Chapter 6
Volume of $SU\left(2^K\right)$

Let's make a more refined calculation of the number of operators in $SU(N)$ by dividing its volume by the volume of an epsilon ball of the same dimensionality (the dimension of $SU(N)$ is N^2-1). The volume of $SU(N)$ is https://arxiv.org/abs/math-ph/0210033

$$V[SU(N)] = \frac{2\pi^{\frac{(N+2)(N-1)}{2}}}{1!2!3!\ldots(N-1)!} \tag{6.1}$$

The volume of an epsilon-ball of dimension N^2-1 is

$$\frac{\pi^{\frac{N^2-1}{2}}}{\left(\frac{N^2-1}{2}\right)!}$$

Using Stirling's formula, and identifying the number of unitary operators with the number of epsilon-balls in $SU(N)$

$$\begin{aligned} \#unitaries &\approx \left(\frac{N}{\epsilon^2}\right)^{\frac{N^2}{2}} \\ &= \left(\frac{2^K}{\epsilon^2}\right)^{4^K/2} \end{aligned} \tag{6.2}$$

Taking the logarithm,

$$\log\left(\#unitaries\right) \approx \frac{4^K}{2} K \log 2 + 4^K \log\frac{1}{\epsilon} \tag{6.3}$$

which is comparable to (5.3). Again, we see the strong exponential dependence on K and the weak logarithmic dependence on ϵ. The $\log\frac{1}{\epsilon}$ term is multiplied by the dimension of the space.

L. Susskind, *Three Lectures on Complexity and Black Holes*,
SpringerBriefs in Physics, https://doi.org/10.1007/978-3-030-45109-7_6