By a symmetric argument, the elements of the form $\pi_i+i-1$ cannot be embedded into the $i-1$ lowest rows nor the $n-j$ highest rows of σ. Therefore these elements, which form a pattern of type π, must be embeddable into σ inclusively between column $n-j+1$, column $n-i+1$, row $i$, and row $j$. Since π was arbitrary, this part of σ must be universal for permutations of length $j-i+1$, as claimed. □

We define a permutation $\mu_n$, which will be shown to be a $S_n(213)$-superpattern, by describing chessboard$(\mu_n) = M_n$. In our construction $M_n$ and $\mu_n$ have $n$ columns and

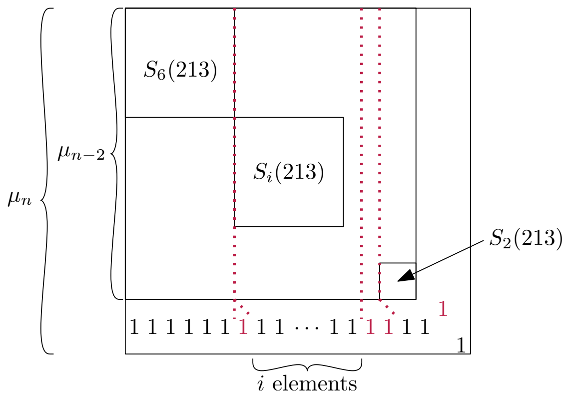



**Fig. 5.** A partial embedding of the red elements, showing where the remaining blocks can be fit into the columns of $\mu_{n-2}$

$n$ rows. The bottom two rows of $M_n$ contain the values $M_n(n,1) = M_n(i,2) = 1$ for $1 \le i \le n-2$, and $M_n(n-1,2) = 2$, with all other values in these rows zero. The values in the top $n-2$ rows are given recursively by $M_n(1:n-2, 3:n) = M_{n-2}$, again with all values outside this submatrix zero. The base cases of $\mu_1$ and $\mu_2$ are shown in Figure 3 and the inductive definition and an example are shown in Figure 4.

**Theorem 2.** *The permutation $\mu_n$ is a $S_n(213)$-superpattern. Thus there exists a $S_n(213)$-superpattern whose size is $n^2/4 + n + ((-1)^n - 1)/8$.*

*Proof.* It can be easily verified that $\mu_i$ is a $S_i(213)$-superpattern when $1 \le i \le 2$. Let $\pi$ be an arbitrary 213-avoiding permutation of length $n$. We will show that $\pi$ can be embedded into $\mu_n$.

**Case 1:** $\pi_n = 1$

Let $\pi_{i_1} \dots \pi_{i_k}$ be the second lowest row of $\pi$. Because $\pi$ is 213-avoiding, $i_k = n-1$ and for all $j$, $\pi_{i_j} = j+1$. We embed this bottom row by mapping $\pi_n$ to the bottom right element of $\mu_n$ and $\pi_{i_j}$ to the $i_j$-th position of the second lowest row of $\mu_n$.

**Case 2:** $\pi_n \neq 1$

Let $\pi_{i_1} \dots \pi_{i_k}$ be the lowest row of $\pi$. Similarly to Case 1, because $\pi$ is 213-avoiding, $i_k = n$ and for all $j$, $\pi_{i_j} = j$. We embed this bottom row by mapping $\pi_{i_j}$ to the $i_j$-th position of the second lowest row of $\mu_n$.

These partial embeddings maintain the ordering of the lowest and possibly second lowest row of $\pi$. To finish the embedding, the remaining elements need to be fit into the copy of $\mu_{n-2}$. Recall that a block of a permutation is a contiguous subsequence of consecutive values. Because $\pi$ is 213-avoiding, the remaining elements of $\pi$ form disjoint blocks that fit between the elements embedded so far. If one block is to the right of another in $\pi$, it has smaller values. Let $\pi_{i_1}$ be the leftmost element that has been embedded on the second row of $\mu_n$. Then there are $i_1 - 1$ elements before it in $\pi$ and $i_1 - 1$ columns before where it was embedded in $\mu_n$. By Lemma 6, these elements can fit into the last $i_1 - 1$ rows of these columns. Let $\pi_{i_j}$ and $\pi_{i_{j+1}}$ be two adjacent elements embedded in the second lowest row of $\mu_n$. Between these two there are at least $i_{j+1} - i_j - 1$ columns of $\mu_{n-2}$ available: the column above $\pi_{i_s}$ to the column before $\pi_{i_{j+1}}$. So again by Lemma 6, the $i_{j+1} - i_j - 1$ elements between $\pi_{i_j}$ and $\pi_{i_{j+1}}$ can be fit into the last $i_{j+1} - i_j - 1$ rows of those columns. (See Fig. 5) Because $i_k = n - 1$ in Case 1 and $n$ in Case 2, there is no block after $\pi_{i_k}$. Therefore $\pi$ can be embedded into $\mu_n$ and $\mu_n$ is a $S_n(213)$-superpattern. □

Combining Theorem 2 with Theorem 1, the following is immediate:

**Theorem 3.** *The n-vertex planar graphs have universal point sets of size* $n^2/4 - \Theta(n)$.

## 5 Dominance Drawing

A *dominance drawing* of a directed acyclic graph [22] is a drawing of the graph in the plane such that each edge is directed upwards and to the right, such that the axis-aligned bounding box of every edge contains no vertices other than its endpoints, and such that no edge can be added to the drawing preserving these properties. The graphs with planar dominance drawings are exactly the transitively reduced *st*-planar graphs, i.e. the planar directed acyclic graphs in which there is one source and one sink, both on the outer face, and in which each edge forms the only directed path connecting its two endpoints.

If a graph has a dominance drawing $D$, then it has a drawing in which the points are in general position, and the points in this case can be thought of as representing a permutation $\pi_D$, where the positions of the elements in the permutation are the positions of the points in the sorted order by their $x$ coordinates and the values of these elements are the positions in the sorted order by the $y$ coordinates. Any two point sets with the same two sorted orders may be used as the basis for a dominance drawing combinatorially equivalent to $D$. In particular, if $\pi_D$ appears as a pattern in another permutation $\sigma$, then the subset of the points $(i, \sigma_i)$ corresponding to elements of $\pi_D$ may be used to draw the same graph. This gives us the following result:

**Theorem 4.** *If $\sigma$ is a superpattern for the length-n permutations, then the set of points $(i, \sigma_i)$ is universal for dominance drawings of n-vertex transitively reduced st-planar graphs.*

Combining this result with Miller's bound on superpatterns [16] shows that dominance drawings have universal point sets of size $n^2/2 + \Theta(n)$, half the size of the point sets given by previous methods based on $n \times n$ grids.

Not every permutation is of the form $\pi_D$ for a planar dominance drawing $D$; for instance the permutation 2143 corresponds to a drawing that has a crossing. However, every permutation $\pi$ forms a pattern in a larger permutation $\sigma$ that does define a planar dominance drawing, constructed from the Dedekind–MacNeille completion of a partially ordered set associated to $\pi$ [23]. For this reason, the permutations that define planar dominance drawings have no forbidden patterns. However, in later research, we have shown that the dominance drawings of some other classes of graphs have forbidden patterns, leading to smaller universal sets for these drawings [24].

## 6 Additional Results

In the full version of the paper, we provide the following results.

- We prove that, for every 213-avoiding permutation $\phi$, the $\{213,\phi\}$-avoiding permutations have superpatterns of near-linear size. In particular, the $\{213,312\}$-avoiding permutations and the $\{213,3412\}$-avoiding permutations have superpatterns of linear size, and the minimum size of a superpattern for the $\{213,132\}$-avoiding permutations is $\Theta(n\log n)$ (despite these permutations being equinumerous with the $\{213,312\}$-avoiding permutations). We define the *Strahler number* of any 213-avoiding permutation from a forest derived from its chessboard representation, and we show that if a permutation $\phi$ has Strahler number $s$ then the $\{213,\phi\}$-avoiding permutations have superpatterns of size $O(n\log^{s-1} n)$.
- We prove that, for every integer $w$, there exists a pattern $\phi$ such that the planar graphs of pathwidth at most $w$ correspond to a $\{213,\phi\}$-avoiding permutation (using the same correspondence between graphs and permutations as in Section 3). As a consequence, the planar graphs of bounded pathwidth have universal point sets of size $O(n\log^{O(1)} n)$.
- We improve the bound of Angelini et al. [10] on universal point sets for simply-nested planar graphs from $O\left(n(\log n/\log\log n)^2\right)$ to $O(n\log n)$.

For space reasons we defer the proofs of these results to the full version of the paper.

## 7 Conclusion

In this paper we have constructed universal point sets for planar graphs of size $n^2/4-\Theta(n)$, and of subquadratic size for graphs of bounded pathwidth. In the process of building these constructions we have provided a new connection between universal point sets and permutation superpatterns. We have also, for the the first time, provided nontrivial upper bounds and lower bounds on the size of superpatterns for restricted classes of permutations. We leave the following problems open for future research:

- Which natural subclasses of planar graphs (beyond the bounded-pathwdith graphs) can be represented by permutations in a proper subclass of $S_n(213)$?
- Can we reduce the gap between our $O(n^2)$ upper bound and $\Omega(n\log n)$ lower bound for $S_n(213)$-superpatterns?
- Our construction uses area exponential in $n^2$; how does constraining the area to a smaller bound affect the number of points in a universal point set?

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