Quantitative Coding and Complexity Theory of Compact Metric Spaces

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Abstract. Specifying a computational problem requires fixing encodings for input and output: encoding graphs as adjacency matrices, characters as integers, integers as bit strings, and vice versa. For such discrete data, the actual encoding is usually straightforward and/or complexity-theoretically inessential (up to polynomial time, say); but concerning continuous data, already real numbers naturally suggest various encodings with very different computational properties. With respect to qualitative computability, Kreitz and Weihrauch (1985) had identified *admissibility* as crucial property for "reasonable" encodings over the Cantor space of infinite binary sequences, so-called representations. For (precisely) these does the Kreitz-Weihrauch representation (aka *Main*) Theorem apply, characterizing continuity of functions in terms of continuous realizers. We similarly identify refined criteria for representations suitable for quantitative complexity investigations. Higher type complexity is captured by replacing Cantor's as ground space with more general compact metric spaces, similar to equilogical spaces in computability.

1 Introduction

Machine models formalize computation: they specify means of input, operations, and output of elements from some fixed set Γ; as well as measures of cost and of input/output 'size'; such that Complexity Theory can investigate the dependence of the former on the latter. Problems over spaces X other than Γ are treated by encoding its elements/instances over Γ.

Example 1. a) Recall the Turing machine model operating on the set Γ of finite (e.g. decimal or binary) sequences, and consider the space X of graphs: encoded for example as adjacency matrices' binary entries. Operations amount to local transformations of, and in local dependence of, the tape

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contents. Size here is an integer: n commonly denotes the number of nodes of the graph, or the binary length of the encoded matrix, both polynomially related to each other.

b) Consider the space $X = \mathbb{N}$ of natural numbers, either encoded in binary or in unary: their lengths are computably but *not* polynomially related, and induce computably equivalent but significantly different notions of computational complexity.

c) Recall the type-2 machine model [Wei00, §2.1] operating on the Cantor space $\mathcal{C} := \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ of infinite binary sequences; and the real unit interval $X = [0;1]$, equipped with various so-called *representations* [Wei00, §4.1]: surjective partial mappings from $\mathcal{C}$ onto X that formalize (sequences of) approximations up to any given absolute error bound $1/2^n$, $n \in \mathbb{N}$. Different representations of X may induce non-/equivalent notions of computability [Wei00, §4.2].

d) Computational cost of a type-2 computation is commonly gauged in dependence of the index position n within the binary input/output sequence, that is, the length of the finite initial segment read/written so far [Wei00, §7.1]. For $X = [0;1]$ and for some of the representations, this notion of 'size' is polynomially (and for some even linearly) related to n occurring in the error bound $1/2^n$ [Wei00, §7.1]; for other computably equivalent representations it is not [Wei00, Examples 7.2.1+7.2.3].

e) Recall the Turing machine model with 'variable' oracles [KC12, §3], operating on a certain subset $\mathcal{B}$ of string functions

$$\{\mathtt{0},\mathtt{1}\}^{**} \;:=\; \big\{\varphi : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*\big\} \ .$$

The 'size' of $\varphi \in \mathcal{B}$ here is captured by an integer function $\ell : \mathbb{N} \ni n \mapsto \max\big\{|\varphi(\vec{x})| : |\vec{x}| \leq n\big\} \in \mathbb{N}$ [KC96]; and polynomial complexity means bounded by a *second-order* polynomial in $\ell \in \mathbb{N}^{\mathbb{N}}$ and in $n \in \mathbb{N}$ [Meh76].

f) Equip the space $X = \mathcal{C}[0;1]$ of continuous functions $f : [0;1] \to \mathbb{R}$ with the surjective partial mapping $\delta_\square :\subseteq \mathcal{B} \twoheadrightarrow X$ from [KC12, §4.3]. Then, up to a second-order polynomial, the 'size' $\ell = \ell(\varphi)$ from (e) is related to a modulus of continuity (cmp. Subsect. 3.1 below) of $f = \delta_\square(\varphi) \in \mathcal{C}[0;1]$ and to the computational complexity of the application operator $(f,r) \mapsto f(r)$ [KS17, KS20, NS20].

g) Spaces X of continuum cardinality beyond real numbers are also commonly encoded over Cantor space [Wei00, §3], or over 'Baire' space $\{\mathtt{0},\mathtt{1}\}^{**}$ [KC12, §3]. Matthias Schröder has recommended the Hilbert Cube as domain for partial surjections onto suitable X. Also *equilogical* spaces serve as such domains [BBS04].

To summarize, computation on various spaces is commonly formalized by various models of computation (Turing machine, type-2 machine, oracle machine) using encodings over various domains (Cantor space, 'Baire' space, Hilbert Cube, etc.) with various notions of 'size' and of polynomial time.

Question 2. Fix two mathematical structures X and Y, expansions over topological spaces. What machine models, what encodings, what notions of size and polynomial time, are suitable to formalize computation of (multi)functions f from X to Y?

In the sequel we will focus on the part of the question concerned with encoding continuous data. Section 2 recalls classical criteria and notions: qualitative *admissibility* of computably 'reasonable' representations for the Kreitz-Weihrauch *Main Theorem* (Subsect. 2.1), and complexity parameters for a quantitative *Main Theorem* in the real case (Subsect. 2.2). Section 4 combines both towards generic quantitative admissibility and an intrinsic complexity-theoretic *Main Theorem*. The key is to consider metric properties of the *inverse* of a representation, which is inherently multivalued a 'function'. To this end Sect. 3 adopts from [PZ13] a notion of quantitative (uniform) continuity multifunctions (Subsect. 3.1) and establishes important properties (Subsect. 3.2), including closure under a generalized conception of *restriction*. We close with applications to higher-type complexity.

2 Coding Theory of Continuous Data

Common models of computation naturally operate on some particular domain $\varGamma$ (e.g., in/finite binary sequences, string functions, etc.); processing data from another domain X (graphs, real numbers, continuous functions) requires agreeing on some way of encoding (the elements x of) X over $\varGamma$.

Formally, a *representation* is a surjective partial mapping $\xi :\subseteq \varGamma \twoheadrightarrow X$; any $\gamma \in \mathrm{dom}(\xi)$ is called a *name* of $x = \xi(\gamma) \in X$; and for another representation υ of Y, *computing* a total function $f : X \to Y$ means to compute some (ξ, υ)-*realizer*: a transformation $F : \mathrm{dom}(\xi) \to \mathrm{dom}(\upsilon)$ on names such that $f \circ \xi = \upsilon \circ F$.

Some representations are computably 'unsuitable' [Tur37], including the binary expansion $\varGamma = \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}} \ni \bar{b} \mapsto \sum_{n=0}^{\infty} b_n 2^{-n-1} \in [0;1]$; cmp. [Wei00, Exercise 7.2.7]. Others are suitable for computability investigations [Wei00, Theorem 4.3.2], but not for complexity purposes [Wei00, Examples 7.2.1+7.2.3].

Example 3. The *signed digit representation* of $[0;1]$ is the partial map

$$\sigma :\subseteq \{\mathtt{00},\mathtt{01},\mathtt{10}\}^{\mathbb{N}} \subseteq \mathcal{C} \ni \bar{b} \mapsto \tfrac{1}{2} + \sum_{m=0}^{\infty} (2b_{2m} + b_{2m+1} - 1) \cdot 2^{-m-2} \in [0;1]$$

Already for the case $X = [0;1]$ of real numbers, it thus takes particular care to arrive at a complexity-theoretically 'reasonable' representation [Wei00, Theorem 7.3.1]; and even more so for continuous real functions [KC12], not to mention for more involved spaces [Ste17].

2.1 Qualitative Admissibility and Computability

Regarding computability on a large class of topological spaces X, an important criterion for a representation is *admissibility* [KW85, Sch02]:

Definition 4. *Call* $\xi :\subseteq \Gamma \twoheadrightarrow X$ *admissible iff it is (i) continuous and satisfies (ii):*
(ii) To every continuous $\zeta :\subseteq \Gamma \twoheadrightarrow X$ *there exists a continuous mapping* $G : \mathrm{dom}(\zeta) \to \mathrm{dom}(\xi)$ *with* $\zeta = \xi \circ G$*; see* [Wei00, Theorem 3.2.9.2].

Admissible representations exist (at least) for T_0 spaces; they are Cartesian closed; and yield the Kreitz-Weihrauch (aka *Main*) Theorem [Wei00, Theorem 3.2.11]:

Fact 5. *Let* $\xi :\subseteq \Gamma \twoheadrightarrow X$ *and* $\upsilon :\subseteq \Gamma \twoheadrightarrow Y$ *be admissible. Then* $f : X \to Y$ *is continuous iff it admits a continuous* (ξ, υ)*-realizer* $F : \mathrm{dom}(\xi) \to \mathrm{dom}(\upsilon)$*.*

In particular *dis*continuous functions are *in*computable.

2.2 Real Quantitative Admissibility

The search for quantitative versions of admissibility and the Main Theorem is guided by above notion of qualitative admissibility. It revolves around quantitative metric versions of qualitative topological properties, such as continuity and compactness, obtained via *Skolemization*. Further guidance comes from reviewing the real case.

Recall that a *modulus of continuity* of a function $f : X \to Y$ between compact metric spaces (X, d) and (Y, e) is a strictly increasing mapping $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$d(x, x') \leq 2^{-\mu(n)} \quad \Rightarrow \quad e\big(f(x), f(x')\big) \leq 2^{-n} \ . \tag{1}$$

In this case one says that f is μ-continuous. Actually we shall occasionally slightly weaken this notion and require Condition (1) only for all sufficiently large n.

Example 6. The signed digit representation $\sigma :\subseteq \mathcal{C} \twoheadrightarrow [0; 1]$ from Example 3 has modulus of continuity $\kappa(n) = 2n$.

Proposition 11d) below provides a converse. Together with Theorem 13 and Lemma 10 below, they yield the following quantitative strengthening of Fact 5 aka qualitative *Main Theorem*, where $\mathcal{O}()$ of refers to the asymptotic Landau symbol:

Theorem 7. *Fix strictly increasing* $\mu : \mathbb{N} \to \mathbb{N}$*. A function* $f : [0; 1] \to [0; 1]$ *has modulus of continuity* $\mathcal{O}\Big(\mu\big(\mathcal{O}(n)\big)\Big)$ *iff it has a* (σ, σ)*-realizer with modulus of continuity* $\mathcal{O}\Big(\mu\big(\mathcal{O}(n)\big)\Big)$*.*

In particular functions f with (only) 'large' modulus of continuity are inherently 'hard' to compute; cmp. [Ko91, Theorem 2.19]. This suggests gauging the efficiency of some actual computation of f *relative* to it modulus of continuity, rather than absolutely [KC12, KS17, KS20, NS20]:

Definition 8. *Function $f : [0;1] \to [0;1]$ is* polynomial-time computable *iff it can be computed in time bounded by a (first or second order) polynomial in the output precision parameter n and in f's modulus of continuity.*

In the sequel we consider continuous total (multi)functions whose domains are compact: The latter condition ensures them to have a modulus of (uniform) continuity. Moreover computable functions with compact domains admit complexity bounds depending only on the output precision parameter n; cmp. [Ko91, Theorem 2.19] or [Wei00, Theorems 7.1.5+7.2.7] or [Sch03].

3 Multifunctions

Multifunctions are unavoidable in real computation [Luc77]. Their introduction simplifies several considerations; for example, every function $f : X \to Y$ has a (possibly multivalued) inverse $f^{-1} : Y \rightrightarrows X$.

Formally, a partial multivalued function (multifunction) F between sets X, Y is a relation $F \subseteq X \times Y$ that models a computational *search* problem: Given (any name of) $x \in X$, return some (name of some) $y \in Y$ with $(x, y) \in F$. One may identify the relation f with the single-valued total function $F : X \ni x \mapsto \{y \in Y \mid (x, y) \in F\}$ from X to the powerset 2^Y; but we prefer the notation $f :\subseteq X \rightrightarrows Y$ to emphasize that not every $y \in F(x)$ needs to occur as output. Letting the answer y depend on the code of x means dropping the requirement for ordinary functions to be extensional; hence, in spite of the oxymoron, such F is also called a *non*-extensional function. Note that no output is feasible in case $F(x) = \emptyset$.

Definition 9. *Abbreviate with* $\mathrm{dom}(F) := \{x \mid F(x) \neq \emptyset\}$ *for the domain of F; and* $\mathrm{range}(F) := \{y \mid \exists x : (x, y) \in F\}$. *$F$ is* total *in case* $\mathrm{dom}(F) = X$*;* surjective *in case* $\mathrm{range}(F) = Y$. *The composition of multifunctions $F :\subseteq X \rightrightarrows Y$ and $G :\subseteq Y \rightrightarrows Z$ is $G \circ F =$*

$$\big\{(x, z) \bigm| x \in X, z \in Z, F(x) \subseteq \mathrm{dom}(G),\ \exists y \in Y : (x, y) \in F \wedge (y, z) \in G\big\}$$

Call F pointwise compact *if $F(x) \subseteq Y$ is compact for every $x \in \mathrm{dom}(F)$.*

Note that every (single-valued) function is pointwise compact. A computational problem, considered as total single-valued function $f : X \to Y$, becomes 'easier' when *restricting* arguments to $x \in X' \subseteq X$, that is, when proceeding to $f' = f|_{X'}$ for some $X' \subseteq X$. A search problem, considered as total multifunction $F : X \rightrightarrows Y$, additionally becomes 'easier' when proceeding to any $F' \subseteq X \rightrightarrows Y$ satisfying the following: $F'(x) \supseteq F(x)$ for every $x \in \mathrm{dom}(F')$. We call such F' also a *restriction* of F, and write $F' \sqsubseteq F$. A single-valued function $f : \mathrm{dom}(F) \to Y$ is a *selection* of $F :\subseteq X \rightrightarrows Y$ if F is a restriction of f.

Lemma 10. *Fix partial multifunctions $F :\subseteq X \rightrightarrows Y$ and $G :\subseteq Y \rightrightarrows Z$.*

a) If both F and G are pointwise compact, then so is their composition $G \circ F$.

b) The composition of restrictions $F' \sqsubseteq F$ and $G' \sqsubseteq G$, is again a restriction $G' \circ F' \sqsubseteq G \circ F$.

c) It holds $F^{-1} \circ F \sqsubseteq \mathrm{id}_X : X \to X$. Single-valued surjective partial $g :\subseteq X \twoheadrightarrow Y$ furthermore satisfy $g \circ g^{-1} = \mathrm{id}_Y$.

d) For representations ξ of X and υ of Y, the following are equivalent: (i) $f \circ \xi$ is a restriction of $\upsilon \circ F$ (ii) f is a restriction of $\upsilon \circ F \circ \xi^{-1}$ (iii) $\upsilon^{-1} \circ f \circ \xi$ is a restriction of F.

3.1 Quantitative Continuity for Multifunctions

Every restriction f' of a single-valued continuous function f is again continuous. This is not true for multifunctions with respect to *hemi*continuity. Instead Definition 12 below adapts, and quantitatively refines, a notion of continuity for multifunctions from [PZ13] such as to satisfy the following properties:

Proposition 11. *a) A single-valued function is μ-continuous iff it is μ-continuous when considered as a multifunction.*

b) Suppose that $F :\subseteq X \rightrightarrows Y$ is μ-continuous. Then every restriction $F' \sqsubseteq F$ is again μ-continuous.

c) If additionally $G :\subseteq Y \rightrightarrows Z$ is ν-continuous, then $G \circ F$ is $\nu \circ \mu$-continuous

d) The multivalued inverse of the signed digit representation σ^{-1} is $\mathcal{O}(n)$-continuous.

e) For every $\varepsilon > 0$, the soft *Heaviside 'function' h_ε is* id*-continuous, but not for $\varepsilon = 0$:*

$$h_\varepsilon(t) \;:=\; \begin{cases} 0 : t \le \varepsilon \\ 1 : t \ge -\varepsilon \end{cases}$$

Our notion of quantitative (uniform) continuity is inspired by [BH94] and [PZ13, §4+§6]:

Definition 12. *Fix metric spaces (X, d) and (Y, e) and strictly increasing $\mu : \mathbb{N} \to \mathbb{N}$. A total multifunction $F : X \rightrightarrows Y$ is called μ-*continuous *if there exists some $n_0 \in \mathbb{N}$, and to every $x_0 \in X$ there exists some $y_0 \in F(x_0)$, such that the following holds for every $k \in \mathbb{N}$:*

$$\begin{aligned} &\forall n_1 \ge n_0 \;\; \forall x_1 \in \overline{\mathrm{B}}_{\mu(n_1)}(x_0) \;\; \exists y_1 \in F(x_1) \cap \overline{\mathrm{B}}_{n_1}(y_0) \\ &\qquad \forall n_2 \ge n_1 + n_0 \;\; \forall x_2 \in \overline{\mathrm{B}}_{\mu(n_2)}(x_1) \;\; \exists y_2 \in F(x_2) \cap \overline{\mathrm{B}}_{n_2}(y_1) \qquad \ldots \\ &\qquad \forall n_{k+1} \ge n_k + n_0 \;\; \forall x_{k+1} \in \overline{\mathrm{B}}_{\mu(n_{k+1})}(x_k) \exists y_{k+1} \in F(x_{k+1}) \cap \overline{\mathrm{B}}_{n_{k+1}}(y_k) \;\; . \end{aligned}$$

The parameter n_0 is introduced for the purpose of Proposition 11d+e). Recall that also in the single-valued case we sometimes understand Eq. (1) to hold only for all $n \ge n_0$.

A continuous multifunction on Cantor space, unlike one for example on the reals [PZ13, Fig. 5], does admit a continuous selection, and even a bound on the modulus:

Theorem 13. *Suppose* (Y, e) *is compact of diameter* $\mathrm{diam}(Y) \leq 1$ *and satisfies the* strong *triangle inequality*

$$e(x, z) \;\leq\; \max\{e(x, y), e(y, z)\} \quad\leq\; e(x, y) \;+\; e(y, z). \tag{2}$$

If $G :\subseteq \mathcal{C} \rightrightarrows Y$ *is* μ*-continuous and pointwise compact with compact domain* $\mathrm{dom}(G) \subseteq \mathcal{C}$*, then* G *admits a* $\mu\big(n + \mathcal{O}(1)\big)$*-continuous selection.*

3.2 Generic Quantitative *Main Theorem*

Generalizing both Fact 5 and Theorem 7, Lemma 10 and Proposition 11 and Theorem 13 together in fact yields the following quantitative counterpart to the qualitative *Main Theorem* for generic compact metric spaces:

Theorem 14. *Fix compact metric spaces* (X, d) *and* (Y, e) *of* $\mathrm{diam}(X)$, $\mathrm{diam}(Y) \leq 1$*. Consider representations* $\xi :\subseteq \mathcal{C} \twoheadrightarrow X$ *and* $\upsilon :\subseteq \mathcal{C} \twoheadrightarrow Y$*.*

Let $\mu, \mu', \nu, \nu', \kappa, K : \mathbb{N} \to \mathbb{N}$ *be strictly increasing such that* ξ *is* μ*-continuous with compact domain and* μ'*-continuous multivalued inverse* $\xi^{-1} : X \rightrightarrows \mathcal{C}$*;* υ *is* ν*-continuous with compact domain and* ν'*-continuous multivalued inverse* $\upsilon^{-1} : Y \rightrightarrows \mathcal{C}$*.*

a) If total multifunction $g : X \rightrightarrows Y$ *has a* K*-continuous* (ξ, υ)*-realizer* G*, then* g *is* $(\nu \circ K \circ \mu')$*-continuous.*
b) If total multifunction $g : X \rightrightarrows Y$ *is* κ*-continuous and pointwise compact, then it has a* $\nu' \circ \kappa \circ \mu\big(n + \mathcal{O}(1)\big)$*-continuous* (ξ, υ)*-realizer* G*.*

Following up on Definition 8, this suggests gauging the efficiency of some actual computation of g *relative* to both it modulus of continuity and moduli of continuity of the representations (and their multivalued inverses) involved.

4 Generic Quantitative Admissibility

According to Theorem 14, quantitative continuity of a (multi)function g is connected to that of a (single-valued) realizer G, subject to properties of the representations ξ, υ under consideration.

A 'true' quantitative *Main Theorem* should replace these extrinsic parameters with ones intrinsic to the co/domains X, Y: by imposing suitable conditions on the representations as quantitative variant of qualitative admissibility [Lim19].

Definition 15. *The* entropy *of a compact metric space* (X, d) *is the mapping* $\eta = \eta_X : \mathbb{N} \to \mathbb{N}$ *such that* X *can be covered by* $2^{\eta(n)}$ *closed balls* $\overline{\mathrm{B}}_n(x)$ *of radius* 2^{-n}*, but not by* $2^{\eta(n)-1}$*.*

Introduced by Kolmogorov [KT59], η thus quantitatively captures total boundedness [Koh08, Definition 18.52]. Its connections to computational complexity are well-known [Wei03, KSZ16].

Example 16. a) The d-dimensional real unit cube $X = [0;1]^d$ has linear entropy $\eta(n) = \Theta(dn)$. Cantor space $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, equipped with the metric $d(\bar{x}, \bar{y}) = 2^{-\min\{n: x_n \neq y_n\}}$, has linear entropy $\eta(n) = \Theta(n)$.

b) The space $[0;1]'$ of non-expansive (aka 1-Lipschitz) functions $f : [0;1] \to [0;1]$ is compact when equipped with the supremum norm and has entropy $\eta(n) = \Theta(2^n)$.

c) More generally fix a connected compact metric space (X, d) of diameter $\operatorname{diam}(X) := \sup\{(d(x, x') : x, x' \in X\}$ with entropy η. Then the space X' of non-expansive function*als* $\Lambda : X \to [0;1]$ is compact when equipped with the supremum norm and has entropy $\eta'(n) = 2^{\eta(n \pm \mathcal{O}(1))}$.

Items (b) and (c) are relevant for higher-type complexity theory.

Since computational efficiency is connected to quantitative continuity (Subsect. 2.2), in Theorem 14 one prefers ξ and ξ^{-1} with 'small' moduli; similarly for υ and υ^{-1}. A simple but important constraint has been identified in [Ste16, Lemma 3.1.13]—originally for single-valued functions, but its proof immediately extends to multifunctions.

Lemma 17. *If surjective (multi)function $g : X \rightrightarrows Y$ is μ-continuous, then it holds $\eta_Y(n) \leq \eta_X \circ \mu(n)$ (for all sufficiently large n).*

This suggests the following tentative definition:

Definition 18. *Fix some compact metric space Γ, and recall Example 1g).*

a) A representation *of compact metric space (X, d) is a continuous partial surjective (single-valued) mapping $\xi :\subseteq \Gamma \twoheadrightarrow X$.*

b) Fix another compact metric space Δ and representation $\upsilon :\subseteq \Delta \twoheadrightarrow Y$. A (ξ, υ)-realizer *of a total (multi)function $f : X \rightrightarrows Y$ is a (single-valued) function $F : \operatorname{dom}(\xi) \to \operatorname{dom}(\upsilon)$ satisfying any/all conditions of Lemma 10d).*

c) Representation $\xi :\subseteq \Gamma \twoheadrightarrow X$ is polynomially admissible *if (i) It has a modulus of continuity μ such that $\eta_\Gamma \circ \mu$ is bounded by a (first or second order) polynomial in the precision parameter n* and *in the entropy η of X. (ii) Its multivalued inverse ξ^{-1} has polynomial modulus of continuity μ'.*

d) Call total (multi)function $f : X \rightrightarrows Y$ polynomial-time computable *iff it can be computed in time bounded by a (first or second order) polynomial in the output precision parameter n* and *in the entropy η of X.*

In view of Lemma 10c+d) we deliberately consider only single-valued representations [Wei05]. Item d) includes Definition 8 as well as higher types, such as Example 16b) and c). Note that Item (c i) indeed quantitatively strengthens Definition 4i). And Item (c ii) quantitatively strengthens Definition 4ii): For ν-continuous ζ, Theorem 13 yields a $\nu \circ \mu'$-continuous selection G of $\xi^{-1} \circ \zeta$, that is, with $\zeta = \xi \circ G$ according to Lemma 10.

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