Spectral Element Discretization for the Vorticity, the Velocity and the Pressure Formulation of the Axisymmetric Navier-Stokes Problem

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Abstract We deal with the Navier-Stokes equations set in a three-dimensional axisymmetric bounded domain with non standard boundary conditions which involve the normal component of the velocity and tangential component of the vorticity. The axisymmetric property of the domain allows to reduce the three-dimensional problem into a two-dimensional one. We write a variational formulation with three independent unknowns: the vorticity, the velocity and the pressure. For the discretization, we use the spectral element methods, which are well-adapted here. We show the well-posedness of the obtained formulations and we establish error estimates for the three unknowns which proves the convergence of the method.

1 Introduction

We consider, in this paper, the Navier-Stokes problem set in a three-dimensional axisymmetric bounded domain and provided with non standard boundary conditions, which are given on the normal component of the velocity and tangential component of the vorticity. This problem reads:

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$$\begin{cases} -\nu \Delta \tilde{u} + (\tilde{u}.\nabla)\tilde{u} + \nabla \tilde{P} = \tilde{f} \text{ in } \tilde{\Omega}, \\ \text{div}\tilde{u} = 0 \text{ in } \tilde{\Omega}, \\ \tilde{u}.\tilde{n} = 0 \text{ on } \partial\tilde{\Omega}, \\ \text{curl}\tilde{u} \wedge \tilde{n} = 0 \text{ on } \partial\tilde{\Omega}. \end{cases} \tag{1}$$

where $\tilde{\Omega}$ is a bounded connected three-dimensional axisymmetric domain, the generic point in $\tilde{\Omega}$ is given by cylindrical components $(r, \theta, z) \in \mathbb{R}_+ \times]-\pi, \pi] \times \mathbb{R}$. ν is the viscosity of the fluid, $\tilde{u} = (u_r, u_\theta, u_z)$ the velocity, $\tilde{P}$ the pressure and $\tilde{f}$ is the data, which represent the density of body forces. When the data is axisymmetric, problem (1) is equivalent to two decoupled systems [9]. In the first one, the unknowns are the components u_r and u_z of the velocity and pressure P, we will focus on. The second is a Laplace problem where the unknown is the velocity component u_θ.

At first, this problem was studied in [1] but in an unspecified bounded domain, then it was taken again by Azaiez et al. [10] in a bounded domain included in $\mathbb{R}^2$ or $\mathbb{R}^3$ in formulation (u, p), though the formulation that we consider here deals with three unknowns: vorticity, velocity and pressure. The first numerical analysis relying on this formulation has been realized in [13] and [8] for finite element methods and it has been extended to the case of spectral methods in [3] and [10], using analogues of Nédélec's finite elements [6].

The discretization method which we use here is the spectral element methods, which are well adapted in domain decomposition. The main tool for the analysis of the nonlinear discrete problem is the theorem of Brezzi, Rappaz and Raviart [5]. We first prove the existence of a discrete solution. Then, by combining the results in [5, 11] and [7], we establish error estimates between the continuous solution and the discrete one, for the three unknowns.

The paper is organized as follows. In the next section, we introduce the variational formulation corresponding to the Navier-Stokes problem and we derive the existence of a solution. In Sect. 3, we study the discrete problem and we prove the well-posedness of this problem. We derive error estimates between the continuous solution and the discrete one in Sect. 4.

2 The Vorticity, Velocity and Pressure Formulation

The domain $\tilde{\Omega}$ is obtained by rotating a two-dimensional domain Ω around the axis $\{r = 0\}$. We note by Γ_0 the intersection of the boundary $\partial\Omega$ with the axis $r = 0$, $\Gamma = \partial\Omega \backslash \Gamma_0$ and by n the normal to Γ in the plane (r, z). We introduce the vorticity ω as a new unknown: $\omega = \text{curl}u$. The bidimensional problem resulting from (1) reads:

$$\begin{cases} \nu \text{curl}_r \omega + \omega \times u + \nabla P = f & \text{in } \Omega, \\ \text{div}_r u = 0 & \text{in } \Omega, \\ \omega = \text{curl} u & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ \omega = 0 & \text{on } \Gamma. \end{cases} \tag{2}$$

The operators div_r, curl and curl_r are given by: for $u = (u_r, u_z)$, $\text{div}_r u = \partial_r u_r + r^{-1} u_r + \partial_z u_z$ and $\text{curl} u = \partial_r u_z - \partial_z u_r$. And for any scalar function φ, we define $\text{curl}_r \varphi = \big(\partial_z \varphi, -r^{-1} \partial_r (r\varphi)\big)$. We refer to [11], for details.

In order to write the variational formulation of problem (2), we define the following weighted Sobolev spaces: For all s in $\mathbb{Z}$ and m in $\mathbb{N}$:

$$L_s^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} \quad \text{measurable} \quad / \int_\Omega |v(r,z)|^2 r^s \, dr dz < \infty \right\}$$

$$H_1^m(\Omega) = \left\{ v \in L_1^2(\Omega) \quad / \quad \partial_r^l \partial_z^{m-l} v \in L_1^2(\Omega) \quad \forall 0 \le l \le m \right\},$$

$$H_1(\text{curl}, \Omega) = \left\{ v \in L_1^2(\Omega)^2 \quad / \quad \text{curl} v \in L_1^2(\Omega) \right\},$$

$$H_1(\text{div}_r, \Omega) = \left\{ v \in L_1^2(\Omega)^2 \quad / \quad \text{div}_r v \in L_1^2(\Omega) \right\},$$

$$H_1^\diamond(\text{div}_r, \Omega) = \{ v \in H_1(\text{div}_r, \Omega) / \quad v \cdot n = 0 \quad \text{on} \quad \Gamma \},$$

$$H_1(\text{curl}_r, \Omega) = \left\{ \varphi \in L_1^2(\Omega) \quad / \quad \text{curl}_r \varphi \in L_1^2(\Omega)^2 \right\},$$

$$V_1^1(\Omega) = H_1^1(\Omega) \cap L_{-1}^2(\Omega) \quad \text{and} \quad V_{1\diamond}^1(\Omega) = \left\{ v \in V_1^1(\Omega) / \quad v = 0 \quad \text{on} \quad \Gamma \right\}.$$

The spaces $V_1^1(\Omega)$, $H_1(\text{div}_r, \Omega)$ and $H_1(\text{curl}_r, \Omega)$ are respectively provided with:

$$\|v\|_{V_1^1(\Omega)} = (\|\partial_r v\|^2_{L_1^2(\Omega)} + \|\partial_z v\|^2_{L_1^2(\Omega)} + \|v\|^2_{L_{-1}^2(\Omega)})^{\frac{1}{2}},$$

$$\|v\|_{H_1(\text{div}_r, \Omega)} = \left(\|v\|^2_{L_1^2(\Omega)} + \|\text{div}_r v\|^2_{L_1^2(\Omega)} \right)^{\frac{1}{2}},$$

$$\|\varphi\|_{H_1(\text{curl}_r, \Omega)} = \left(\|\varphi\|^2_{L_1^2(\Omega)} + \|\text{curl}_r \varphi\|^2_{L_1^2(\Omega)^2} \right)^{\frac{1}{2}}.$$

We note that the two norms $\|.\|_{H_1(\text{curl}_r, \Omega)}$ and $\|.\|_{V_1^1(\Omega)}$ are equivalent on $V_1^1(\Omega)$.
The variational problem reads:
Find $(\omega, u, p) \in V_{1\diamond}^1(\Omega) \times H_1^\diamond(\text{div}_r, \Omega) \times L_{1,0}^2(\Omega)$ such that:

$$\begin{cases} a(\omega, u; v) + K(\omega, u; v) + b(v, p) = \langle f, v \rangle, & \forall v \in H_1^\diamond(\text{div}_r, \Omega), \\ b(u, q) = 0, & \forall q \in L_{1,0}^2(\Omega), \\ c(\omega, u, \varphi) = 0, & \forall \varphi \in V_{1\diamond}^1(\Omega). \end{cases} \tag{3}$$

where $\langle ., . \rangle$ is the duality pairing between $H_1^\diamond(\text{div}_r, \Omega)$ and its dual space. The forms $a(., .; .)$, $b(., .)$ and $c(., .; .)$ are defined by:

$$a(\omega, u, \theta) = \nu \int_\Omega (\theta.\text{curl}_r\omega)(r, z) r dr dz, \quad b(v, q) = -\int_\Omega (\text{div}_r v) q(r, z) r dr dz,$$

$$c(\omega, u, \varphi) = \int_\Omega \omega(r, z)\varphi(r, z) r dr dz - \int_\Omega (u.\text{curl}_r\varphi)(r, z) r dr dz.$$

and K is the trilinear form given by: $K(\omega, u; v) = \int_\Omega (\omega \times u) .v(r, z) r dr dz.$
Using density results, we first prove that problems (2) and (3) are equivalent. To prove the existence and the uniqueness of the solution of problem (3), we define the two following kernels V and W:

$$V = \left\{ v \in H_1^\diamond(\text{div}_r, \Omega), \quad \forall q \in L^2_{1,0}(\Omega) \quad / \quad b(v, q) = 0 \right\},$$
$$W = \left\{ (\vartheta, v) \in V^1_{1\diamond}(\Omega) \times V \quad / \quad \forall \varphi \in V^1_{1\diamond}(\Omega), \qquad c(\vartheta, v; \varphi) = 0 \right\},$$

and the reduced problem: Find (ω, u) in W such that:

$$\forall v \in V, \quad a(\omega, u; v) + K(\omega, u; v) = \langle f, v \rangle . \tag{4}$$

By using standard arguments and properties on the linear forms, proven in [3] and [11], we can prove the existence and uniqueness of a solution for problem (4). So for any function f in $H_1^\diamond(\text{div}_r, \Omega)'$ such that

$$c_\diamond \nu^{-2} \| f \|_{H_1^\diamond(\text{div}_r, \Omega)'} < 1, \tag{5}$$

Problem (3) admits a unique solution $(\omega, u; p)$ in $V^1_{1\diamond}(\Omega) \times H_1^\diamond(\text{div}_r, \Omega) \times L^2_{1,0}(\Omega)$, such that

$$\|\omega\|_{V_1^1(\Omega)} + \|u\|_{H_1(\text{div}_r, \Omega)} + \nu^{-1} \|p\|_{L_1^2(\Omega)}$$
$$\leq c\nu^{-1} \| f \|_{H_1^\diamond(\text{div}_r, \Omega)'} \left(1 + \nu^{-2} \| f \|_{H_1^\diamond(\text{div}_r, \Omega)'} \right). \tag{6}$$

3 Discrete Navier-Stokes Problem

From now on, we assume that Ω is the rectangle $]0, 1[\times]-1, 1[$ and admits a partition without overlap into a finite number of subdomains:

$$\overline{\Omega} = \bigcup_{k=1}^{K} \Omega_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset \quad , \quad 1 \leq k < k' \leq K, \text{ such that:}$$

1. Each Ω_k , $1 \le k \le K$ is a rectangle.
2. The intersection between two subdomains $\overline{\Omega}_k$ and $\overline{\Omega}_{k'}$, $1 \le k < k' \le K$, if not empty, is either a vertex or a whole edge of both Ω_k and $\Omega_{k'}$.

The discrete spaces $\mathbb{D}_N$, $\mathbb{C}_N$ and $\mathbb{M}_N$ which approximate, respectively, $H_1^\diamond(\text{div}_\text{r}, \Omega)$, $V^1_{1\diamond}(\Omega)$ and $L^2_{1,0}(\Omega)$ are defined from local discrete ones, for an integer $N \ge 2$ and $1 \le k \le K$, by:

$$\mathbb{D}_N = \left\{v_N \in H_1^\diamond(\text{div}_\text{r}, \Omega);\ v_N|_{\Omega_k} \in \mathbb{P}_{N,N-1}(\Omega_k) \times \mathbb{P}_{N-1,N}(\Omega_k),\ 1 \le k \le K\right\},$$
$$\mathbb{C}_N = \left\{\varphi_N \in V^1_{1\diamond}(\Omega);\ \varphi_N|_{\Omega_k} \in \mathbb{P}_N(\Omega_k),\ 1 \le k \le K\right\} \text{ and}$$
$$\mathbb{M}_N = \left\{q_N \in L^2_{1,0}(\Omega);\ q_N|_{\Omega_k} \in \mathbb{P}_{N-1}(\Omega_k),\ 1 \le k \le K\right\}.$$

where $\mathbb{P}_{n,m}(\Omega_k)$ is the space of restrictions to Ω_k of polynomials with degree $\le n$ with respect to r and $\le m$ with respect to z, for any nonnegative integers n and m. To calculate the integrals involved in the discrete forms, we define (ξ_i, ρ_i), $0 \le i \le N$ the nodes and weights of the Gauss-Lobatto quadrature formula on $[-1, 1]$ for the measure $d\zeta$ and (ζ_j, ω_j), $1 \le j \le N+1$ their analogues for the measure $(1+\zeta)d\zeta$, see [9] for a more explicit definition, we need two different quadrature formulas. The quadrature formula on $[-1, 1]$ is given by:

$$\forall \phi \in \mathbb{P}_{2N-1}([-1,1]), \quad \int_{-1}^{1} \phi(\xi)d\xi = \sum_{i=0}^{N} \phi(\xi_i)\rho_i, \tag{7}$$

and by setting $r = \frac{1}{2}(1+\zeta)$, we define the quadrature formula with the measure rdr:

$$\forall \phi \in \mathbb{P}_{2N-1}([0,1]), \quad \int_{0}^{1} \phi(r)rdr = \frac{1}{4}\sum_{j=1}^{N+1} \phi(r_j)\omega_j. \tag{8}$$

We denote by $(\Omega_k)_{1\le k\le K_0}$ the rectangles such that $\partial\overline{\Omega}_k \cap \Gamma_0 \neq \varnothing$ and by $(\Omega_k)_{K_0+1\le k\le K}$ those such that $\partial\overline{\Omega}_k \cap \Gamma_0 = \varnothing$. Denoting by F_k the affine mapping that sends $]0,1[\times]-1,1[$ onto Ω_k, $1 \le k \le K_0$ and sends $]-1,1[^2$ onto Ω_k, $K_0+1 \le k \le K$. We define the discrete scalar product: For all functions u and v such that $u_k = u|_{\Omega_k}$ and $v_k = v|_{\Omega_k}$ are continuous on $\overline{\Omega}_k$, $1 \le k \le K$, by:

$$((u,v))_N = \sum_{k=1}^{K_0} \frac{mes(\Omega_k)}{4} \sum_{i=0}^{N}\sum_{j=1}^{N+1} u \circ F_k(r_j, \xi_i).v \circ F_k(r_j, \xi_i)\rho_i\omega_j.$$
$$+ \sum_{k=K_0+1}^{K} \frac{mes(\Omega_k)}{4} \sum_{i=0}^{N}\sum_{j=0}^{N} u \circ F_k(\xi_j, \xi_i).v \circ F_k(\xi_j, \xi_i)\rho_i\rho_j.$$

We denote by I_N^k, $1 \le k \le K$, the Lagrange interpolation operators associated with the nodes $F_k(r_j, \xi_i)_{1\le j\le N+1, 0\le i\le N}$ for $1 \le k \le K_0$ and with $F_k(\xi_j, \xi_i)_{0\le j,i\le N}$ for $K_0+1 \le k \le K$, with values in $\mathbb{P}_N(\Omega_k)$, $1 \le k \le K$. For each function ϕ continuous on $\bar{\Omega}$, $I_N\phi$ denotes the function such that $I_N\phi|_{\Omega_k} = I_N^k\phi$,

$1 \leq k \leq K$. Using the Galerkin method with numerical integration, we build from the continuous problem (3) the following discrete problem:

Find $(\omega_N, u_N; p_N)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ such that

$$\begin{cases} a_N(\omega_N, u_N, v_N) + K_N(\omega_N, u_N, v_N), \\ \qquad\qquad + b_N(v_N, p_N) = ((f, v_N))_N, \ \forall v_N \in \mathbb{D}_N, \\ \qquad\qquad b_N(u_N, q_N) = 0, \qquad \forall q_N \in \mathbb{M}_N, \\ \qquad\qquad c_N(\omega_N, u_N, \varphi_N) = 0, \qquad \forall \varphi_N \in \mathbb{C}_N. \end{cases} \tag{9}$$

where the bilinear forms $a_N(.,.;.)$, $b_N(.,.)$ and $c_N(.,.;.)$ are defined by: $a_N(\omega_N, u_N; v_N) = \nu((\text{curl}_r\omega_N, v_N))_N$, $\quad b_N(v_N, q_N) = -((\text{div}_r v_N, q_N))_N$, $c_N(\omega_N, u_N, \varphi_N) = ((\omega_N, \varphi_N))_N - ((u_N, \text{curl}_r\varphi_N))_N$, while the trilinear form $K_N(.,.;.)$ is given by: $K_N(\omega_N, u_N; v_N) = ((\omega_N \times u_N, v_N))_N$. In order to prove the well-posedness of the discrete problem, we need to introduce the kernels:

$$V_N = \{v_N \in \mathbb{D}_N / \forall q_N \in \mathbb{M}_N \quad , \quad b_N(vN, q_N) = 0\},$$
$$W_N = \{(\omega_N, u_N) \in \mathbb{C}_N \times V_N / \forall \theta_N \in \mathbb{C}_N, c_N(\omega_N, u_N, \theta_N) = 0\}.$$

We observe that, for any solution (ω_N, u_N, p_N) of problem (9), the pair (ω_N, u_N) is a solution of the reduced problem: Find $(\omega_N, u_N) \in W_N$ such that:

$$\forall v_N \in V_N \quad , \quad a_N(\omega_N, u_N; v_N) + K_N(\omega_N, u_N; v_N) = ((f, v_N))_N. \tag{10}$$

We recall from [4] and [7] that the bilinear form $a_N(.,.;.)$ satisfies, on the discrete spaces, a positivity property and an inf − sup condition with constants independent of N. We also refer to [4], for a discrete inf − sup condition on the form $b_N(.,.)$. Using the fixed point theorem of Brower, we can prove the wellposedness of problem (10) and then derive the:

Theorem 1. *For any data f continuous on $\overline{\Omega}$, the discrete problem (9) admits a solution $(\omega_N, u_N; p_N)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$. Moreover, (ω_N, u_N) satisfies:*

$$\|\omega_N\|_{L^2_1(\Omega)} + \|u_N\|_{L^2_1(\Omega)^2} \leq c\nu^{-1} \|I_N f\|_{L^2_1(\Omega)^2}. \tag{11}$$

4 Error Estimates

We now intend to prove an error estimate between the solutions of problems (3) and (9). Since the error analysis of the discrete problem relies on the theory of Brezzi, Rappaz and Raviart [5], we express both problems (4) and (10) in a different form and we set $X = V^1_{1\diamond}(\Omega) \times (V \cap H_1(\text{curl}, \Omega))$. We denote by S the linear

operator of Stokes which for any f in the dual space of $H_1^\diamond(\mathrm{div_r}, \Omega)$, associates the solution (ω, u) of the following reduced problem:
Find $(\omega, u) \in W$ such that $\quad \forall v \in V, \quad a(\omega, u; v) = \langle f, v\rangle$.
We introduce the mapping G defined from X into the dual space of $H_1^\diamond(\mathrm{div_r}, \Omega)$ by: $\forall(\omega, u) \in X, \quad \forall v \in H_1^\diamond(\mathrm{div_r}, \Omega), \langle G(\omega, u), v\rangle = K(\omega, u; v) - \langle f, v\rangle$.
Then, problem (4) can be equivalently written as: Find $(\omega, u) \in X$ such that

$$(\omega, u) + SG(\omega, u) = 0. \tag{12}$$

Similarly, we define the discrete space $X_N = \mathbb{C}_N \times (V_N \cap H_1(\mathrm{curl}, \Omega))$. We thus define the discrete Stokes operator S_N: for any f in the dual space of $H_1^\diamond(\mathrm{div_r}, \Omega)$, $S_N f$ denotes the solution (ω_N, u_N) of problem: Find $(\omega_N, u_N) \in W_N$ such that

$$\forall v_N \in V_N, \quad a_N(\omega_N, u_N; v_N) = \langle f, v_N\rangle\,. \tag{13}$$

The well-posedness of problem (13) is proven in [4], for a slightly different right-hand side. Finally, we consider the mapping G_N defined from X_N in the dual space of $\mathbb{D}_N$ by $\quad \forall(\omega_N, u_N) \in X_N, \quad \forall v_N \in \mathbb{D}_N$

$$\langle G_N(\omega_N, u_N), v_N\rangle = K_N(\omega_N, u_N; v_N) - ((f, v_N))_N. \tag{14}$$

Problem (10) can equivalently be written as: $\quad$ Find $(\omega_N, u_N) \in X_N$ such that

$$(\omega_N, u_N) + S_N G_N\,(\omega_N, u_N) = 0. \tag{15}$$

Using analogous arguments to those in [4], we easily derive that the operator S_N satisfies a stability property, with a constant independent of N and that, the following error estimate holds for all f in $H_1^{s+1}(\Omega) \times H_1^s(\Omega)^2$, $s > 1$,

$$\|(S - S_N) f\|_X \le cN^{-s}\, \|Sf\|_{H_1^{s+1}(\Omega)\times H_1^s(\Omega)^2}\,. \tag{16}$$

We are led to make the following assumptions. Here, D is the differential operator.

Assumption 1. The triplet (ω, u, p) is a solution of the problem (3) such that the operator $Id + SDG(\omega, u)$ is an isomorphism of X.

This assumption can equivalently be written as: For any data g in $H_1^\diamond(\mathrm{div_r}, \Omega)'$, the linearized problem
Find (ϑ, w, r) in $V^1_{1\diamond}(\Omega) \times \big(H_1^\diamond(\mathrm{div_r}, \Omega) \cap H_1(\mathrm{curl}, \Omega)\big) \times L^2_{1,0}(\Omega)$ such that:

$$\begin{cases} a(\vartheta, w, v) + K(\omega, w, v) + K(\vartheta, u; v) \\ \qquad\qquad + b(v, r) = \langle g, v\rangle\,, & \forall v \in H_1^\diamond(\mathrm{div_r}, \Omega) \cap H_1(\mathrm{curl}, \Omega), \\ \qquad\qquad b(w, q) = 0, & \forall q \in L^2_{1,0}(\Omega), \\ \qquad\qquad c(\vartheta, w, \varphi) = 0, & \forall \varphi \in V^1_{1\diamond}(\Omega). \end{cases} \tag{17}$$

has a unique solution with norm bounded by a constant times $\|g\|_{H_1^\diamond(\mathrm{div_r},\Omega)}$. It yields the local uniqueness of the solution (ω, u, p) but is much less restrictive than the global uniqueness condition. We need to prove a few technical results in order to derive the error estimate. For this, we make the:

Assumption 2. The solution (ω, u, p) of problem (3) satisfying Assumption 1, belongs to $H_1^{s+1}(\Omega) \times H_1^s(\Omega)^2 \times H_1^s(\Omega)$, $s > 1$.

Then, we prove:

Lemma 1. *For any* $(\omega_N, u_N; v_N)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{D}_N$,

$$|K(\omega_N, u_N; v_N)| \leq c_1 N \,\|\omega_N\|_{V_1^1(\Omega)} \|u_N\|_{L_1^2(\Omega)^2} \|v_N\|_{L_1^2(\Omega)^2}\,, \tag{18}$$

$$|K_N(\omega_N, u_N; v_N)| \leq c_2 N \,\|\omega_N\|_{V_1^1(\Omega)} \|u_N\|_{L_1^2(\Omega)^2} \|v_N\|_{L_1^2(\Omega)^2}\,. \tag{19}$$

the constants c_1 *and* c_2 *are independent of* N.

Proof. According to the Cauchy-Schwarz inequality we have:

$$|K(\omega_N, u_N; v_N)| = \left|\int_\Omega (\omega_N \times u_N).v_N\, r dr dz\right| \leq \|\omega_N\|_{L_1^4(\Omega)} \|u_N\|_{L_1^4(\Omega)^2} \|v_N\|_{L_1^2(\Omega)^2}\,.$$

Using the inclusion of $V_1^1(\Omega)$ in $L_1^4(\Omega)$ and inequality:

$$\forall z_N \in \mathbb{P}_N(\Omega), \qquad \|z_N\|_{L_1^4(\Omega)} \leq cN \,\|z_N\|_{L_1^2(\Omega)}\,, \tag{20}$$

see [2], we have the first previous result. For the second one, we have with obvious notation,

$$\begin{aligned} K_N(\omega_N, u_N; v_N) &= ((\omega_N u_{Nr}, v_{Nz}))_N - ((\omega_N u_{Nz}, v_{Nr}))_N \\ &= ((I_N(\omega_N u_{Nr}), v_{Nz}))_N - ((I_N(\omega_N u_{Nz}), v_{Nr}))_N\,. \end{aligned}$$

By combining the Cauchy-Schwarz inequalities with inequality (3.7) in [7], we obtain

$$|K_N(\omega_N, u_N; v_N)| \leq \|I_N(\omega_N u_N)\|_{L_1^2(\Omega)^2} \|v_N\|_{L_1^2(\Omega)^2}\,.$$

Then, we use the following result which can be derived from its one-dimensional analogue [7],

$$\forall \varphi_M \in \mathbb{P}_M(\Omega_k), \quad \left\|I_N^k \varphi_M\right\|_{L_1^2(\Omega_k)} \leq c(1 + \frac{M}{m(N)})^2 \,\|\varphi_M\|_{L_1^2(\Omega_k)}\,,$$

with $m(N) = E((1+\delta)N)$ and δ a real number between 0 and 1. We conclude, by using the inequalities (20) and $\|\omega_N u_N\|_{L^2_1(\Omega)^2} \leq \|\omega_N\|_{L^4_1(\Omega)} \|u_N\|_{L^4_1(\Omega)^2}$, together with the continuous inclusion of $V^1_1(\Omega)$ in $L^4_1(\Omega)$, that

$$\|\omega_N u_N\|_{L^2_1(\Omega)^2} \leq cN \|\omega_N\|_{V^1_1(\Omega)} \|u_N\|_{L^2_1(\Omega)^2}.$$

Remark 1. Similar arguments lead to estimate (18), if at most two of the three functions ω_N, u_N and v_N are replaced by their analogues ω in $V^1_{1\diamond}(\Omega)$, u and v in $\mathbb{D}(\Omega)$.

Remark 2. Under Assumption 2 and taking $\tilde{N} = E(2\delta N - 1)$, we can find $(\tilde{\omega}_N, \tilde{u}_N)$ in $\mathbb{C}_{\tilde{N}} \times V_{\tilde{N}}$ such that:

$$\|(\omega - \tilde{\omega}_N, u - \tilde{u}_N)\|_X \leq c\tilde{N}^{-s} \|(\omega, u)\|_{H^{s+1}_1(\Omega) \times H^s_1(\Omega)^2}, s > 1. \tag{21}$$

Note that estimate (21) makes sense only when $\tilde{N} \geq 2$.

Lemma 2. *If Assumptions 1 and 2 hold, there exists an integer N_0 such that, for all $N \geq N_0$, the operator $Id + S_N DG_N(\tilde{\omega}_N, \tilde{u}_N)$ is an isomorphism of X_N. Moreover, the norm of its inverse operator is bounded independently of N.*

Proof. We can write that:

$$\begin{aligned} Id + S_N DG_N(\tilde{\omega}_N, \tilde{u}_N) &= Id + SDG(\omega, u) - (S - S_N)DG(\omega, u) \\ &- S_N(DG(\omega, u) - DG(\tilde{\omega}_N, \tilde{u}_N)) - S_N(DG(\tilde{\omega}_N, \tilde{u}_N) - DG_N(\tilde{\omega}_N, \tilde{u}_N)). \end{aligned} \tag{22}$$

It follows from the definition of G and G_N that, for all (θ_N, ω_N) in X_N and v_N in V_N:
$\langle DG(\tilde{\omega}_N, \tilde{u}_N).(\theta_N, w_N), v_N\rangle = K(\tilde{\omega}_N, w_N; v_N) + K(\theta_N, \tilde{u}_N; v_N)$, and
$\langle DG_N(\tilde{\omega}_N, \tilde{u}_N).(\theta_N, w_N), v_N\rangle = K_N(\tilde{\omega}_N, w_N; v_N) + K_N(\theta_N, \tilde{u}_N; v_N)$.
Thanks to the choice of $(\tilde{\omega}_N, \tilde{u}_N)$, the term $S_N(DG(\tilde{\omega}_N, \tilde{u}_N) - DG_N(\tilde{\omega}_N, \tilde{u}_N))$ vanishes. Then, using the stability of S_N, we can derive that:

$$\begin{aligned} &\|S_N(DG(\omega, u) - DG(\tilde{\omega}_N.\tilde{u})).(\theta_N, w_N)\|_X \\ &\qquad \leq c \sup_{v_N \in V_N} \frac{K(\omega - \tilde{\omega}_N, w_N, v_N) + K(\theta_N, u - \tilde{u}_N, v_N)}{\|v_N\|_{L^2_1(\Omega)^2}}. \end{aligned}$$

By Lemma 1, we have:

$$\begin{aligned} &\|S_N(DG(\omega, u) - DG(\tilde{\omega}_N.\tilde{u}_N)).(\theta_N, w_N)\|_X \\ &\qquad \leq cN \left(\|\omega - \tilde{\omega}_N\|_{V^1_1(\Omega)} \|w_N\|_{L^2_1(\Omega)^2} + \|\theta_N\|_{V^1_1(\Omega)} \|u - \tilde{u}_N\|_{L^2_1(\Omega)^2} \right). \end{aligned} \tag{23}$$

Estimate (21) leads to

$$\lim_{N\to+\infty} \|S_N(DG(\omega,u) - DG(\tilde{\omega}_N.\tilde{u}_N))\|_{L(X_N)} = 0. \tag{24}$$

Finally, it follows from Assumption 2 that, when (θ, w) runs through the unit ball of X, $DG(\omega,u)(\theta,w)$ belongs to a compact subset of $L_1^2(\Omega)^2$. So, the next property is derived from the stability of S_N and from inequality (16) by standard arguments:

$$\lim_{N\to+\infty} \|(S - S_N)\, DG(\omega,u)\|_{L(X_N)} = 0. \tag{25}$$

Thanks to Assumption 1, for $\gamma = \left\|(Id + SDG(\omega,u))^{-1}\right\|_{L(X)}$, and by choosing N large enough so that the quantities in (24) and (25) are smaller than $\frac{1}{4\gamma}$, we obtain the desired property with $\left\|(Id + S_N DG_N(\tilde{\omega}_N, \tilde{u}_N))^{-1}\right\|_{L(X_N)} < 2\gamma$.

Lemma 3. *The following Lipschitz property holds:* $\forall(\omega_N^*, u_N^*) \in X_N$,

$$\left\|S_N\left(DG_N(\tilde{\omega}_N,\tilde{u}_N) - DG_N(\omega_N^*,u_N^*)\right)\right\|_{L(X_N)} \le cN\left\|\left(\tilde{\omega}_N - \omega_N^*, \tilde{u}_N - u_N^*\right)\right\|_X. \tag{26}$$

Proof. We just note that

$$\begin{aligned}&\left\langle\left(DG_N(\tilde{\omega}_N,\tilde{u}_N) - DG_N(\omega_N^*,u_N^*)\right).(\theta_N,w_N),v_N\right\rangle\\ &\qquad = K_N(\tilde{\omega}_N - \omega_N^*, w_N; v_N) + K_N(\theta_N, \tilde{u}_N - u_N^*; v_N).\end{aligned}$$

Lemma 1 leads to the desired property.

Lemma 4. *Assume that the data* $f \in H_1^\sigma(\Omega)^2$, $\sigma > \frac{3}{2}$. *Under Assumption 2,*

$$\begin{aligned}&\|(\tilde{\omega}_N,\tilde{u}_N) + S_N G_N(\tilde{\omega}_N,\tilde{u}_N)\|_X\\ &\qquad \le c(\omega,u)\left(N^{-s}\,\|(\omega,u)\|_{H_1^{s+1}(\Omega)\times H_1^s(\Omega)^2} + N^{-\sigma}\,\|f\|_{H_1^\sigma(\Omega)^2}\right),\end{aligned}$$

for a constant $c(\omega,u)$ *only depending on the solution* (ω,u).

Proof. From (12), we derive

$$\begin{aligned}\|(\tilde{\omega}_N,\tilde{u}_N) + S_N G_N(\tilde{\omega}_N,\tilde{u}_N)\|_X &\le \|(\omega-\tilde{\omega}_N, u-\tilde{u}_N)\|_X + \|(S-S_N)\,G(\omega,u)\|_X\\ &+ \|S_N(G(\omega,u) - G(\tilde{\omega}_N,\tilde{u}_N))\|_X + \|S_N(G(\tilde{\omega}_N,\tilde{u}_N) - G_N(\tilde{\omega}_N,\tilde{u}_N))\|_X\end{aligned}$$

The bound for the first term in the right-hand side obviously follows from (21). From estimate (16) with Assumption 2, we also derive

$$\|(S-S_N)\,G(\omega,u)\|_X \le cN^{-s}\,\|(\omega,u)\|_{H_1^{s+1}(\Omega)\times H_1^s(\Omega)^2}.$$

On the other hand,

$$K(\omega,u;v_N) - K(\tilde{\omega}_N,\tilde{u}_N;v_N) = K(\omega-\tilde{\omega}_N,u;v_N) + K(\omega,u-\tilde{u}_N;v_N) - K(\omega-\tilde{\omega}_N,u-\tilde{u}_N;v_N).$$

So, we have from the stability property on S_N

$$\|S_N\left(G\left(\omega,u\right)-G\left(\tilde{\omega}_N,\tilde{u}_N\right)\right)\|_X \leq c \sup_{v_N\in V_N} \frac{\langle K(\omega,u;v_N)-K(\tilde{\omega}_N,\tilde{u}_N;v_N),v_N\rangle}{\|v_N\|_{L^2_1(\Omega)^2}},$$

From (21), Remarks 1 and 2, we have

$$\|S_N\left(G\left(\omega,u\right)-G\left(\tilde{\omega}_N,\tilde{u}_N\right)\right)\|_X \leq c(\omega,u)N^{-s}\,\|(\omega,u)\|_{H^{s+1}_1(\Omega)\times H^s_1(\Omega)^2}.$$

We note that $\forall v_N \in \mathbb{D}_N$, the quantities $K(\tilde{\omega}_N,\tilde{u}_N;v_N)$ and $K_N(\tilde{\omega}_N,\tilde{u}_N;v_N)$ coincide. Then, if Π_{N-1} denotes the orthogonal projection operator from $L^2_1(\Omega)$ onto the space of functions such that their restrictions to all Ω_k, $1\leq k\leq K$, belong to $\mathbb{P}_{N-1}(\Omega_k)$, and by adding and subtracting the quantity $\Pi_{N-1}f$ in $\|S_N\left(G\left(\tilde{\omega}_N,\tilde{u}_N\right)-G_N\left(\tilde{\omega}_N,\tilde{u}_N\right)\right)\|_X$, we can prove that
$\|S_N\left(G\left(\tilde{\omega}_N,\tilde{u}_N\right)-G_N\left(\tilde{\omega}_N,\tilde{u}_N\right)\right)\|_X \leq c\Big(\|f-\Pi_{N-1}f\|_{L^2_1(\Omega)^2}+\|f-I_Nf\|_{L^2_1(\Omega)^2}\Big).$
Finally, the standard approximation properties of the operators Π_{N-1} and I_N, lead to

$$\|S_N\left(G\left(\tilde{\omega}_N,\tilde{u}_N\right)-G_N\left(\tilde{\omega}_N,\tilde{u}_N\right)\right)\|_X \leq cN^{-\sigma}\,\|f\|_{H^\sigma(\Omega)^2}.$$

The desired bound is then derived by combining the previous estimates.

We are now in a position to prove the error estimate.

Theorem 2. *We assume that the data f is in $H^\sigma_1(\Omega)^2$, $\sigma>\frac{3}{2}$, and that the solution (ω,u,p), of problem (3) satisfies Assumptions 1 and 2.*

Then, there exists an integer $N_\diamond$ and a constant $c_\diamond$ such that for any $N\geq N_\diamond$, the problem (9) has a unique solution (ω_N,u_N,p_N) satisfying the following estimate:

$$\|\omega-\omega_N\|_{V^1_1(\Omega)}+\|u-u_N\|_{H_1(\mathrm{div}_r,\Omega)}+\|p-p_N\|_{L^2_1(\Omega)} \leq c(\omega,u)\Big[N^{1-s}\Big(\|\omega\|_{H^{s+1}_1(\Omega)}+\|u\|_{H^s_1(\Omega)^2}+\|p\|_{H^s_1(\Omega)}\Big)+N^{-\sigma}\,\|f\|_{H^\sigma_1(\Omega)^2}\Big]. \tag{27}$$

Proof. Combining Lemmas 2–4 with the Brezzi-Rappaz-Raviart theorem [5], yields that, for N sufficiently large, problem (10) has a unique solution (ω_N,u_N).

Moreover, thanks to the discrete inf-sup condition of $b_N(.,.)$, there exists a unique p_N in $\mathbb{M}_N$ such that

$$\forall v_N \in \mathbb{D}_N, \quad b_N(v_N, p_N) = ((f, v_N))_N - a_N(\omega_N, u_N; v_N) - K_N(\omega_N, u_N; v_N).$$

Hence, the existence and local uniqueness result follows. Moreover,

$$\begin{aligned} \forall q_N \text{ in } \mathbb{M}_N,\ b_N(v_N, p_N - q_N) &= b(v_N, p - q_N) - \langle f, v_N\rangle + ((f, v_N))_N \\ &+ a(\omega - \omega_N, u - u_N; v_N) + (a - a_N)(\omega_N, u_N; v_N) \quad (28) \\ &+ K(\omega, u; v_N) - K_N(\omega_N, u_N; v_N). \end{aligned}$$

so that the estimate for $\|p - p_N\|_{L^2_1(\Omega)}$ follows from the discrete inf-sup condition of $b_N(.,.)$, a triangle inequality and the same arguments as in the proof of Lemma 4.

To conclude, the vorticity-velocity and pressure formulation allows to decouple the calculus of the velocity and the pressure, to handle easily non standard boundary conditions and leads to a more accurate approximation of the pressure. The axisymmetric property of domain allows to move from a three-dimensional problem to a two-dimensional one, which reduces the cost of the resolution. In addition, the tensorization properties of the polynomial spaces, which characterize the spectral methods, enable to inverse the obtained system matrix with a raisonable cost.

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