Chapter 5

Boundary Triplets and Boundary Pairs for Semibounded Relations

Semibounded relations in a Hilbert space automatically have equal defect numbers, so that there are always self-adjoint extensions. In this chapter the semibounded self-adjoint extensions of a semibounded relation will be investigated. Special attention will be paid to the Friedrichs extension, which is introduced with the help of closed semibounded forms. Section 5.1 provides a self-contained introduction to closed semibounded forms and their representations via semibounded self-adjoint relations. Closely related is the discussion of the ordering for closed semibounded forms and for semibounded self-adjoint relations in Section 5.2; this section also contains a general monotonicity principle about monotone sequences of semibounded relations. The Friedrichs extension of a semibounded relation is defined and its central properties are studied in Section 5.3. Particular attention is paid to semibounded self-adjoint extensions which are transversal to the Friedrichs extension. Section 5.4 is devoted to special semibounded extensions, namely the Kreĭn type extensions. In the nonnegative case these extensions include the well-known Kreĭn–von Neumann extension. The Friedrichs extension and the Kreĭn type extensions act as extremal elements to describe the semibounded self-adjoint extensions with a given lower bound. In Section 5.5 there is a return to boundary triplets and Weyl functions for symmetric relations which are semibounded. Of special interest is the case where the self-adjoint extensions determined by the boundary triplet are semibounded and one of them coincides with the Friedrichs extension. In particular, this leads to a useful abstract version of the first Green formula. The notion of a boundary pair for semibounded relations is developed in Section 5.6. In conjunction with the above first Green formula, this notion serves as a link between boundary triplet methods and form methods when semibounded self-adjoint extensions are described; in a wider sense it establishes the connection with the Birman–Kreĭn–Vishik method.

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5.1 Closed semibounded forms and their representations

A *sesquilinear form* $\mathfrak{t}[\cdot,\cdot]$ in a Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$ is a mapping from $\mathfrak{D}\times\mathfrak{D}$ to $\mathbb{C}$, where $\mathfrak{D}$ is a linear subspace of $\mathfrak{H}$, such that $\mathfrak{t}[\cdot,\cdot]$ is linear in the first entry and anti-linear in the second entry. The domain $\mathrm{dom}\,\mathfrak{t}$ is defined by $\mathrm{dom}\,\mathfrak{t}=\mathfrak{D}$. The form is said to be *symmetric* if $\mathfrak{t}[\varphi,\psi]=\overline{\mathfrak{t}[\psi,\varphi]}$ for all $\varphi,\psi\in\mathrm{dom}\,\mathfrak{t}$. The corresponding *quadratic form* $\mathfrak{t}[\cdot]$ is defined by $\mathfrak{t}[\varphi]=\mathfrak{t}[\varphi,\varphi]$, $\varphi\in\mathrm{dom}\,\mathfrak{t}$. The polarization formula

$$\mathfrak{t}[\varphi,\psi]=\frac{1}{4}\big(\mathfrak{t}[\varphi+\psi]-\mathfrak{t}[\varphi-\psi]\big)+\frac{i}{4}\big(\mathfrak{t}[\varphi+i\psi]-\mathfrak{t}[\varphi-i\psi]\big) \tag{5.1.1}$$

for $\varphi,\psi\in\mathrm{dom}\,\mathfrak{t}$ is easily checked. In the following the term sesquilinear will be dropped; whenever a form $\mathfrak{t}[\cdot,\cdot]$ is mentioned it is assumed to be sesquilinear and it will be denoted by $\mathfrak{t}$. For instance, the inner product $(\cdot,\cdot)$ is a form defined on all of $\mathfrak{H}$.

Definition 5.1.1. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be forms in $\mathfrak{H}$. Then the *inclusion* $\mathfrak{t}_2\subset\mathfrak{t}_1$ means that

$$\mathrm{dom}\,\mathfrak{t}_2\subset\mathrm{dom}\,\mathfrak{t}_1,\quad \mathfrak{t}_2[\varphi]=\mathfrak{t}_1[\varphi],\quad \varphi\in\mathrm{dom}\,\mathfrak{t}_2. \tag{5.1.2}$$

If $\mathfrak{t}_2\subset\mathfrak{t}_1$, then $\mathfrak{t}_2$ is said to be a *restriction* of $\mathfrak{t}_1$ and $\mathfrak{t}_1$ is said to be an *extension* of $\mathfrak{t}_2$. The *sum* $\mathfrak{t}_1+\mathfrak{t}_2$ is defined by

$$(\mathfrak{t}_1+\mathfrak{t}_2)[\varphi,\psi]=\mathfrak{t}_1[\varphi,\psi]+\mathfrak{t}_2[\varphi,\psi],\quad \varphi,\psi\in\mathrm{dom}\,(\mathfrak{t}_1+\mathfrak{t}_2),$$

where $\mathrm{dom}\,(\mathfrak{t}_1+\mathfrak{t}_2)=\mathrm{dom}\,\mathfrak{t}_1\cap\mathrm{dom}\,\mathfrak{t}_2$.

If $\alpha\in\mathbb{C}$ the sum $\mathfrak{t}[\cdot,\cdot]+\alpha(\cdot,\cdot)$ is given by

$$\mathfrak{t}[\varphi,\psi]+\alpha(\varphi,\psi),\quad \varphi,\psi\in\mathrm{dom}\,\mathfrak{t}.$$

This sum will be denoted by $\mathfrak{t}+\alpha$. It is symmetric when $\mathfrak{t}$ is symmetric and $\alpha\in\mathbb{R}$.

Definition 5.1.2. A symmetric form $\mathfrak{t}$ in $\mathfrak{H}$ is *bounded from below* if there exists a constant $c\in\mathbb{R}$ such that

$$\mathfrak{t}[\varphi]\geq c\|\varphi\|^2,\quad \varphi\in\mathrm{dom}\,\mathfrak{t}.$$

This inequality will be denoted by $\mathfrak{t}\geq c$. The *lower bound* $m(\mathfrak{t})$ is the largest of such numbers $c\in\mathbb{R}$:

$$m(\mathfrak{t})=\inf\left\{\frac{\mathfrak{t}[\varphi]}{\|\varphi\|^2}:\varphi\in\mathrm{dom}\,\mathfrak{t},\ \varphi\neq 0\right\}.$$

If $m(\mathfrak{t})\geq 0$, then $\mathfrak{t}$ is called *nonnegative*.

In the following the terminology *semibounded form* is used for a symmetric form which is bounded from below. Note that $\mathfrak{t}$ is a semibounded form if and only if for some, and hence for all $\alpha \in \mathbb{R}$ the form $\mathfrak{t} + \alpha$ is semibounded. For a semibounded form $\mathfrak{t}$ the lower bound $m(\mathfrak{t})$ will often be denoted by γ. Note that the form $\mathfrak{t} - \gamma$, $\gamma = m(\mathfrak{t})$, is nonnegative with lower bound 0. Therefore, one has the Cauchy–Schwarz inequality

$$|(\mathfrak{t} - \gamma)[\varphi, \psi]| \leq (\mathfrak{t} - \gamma)[\varphi]^{\frac{1}{2}} (\mathfrak{t} - \gamma)[\psi]^{\frac{1}{2}}, \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}, \tag{5.1.3}$$

and, hence the triangle inequality

$$(\mathfrak{t} - \gamma)[\varphi + \psi]^{\frac{1}{2}} \leq (\mathfrak{t} - \gamma)[\varphi]^{\frac{1}{2}} + (\mathfrak{t} - \gamma)[\psi]^{\frac{1}{2}}, \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}. \tag{5.1.4}$$

It follows from (5.1.4) that

$$\left|(\mathfrak{t} - \gamma)[\varphi]^{\frac{1}{2}} - (\mathfrak{t} - \gamma)[\psi]^{\frac{1}{2}}\right| \leq (\mathfrak{t} - \gamma)[\varphi - \psi]^{\frac{1}{2}}, \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}. \tag{5.1.5}$$

The following continuity property is a simple consequence of (5.1.5). For a sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ and $\varphi \in \operatorname{dom} \mathfrak{t}$ one has

$$(\mathfrak{t} - \gamma)[\varphi - \varphi_n] \to 0 \quad \Rightarrow \quad (\mathfrak{t} - \gamma)[\varphi_n] \to (\mathfrak{t} - \gamma)[\varphi]. \tag{5.1.6}$$

Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $a < \gamma$. Equip the space $\operatorname{dom} \mathfrak{t} \subset \mathfrak{H}$ with the form

$$(\varphi, \psi)_{\mathfrak{t}-a} = \mathfrak{t}[\varphi, \psi] - a(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}. \tag{5.1.7}$$

By rewriting this definition as

$$(\varphi, \psi)_{\mathfrak{t}-a} = (\mathfrak{t} - \gamma)[\varphi, \psi] + (\gamma - a)(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}, \tag{5.1.8}$$

one sees that $(\cdot, \cdot)_{\mathfrak{t}-a}$ is the sum of the semidefinite inner product $\mathfrak{t} - \gamma$ and the inner product $(\gamma - a)(\cdot, \cdot)$. Hence, $(\cdot, \cdot)_{\mathfrak{t}-a}$ is an inner product on $\operatorname{dom} \mathfrak{t}$ and the corresponding norm $\|\cdot\|_{\mathfrak{t}-a}$ satisfies the inequality

$$\|\varphi\|^2_{\mathfrak{t}-a} \geq (\gamma - a)\|\varphi\|^2, \quad \varphi \in \operatorname{dom} \mathfrak{t}. \tag{5.1.9}$$

When $\operatorname{dom} \mathfrak{t}$ is equipped with the inner product $(\cdot, \cdot)_{\mathfrak{t}-a}$, the resulting inner product space will be denoted by $\mathfrak{H}_{\mathfrak{t}-a}$. Note that if $\gamma > 0$, then obviously $a = 0$ is a natural choice in the above and the following arguments.

Lemma 5.1.3. *Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $a < \gamma$. Let (φ_n) be a sequence in $\operatorname{dom} \mathfrak{t}$. Then (φ_n) is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$ if and only if*

$$\mathfrak{t}[\varphi_n - \varphi_m] \to 0 \quad \textit{and} \quad \|\varphi_n - \varphi_m\| \to 0. \tag{5.1.10}$$

Proof. According to (5.1.8), (φ_n) is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$ if and only if

$$(\mathfrak{t}-\gamma)[\varphi_n-\varphi_m]\to 0 \quad \text{and} \quad \|\varphi_n-\varphi_m\|^2\to 0. \tag{5.1.11}$$

Now assume that (φ_n) is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. Then it follows from (5.1.11) that (φ_n) is a Cauchy sequence in $\mathfrak{H}$ and that

$$\mathfrak{t}[\varphi_n-\varphi_m]=(\mathfrak{t}-\gamma)[\varphi_n-\varphi_m]+\gamma\|\varphi_n-\varphi_m\|^2\to 0,$$

which shows (5.1.10). Conversely, if the sequence (φ_n) satisfies (5.1.10), then it follows from (5.1.7) that (φ_n) is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. □

Let (φ_n) be a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. Since $\mathfrak{H}$ is a Hilbert space, it follows from Lemma 5.1.3 that there is an element $\varphi\in\mathfrak{H}$ such that $\varphi_n\to\varphi$ in $\mathfrak{H}$.

Definition 5.1.4. Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$. A sequence (φ_n) in $\operatorname{dom}\mathfrak{t}$ is said to be $\mathfrak{t}$-*convergent* to an element $\varphi\in\mathfrak{H}$, not necessarily belonging to $\operatorname{dom}\mathfrak{t}$, if

$$\varphi_n\to\varphi \text{ in } \mathfrak{H} \quad \text{and} \quad \mathfrak{t}[\varphi_n-\varphi_m]\to 0, \quad n,m\to\infty.$$

This type of convergence will be denoted by $\varphi_n\to_{\mathfrak{t}}\varphi$.

The following result is a direct consequence of Lemma 5.1.3 and the completeness of $\mathfrak{H}$.

Corollary 5.1.5. *Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $a<\gamma$. Then any Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$ is $\mathfrak{t}$-convergent. Conversely, any $\mathfrak{t}$-convergent sequence in* $\operatorname{dom}\mathfrak{t}$ *is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$.*

If the sequence (φ_n) in $\operatorname{dom}\mathfrak{t}$ is $\mathfrak{t}$-convergent, then by Definition 5.1.4

$$(\mathfrak{t}-\gamma)[\varphi_n-\varphi_m]\to 0 \quad \text{and} \quad \|\varphi_n-\varphi_m\|\to 0.$$

Thus, one has the following result.

Corollary 5.1.6. *Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let the sequence (φ_n) in* $\operatorname{dom}\mathfrak{t}$ *be $\mathfrak{t}$-convergent. Then the sequences $((\mathfrak{t}-\gamma)[\varphi_n])$, $(\mathfrak{t}[\varphi_n])$, and $(\|\varphi_n\|)$ converge and, consequently, they are bounded.*

Proof. Since γ is the lower bound of $\mathfrak{t}$, one has $(\mathfrak{t}-\gamma)[\varphi_n-\varphi_m]\to 0$. Hence, (5.1.5) shows that $((\mathfrak{t}-\gamma)[\varphi_n])$ is a Cauchy sequence. Then the same is true for the sequence $(\mathfrak{t}[\varphi_n])$ and it is also clear that $(\|\varphi_n\|)$ is a Cauchy sequence. In particular, the sequences $((\mathfrak{t}-\gamma)[\varphi_n])$, $(\mathfrak{t}[\varphi_n])$, and $(\|\varphi_n\|)$ are bounded. □

The $\mathfrak{t}$-convergence is preserved when one takes a sum of sequences. To see this, let (φ_n) and (ψ_n) be sequences in $\operatorname{dom}\mathfrak{t}$ such that

$$\varphi_n\to_{\mathfrak{t}}\varphi \quad \text{and} \quad \psi_n\to_{\mathfrak{t}}\psi$$

for some $\varphi, \psi \in \mathfrak{H}$. Then clearly $\varphi_n + \psi_n \to \varphi + \psi$ in $\mathfrak{H}$ and

$$\begin{aligned}(\mathfrak{t}-\gamma)[\varphi_n + \psi_n - (\varphi_m + \psi_m)]^{\frac{1}{2}} \\ \leq (\mathfrak{t}-\gamma)[\varphi_n - \varphi_m]^{\frac{1}{2}} + (\mathfrak{t}-\gamma)[\psi_n - \psi_m]^{\frac{1}{2}},\end{aligned}$$

by the triangle inequality in (5.1.4). Therefore,

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \text{and} \quad \psi_n \to_{\mathfrak{t}} \psi \quad \Rightarrow \quad \varphi_n + \psi_n \to_{\mathfrak{t}} \varphi + \psi. \tag{5.1.12}$$

As a consequence, one sees that

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \text{and} \quad \psi_n \to_{\mathfrak{t}} \psi \quad \Rightarrow \quad \lim_{n\to\infty} \mathfrak{t}[\varphi_n, \psi_n] \quad \text{exists.} \tag{5.1.13}$$

This last implication follows easily from Corollary 5.1.6 and (5.1.12) by the polarization formula in (5.1.1).

Assume that $\varphi_n \in \operatorname{dom} \mathfrak{t}$ and that $\varphi_n \to_{\mathfrak{t}} \varphi$ for some $\varphi \in \mathfrak{H}$. Now the question is when $\varphi \in \operatorname{dom} \mathfrak{t}$ and, if this is the case, when $\mathfrak{t}[\varphi_n - \varphi] \to 0$? This question gives rise to the notions of closed form and closable form in Definition 5.1.7 and Definition 5.1.11.

Definition 5.1.7. A semibounded form $\mathfrak{t}$ in $\mathfrak{H}$ is said to be *closed* if

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \Rightarrow \quad \varphi \in \operatorname{dom} \mathfrak{t} \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi] \to 0.$$

The statement in (5.1.13) can now be made more precise when the form is closed.

Lemma 5.1.8. *Let $\mathfrak{t}$ be a closed semibounded form in $\mathfrak{H}$. Then*

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \Rightarrow \quad \varphi \in \operatorname{dom} \mathfrak{t} \quad \textit{and} \quad \mathfrak{t}[\varphi_n] \to \mathfrak{t}[\varphi], \tag{5.1.14}$$

and, consequently,

$$\varphi_n \to_{\mathfrak{t}} \varphi, \quad \psi_n \to_{\mathfrak{t}} \psi \quad \Rightarrow \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} \quad \textit{and} \quad \mathfrak{t}[\varphi_n, \psi_n] \to \mathfrak{t}[\varphi, \psi]. \tag{5.1.15}$$

Proof. Assume that $\mathfrak{t}$ is a closed semibounded form and $\varphi_n \to_{\mathfrak{t}} \varphi$. Then $\varphi \in \operatorname{dom} \mathfrak{t}$ and $\mathfrak{t}[\varphi_n - \varphi] \to 0$, and since $\varphi_n \to \varphi$ it follows that $(\mathfrak{t}-\gamma)[\varphi_n - \varphi] \to 0$. Hence, $(\mathfrak{t}-\gamma)[\varphi_n] \to (\mathfrak{t}-\gamma)[\varphi]$ by (5.1.6), and therefore $\mathfrak{t}[\varphi_n] \to \mathfrak{t}[\varphi]$. This shows (5.1.14). Now polarization and (5.1.12) yield the assertion (5.1.15). □

Lemma 5.1.9. *Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $a < \gamma$. Then the following statements are equivalent:*

(i) $\mathfrak{H}_{\mathfrak{t}-a}$ *is a Hilbert space;*

(ii) $\mathfrak{t}$ *is closed.*

In particular, $\mathfrak{t}$ is closed if and only if $\mathfrak{t} - x$ is closed for some, and hence for all $x \in \mathbb{R}$.

Proof. (i) ⇒ (ii) Assume that $\mathfrak{H}_{\mathfrak{t}-a}$ is complete. To show that $\mathfrak{t}$ is closed, assume that $\varphi_n \to_{\mathfrak{t}} \varphi$, so

$$\varphi_n \to \varphi \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi_m] \to 0.$$

In particular, this implies by Lemma 5.1.3 that $\|\varphi_n - \varphi_m\|_{\mathfrak{t}-a} \to 0$. Since $\mathfrak{H}_{\mathfrak{t}-a}$ is complete there is an element $\varphi_0 \in \mathfrak{H}_{\mathfrak{t}-a} = \operatorname{dom} \mathfrak{t}$ such that $\|\varphi_n - \varphi_0\|_{\mathfrak{t}-a} \to 0$. Hence, by (5.1.9),

$$\|\varphi_n - \varphi_0\| \to 0.$$

Thus $\varphi = \varphi_0 \in \operatorname{dom} \mathfrak{t}$. Therefore, $\|\varphi_n - \varphi\|_{\mathfrak{t}-a} \to 0$ and by (5.1.7) one sees that $\mathfrak{t}[\varphi_n - \varphi] \to 0$. This proves that $\mathfrak{t}$ is closed.

(ii) ⇐ (i) Assume that $\mathfrak{t}$ is closed. To show that $\mathfrak{H}_{\mathfrak{t}-a}$ is complete, let (φ_n) be a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. This implies that $\varphi_n \to_{\mathfrak{t}} \varphi$ for some $\varphi \in \mathfrak{H}$; cf. Corollary 5.1.5. The closedness of $\mathfrak{t}$ gives that $\varphi \in \operatorname{dom} \mathfrak{t} = \mathfrak{H}_{\mathfrak{t}-a}$ and $\mathfrak{t}[\varphi_n - \varphi] \to 0$. By (5.1.7) this leads to $\|\varphi_n - \varphi\|_{\mathfrak{t}-a} \to 0$, so that $\mathfrak{H}_{\mathfrak{t}-a}$ is complete.

Since $\mathfrak{t} - x$ is a semibounded form in $\mathfrak{H}$ with lower bound $\gamma - x$, the last statement follows from $\mathfrak{H}_{\mathfrak{t}-a} = \mathfrak{H}_{\mathfrak{t}-x-(a-x)}$ and the equivalence of (i) and (ii). □

Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $\mathfrak{H}_{\mathfrak{t}-a}$ be the corresponding inner product space with $a < \gamma$. In general $\mathfrak{t}$ is not closed and hence $\mathfrak{H}_{\mathfrak{t}-a}$ is not complete; cf. Lemma 5.1.9. If $\mathfrak{t}_1$ is a semibounded form with lower bound γ_1, which extends the semibounded form $\mathfrak{t}$ with lower bound γ, then

$$\gamma_1 \leq \gamma.$$

Note that for $a < \gamma_1$ one has that $\mathfrak{t}_1$ is closed if and only if $\mathfrak{H}_{\mathfrak{t}_1-a}$ is a Hilbert space. The question is when such a closed extension $\mathfrak{t}_1$ exists and, if so, to determine the smallest such extension of $\mathfrak{t}$. In order to construct an extension of $\mathfrak{t}$, note that Lemma 5.1.8 suggests the following definition.

Definition 5.1.10. Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$. The linear subspace $\operatorname{dom} \widetilde{\mathfrak{t}}$ is the set of all $\varphi \in \mathfrak{H}$ for which there exists a sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ such that $\varphi_n \to_{\mathfrak{t}} \varphi$.

It is clear that $\operatorname{dom} \widetilde{\mathfrak{t}}$ is an extension of $\operatorname{dom} \mathfrak{t}$. To establish the linearity of $\operatorname{dom} \widetilde{\mathfrak{t}}$, recall the property (5.1.12). According to (5.1.13), it would now be natural to define the form $\widetilde{\mathfrak{t}}$ on $\operatorname{dom} \widetilde{\mathfrak{t}}$ as an extension of $\mathfrak{t}$ by

$$\widetilde{\mathfrak{t}}[\varphi, \psi] = \lim_{n\to\infty} \mathfrak{t}[\varphi_n, \psi_n] \quad \text{for any} \quad \varphi_n \to_{\mathfrak{t}} \varphi, \quad \psi_n \to_{\mathfrak{t}} \psi, \tag{5.1.16}$$

as the limit on the right-hand side exists. However, in general the limit on the right-hand side of (5.1.16) depends on the choice of the sequences (φ_n) and (ψ_n), so that $\widetilde{\mathfrak{t}}$ may not be well defined as a form.

Definition 5.1.11. A semibounded form $\mathfrak{t}$ in $\mathfrak{H}$ is said to be *closable* if for any sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$

$$\varphi_n \to_{\mathfrak{t}} 0 \quad \Rightarrow \quad \mathfrak{t}[\varphi_n] \to 0.$$

It will be shown that the extension procedure in (5.1.16) defines a form extension of $\mathfrak{t}$ if $\mathfrak{t}$ is closable. In fact, in this case the resulting form $\widetilde{\mathfrak{t}}$ is unique, being the smallest closed extension, and will be called the *closure* of $\mathfrak{t}$.

Theorem 5.1.12. *Let $\mathfrak{t}$ be a semibounded form in $\mathfrak{H}$ with lower bound γ and let $a < \gamma$. Then $\mathfrak{t}$ has a closed extension if and only if $\mathfrak{t}$ is closable. In fact, if $\mathfrak{t}$ is closable, then*

(i) *the* closure $\widetilde{\mathfrak{t}}$ *in* (5.1.16) *is a well-defined form which extends $\mathfrak{t}$;*
(ii) *$\widetilde{\mathfrak{t}}$ has the same lower bound as $\mathfrak{t}$;*
(iii) *$\widetilde{\mathfrak{t}}$ is the smallest closed extension of $\mathfrak{t}$,*

and the inner product space $\mathfrak{H}_{\mathfrak{t}-a}$ is dense in the Hilbert space $\mathfrak{H}_{\widetilde{\mathfrak{t}}-a}$. Moreover, $\mathfrak{t}$ is closable if and only $\mathfrak{t} - x$ is closable for some, and hence for all $x \in \mathbb{R}$, in which case

$$\widetilde{\mathfrak{t} - x} = \widetilde{\mathfrak{t}} - x. \tag{5.1.17}$$

Proof. ($\Rightarrow$) Let $\mathfrak{t}_1$ be a closed extension of $\mathfrak{t}$. In order to show that $\mathfrak{t}$ is closable, assume that $\varphi_n \to_{\mathfrak{t}} 0$. The form $\mathfrak{t}_1$ is an extension of $\mathfrak{t}$ and this implies $\varphi_n \to_{\mathfrak{t}_1} 0$. Since $\mathfrak{t}_1$ is closed, it follows that

$$\mathfrak{t}[\varphi_n] = \mathfrak{t}_1[\varphi_n] \to 0.$$

Hence, $\mathfrak{t}$ is closable.

($\Leftarrow$) Assume that $\mathfrak{t}$ is closable. It will be shown that $\widetilde{\mathfrak{t}}$ in (5.1.16) is a well-defined form on $\operatorname{dom} \widetilde{\mathfrak{t}}$. It is clear from (5.1.13) that the limit on the right-hand side of (5.1.16) exists. To verify that this limit depends only on the elements φ, ψ and not on the particular sequences (φ_n), (ψ_n), let (φ_n'), (ψ_n') be other sequences such that $\varphi_n' \to_{\mathfrak{t}} \varphi$ and $\psi_n' \to_{\mathfrak{t}} \psi$. Then

$$\varphi_n' - \varphi_n \to_{\mathfrak{t}} 0 \quad \text{and} \quad \psi_n' - \psi_n \to_{\mathfrak{t}} 0;$$

cf. (5.1.12). In particular, this gives

$$\varphi_n' - \varphi_n \to 0 \quad \text{and} \quad \psi_n' - \psi_n \to 0,$$

while the closability of $\mathfrak{t}$ implies that

$$\mathfrak{t}[\varphi_n' - \varphi_n] \to 0 \quad \text{and} \quad \mathfrak{t}[\psi_n' - \psi_n] \to 0.$$

To see that the sequences $\mathfrak{t}[\varphi_n', \psi_n']$ and $\mathfrak{t}[\varphi_n, \psi_n]$ have the same limit, consider the inequalities

$$\begin{aligned}
&|(\mathfrak{t} - \gamma)[\varphi_n', \psi_n'] - (\mathfrak{t} - \gamma)[\varphi_n, \psi_n]| \\
&\quad = |(\mathfrak{t} - \gamma)[\varphi_n' - \varphi_n, \psi_n'] + (\mathfrak{t} - \gamma)[\varphi_n, \psi_n' - \psi_n]| \\
&\quad \leq |(\mathfrak{t} - \gamma)[\varphi_n' - \varphi_n, \psi_n']| + |(\mathfrak{t} - \gamma)[\varphi_n, \psi_n' - \psi_n]| \\
&\quad \leq (\mathfrak{t} - \gamma)[\varphi_n' - \varphi_n]^{\frac{1}{2}} (\mathfrak{t} - \gamma)[\psi_n']^{\frac{1}{2}} + (\mathfrak{t} - \gamma)[\varphi_n]^{\frac{1}{2}} (\mathfrak{t} - \gamma)[\psi_n' - \psi_n]^{\frac{1}{2}}.
\end{aligned}$$

Clearly, due to the closability assumption, the terms

$$(\mathfrak{t}-\gamma)[\varphi_n'-\varphi_n] \quad \text{and} \quad (\mathfrak{t}-\gamma)[\psi_n'-\psi_n]$$

converge to 0 as $n \to \infty$, while the terms

$$(\mathfrak{t}-\gamma)[\psi_n'] \quad \text{and} \quad (\mathfrak{t}-\gamma)[\varphi_n]$$

are bounded since $\psi_n' \to_{\mathfrak{t}} \psi$ and $\varphi_n \to_{\mathfrak{t}} \varphi$, respectively; cf. Corollary 5.1.6. It follows that $\mathfrak{t}[\varphi_n', \psi_n'] - \mathfrak{t}[\varphi_n, \psi_n] \to 0$ and hence $\widetilde{\mathfrak{t}}$ in (5.1.16) is a well-defined form. Moreover, it is clear that $\widetilde{\mathfrak{t}}$ extends $\mathfrak{t}$: $\mathfrak{t} \subset \widetilde{\mathfrak{t}}$.

The form $\widetilde{\mathfrak{t}}$ is semibounded. To see this, let $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}}$. Then there exists a sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ such that $\varphi_n \to_{\mathfrak{t}} \varphi$. In particular, $\varphi_n \to \varphi$ and hence $\|\varphi_n\| \to \|\varphi\|$. According to (5.1.16),

$$\widetilde{\mathfrak{t}}[\varphi] = \lim_{n\to\infty} \mathfrak{t}[\varphi_n],$$

where $\mathfrak{t}[\varphi_n] \geq \gamma\|\varphi_n\|^2$. Therefore,

$$\widetilde{\mathfrak{t}}[\varphi] \geq \gamma\|\varphi\|^2, \quad \varphi \in \operatorname{dom} \widetilde{\mathfrak{t}},$$

so that $\widetilde{\mathfrak{t}}$ is semibounded. Moreover, this argument shows that the lower bound of the extension is at least γ. Hence, $\widetilde{\mathfrak{t}}$ and $\mathfrak{t}$ have the same lower bound.

The argument to show that $\widetilde{\mathfrak{t}}$ is closed, is based on the observation that for the extension $\widetilde{\mathfrak{t}}$:

$$\varphi_n \to_{\mathfrak{t}} \varphi \quad \Rightarrow \quad \widetilde{\mathfrak{t}}[\varphi - \varphi_n] \to 0. \tag{5.1.18}$$

To see this, let $\varphi_n \to_{\mathfrak{t}} \varphi$, that is $\varphi_n \to \varphi$ and $\lim_{m,n\to\infty} \mathfrak{t}[\varphi_n - \varphi_m] = 0$. Now fix $n \in \mathbb{N}$, then $\varphi_m \to_{\mathfrak{t}} \varphi$ implies that

$$\varphi_m - \varphi_n \to_{\mathfrak{t}} \varphi - \varphi_n \quad \text{as} \quad m \to \infty,$$

so that, by definition,

$$\widetilde{\mathfrak{t}}[\varphi - \varphi_n] = \lim_{m\to\infty} \mathfrak{t}[\varphi_m - \varphi_n].$$

Now taking $n \to \infty$ gives (5.1.18).

The following three steps will establish that $\widetilde{\mathfrak{t}}$ is closed or, equivalently, that $\mathfrak{H}_{\widetilde{\mathfrak{t}}-a}$, $a < \gamma$, is complete.

Step 1. $\mathfrak{H}_{\mathfrak{t}-a}$ is dense in $\mathfrak{H}_{\widetilde{\mathfrak{t}}-a}$. Indeed, let $\varphi \in \mathfrak{H}_{\widetilde{\mathfrak{t}}-a} = \operatorname{dom} \widetilde{\mathfrak{t}}$. Then there is a sequence (φ_n) in $\mathfrak{H}_{\mathfrak{t}-a} = \operatorname{dom} \mathfrak{t}$ such that $\varphi_n \to_{\mathfrak{t}} \varphi$. It follows from this assumption and (5.1.18) that

$$\varphi_n \to \varphi \quad \text{and} \quad \widetilde{\mathfrak{t}}[\varphi - \varphi_n] \to 0,$$

in other words,

$$\|\varphi - \varphi_n\|^2_{\widetilde{\mathfrak{t}}-a} = \widetilde{\mathfrak{t}}[\varphi - \varphi_n] - a\|\varphi - \varphi_n\|^2 \to 0.$$

This shows that $\mathfrak{H}_{\mathfrak{t}-a}$ is dense in $\mathfrak{H}_{\widetilde{\mathfrak{t}}-a}$.

Step 2. Every Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$ is convergent in $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$. To see this, let (φ_n) be a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. Then clearly there exists an element $\varphi \in \mathfrak{H}$ such that $\varphi_n \to_{\mathfrak{t}} \varphi$; cf. Corollary 5.1.5. Again by (5.1.18) it follows that

$$\|\varphi - \varphi_n\|_{\tilde{\mathfrak{t}}-a} \to 0,$$

which now shows that the Cauchy sequence (φ_n) in $\mathfrak{H}_{\mathfrak{t}-a}$ is convergent in $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$ to, in fact, $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}} = \mathfrak{H}_{\tilde{\mathfrak{t}}-a}$.

Step 3. $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$ is a Hilbert space. To see this, let (χ_n) be a Cauchy sequence in $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$. By Step 1, there is an element $\varphi_n \in \mathfrak{H}_{\mathfrak{t}-a}$ such that

$$\|\chi_n - \varphi_n\|_{\tilde{\mathfrak{t}}-a} \leq \frac{1}{n}.$$

Hence, the approximating sequence (φ_n) is a Cauchy sequence in $\mathfrak{H}_{\mathfrak{t}-a}$. By Step 2, (φ_n) converges in $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$, which implies that the original sequence (χ_n) converges in $\mathfrak{H}_{\tilde{\mathfrak{t}}-a}$.

Next it will be shown that $\widetilde{\mathfrak{t}}$ is the smallest closed extension of $\mathfrak{t}$. Assume that $\mathfrak{t}_1$ is a closed extension of $\mathfrak{t}$: $\mathfrak{t} \subset \mathfrak{t}_1$. Let $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}}$; then there exists a sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ with $\varphi_n \to_{\mathfrak{t}} \varphi$. Then also $\varphi_n \to_{\mathfrak{t}_1} \varphi$ and hence $\varphi \in \operatorname{dom} \mathfrak{t}_1$. Therefore, $\operatorname{dom} \widetilde{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}_1$. For every $\varphi, \psi \in \operatorname{dom} \widetilde{\mathfrak{t}}$ it follows via corresponding sequences (φ_n), (ψ_n) in $\operatorname{dom} \mathfrak{t}$ with $\varphi_n \to_{\mathfrak{t}} \varphi$ and $\psi_n \to_{\mathfrak{t}} \psi$ that

$$\widetilde{\mathfrak{t}}[\varphi, \psi] = \lim_{n \to \infty} \mathfrak{t}[\varphi_n, \psi_n] = \lim_{n \to \infty} \mathfrak{t}_1[\varphi_n, \psi_n] = \mathfrak{t}_1[\varphi, \psi],$$

where the first equality follows from (5.1.16), the second equality is valid as $\mathfrak{t}_1$ extends $\mathfrak{t}$, and the third equality follows from (5.1.15). Therefore, $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_1$, and $\widetilde{\mathfrak{t}}$ is the smallest closed extension of $\mathfrak{t}$.

As to the last statement, observe that Definition 5.1.11 implies that $\mathfrak{t}$ is closable if and only $\mathfrak{t} - x$ is closable for some, and hence for all $x \in \mathbb{R}$. Finally, (5.1.17) follows from (5.1.16). □

Thus, a closed semibounded form $\mathfrak{t}_1$ which extends $\mathfrak{t}$ contains the closure $\widetilde{\mathfrak{t}}$. The next corollary is a simple but useful description of the gap between $\mathfrak{t}_1$ and $\widetilde{\mathfrak{t}}$.

Corollary 5.1.13. *Let the semibounded form* $\mathfrak{t}$ *with lower bound* γ *be closable and let the closed form* $\mathfrak{t}_1$ *with lower bound* γ_1 *be an extension of* $\mathfrak{t}$*, so that* $\gamma_1 \leq \gamma$*. Assume that* $a < \gamma_1$*, then*

$$\mathfrak{H}_{\mathfrak{t}_1-a} = \left\{ \varphi \in \mathfrak{H}_{\mathfrak{t}_1-a} : (\varphi, \psi)_{\mathfrak{t}_1-a} = 0, \psi \in \mathfrak{H}_{\tilde{\mathfrak{t}}-a} \right\} \oplus_{\mathfrak{t}_1-a} \mathfrak{H}_{\tilde{\mathfrak{t}}-a}.$$

Let $\mathfrak{t}$ be a closed semibounded form in $\mathfrak{H}$. Let $\mathfrak{D} \subset \operatorname{dom} \mathfrak{t}$ be a linear subspace and consider the restriction $\mathfrak{t}_{\mathfrak{D}}$ of $\mathfrak{t}$ to $\mathfrak{D}$,

$$\mathfrak{t}_{\mathfrak{D}}[\varphi, \psi] = \mathfrak{t}[\varphi, \psi], \quad \varphi, \psi \in \mathfrak{D}.$$

Since $\mathfrak{t}_{\mathfrak{D}}$ is a restriction of a closed form, it is closable, see Theorem 5.1.12. Let $\widetilde{\mathfrak{t}}_{\mathfrak{D}}$ be the closure of $\mathfrak{t}_{\mathfrak{D}}$. Then by definition $\operatorname{dom}\widetilde{\mathfrak{t}}_{\mathfrak{D}}$ is the set of all $\varphi \in \mathfrak{H}$ for which there exists a sequence (φ_n) in $\mathfrak{D}$ with $\varphi_n \to_{\mathfrak{t}_{\mathfrak{D}}} \varphi$, which means $\varphi_n \to_{\mathfrak{t}} \varphi$. Since $\mathfrak{t}$ is closed, one sees in particular that $\operatorname{dom}\widetilde{\mathfrak{t}}_{\mathfrak{D}} \subset \operatorname{dom}\mathfrak{t}$. Moreover, one has

$$\widetilde{\mathfrak{t}}_{\mathfrak{D}}[\varphi,\psi] = \lim_{n\to\infty} \mathfrak{t}_{\mathfrak{D}}[\varphi_n,\psi_n] = \lim_{n\to\infty} \mathfrak{t}[\varphi_n,\psi_n] = \mathfrak{t}[\varphi,\psi]$$

for $\varphi,\psi \in \operatorname{dom}\widetilde{\mathfrak{t}}_{\mathfrak{D}}$, where the first equality is by definition, and the third equality follows from Lemma 5.1.8. Hence, the closure $\widetilde{\mathfrak{t}}_{\mathfrak{D}}$ of $\mathfrak{t}_{\mathfrak{D}}$ is the restriction of $\mathfrak{t}$ to $\operatorname{dom}\widetilde{\mathfrak{t}}_{\mathfrak{D}}$. Since $\mathfrak{t}$ is closed, it follows that $\varphi \in \operatorname{dom}\widetilde{\mathfrak{t}}_{\mathfrak{D}}$ if and only if there is a sequence (φ_n) in $\mathfrak{D}$ such that

$$\varphi_n \to \varphi \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi] \to 0.$$

Definition 5.1.14. Let $\mathfrak{t}$ be a closed semibounded form in $\mathfrak{H}$. A linear subspace $\mathfrak{D}$ of $\operatorname{dom}\mathfrak{t}$ is said to be a *core* of $\mathfrak{t}$ if the closure $\widetilde{\mathfrak{t}}_{\mathfrak{D}}$ of the restriction $\mathfrak{t}_{\mathfrak{D}}$ of $\mathfrak{t}$ to $\mathfrak{D}$ coincides with $\mathfrak{t}$.

Therefore, $\mathfrak{D} \subset \operatorname{dom}\mathfrak{t}$ is a core of $\mathfrak{t}$ if and only if for every $\varphi \in \operatorname{dom}\mathfrak{t}$ there is a sequence (φ_n) in $\mathfrak{D}$ such that

$$\varphi_n \to \varphi \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi] \to 0. \tag{5.1.19}$$

This leads to the following corollary.

Corollary 5.1.15. *Let $\mathfrak{t}$ be a closed semibounded form in $\mathfrak{H}$ with lower bound γ, let $a < \gamma$, and let $\mathfrak{D} \subset \operatorname{dom}\mathfrak{t}$ be a linear subspace. Then $\mathfrak{D}$ is a core of $\mathfrak{t}$ if and only if $\mathfrak{D}$ is dense in the Hilbert space $\mathfrak{H}_{\mathfrak{t}-a}$.*

Note that in the situation of Theorem 5.1.12 the original domain $\operatorname{dom}\mathfrak{t}$ is a core of the closure $\widetilde{\mathfrak{t}}$ of $\mathfrak{t}$ (recall that the form $\widetilde{\mathfrak{t}}$ is closed). The following fact is useful: If $\mathfrak{t}$ and $\mathfrak{s}$ are closed semibounded forms in $\mathfrak{H}$ which coincide on $\mathfrak{D} \subset \operatorname{dom}\mathfrak{t} \cap \operatorname{dom}\mathfrak{s}$ and $\mathfrak{D}$ is a core of both $\mathfrak{t}$ and $\mathfrak{s}$, then $\mathfrak{t} = \mathfrak{s}$.

Recall the definition of the sum of two forms in Definition 5.1.1 and observe that a sum of semibounded forms is also semibounded. The following result is concerned with additive perturbations of forms: it provides a sufficient condition so that the sum of a closed semibounded form and a symmetric form remains closed and semibounded. Sometimes this result is referred to as KLMN theorem, named after Kato, Lions, Lax, Milgram, and Nelson. For a typical application to Sturm–Liouville operators, see, e.g., Lemma 6.8.3.

Theorem 5.1.16. *Assume that $\mathfrak{t}$ is a closed semibounded form in $\mathfrak{H}$ and let $\mathfrak{s}$ be a symmetric form in $\mathfrak{H}$ such that* $\operatorname{dom}\mathfrak{t} \subset \operatorname{dom}\mathfrak{s}$ *and*

$$|\mathfrak{s}[\varphi]| \le a\|\varphi\|^2 + b\,\mathfrak{t}[\varphi], \qquad \varphi \in \operatorname{dom}\mathfrak{t}, \tag{5.1.20}$$

holds for some $a \geq 0$ *and* $b \in [0,1)$*. Then the symmetric form*

$$\mathfrak{t} + \mathfrak{s}, \qquad \operatorname{dom}(\mathfrak{t} + \mathfrak{s}) = \operatorname{dom} \mathfrak{t},$$

is closed and semibounded in $\mathfrak{H}$*. Furthermore, if* $\mathfrak{D}$ *is a core of* $\mathfrak{t}$*, then* $\mathfrak{D}$ *is also a core of* $\mathfrak{t} + \mathfrak{s}$*.*

Proof. Let γ be the lower bound of $\mathfrak{t}$. Fix some $a' < \gamma$ and assume $a' < 0$. For all $\varphi \in \operatorname{dom} \mathfrak{t}$, $\varphi \neq 0$, one obtains from (5.1.20) that

$$\mathfrak{s}[\varphi] \geq -a\|\varphi\|^2 - b\mathfrak{t}[\varphi], \tag{5.1.21}$$

and hence

$$(\mathfrak{t} + \mathfrak{s})[\varphi] \geq (1-b)\mathfrak{t}[\varphi] - a\|\varphi\|^2 > \big((1-b)a' - a\big)\|\varphi\|^2 = c'\|\varphi\|^2,$$

where $c' = (1-b)a' - a < 0$. This shows that $\mathfrak{t} + \mathfrak{s}$ is semibounded from below. Furthermore, the estimate (5.1.21) also shows that

$$\begin{aligned}
(1-b)\|\varphi\|^2_{\mathfrak{t}-a'} &= (1-b)\mathfrak{t}[\varphi] - (1-b)a'\|\varphi\|^2 \\
&= \mathfrak{t}[\varphi] - b\mathfrak{t}[\varphi] - a\|\varphi\|^2 - \big((1-b)a' - a\big)\|\varphi\|^2 \\
&\leq (\mathfrak{t} + \mathfrak{s})[\varphi] - \big((1-b)a' - a\big)\|\varphi\|^2 \\
&= \|\varphi\|^2_{\mathfrak{t}+\mathfrak{s}-c'}.
\end{aligned}$$

Using (5.1.20) one obtains

$$\begin{aligned}
\|\varphi\|^2_{\mathfrak{t}+\mathfrak{s}-c'} &= \mathfrak{t}[\varphi] + \mathfrak{s}[\varphi] - c'\|\varphi\|^2 \\
&\leq (1+b)\mathfrak{t}[\varphi] - (c'-a)\|\varphi\|^2 \\
&\leq b'\|\varphi\|^2_{\mathfrak{t}-a'},
\end{aligned}$$

where $b' = \max\{(1+b), (c'-a)/a'\}$. Therefore, the above estimates imply that the norms $\|\cdot\|^2_{\mathfrak{t}-a'}$ and $\|\cdot\|^2_{\mathfrak{t}+\mathfrak{s}-c'}$ are equivalent on $\operatorname{dom} \mathfrak{t} = \operatorname{dom}(\mathfrak{s} + \mathfrak{t})$. Since $\mathfrak{t}$ is closed, $\mathfrak{H}_{\mathfrak{t}-a'}$ is a Hilbert space and hence $\mathfrak{H}_{\mathfrak{t}+\mathfrak{s}-c'}$ is a Hilbert space, that is, the form $\mathfrak{t}+\mathfrak{s}$ is closed; cf. Lemma 5.1.9. The assertion about the core $\mathfrak{D}$ is clear from Corollary 5.1.15. □

Semibounded relations in a Hilbert space generate closable semibounded forms as will be shown in the following lemma. Note that if a relation is semibounded, then so is its closure, with the same lower bound; this follows directly from Definition 1.4.5. Furthermore, the closure will generate the same form. The particular situation of semibounded self-adjoint relations will be considered in detail in Theorem 5.1.18 and Proposition 5.1.19.

Lemma 5.1.17. *Let* S *be a semibounded relation in* $\mathfrak{H}$ *with lower bound* $m(S)$*. Then the form* $\mathfrak{t}_S$ *given by*

$$\mathfrak{t}_S[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in S, \tag{5.1.22}$$

with $\operatorname{dom} \mathfrak{t}_S = \operatorname{dom} S$, *is well defined, semibounded with the lower bound* $m(S)$, *and closable. The closure* $\widetilde{\mathfrak{t}}_S$ *of* $\mathfrak{t}_S$ *is a semibounded closed form whose lower bound is equal to* $m(S)$, *and*

$$\operatorname{dom} \widetilde{\mathfrak{t}}_S \subset \overline{\operatorname{dom} S}. \tag{5.1.23}$$

Moreover, $\operatorname{dom} S = \operatorname{dom} \mathfrak{t}_S$ *is a core of* $\widetilde{\mathfrak{t}}_S$. *Furthermore, with the closure* $\overline{S}$ *of* S *one has*

$$\widetilde{\mathfrak{t}}_S = \widetilde{\mathfrak{t}}_{\overline{S}}. \tag{5.1.24}$$

Proof. As a semibounded relation S is automatically symmetric, it follows that $\operatorname{mul} S \subset \operatorname{mul} S^* = (\operatorname{dom} S)^\perp$, and hence

$$(\varphi', \psi) = (\varphi'', \psi), \quad \{\varphi, \varphi'\}, \{\varphi, \varphi''\}, \{\psi, \psi'\} \in S.$$

Thus, the form in (5.1.22) is well defined with $\operatorname{dom} \mathfrak{t}_S = \operatorname{dom} S$. By definition $\mathfrak{t}_S$ is semibounded and its lower bound is clearly equal to $\gamma = m(S)$.

In order to show that $\mathfrak{t}_S$ is closable, let $\varphi_n \to_{\mathfrak{t}_S} 0$. Then, equivalently,

$$\varphi_n \to 0 \quad \text{and} \quad (\mathfrak{t}_S - \gamma)[\varphi_n - \varphi_m] \to 0.$$

It suffices to verify that $(\mathfrak{t}_S - \gamma)[\varphi_n] \to 0$. Note that there exists $\varphi_n' \in \mathfrak{H}$ such that $\{\varphi_n, \varphi_n'\} \in S$. Then

$$(\mathfrak{t}_S - \gamma)[\varphi_n] = (\mathfrak{t}_S - \gamma)[\varphi_n, \varphi_n] = (\mathfrak{t}_S - \gamma)[\varphi_n, \varphi_n - \varphi_m] + (\mathfrak{t}_S - \gamma)[\varphi_n, \varphi_m],$$

and it follows with the help of the Cauchy–Schwarz inequality (5.1.3) for the nonnegative form $(\mathfrak{t}_S - \gamma)$ and (5.1.22) that

$$\begin{aligned} |(\mathfrak{t}_S - \gamma)[\varphi_n]| &\le |(\mathfrak{t}_S - \gamma)[\varphi_n, \varphi_n - \varphi_m]| + |(\mathfrak{t}_S - \gamma)[\varphi_n, \varphi_m]| \\ &\le (\mathfrak{t}_S - \gamma)[\varphi_n]^{\frac{1}{2}} (\mathfrak{t}_S - \gamma)[\varphi_n - \varphi_m]^{\frac{1}{2}} + |(\varphi_n' - \gamma\varphi_n, \varphi_m)|. \end{aligned}$$

By Corollary 5.1.6, the sequence $((\mathfrak{t}_S - \gamma)[\varphi_n])$ is bounded by M^2 for some $M > 0$. Moreover, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $(\mathfrak{t}_S - \gamma)[\varphi_n - \varphi_m] \le \varepsilon^2$ for $n, m \ge N$. Therefore,

$$|(\mathfrak{t}_S - \gamma)[\varphi_n]| \le M\varepsilon + |(\varphi_n' - \gamma\varphi_n, \varphi_m)|, \qquad n, m \ge N.$$

Fix $n \ge N$ and let $m \to \infty$. From $|(\varphi_n' - \gamma\varphi_n, \varphi_m)| \le \|\varphi_n' - \gamma\varphi_n\| \|\varphi_m\|$ and $\|\varphi_m\| \to 0$ it follows that $|(\mathfrak{t}_S - \gamma)[\varphi_n]| \le M\varepsilon$ for $n \ge N$. This shows that $(\mathfrak{t}_S - \gamma)[\varphi_n] \to 0$ as $n \to \infty$, and hence $\mathfrak{t}_S$ is closable.

By Theorem 5.1.12, it is clear that the closure $\widetilde{\mathfrak{t}}_S$ of $\mathfrak{t}_S$ is a semibounded closed form whose lower bound is equal to $m(S)$. It also follows from the definition of $\widetilde{\mathfrak{t}}$ that the inclusion (5.1.23) holds. Furthermore, $\operatorname{dom} S = \operatorname{dom} \mathfrak{t}_S$ is a core of $\widetilde{\mathfrak{t}}_S$.

It remains to show (5.1.24). The inclusion $\widetilde{\mathfrak{t}}_S \subset \widetilde{\mathfrak{t}}_{\overline{S}}$ is clear. For the opposite inclusion, let $\varphi \in \operatorname{dom} \mathfrak{t}_{\overline{S}} = \operatorname{dom} \overline{S}$ and $\varphi' \in \mathfrak{H}$ such that $\{\varphi, \varphi'\} \in \overline{S}$, in which case

$$\mathfrak{t}_{\overline{S}}[\varphi, \varphi] = (\varphi', \varphi).$$

Then there exists a sequence $(\{\varphi_n, \varphi_n'\})$ in S with $\varphi_n \to \varphi$ and $\varphi_n' \to \varphi'$, and hence

$$\mathfrak{t}_S[\varphi_n - \varphi_m] = (\varphi_n' - \varphi_m', \varphi_n - \varphi_m) \to 0.$$

Therefore, $\varphi_n \to_{\mathfrak{t}_S} \varphi$, so that $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}}_S$. Moreover,

$$\mathfrak{t}_{\overline{S}}[\varphi, \varphi] = (\varphi', \varphi) = \lim_{n\to\infty} (\varphi_n', \varphi_n) = \lim_{n\to\infty} \mathfrak{t}_S[\varphi_n, \varphi_n] = \widetilde{\mathfrak{t}}_S[\varphi, \varphi],$$

where in the last equality the definition of the closure in Theorem 5.1.12 was used. This implies $\mathfrak{t}_{\overline{S}} \subset \widetilde{\mathfrak{t}}_S$ and hence $\widetilde{\mathfrak{t}}_{\overline{S}} \subset \widetilde{\mathfrak{t}}_S$. Therefore, $\widetilde{\mathfrak{t}}_S = \widetilde{\mathfrak{t}}_{\overline{S}}$. □

In the next theorem it is shown that every closed semibounded form can be represented by a semibounded self-adjoint relation.

Theorem 5.1.18 (First representation theorem)**.** *Assume that* $\mathfrak{t}$ *is a closed semibounded form in* $\mathfrak{H}$*. Then there exists a semibounded self-adjoint relation* H *in* $\mathfrak{H}$ *such that the following statements hold:*

(i) $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ *and*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \tag{5.1.25}$$

for every $\{\varphi, \varphi'\} \in H$ *and* $\psi \in \operatorname{dom} \mathfrak{t}$*;*

(ii) $\operatorname{dom} H$ *is a core of* $\mathfrak{t}$*;*

(iii) *if* $\varphi \in \operatorname{dom} \mathfrak{t}$*,* $\varphi' \in \mathfrak{H}$*, and*

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) \tag{5.1.26}$$

for every ψ *in a core of* $\mathfrak{t}$*, then* $\{\varphi, \varphi'\} \in H$*;*

(iv) $\operatorname{mul} H = (\operatorname{dom} \mathfrak{t})^\perp$ *and*

$$\mathfrak{t}[\varphi, \psi] = (H_{\rm op}\, \varphi, \psi) \tag{5.1.27}$$

for every $\varphi \in \operatorname{dom} H$ *and* $\psi \in \operatorname{dom} \mathfrak{t}$.

The semibounded self-adjoint relation H *is uniquely determined by* (i)*. The closed form* $\mathfrak{t}$ *and the corresponding semibounded self-adjoint relation* H *have the same lower bound:* $m(\mathfrak{t}) = m(H)$*. Moreover, for each* $x \in \mathbb{R}$ *the closed semibounded form* $\mathfrak{t} - x$ *corresponds to the semibounded self-adjoint relation* $H - x$*.*

Proof. (i) Let $m(\mathfrak{t}) = \gamma$ and choose $a < \gamma$. Then the assumption that $\mathfrak{t}$ is closed is equivalent to the inner product space $\mathfrak{H}_{\mathfrak{t}-a}$ being complete, where $\mathfrak{H}_{\mathfrak{t}-a} = \operatorname{dom} \mathfrak{t}$ is equipped with the inner product of $(\cdot,\cdot)_{\mathfrak{t}-a}$ as in (5.1.7)–(5.1.8); cf. Lemma 5.1.9. For any fixed $\omega \in \mathfrak{H}$ consider the linear functional

$$\psi \mapsto (\psi, \omega)$$

defined for all $\psi \in \mathfrak{H}_{\mathfrak{t}-a} = \operatorname{dom} \mathfrak{t} \subset \mathfrak{H}$. It follows from (5.1.9) that

$$|(\psi, \omega)| \leq \|\psi\|\|\omega\| \leq \left(\frac{1}{\sqrt{\gamma - a}}\|\omega\|\right) \|\psi\|_{\mathfrak{t}-a}, \quad \psi \in \mathfrak{H}_{\mathfrak{t}-a}.$$

Hence, the mapping $\psi \mapsto (\psi, \omega)$ from $\mathfrak{H}_{\mathfrak{t}-a}$ to $\mathbb{C}$ is bounded with bound at most $\|\omega\|/\sqrt{\gamma - a}$. Therefore, by the Riesz representation theorem, there exists an element $\widehat{\omega}$ in $\mathfrak{H}_{\mathfrak{t}-a}$ such that for all $\psi \in \mathfrak{H}_{\mathfrak{t}-a}$:

$$(\psi, \omega) = (\psi, \widehat{\omega})_{\mathfrak{t}-a}, \quad \|\widehat{\omega}\|_{\mathfrak{t}-a} \leq \frac{1}{\sqrt{\gamma - a}} \|\omega\|.$$

Taking conjugates for convenience, it follows from the definition (5.1.7) of $(\cdot, \cdot)_{\mathfrak{t}-a}$ that

$$(\omega, \psi) = (\widehat{\omega}, \psi)_{\mathfrak{t}-a} = \mathfrak{t}[\widehat{\omega}, \psi] - a(\widehat{\omega}, \psi), \tag{5.1.28}$$

or, in other words,

$$\mathfrak{t}[\widehat{\omega}, \psi] = (\omega + a\widehat{\omega}, \psi), \quad \psi \in \mathfrak{H}_{\mathfrak{t}-a}. \tag{5.1.29}$$

Note that the linear mapping A from $\mathfrak{H}$ to $\mathfrak{H}_{\mathfrak{t}-a}$ defined by $A\omega = \widehat{\omega}$ satisfies

$$\sqrt{\gamma - a}\, \|A\omega\| \leq \|A\omega\|_{\mathfrak{t}-a} \leq \frac{1}{\sqrt{\gamma - a}} \|\omega\|;$$

where in the first inequality (5.1.9) was used. In other words, if A is interpreted as a mapping from $\mathfrak{H}$ to $\mathfrak{H}$, then

$$\|A\omega\| \leq \frac{1}{\gamma - a} \|\omega\|.$$

By means of A define the linear relation H in $\mathfrak{H}$ by

$$H = \bigl\{\{A\omega, \omega + aA\omega\} : \omega \in \mathfrak{H}\bigr\},$$

so that

$$A = (H - a)^{-1}.$$

One sees that $\operatorname{dom} H = \operatorname{ran} A \subset \operatorname{dom} \mathfrak{t}$ and $\operatorname{mul} H = \ker A$. Moreover, every element $\{\varphi, \varphi'\} \in H$ can be written as $\{\varphi, \varphi'\} = \{\widehat{\omega}, \omega + a\widehat{\omega}\}$ for some ω, so that by the identity (5.1.29) one obtains

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\} \in H, \qquad \psi \in \mathfrak{H}_{\mathfrak{t}-a} = \operatorname{dom} \mathfrak{t}. \tag{5.1.30}$$

It follows from (5.1.30) with $\psi = \varphi$ that H is a semibounded relation with lower bound

$$m(H) \geq m(\mathfrak{t}) = \gamma. \tag{5.1.31}$$

It is clear that H is symmetric. According to the definition of H one sees that $\operatorname{ran}(H - a) = \mathfrak{H}$, which, since $a < \gamma$, implies that H is self-adjoint; cf. Proposition 1.5.6. Thus, (i) has been proved.

(ii) The statement that $\operatorname{dom} H$ is a core of $\mathfrak{t}$ is equivalent to the statement that $\operatorname{dom} H$ is dense in the Hilbert space $\mathfrak{H}_{\mathfrak{t}-a}$. To verify denseness, assume that the element $\psi \in \mathfrak{H}_{\mathfrak{t}-a}$ is orthogonal to $\operatorname{dom} H = \operatorname{ran} A$, i.e.,

$$0 = (A\omega, \psi)_{\mathfrak{t}-a} = (\widehat{\omega}, \psi)_{\mathfrak{t}-a} = (\omega, \psi),$$

for all $\omega \in \mathfrak{H}$; cf. (5.1.28). This leads to $\psi = 0$. Hence, $\operatorname{dom} H = \operatorname{ran} A$ is dense in the Hilbert space $\mathfrak{H}_{\mathfrak{t}-a}$, and the assertion follows from Corollary 5.1.15.

(iii) Let $\varphi \in \operatorname{dom} \mathfrak{t}$ and $\varphi' \in \mathfrak{H}$ satisfy (5.1.26) for every ψ in a core $\mathfrak{D}$ of the form $\mathfrak{t}$. Then (5.1.26) holds for all $\psi \in \operatorname{dom} \mathfrak{t}$. To see this, let $\psi \in \operatorname{dom} \mathfrak{t}$. Then there exists a sequence (ψ_n) in $\mathfrak{D}$ such that $\psi_n \to_{\mathfrak{t}} \psi$, which implies that $\mathfrak{t}[\psi_n - \psi] \to 0$. Since $\psi_n \in \mathfrak{D}$, the assumption yields

$$\mathfrak{t}[\varphi, \psi] = \lim_{n\to\infty} \mathfrak{t}[\varphi, \psi_n] = \lim_{n\to\infty} (\varphi', \psi_n) = (\varphi', \psi), \quad \psi \in \operatorname{dom} \mathfrak{t},$$

so that (5.1.26) holds for all $\psi \in \operatorname{dom} \mathfrak{t}$. Due to the symmetry of $\mathfrak{t}$ this result may also be written as

$$\mathfrak{t}[\psi, \varphi] = (\psi, \varphi'), \qquad \psi \in \operatorname{dom} \mathfrak{t}. \tag{5.1.32}$$

Now let $\{\psi, \psi'\} \in H$. Then $\psi \in \operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ and, by (i),

$$\mathfrak{t}[\psi, \varphi] = (\psi', \varphi), \tag{5.1.33}$$

because $\varphi \in \operatorname{dom} \mathfrak{t}$. Comparing (5.1.32) and (5.1.33) gives

$$(\psi, \varphi') = (\psi', \varphi) \quad \text{for all} \quad \{\psi, \psi'\} \in H,$$

which leads to $\{\varphi, \varphi'\} \in H^* = H$. This proves (iii).

(iv) It follows from (i) that if $\{0, \varphi'\} \in H$, then $(\varphi', \psi) = 0$ for all $\psi \in \operatorname{dom} \mathfrak{t}$, and hence $\operatorname{mul} H \subset (\operatorname{dom} \mathfrak{t})^\perp$. Conversely, as $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}$ by (i) and H is self-adjoint, $(\operatorname{dom} \mathfrak{t})^\perp \subset (\operatorname{dom} H)^\perp = \operatorname{mul} H$. This shows that $\operatorname{mul} H = (\operatorname{dom} \mathfrak{t})^\perp$.

To see (5.1.27), let $\{\varphi, \varphi'\} \in H$. Then $\varphi' = H_{\rm op}\, \varphi + \chi$, where $\chi \in \operatorname{mul} H$. Hence, from (5.1.25) one obtains

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi) = (H_{\rm op}\, \varphi + \chi, \psi) = (H_{\rm op}\, \varphi, \psi),$$

which gives (5.1.27). This completes the proof of (iv).

To show uniqueness, assume that H' is a semibounded self-adjoint relation in $\mathfrak{H}$ such that $\operatorname{dom} H' \subset \operatorname{dom} \mathfrak{t}$ and

$$\mathfrak{t}[\varphi, \psi] = (\varphi', \psi)$$

for every $\{\varphi, \varphi'\} \in H'$ and $\psi \in \operatorname{dom} \mathfrak{t}$. Then, in particular, one concludes that $\varphi \in \operatorname{dom} H' \subset \operatorname{dom} \mathfrak{t}$ and $\varphi' \in \mathfrak{H}$, so that by (iii) it follows that $\{\varphi, \varphi'\} \in H$. Hence, $H' \subset H$ and one obtains equality as H' and H are both self-adjoint.

Recall that it has been shown in the proof of (i) that $m(H) \geq m(\mathfrak{t})$; cf. (5.1.31). The equality follows from the fact that $\operatorname{dom} H$ is a core of $\mathfrak{t}$; see (ii). In fact, if $\varphi \in \operatorname{dom} \mathfrak{t}$, then there exists a sequence (φ_n) in $\operatorname{dom} H$ such that $\varphi_n \to_{\mathfrak{t}} \varphi$. Therefore, if $\varphi \neq 0$, then

$$\frac{\mathfrak{t}[\varphi]}{\|\varphi\|^2} = \lim_{n\to\infty} \frac{\mathfrak{t}[\varphi_n]}{\|\varphi_n\|^2} = \lim_{n\to\infty} \frac{(H_{\rm op}\, \varphi_n, \varphi_n)}{\|\varphi_n\|^2} \geq m(H).$$

Since this inequality holds for every nontrivial $\varphi \in \operatorname{dom} \mathfrak{t}$ one concludes that

$$m(\mathfrak{t}) = \inf \left\{ \frac{\mathfrak{t}[\varphi]}{\|\varphi\|^2} : \varphi \in \operatorname{dom} \mathfrak{t},\, \varphi \neq 0 \right\} \geq m(H),$$

and so $m(\mathfrak{t}) = m(H)$.

Finally, note that for $x \in \mathbb{R}$ the form $\mathfrak{t} - x$ is semibounded and closed, and the relation $H - x$ is semibounded and self-adjoint. For $\{\varphi, \varphi'\} \in H$,

$$(\mathfrak{t} - x)[\varphi, \psi] = (\varphi', \psi) - x(\varphi, \psi) = (\varphi' - x\varphi, \psi) \tag{5.1.34}$$

for all $\psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} \mathfrak{t} - x$. Observe from (iii) that $\{\varphi, \varphi' - x\varphi\} \in H - x$ belongs to the semibounded self-adjoint relation corresponding to $\mathfrak{t} - x$. As $H - x$ is self-adjoint and contained in the semibounded self-adjoint relation corresponding to $\mathfrak{t} - x$ both coincide, i.e., $H - x$ corresponds to the closed semibounded form $\mathfrak{t} - x$. □

The representation result in Theorem 5.1.18 gives assertions concerning the semibounded self-adjoint relation associated with a given semibounded form. In fact, every semibounded self-adjoint relation appears in such a context, as is shown in the following proposition; cf. Lemma 5.1.17.

Proposition 5.1.19. *Let A be a semibounded self-adjoint relation in $\mathfrak{H}$. Then the semibounded, closable form defined by*

$$\mathfrak{t}_A[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in A,$$

has a closure whose corresponding semibounded self-adjoint relation is given by A.

Proof. Since A is semibounded and self-adjoint, Lemma 5.1.17 shows that the form $\mathfrak{t}_A$ is well defined, semibounded, and closable. Moreover, $\operatorname{dom} A$ is a core of its closure $\widetilde{\mathfrak{t}}$. Let H be the semibounded self-adjoint relation corresponding to $\widetilde{\mathfrak{t}}$. Since $\widetilde{\mathfrak{t}}$ is an extension of $\mathfrak{t}$, one has

$$\widetilde{\mathfrak{t}}[\varphi, \psi] = \mathfrak{t}[\varphi, \psi] = (\varphi', \psi), \quad \{\varphi, \varphi'\}, \{\psi, \psi'\} \in A.$$

Therefore, Theorem 5.1.18 (iii) implies that $\{\varphi, \varphi'\} \in H$, since $\operatorname{dom} A$ is a core of $\widetilde{\mathfrak{t}}$. Consequently, $A \subset H$ and since A and H are both self-adjoint, one concludes $A = H$. □

The following observation, based on Theorem 5.1.18 and Proposition 5.1.19, is included for completeness.

Corollary 5.1.20. *There is a one-to-one correspondence between all closed semibounded forms and all closed semibounded self-adjoint relations via the identity* (5.1.25) *or, equivalently, via the identity* (5.1.27) *in the first representation theorem.*

The correspondence between closed semibounded forms and semibounded self-adjoint relations in Theorem 5.1.18 can be illuminated further in the context of nonnegative forms and nonnegative self-adjoint relations. As a preparation, observe that a typical way to define forms is via linear operators.

Lemma 5.1.21. *Let T be a linear operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and define a nonnegative form $\mathfrak{t}$ in $\mathfrak{H}$ by*

$$\mathfrak{t}[\varphi, \psi] = (T\varphi, T\psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T.$$

Then

$$\mathfrak{t} \text{ is a closable form} \quad \Leftrightarrow \quad T \text{ is a closable operator},$$

and in this case the closure of $\mathfrak{t}$ is given by

$$\widetilde{\mathfrak{t}}[\varphi, \psi] = (\overline{T}\varphi, \overline{T}\psi), \quad \varphi, \psi \in \operatorname{dom} \widetilde{\mathfrak{t}} = \operatorname{dom} \overline{T}. \tag{5.1.35}$$

Proof. ($\Rightarrow$) Assume that $\mathfrak{t}$ is closable. Let (φ_n) be a sequence in $\operatorname{dom} T$ such that $\varphi_n \to 0$ in $\mathfrak{H}$ and $T\varphi_n \to \psi$ in $\mathfrak{K}$. Then

$$\mathfrak{t}[\varphi_n - \varphi_m] = \|T(\varphi_n - \varphi_m)\|^2 \to 0,$$

which implies that $\varphi_n \to_{\mathfrak{t}} 0$. Since $\mathfrak{t}$ is closable, one obtains

$$\|T\varphi_n\|^2 = \mathfrak{t}[\varphi_n] \to 0,$$

so that $T\varphi_n \to 0$. It follows that T is closable.

($\Leftarrow$) Assume that T is closable. Let (φ_n) in $\operatorname{dom} \mathfrak{t}$ with $\varphi_n \to_{\mathfrak{t}} 0$. Then $\varphi_n \to 0$ in $\mathfrak{H}$ and $(T\varphi_n)$ is a Cauchy sequence in $\mathfrak{K}$. Hence, $T\varphi_n \to \psi$ for some $\psi \in \mathfrak{K}$ and since T is closable one sees that $\psi = 0$. Therefore, $\mathfrak{t}[\varphi_n] = \|T\varphi_n\|^2 \to 0$. It follows that $\mathfrak{t}$ is closable.

Finally, assume that $\mathfrak{t}$ or, equivalently, T is closable. Then one has

$$\operatorname{dom} \widetilde{\mathfrak{t}} = \operatorname{dom} \overline{T}. \tag{5.1.36}$$

Indeed, for the inclusion ($\subset$) in (5.1.36) consider $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}}$. Then there exists a sequence (φ_n) in $\operatorname{dom} \mathfrak{t}$ with $\varphi_n \to_{\mathfrak{t}} \varphi$; cf. Theorem 5.1.12. Hence, $\varphi_n \to \varphi$ in $\mathfrak{H}$ and $(T\varphi_n)$ is a Cauchy sequence in $\mathfrak{K}$. Thus, there exists $\varphi' \in \mathfrak{K}$ such that $T\varphi_n \to \varphi'$. Since T is closable it follows that $\varphi \in \operatorname{dom} \overline{T}$ and $\varphi' = \overline{T}\varphi$. Moreover, by Theorem 5.1.12 and (5.1.16) it follows that

$$\widetilde{\mathfrak{t}}[\varphi, \varphi] = \lim_{n\to\infty} \mathfrak{t}[\varphi_n, \varphi_n] = \lim_{n\to\infty} (T\varphi_n, T\varphi_n) = (\overline{T}\varphi, \overline{T}\varphi),$$

and polarization leads to the identity in (5.1.35). For the inclusion ($\supset$) in (5.1.36) let $\varphi \in \operatorname{dom} \overline{T}$. Then $\overline{T}\varphi = \varphi'$ for some $\varphi' \in \mathfrak{K}$, and there exists a sequence (φ_n) in $\operatorname{dom} T$ for which $\varphi_n \to \varphi$ while $T\varphi_n \to \varphi'$. In particular, it follows that $\varphi_n \to_{\mathfrak{t}} \varphi$. Therefore, $\varphi \in \operatorname{dom} \widetilde{\mathfrak{t}}$; this proves (5.1.36). □

The following result specializes the first representation theorem to closed nonnegative forms as in Lemma 5.1.21. For a class of closed nonnegative forms it identifies the associated self-adjoint relations. Recall that for a closed operator R a linear subspace $\mathfrak{D} \subset \operatorname{dom} R$ is a *core* if the closure of the restriction $R \upharpoonright_{\mathfrak{D}}$ of R to $\mathfrak{D}$ coincides with R; cf. Lemma 1.5.10.

Proposition 5.1.22. *Let T be a closed relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and let $T_{\rm op} = PT$ be the closed orthogonal operator part of T, where P is the orthogonal projection in $\mathfrak{K}$ onto $(\operatorname{mul} T)^\perp$; cf. Theorem 1.3.15. Then the rule*

$$\mathfrak{t}[\varphi,\psi] = (T_{\rm op}\,\varphi, T_{\rm op}\,\psi), \quad \varphi,\psi \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} T_{\rm op} = \operatorname{dom} T, \tag{5.1.37}$$

defines a closed nonnegative form $\mathfrak{t}$ in $\mathfrak{H}$. The nonnegative self-adjoint relation corresponding to the form $\mathfrak{t}$ is given by T^*T. Moreover, a subset of $\operatorname{dom}\mathfrak{t} = \operatorname{dom} T$ is a core of the form $\mathfrak{t}$ if and only if it is a core of the operator $T_{\rm op}$.

Proof. Since the operator $T_{\rm op}$ is closed, the nonnegative form $\mathfrak{t}$ in (5.1.37) is closed, with $\operatorname{dom}\mathfrak{t} = \operatorname{dom} T_{\rm op} = \operatorname{dom} T$; cf. Lemma 5.1.21. Recall that T^*T is a nonnegative self-adjoint relation in $\mathfrak{H}$; cf. Lemma 1.5.8. Assume that $\varphi \in \operatorname{dom} T^*T$ and $\psi \in \operatorname{dom} T$. Let $\varphi' \in \mathfrak{H}$ be any element such that $\{\varphi,\varphi'\} \in T^*T$. This implies that $\{\varphi,\eta\} \in T$ and $\{\eta,\varphi'\} \in T^*$ for some $\eta \in \mathfrak{K}$. Clearly, $\eta = T_{\rm op}\,\varphi + \omega$ for some $\omega \in \operatorname{mul} T$. Since $\{T_{\rm op}\,\varphi + \omega, \varphi'\} \in T^*$ and $\{\psi, T_{\rm op}\,\psi\} \in T$, one sees that

$$0 = (\varphi',\psi) - (T_{\rm op}\,\varphi + \omega, T_{\rm op}\,\psi) = (\varphi',\psi) - (T_{\rm op}\,\varphi, T_{\rm op}\,\psi), \quad \psi \in \operatorname{dom} T,$$

i.e.,

$$\mathfrak{t}[\varphi,\psi] = (\varphi',\psi), \quad \{\varphi,\varphi'\} \in T^*T, \quad \psi \in \operatorname{dom} T.$$

Let H be the nonnegative self-adjoint relation associated with $\mathfrak{t}$ via Theorem 5.1.18. According to (iii) of Theorem 5.1.18, the nonnegative self-adjoint relation T^*T satisfies $T^*T \subset H$, which gives $T^*T = H$.

Now let $\mathfrak{D} \subset \operatorname{dom}\mathfrak{t} = \operatorname{dom} T$ be a linear subset. Then $\mathfrak{D}$ is a core of $\mathfrak{t}$ if and only if for every $\varphi \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} T$ there is a sequence (φ_n) in $\mathfrak{D}$ such that

$$\varphi_n \to \varphi \quad \text{and} \quad \mathfrak{t}[\varphi_n - \varphi] \to 0;$$

cf. (5.1.19). In view of the definition of $\mathfrak{t}$, this condition reads as

$$\varphi_n \to \varphi \quad \text{and} \quad T_{\rm op}\,\varphi_n \to T_{\rm op}\,\varphi,$$

in other words $\mathfrak{D}$ is a core of $T_{\rm op}$. □

The so-called second representation theorem may be seen as a corollary of Theorem 5.1.18 and Proposition 5.1.22.

Theorem 5.1.23 (Second representation theorem). *Assume that the closed semibounded form* $\mathfrak{t}$ *and the semibounded self-adjoint relation* H *are connected as in Theorem 5.1.18, so that* $m(H) = m(\mathfrak{t}) = \gamma$*, and let* $x \le \gamma$*. Then*

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} (H - x)^{\frac{1}{2}}$$

and the form $\mathfrak{t}$ *is represented by*

$$\mathfrak{t}[\varphi, \psi] = \big((H_{\rm op} - x)^{\frac{1}{2}}\varphi, (H_{\rm op} - x)^{\frac{1}{2}}\psi\big) + x(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}.$$

Moreover, a subset of $\operatorname{dom} \mathfrak{t} = \operatorname{dom} (H - x)^{\frac{1}{2}}$ *is a core of the form* $\mathfrak{t}$ *if and only if it is a core of the operator* $(H_{\rm op} - x)^{\frac{1}{2}}$*.*

Proof. For $x \le \gamma$ define the form $\mathfrak{s}_x$ by

$$\mathfrak{s}_x[\varphi, \psi] = \big((H_{\rm op} - x)^{\frac{1}{2}}\varphi, (H_{\rm op} - x)^{\frac{1}{2}}\psi\big), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{s}_x,$$

on the domain $\operatorname{dom} \mathfrak{s}_x = \operatorname{dom} (H_{\rm op} - x)^{1/2} = \operatorname{dom} (H - x)^{1/2}$. By Proposition 5.1.22, the form $\mathfrak{s}_x$ is closed and nonnegative. The corresponding nonnegative self-adjoint relation is given by

$$\big((H - x)^{\frac{1}{2}}\big)^* (H - x)^{\frac{1}{2}} = H - x,$$

and hence $\mathfrak{s}_x[\varphi, \psi] = (\varphi', \psi)$ holds for all $\{\varphi, \varphi'\} \in H - x$ and $\psi \in \operatorname{dom} \mathfrak{s}_x$. It follows as in the proof of Theorem 5.1.18 (see (5.1.34)) that the closed semibounded form

$$(\mathfrak{s}_x + x)[\varphi, \psi] = \big((H_{\rm op} - x)^{\frac{1}{2}}\varphi, (H_{\rm op} - x)^{\frac{1}{2}}\psi\big) + x(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{s}_x,$$

is represented by the semibounded self-adjoint relation H. Furthermore,

$$(\mathfrak{s}_x + x)[\varphi, \psi] = ((H_{\rm op} - x)\varphi, \psi) + x(\varphi, \psi) = (H_{\rm op}\varphi, \psi)$$

for all $\varphi, \psi \in \operatorname{dom} H$, and hence the restrictions of the form $\mathfrak{s}_x + x$ and of the form $\mathfrak{t}$ to $\operatorname{dom} H_{\rm op}$ coincide; cf. Theorem 5.1.18 (iv). According to Proposition 5.1.22 and Lemma 1.5.10, $\operatorname{dom} H_{\rm op}$ is a core of $\mathfrak{s}_x$ and hence also of $\mathfrak{s}_x + x$. On the other hand, by Theorem 5.1.18 (ii), $\operatorname{dom} H_{\rm op} = \operatorname{dom} H$ is also a core of $\mathfrak{t}$. Hence, the forms $\mathfrak{s}_x + x$ and $\mathfrak{t}$ coincide on the common core $\operatorname{dom} H_{\rm op}$. This implies that the forms $\mathfrak{s}_x + x$ and $\mathfrak{t}$ coincide. Therefore,

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} (\mathfrak{s}_x + x) = \operatorname{dom} (H - x)^{\frac{1}{2}}, \qquad x \le \gamma,$$

and

$$\mathfrak{t}[\varphi, \psi] = \big((H_{\rm op} - x)^{\frac{1}{2}}\varphi, (H_{\rm op} - x)^{\frac{1}{2}}\psi\big) + x(\varphi, \psi), \quad \varphi, \psi \in \operatorname{dom} \mathfrak{t}.$$

Finally, Proposition 5.1.22 shows that a subset of $\operatorname{dom} \mathfrak{t}$ is a core of $\mathfrak{t}$ if and only if it is a core of the operator $(H_{\rm op} - x)^{\frac{1}{2}}$. This completes the proof. □

5.2 Ordering and monotonicity

In this section an ordering will be introduced for semibounded closed forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$, and for semibounded self-adjoint relations H_1 and H_2 in a Hilbert space $\mathfrak{H}$. It will be shown that these orderings are compatible if $\mathfrak{t}_1$ and H_1, and $\mathfrak{t}_2$ and H_2 are related via the first representation theorem (Theorem 5.1.18), respectively. An alternative formulation of the ordering of semibounded self-adjoint relations will be given in terms of their resolvent operators. The last part of the section is devoted to a general monotonicity principle in the context of semibounded self-adjoint relations or, equivalently, of closed semibounded forms.

First an ordering will be defined for semibounded forms that are not necessarily closed.

Definition 5.2.1. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be semibounded forms in $\mathfrak{H}$ that are not necessarily closed. Then one writes $\mathfrak{t}_1 \leq \mathfrak{t}_2$, if

$$\operatorname{dom} \mathfrak{t}_2 \subset \operatorname{dom} \mathfrak{t}_1, \quad \mathfrak{t}_1[\varphi] \leq \mathfrak{t}_2[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}_2. \tag{5.2.1}$$

Note that if $\mathfrak{t}_1 \leq \mathfrak{t}_2$, then $\mathfrak{t}_2$-convergence implies $\mathfrak{t}_1$-convergence. Indeed, let $\varphi_n \to_{\mathfrak{t}_2} \varphi$. By Definition 5.1.4, this means that

$$\varphi_n \in \operatorname{dom} \mathfrak{t}_2, \quad \varphi_n \to \varphi, \quad \text{and} \quad \mathfrak{t}_2[\varphi_n - \varphi_m] \to 0.$$

Since $\mathfrak{t}_1 \leq \mathfrak{t}_2$, this implies that

$$\varphi_n \in \operatorname{dom} \mathfrak{t}_1 \quad \text{and} \quad \mathfrak{t}_1[\varphi_n - \varphi_m] \to 0,$$

which shows that $\varphi_n \to_{\mathfrak{t}_1} \varphi$. Definition 5.2.1 generates a number of simple but useful observations.

Lemma 5.2.2. *Let $\mathfrak{t}_1$, $\mathfrak{t}_2$, and $\mathfrak{t}_3$ be semibounded forms in $\mathfrak{H}$ that are not necessarily closed. Then the following statements hold:*

(i) $\mathfrak{t}_2 \subset \mathfrak{t}_1 \Rightarrow \mathfrak{t}_1 \leq \mathfrak{t}_2$*;*
(ii) $\mathfrak{t}_1 \leq \mathfrak{t}_2 \Rightarrow m(\mathfrak{t}_1) \leq m(\mathfrak{t}_2)$*;*
(iii) $\mathfrak{t}_1 \leq \mathfrak{t}_2$ *and* $\mathfrak{t}_2 \leq \mathfrak{t}_3 \Rightarrow \mathfrak{t}_1 \leq \mathfrak{t}_3$*;*
(iv) $\mathfrak{t}_1 \leq \mathfrak{t}_2$ *and* $\mathfrak{t}_2 \leq \mathfrak{t}_1 \Rightarrow \mathfrak{t}_1 = \mathfrak{t}_2$*;*
(v) $\mathfrak{t}_1 \leq \mathfrak{t}_2 \Rightarrow \widetilde{\mathfrak{t}}_1 \leq \widetilde{\mathfrak{t}}_2$*, when* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *are closable.*

Proof. (i) This follows from the definition of $\mathfrak{t}_2 \subset \mathfrak{t}_1$; cf. (5.1.2).

(ii) It follows from (5.2.1) that

$$\begin{aligned} \inf\left\{\frac{\mathfrak{t}_1[\varphi]}{\|\varphi\|^2} : \varphi \in \operatorname{dom} \mathfrak{t}_1,\, \varphi \neq 0\right\} &\leq \inf\left\{\frac{\mathfrak{t}_1[\varphi]}{\|\varphi\|^2} : \varphi \in \operatorname{dom} \mathfrak{t}_2,\, \varphi \neq 0\right\} \\ &\leq \inf\left\{\frac{\mathfrak{t}_2[\varphi]}{\|\varphi\|^2} : \varphi \in \operatorname{dom} \mathfrak{t}_2,\, \varphi \neq 0\right\}. \end{aligned}$$

Hence, Definition 5.1.2 implies that $m(\mathfrak{t}_1) \leq m(\mathfrak{t}_2)$.

(iii) This is an immediate consequence of Definition 5.2.1.

(iv) If $\mathfrak{t}_1 \leq \mathfrak{t}_2$ and $\mathfrak{t}_2 \leq \mathfrak{t}_1$, then it follows from (5.2.1) that $\operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$ and that $\mathfrak{t}_1[\varphi] = \mathfrak{t}_2[\varphi]$ for all $\varphi \in \operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$. The conclusion now follows by polarization; cf. (5.1.1).

(v) Assume that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable forms. Let $\varphi \in \operatorname{dom}\widetilde{\mathfrak{t}}_2$; then, by Definition 5.1.10, there exists a sequence (φ_n) in $\operatorname{dom}\mathfrak{t}_2$ such that $\varphi_n \to_{\mathfrak{t}_2} \varphi$. Recall that $\mathfrak{t}_2$-convergence implies $\mathfrak{t}_1$-convergence and thus $\varphi \in \operatorname{dom}\widetilde{\mathfrak{t}}_1$. This shows $\operatorname{dom}\widetilde{\mathfrak{t}}_2 \subset \operatorname{dom}\widetilde{\mathfrak{t}}_1$. Therefore, Theorem 5.1.12 implies that for $\varphi \in \operatorname{dom}\widetilde{\mathfrak{t}}_2$ one has

$$\widetilde{\mathfrak{t}}_1[\varphi] = \lim_{n\to\infty} \mathfrak{t}_1[\varphi_n] \leq \lim_{n\to\infty} \mathfrak{t}_2[\varphi_n] = \widetilde{\mathfrak{t}}_2[\varphi],$$

which shows (v). □

Next an ordering will be defined for semibounded self-adjoint relations. It will be shown in Proposition 5.2.6 below that this ordering is in agreement with the notation $\gamma \leq H$ for a semibounded self-adjoint relation H with lower bound γ; cf. Definition 1.4.5. Note that the following definition relies on Lemma 1.5.10.

Definition 5.2.3. Let H_1 and H_2 be semibounded self-adjoint relations in $\mathfrak{H}$, with lower bounds $m(H_1)$ and $m(H_2)$, respectively. Then the relations H_1 and H_2 are said to be *ordered*, and one writes $H_1 \leq H_2$, if

$$\begin{aligned} &\operatorname{dom}(H_2 - x)^{\frac{1}{2}} \subset \operatorname{dom}(H_1 - x)^{\frac{1}{2}}, \\ &\|(H_{1,\mathrm{op}} - x)^{\frac{1}{2}}\varphi\| \leq \|(H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi\|, \quad \varphi \in \operatorname{dom}(H_2 - x)^{\frac{1}{2}}, \end{aligned} \tag{5.2.2}$$

is satisfied for some, and hence for all $x \leq \min\{m(H_1), m(H_2)\}$.

In the next theorem it is shown that the ordering for semibounded forms in Definition 5.2.1 and the ordering for semibounded self-adjoint relations in Definition 5.2.3 are compatible. Here the second representation theorem (Theorem 5.1.23) plays an essential role.

Theorem 5.2.4. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be closed semibounded forms in $\mathfrak{H}$ and let H_1 and H_2 be the corresponding semibounded self-adjoint relations. Then*

$$\mathfrak{t}_1 \leq \mathfrak{t}_2 \quad \Leftrightarrow \quad H_1 \leq H_2.$$

Proof. Assume first that $\mathfrak{t}_1 \leq \mathfrak{t}_2$. Then, by Definition 5.2.1,

$$\operatorname{dom}\mathfrak{t}_2 \subset \operatorname{dom}\mathfrak{t}_1, \quad \mathfrak{t}_1[\varphi] \leq \mathfrak{t}_2[\varphi], \quad \varphi \in \operatorname{dom}\mathfrak{t}_2,$$

and for all $x \leq \min\{m(\mathfrak{t}_1), m(\mathfrak{t}_2)\}$ it follows from Theorem 5.1.23 that (5.2.2) holds. Hence, $H_1 \leq H_2$ by Definition 5.2.3.

Conversely, assume that $H_1 \leq H_2$. Then, by Definition 5.2.3, (5.2.2) holds for all $x \leq \min\{m(H_1), m(H_2)\}$ and hence Theorem 5.1.23 implies $\mathfrak{t}_1 \leq \mathfrak{t}_2$. □

Lemma 5.2.5. *Let H_1, H_2, and H_3 be semibounded self-adjoint relations in $\mathfrak{H}$. Then the following statements hold:*

(i) $H_1 \leq H_2 \Rightarrow \operatorname{mul} H_1 \subset \operatorname{mul} H_2$;

(ii) $H_1 \leq H_2 \Rightarrow m(H_1) \leq m(H_2)$;

(iii) $H_1 \leq H_2$ *and* $H_2 \leq H_3 \Rightarrow H_1 \leq H_3$;

(iv) $H_1 \leq H_2$ *and* $H_2 \leq H_1 \Rightarrow H_1 = H_2$;

(v) $H_1 \leq H_2 \Leftrightarrow H_1 - x \leq H_2 - x$ *for every* $x \in \mathbb{R}$.

Proof. Let $\mathfrak{t}_i$ be the closed semibounded form corresponding to H_i, $i = 1, 2, 3$. For the proof of (i) it is sufficient to observe that

$$\overline{\operatorname{dom}} H_2 = \overline{\operatorname{dom}}\, \mathfrak{t}_2 \subset \overline{\operatorname{dom}}\, \mathfrak{t}_1 = \overline{\operatorname{dom}} H_1,$$

where Theorem 5.2.4 and Theorem 5.1.18 (iv) were used. Taking orthogonal complements then gives $\operatorname{mul} H_1 \subset \operatorname{mul} H_2$. For (ii) recall that

$$m(H_1) = m(\mathfrak{t}_1) \leq m(\mathfrak{t}_2) = m(H_2),$$

as follows from Lemma 5.2.2 and Theorem 5.1.18. Statements (iii) and (iv) are translations of similar statements in Lemma 5.2.2. The statement (v) is clear from Theorem 5.2.4. □

Assume that in Definition 5.2.3 the self-adjoint relation H_1 has a closed domain $\operatorname{dom} H_1$. Then the operator part $H_{1,\mathrm{op}}$ of H_1 is a bounded operator which implies that $\operatorname{dom} H_1 = \operatorname{dom}(H_1 - x)^{\frac{1}{2}}$. Thus, in this case $H_1 \leq H_2$ if and only if

$$\begin{aligned} &\operatorname{dom}(H_2 - x)^{\frac{1}{2}} \subset \operatorname{dom} H_1, \\ &((H_{1,\mathrm{op}} - x)\varphi, \varphi) \leq \|(H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi\|^2, \quad \varphi \in \operatorname{dom}(H_2 - x)^{\frac{1}{2}}. \end{aligned} \tag{5.2.3}$$

The following proposition gives an alternative version of this statement.

Proposition 5.2.6. *Let H_1 and H_2 be semibounded self-adjoint relations in $\mathfrak{H}$ and assume that* $\operatorname{dom} H_1$ *is closed. Then the following statements are equivalent:*

(i) $H_1 \leq H_2$;

(ii) $\operatorname{dom} H_2 \subset \operatorname{dom} H_1$, $(H_{1,\mathrm{op}}\varphi, \varphi) \leq (H_{2,\mathrm{op}}\varphi, \varphi)$, $\varphi \in \operatorname{dom} H_2$.

Moreover, if $H_1 \in \mathbf{B}(\mathfrak{H})$, then these statements are equivalent to

(iii) $(H_1\varphi, \varphi) \leq (H_{2,\mathrm{op}}\varphi, \varphi)$, $\varphi \in \operatorname{dom} H_2$;

(iv) $(H_1\varphi, \varphi) \leq (\varphi', \varphi)$, $\{\varphi, \varphi'\} \in H_2$;

and in the particular case that $H_1 = \gamma_1 I_{\mathfrak{H}}$ to

(v) $\gamma_1\|\varphi\|^2 \leq (H_{2,\mathrm{op}}\varphi, \varphi)$, $\varphi \in \operatorname{dom} H_2$;

(vi) $\gamma_1\|\varphi\|^2 \leq (\varphi', \varphi)$, $\{\varphi, \varphi'\} \in H_2$.

Proof. (i) $\Rightarrow$ (ii) Let (i) be satisfied. Then $\operatorname{dom} H_2 \subset \operatorname{dom}(H_2 - x)^{\frac{1}{2}} \subset \operatorname{dom} H_1$ by (5.2.3), and for all $\varphi \in \operatorname{dom} H_2$ the inequality in (5.2.3) takes the form

$$((H_{1,\mathrm{op}} - x)\varphi, \varphi) \leq ((H_{2,\mathrm{op}} - x)\varphi, \varphi),$$

which implies (ii).

(ii) $\Rightarrow$ (i) Let (ii) be satisfied and let $\varphi \in \operatorname{dom}(H_2 - x)^{\frac{1}{2}}$. Then there exists a sequence (φ_n) in $\operatorname{dom} H_2$ such that

$$\varphi_n \to \varphi \quad \text{and} \quad (H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi_n \to (H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi, \quad n \to \infty,$$

since $\operatorname{dom} H_2$ is a core of $(H_2 - x)^{\frac{1}{2}}$; see Lemma 1.5.10. Due to the assumption one has $\varphi_n \in \operatorname{dom} H_1$ and

$$((H_{1,\mathrm{op}} - x)\varphi_n, \varphi_n) \leq ((H_{2,\mathrm{op}} - x)\varphi_n, \varphi_n) = \|(H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi_n\|^2.$$

Since $\operatorname{dom} H_1$ is closed it follows by taking the limit that

$$((H_{1,\mathrm{op}} - x)\varphi, \varphi) \leq \|(H_{2,\mathrm{op}} - x)^{\frac{1}{2}}\varphi\|^2, \quad \varphi \in \operatorname{dom}(H_2 - x)^{\frac{1}{2}}.$$

Hence, (5.2.3) is satisfied or, equivalently, $H_1 \leq H_2$.

If $H_1 \in \mathbf{B}(\mathfrak{H})$, then $\operatorname{dom} H_1 = \mathfrak{H}$ and hence the rest of the statements is clear. □

In particular, the inequality in (v)–(vi) of Proposition 5.2.6 shows that the ordering $\gamma I_{\mathfrak{H}} \leq H$ is equivalent to H being semibounded with lower bound γ as defined in Definition 1.4.5. Furthermore, if both H_1 and H_2 are self-adjoint operators in $\mathbf{B}(\mathfrak{H})$, then they are semibounded and Proposition 5.2.6 (iii) shows that $H_1 \leq H_2$ in the sense of Definition 5.2.3 agrees with the usual definition $(H_1\varphi, \varphi) \leq (H_2\varphi, \varphi)$ for all $\varphi \in \mathfrak{H}$.

The ordering for semibounded relations H_1 and H_2 can also be expressed in terms of their resolvent operators. The next proposition is an immediate consequence of Proposition 1.5.11 (for the special case $\rho = 1$).

Proposition 5.2.7. *Let H_1 and H_2 be semibounded self-adjoint relations in $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $H_1 \leq H_2$;

(ii) *for some, and hence for all* $x < \min\{m(H_1), m(H_2)\}$

$$(H_2 - x)^{-1} \leq (H_1 - x)^{-1}.$$

The next corollary slightly extends Proposition 5.2.7 and gives a further interpretation of the inequality $H_1 \leq H_2$ when $x \leq \min\{m(H_1), m(H_2)\}$. The equivalence in (5.2.4) below is an example of the antitonicity property.

Corollary 5.2.8. *Let H_1 and H_2 be semibounded self-adjoint relations in $\mathfrak{H}$. Then*

$$H_1 \le H_2$$

if and only if for $\gamma \le \min\{m(H_1), m(H_2)\}$ one has

$$(H_2 - \gamma)^{-1} \le (H_1 - \gamma)^{-1}.$$

In particular, if H_1 and H_2 are nonnegative self-adjoint relations, then

$$H_1 \le H_2 \quad \Leftrightarrow \quad H_2^{-1} \le H_1^{-1}. \tag{5.2.4}$$

Proof. Let H be a semibounded self-adjoint relation with $\gamma \le m(H)$. Then $H-\gamma$ is nonnegative and hence also $(H-\gamma)^{-1}$ is a nonnegative self-adjoint relation. Now write for $x < \gamma$

$$H - x = H - \gamma - (x - \gamma),$$

and apply Corollary 1.1.12 (with H replaced by $(H-\gamma)^{-1}$ and λ replaced by $(x-\gamma)^{-1}$), obtaining

$$(H-x)^{-1} = -\frac{1}{x-\gamma} - \frac{1}{(x-\gamma)^2}\left((H-\gamma)^{-1} - \frac{1}{x-\gamma}\right)^{-1}.$$

Hence, for the pair of semibounded self-adjoint relations H_1 and H_2 and with $\gamma \le \min\{m(H_1), m(H_2)\}$ one obtains for each $x < \gamma$:

$$\begin{aligned}&(H_1 - x)^{-1} - (H_2 - x)^{-1}\\ &= \frac{1}{(x-\gamma)^2}\left[\left((H_2-\gamma)^{-1} - \frac{1}{x-\gamma}\right)^{-1} - \left((H_1-\gamma)^{-1} - \frac{1}{x-\gamma}\right)^{-1}\right].\end{aligned}$$

Since $x-\gamma < 0$, a repeated application of Proposition 5.2.7 shows the equivalence. In fact, $H_1 \le H_2$ if and only if $(H_2 - x)^{-1} \le (H_1 - x)^{-1}$ by Proposition 5.2.7, which by the above formula is equivalent to

$$\left((H_1-\gamma)^{-1} - \frac{1}{x-\gamma}\right)^{-1} \le \left((H_2-\gamma)^{-1} - \frac{1}{x-\gamma}\right)^{-1}. \tag{5.2.5}$$

Another application of Proposition 5.2.7 shows that the inequality (5.2.5) is equivalent to the inequality $(H_2-\gamma)^{-1} \le (H_1-\gamma)^{-1}$. □

As a corollary to Proposition 5.2.7 it will be shown that in the case $H_1 \le H_2$ the difference $(H_1 - x)^{-1} - (H_2 - x)^{-1}$, $x < \min\{m(H_1), m(H_2)\}$, can be used to describe the gap between the corresponding form domains

$$\operatorname{dom}(H_2 - x)^{\frac{1}{2}} \subset \operatorname{dom}(H_1 - x)^{\frac{1}{2}}.$$

Corollary 5.2.9. *Let H_1 and H_2 be semibounded self-adjoint relations in $\mathfrak{H}$ and assume that*

$$H_1 \le H_2.$$

Then for all $x < \min\{m(H_1), m(H_2)\}$ the operator $(H_1 - x)^{-1} - (H_2 - x)^{-1} \in \mathbf{B}(\mathfrak{H})$ is nonnegative and

$$\operatorname{dom}(H_1 - x)^{\frac{1}{2}} = \operatorname{ran}\left((H_1 - x)^{-1} - (H_2 - x)^{-1}\right)^{\frac{1}{2}} + \operatorname{dom}(H_2 - x)^{\frac{1}{2}}.$$

Proof. Since $H_1 \le H_2$, the operator $R(x) \in \mathbf{B}(\mathfrak{H})$, defined by

$$R(x) = (H_1 - x)^{-1} - (H_2 - x)^{-1},$$

is nonnegative for $x < \min\{m(H_1), m(H_2)\}$; cf. Proposition 5.2.7. Hence, one can write

$$\begin{aligned} (H_1 - x)^{-1} &= R(x) + (H_2 - x)^{-1} \\ &= \begin{pmatrix} R(x)^{\frac{1}{2}} & (H_2 - x)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} R(x)^{\frac{1}{2}} \\ (H_2 - x)^{-\frac{1}{2}} \end{pmatrix}. \end{aligned} \tag{5.2.6}$$

Now recall that if $T = (A \;\; B)$ is a row operator with $A, B \in \mathbf{B}(\mathfrak{H})$, then it follows from $\operatorname{ran}(TT^*)^{\frac{1}{2}} = \operatorname{ran}|T^*| = \operatorname{ran} T$, cf. Corollary D.6, that

$$\operatorname{ran}(AA^* + BB^*)^{\frac{1}{2}} = \operatorname{ran}(A \;\; B) = \operatorname{ran} A + \operatorname{ran} B. \tag{5.2.7}$$

Hence, taking square roots in the identity (5.2.6) and applying (5.2.7) shows that

$$\operatorname{ran}(H_1 - x)^{-\frac{1}{2}} = \operatorname{ran} R(x)^{\frac{1}{2}} + \operatorname{ran}(H_2 - x)^{-\frac{1}{2}},$$

which yields the desired decomposition

$$\operatorname{dom}(H_1 - x)^{\frac{1}{2}} = \operatorname{ran} R(x)^{\frac{1}{2}} + \operatorname{dom}(H_2 - x)^{\frac{1}{2}}$$

for $x < \min\{m(H_1), m(H_2)\}$. □

Now the ordering for semibounded self-adjoint relations and for semibounded closed forms will be used to reinterpret and extend the monotonicity result in Proposition 1.9.9

For the proof of the following theorem it is useful to have available an auxiliary result concerning the interchange of limits. Let (f_n) be a nondecreasing sequence of real nondecreasing functions defined on an open interval (a, b). Thus, for all $x \in (a, b)$ one has

$$f_m(x) \le f_n(x), \quad m \le n, \tag{5.2.8}$$

and for all $n \in \mathbb{N}$

$$f_n(x) \le f_n(y), \quad a < x \le y < b. \tag{5.2.9}$$

In view of (5.2.8) the pointwise limit

$$f_\infty(x) = \lim_{n \to \infty} f_n(x), \quad x \in (a, b), \tag{5.2.10}$$

gives a function $f_\infty : (a,b) \to \mathbb{R} \cup \{\infty\}$ that is nondecreasing, thanks to (5.2.9). This is clear when all $f_\infty(x)$ are finite, in which case $\lim_{x\to b} f_\infty(x)$ is proper or improper. However, if $f_\infty(x_0) = \infty$ for some $x_0 \in (a,b)$, then (5.2.9) shows that $f_\infty(x) = \infty$ for all $x_0 \le x < b$. In this case the function f_∞ is also called nondecreasing (in the sense of $\mathbb{R} \cup \{\infty\}$) and one defines $\lim_{x\to b} f_\infty(x) = \infty$. In view of (5.2.9) the limit

$$f_n(b) = \lim_{x\to b} f_n(x), \quad n \in \mathbb{N}, \tag{5.2.11}$$

gives a sequence with values in $\mathbb{R} \cup \{\infty\}$ that is nondecreasing, thanks to (5.2.8). This is again clear when all limits $f_n(b)$ are finite in which case $\lim_{n\to\infty} f_n(b)$ is proper or improper. However, if there exists some $m \in \mathbb{N}$ for which $f_m(b) = \infty$, then for all $n \ge m$ one has $f_n(b) = \infty$. In this case one defines $\lim_{n\to\infty} f_n(b) = \infty$.

Lemma 5.2.10. *Let (f_n) be a nondecreasing sequence of nondecreasing functions defined on some open interval (a,b). Let f_∞ be the nondecreasing limit function in (5.2.10) and let $(f_n(b))$ be the nondecreasing sequence of limits in (5.2.11). Then*

$$\lim_{x\to b} f_\infty(x) = \lim_{n\to\infty} f_n(b). \tag{5.2.12}$$

In particular, both limits in (5.2.12) are finite or infinite simultaneously.

Proof. Consider the case that all values of f_∞ are real. Since $f_n(x) \le f_\infty(x)$ for all $x \in (a,b)$, it follows that for any $n \in \mathbb{N}$

$$f_n(b) = \lim_{x\to b} f_n(x) \le \lim_{x\to b} f_\infty(x).$$

This implies

$$\lim_{n\to\infty} f_n(b) \le \lim_{x\to b} f_\infty(x), \tag{5.2.13}$$

where the limits may be infinite. Assume that there is strict inequality in (5.2.13). First consider the case $\lim_{x\to b} f_\infty(x) < \infty$. Then clearly there exists some $\delta > 0$ for which

$$\delta + \lim_{n\to\infty} f_n(b) < \lim_{x\to b} f_\infty(x). \tag{5.2.14}$$

Next consider the case $\lim_{x\to b} f_\infty(x) = \infty$. Then $\lim_{n\to\infty} f_n(b) < \infty$ (otherwise there would be equality in (5.2.13)) and (5.2.14) holds for any $\delta > 0$. In each case, there exists some $x \in (a,b)$ such that

$$\delta + \lim_{n\to\infty} f_n(b) < f_\infty(x).$$

From this one concludes

$$\delta + f_\infty(x) = \delta + \lim_{n\to\infty} f_n(x) \le \delta + \lim_{n\to\infty} f_n(b) < f_\infty(x);$$

a contradiction. Hence, there is equality in (5.2.13). It remains to consider the situation where $f_\infty(x_0) = \infty$ for some $a < x_0 < b$. In this case $f_\infty(x) = \infty$ for all $x_0 < x < b$ and $\lim_{x\to b} f_\infty(x) = \infty$. Assume that

$$L = \lim_{n\to\infty} f_n(b) < \infty.$$

For any $x_0 \le x < b$ one has

$$f_n(x) \le f_n(b) \le L,$$

which implies that $\lim_{n\to\infty} f_n(x) \le L$; a contradiction. Again, there is equality in (5.2.13). □

Theorem 5.2.11 (Monotonicity principle). *Let (H_n) be a nondecreasing sequence of semibounded self-adjoint relations in $\mathfrak{H}$ and let $\gamma \le m(H_1)$. Then there exists a semibounded self-adjoint relation H_∞ with $\gamma \le m(H_\infty)$ and $H_n \le H_\infty$ such that $H_n \to H_\infty$ in the strong resolvent sense, i.e.,*

$$(H_n - \lambda)^{-1}\varphi \to (H_\infty - \lambda)^{-1}\varphi, \quad \varphi \in \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus [\gamma, \infty). \tag{5.2.15}$$

Furthermore, H_∞ satisfies

$$\begin{aligned} &\operatorname{dom}(H_\infty - \gamma)^{\frac{1}{2}} \\ &= \left\{ \varphi \in \bigcap_{n=1}^{\infty} \operatorname{dom}(H_n - \gamma)^{\frac{1}{2}} : \lim_{n\to\infty} \|(H_{n,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\| < \infty \right\} \end{aligned} \tag{5.2.16}$$

and for all $\varphi \in \operatorname{dom}(H_\infty - \gamma)^{\frac{1}{2}}$ it holds that

$$\|(H_{\infty,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\| = \lim_{n\to\infty} \|(H_{n,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\|. \tag{5.2.17}$$

Proof. The assumption $H_n \le H_m$ for $n \le m$ and Proposition 5.2.7 lead to

$$0 \le (H_m - x)^{-1} \le (H_n - x)^{-1}, \quad x < \gamma,$$

where $\gamma \le m(H_1)$. Hence, by Proposition 1.9.14, there exists a semibounded self-adjoint relation H_∞ with $\gamma \le m(H_\infty)$ such that

$$0 \le (H_\infty - x)^{-1} \le (H_n - x)^{-1}, \quad x < \gamma, \tag{5.2.18}$$

and H_n converges to H_∞ in the strong resolvent sense on $\mathbb{C} \setminus [\gamma, \infty)$, that is, (5.2.15) holds.

It remains to prove (5.2.16) and (5.2.17). It follows from Corollary 1.1.12 with H replaced by $H_n - \gamma$ and $H_\infty - \gamma$, respectively, that for $x < 0$ one has

$$\begin{aligned} &\big(\big((H_n - \gamma)^{-1} - x\big)^{-1}\varphi, \varphi\big) - \big(\big((H_\infty - \gamma)^{-1} - x\big)^{-1}\varphi, \varphi\big) \\ &= \frac{1}{x^2}\left[\left(\left((H_\infty - \gamma) - \frac{1}{x}\right)^{-1}\varphi, \varphi\right) - \left(\left((H_n - \gamma) - \frac{1}{x}\right)^{-1}\varphi, \varphi\right)\right]. \end{aligned}$$

Since $\gamma + 1/x < \gamma$, the right-hand side tends to zero monotonically from below for $n \to \infty$, as follows from (5.2.15) and (5.2.18); but then also the left-hand side tends to zero monotonically from below.

To complete the proof, consider the functions, defined for $\varphi \in \mathfrak{H}$ and $x < 0$ by

$$f_n(x) = \big(\big((H_n - \gamma)^{-1} - x\big)^{-1}\varphi, \varphi\big)$$

and

$$f_\infty(x) = \big(\big((H_\infty - \gamma)^{-1} - x\big)^{-1}\varphi, \varphi\big).$$

The above argument shows that the sequence f_n is nondecreasing with f_∞ as pointwise limit. It follows from Lemma 1.5.12 (with H replaced by $H_n - \gamma$ and $H_\infty - \gamma$, respectively), that both functions f_n and f_∞ are nondecreasing on the interval $(-\infty, 0)$ and that

$$
\begin{aligned}
f_n(0) &= \lim_{x \uparrow 0} \big(((H_n - \gamma)^{-1} - x)^{-1}\varphi, \varphi\big) \\
&= \begin{cases} \|(H_{n,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\|^2, & \varphi \in \mathrm{dom}\,(H_n - \gamma)^{\frac{1}{2}}, \\ \infty, & \text{otherwise}, \end{cases}
\end{aligned}
\tag{5.2.19}
$$

while

$$
\begin{aligned}
f_\infty(0) &= \lim_{x \uparrow 0} \big(((H_\infty - \gamma)^{-1} - x)^{-1}\varphi, \varphi\big) \\
&= \begin{cases} \|(H_{\infty,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\|^2, & \varphi \in \mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}}, \\ \infty, & \text{otherwise}. \end{cases}
\end{aligned}
\tag{5.2.20}
$$

Hence, by Lemma 5.2.10,

$$
\lim_{n\to\infty} f_n(0) = f_\infty(0), \tag{5.2.21}
$$

where the limits in (5.2.21) are finite or infinite simultaneously.

Assume that $\varphi \in \mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}}$. Then, by (5.2.20), $f_\infty(0) < \infty$, which, in view of (5.2.21), implies that all $f_n(0) < \infty$. Hence, $\varphi \in \bigcap_{n=1}^\infty \mathrm{dom}\,(H_n - \gamma)^{\frac{1}{2}}$ by (5.2.19), and (5.2.21) reads

$$
\lim_{n\to\infty} \|(H_{n,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\|^2 = \|(H_{\infty,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\|^2. \tag{5.2.22}
$$

Thus, φ belongs to the right-hand side of (5.2.16). This shows the inclusion $(\subset)$ in (5.2.16), and (5.2.22) gives (5.2.17).

Conversely, assume that φ belongs to the right-hand side of (5.2.16), that is, $\varphi \in \bigcap_{n=1}^\infty \mathrm{dom}\,(H_n - \gamma)^{\frac{1}{2}}$ and

$$
\lim_{n\to\infty} \|(H_{n,\mathrm{op}} - \gamma)^{\frac{1}{2}}\varphi\| < \infty.
$$

By (5.2.19) one sees that $f_n(0) < \infty$ and that $\lim_{n\to\infty} f_n(0) < \infty$. It follows from (5.2.21) that $f_\infty(0) < \infty$. Now apply (5.2.20) to conclude that $\varphi \in \mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}}$. This shows the inclusion $(\supset)$ in (5.2.16). □

Corollary 5.2.12. *Let (H_n) be a nondecreasing sequence of semibounded self-adjoint relations and let H_∞ be the strong resolvent limit as in Theorem 5.2.11. Then the following statements hold:*

(i) *If K is a semibounded self-adjoint relation such that $H_n \le K$ for all $n \in \mathbb{N}$, then also $H_\infty \le K$.*

(ii) *If S is a symmetric relation such that $S \subset H_n$ for all $n \in \mathbb{N}$, then also $S \subset H_\infty$.*

Proof. (i) Assume that $H_n \leq K$. Then for all $x < \gamma \leq m(H_1)$

$$0 \leq ((K-x)^{-1}\varphi, \varphi) \leq ((H_n - x)^{-1}\varphi, \varphi), \quad \varphi \in \mathfrak{H}.$$

By (5.2.15), $(H_n - x)^{-1}\varphi \to (H_\infty - x)^{-1}\varphi$ for $\varphi \in \mathfrak{H}$ and one concludes that

$$0 \leq ((K-x)^{-1}\varphi, \varphi) \leq ((H_\infty - x)^{-1}\varphi, \varphi), \quad \varphi \in \mathfrak{H}.$$

Hence, by Proposition 5.2.7 it follows that $H_\infty \leq K$.

(ii) Assume that $\{\varphi, \varphi'\} \in S$. Then $\{\varphi, \varphi'\} \in H_n$ by assumption and hence for all $n \in \mathbb{N}$ one has

$$(H_n - \lambda)^{-1}(\varphi' - \lambda\varphi) = \varphi, \quad \lambda \in \mathbb{C} \setminus [\gamma, \infty).$$

By (5.2.15), $(H_n - x)^{-1}\psi \to (H_\infty - x)^{-1}\psi$ for $\psi \in \mathfrak{H}$ and one concludes that

$$(H_\infty - \lambda)^{-1}(\varphi' - \lambda\varphi) = \lim_{n\to\infty} (H_n - \lambda)^{-1}(\varphi' - \lambda\varphi) = \varphi,$$

which gives $\{\varphi, \varphi'\} \in H_\infty$. Hence, $S \subset H_\infty$. □

Now consider the special case of a nondecreasing sequence of self-adjoint operators (H_n) in $\mathbf{B}(\mathfrak{H})$. Then it is clear that H_∞ is a semibounded self-adjoint relation which is an operator in $\mathbf{B}(\mathfrak{H})$ if and only if the sequence (H_n) is uniformly bounded; cf. Corollary 1.9.10 and the beginning of Section 1.9. The following corollary shows that the domain of the square root of $H_\infty - \gamma$, $\gamma \leq m(H_1)$, is given by those $\varphi \in \mathfrak{H}$ for which $(H_n\varphi, \varphi)$ has a finite limit as $n \to \infty$.

Corollary 5.2.13. *Let (H_n) be a nondecreasing sequence of self-adjoint operators in $\mathbf{B}(\mathfrak{H})$ with $\gamma \leq m(H_1)$ and define*

$$\mathfrak{E} = \left\{ \varphi \in \mathfrak{H} : \lim_{n\to\infty} (H_n\varphi, \varphi) < \infty \right\}.$$

Let H_∞ be the semibounded self-adjoint limit of the sequence H_n. Then

$$\mathfrak{E} = \operatorname{dom}(H_\infty - \gamma)^{\frac{1}{2}}.$$

In particular, one has

(i) $\mathfrak{E} = \mathfrak{H} \Leftrightarrow H_\infty \in \mathbf{B}(\mathfrak{H})$;
(ii) $\mathfrak{E}$ *is closed* $\Leftrightarrow H_{\infty,\mathrm{op}}$ *is a bounded operator;*
(iii) $\operatorname{clos}\mathfrak{E} = \mathfrak{H} \Leftrightarrow H_\infty$ *is an operator;*
(iv) $\mathfrak{E} = \{0\} \Leftrightarrow H_\infty = \{0\} \times \mathfrak{H}$.

A useful variant of Corollary 5.2.13 is concerned with a nondecreasing function $M : (a,b) \to \mathbf{B}(\mathfrak{H})$, whose values are self-adjoint operators; cf. Corollary 2.3.8. Then there exists a self-adjoint limit at the right endpoint b, which can be retrieved via sequences converging to b. For the existence of the self-adjoint limit at the left endpoint a consider the function $x \mapsto -M(x)$, which is nondecreasing when $x \in (a,b)$ tends to a.

Corollary 5.2.14. *Let $M : (a,b) \to \mathbf{B}(\mathfrak{H})$ be a nondecreasing function, whose values are self-adjoint operators. Then there exist self-adjoint relations $M(a)$ and $M(b)$ in $\mathfrak{H}$ such that $M(x) \to M(b)$ in the strong resolvent sense when $x \to b$ and $M(x) \to M(a)$ in the strong resolvent sense when $x \to a$. Furthermore,*

$$M(x) \leq M(b) \quad \text{and} \quad -M(x) \leq -M(a), \qquad x \in (a,b).$$

Define

$$\mathfrak{E}_b = \left\{ \varphi \in \mathfrak{H} : \lim_{x \uparrow b} (M(x)\varphi, \varphi) < \infty \right\}$$

and

$$\mathfrak{E}_a = \left\{ \varphi \in \mathfrak{H} : \lim_{x \downarrow a} (M(x)\varphi, \varphi) > -\infty \right\}.$$

Then for $c = a$ or $c = b$ one has

(i) $\mathfrak{E}_c = \mathfrak{H} \Leftrightarrow M(c) \in \mathbf{B}(\mathfrak{H})$;

(ii) $\mathfrak{E}_c$ *is closed* $\Leftrightarrow M(c)_{\rm op}$ *is a bounded operator;*

(iii) $\operatorname{clos} \mathfrak{E}_c = \mathfrak{H} \Leftrightarrow M(c)$ *is an operator;*

(iv) $\mathfrak{E}_c = \{0\} \Leftrightarrow M(c) = \{0\} \times \mathfrak{H}$.

Let $(\mathfrak{t}_n)$ be a nondecreasing sequence of closed semibounded forms in $\mathfrak{H}$ which satisfy $\gamma \leq m(\mathfrak{t}_1)$. By Theorem 5.1.18, there exist unique semibounded self-adjoint relations H_n, bounded from below by γ, which correspond to $\mathfrak{t}_n$. According to Theorem 5.2.4, the sequence (H_n) is nondecreasing. By the monotonicity principle in Theorem 5.2.11 the strong resolvent limit of the sequence (H_n) exists as a semibounded self-adjoint relation H_∞ with lower bound γ such that $H_n \leq H_\infty$. Let $\mathfrak{t}_\infty$ be the form corresponding to H_∞ by Proposition 5.1.19. Then $\mathfrak{t}_\infty$ is bounded below by γ and $\mathfrak{t}_n \leq \mathfrak{t}_\infty$ by Theorem 5.2.4. Therefore, the following theorem concerning a nondecreasing sequence of forms may be seen as a direct consequence of Theorem 5.2.11.

Theorem 5.2.15 (Monotonicity principle)**.** *Let $(\mathfrak{t}_n)$ be a nondecreasing sequence of closed semibounded forms in $\mathfrak{H}$ and let $\gamma \leq m(\mathfrak{t}_1)$. Then there exists a closed semibounded form $\mathfrak{t}_\infty$ with $\gamma \leq m(\mathfrak{t}_\infty)$ such that $\mathfrak{t}_n \leq \mathfrak{t}_\infty$ and*

$$\operatorname{dom} \mathfrak{t}_\infty = \left\{ \varphi \in \bigcap_{n=1}^{\infty} \operatorname{dom} \mathfrak{t}_n : \lim_{n \to \infty} \mathfrak{t}_n[\varphi] < \infty \right\} \tag{5.2.23}$$

and

$$\mathfrak{t}_\infty[\varphi] = \lim_{n \to \infty} \mathfrak{t}_n[\varphi], \quad \varphi \in \operatorname{dom} \mathfrak{t}_\infty. \tag{5.2.24}$$

Moreover, the relations H_n corresponding to the forms $\mathfrak{t}_n$ converge in the strong resolvent sense to the relation H_∞ corresponding to the form $\mathfrak{t}_\infty$.

Proof. It is clear from Theorem 5.1.23 and the formulas (5.2.16) and (5.2.17) in Theorem 5.2.11 that the limit form $\mathfrak{t}_\infty$ satisfies (5.2.23) and (5.2.24). □

5.3 Friedrichs extensions of semibounded relations

A semibounded, not necessarily closed, relation S in a Hilbert space $\mathfrak{H}$ has equal defect numbers, and hence admits self-adjoint extensions in $\mathfrak{H}$. It will be shown that such a relation S has a distinguished semibounded self-adjoint extension S_{F}, the so-called Friedrichs extension of S, with $m(S_{\mathrm{F}}) = m(S)$. The construction of this extension involves a closed semibounded form associated with S. The characteristic properties of this extension will be investigated in detail.

Let S be a semibounded relation in $\mathfrak{H}$. Recall from Lemma 5.1.17 that the form $\mathfrak{t}_S$ given by

$$\mathfrak{t}_S[f,g] = (f',g), \quad \{f,f'\},\{g,g'\} \in S, \tag{5.3.1}$$

is well defined and that it is semibounded with the lower bound $m(S)$. Moreover, it has been shown that $\mathfrak{t}_S$ is closable and that the closure $\widetilde{\mathfrak{t}}_S$ of $\mathfrak{t}_S$ is a semibounded closed form whose lower bound is equal to $m(S)$. Also, $\operatorname{dom} \mathfrak{t}_S = \operatorname{dom} S$ is a core of $\widetilde{\mathfrak{t}}_S$.

Lemma 5.3.1. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound $m(S)$. Let $\widetilde{\mathfrak{t}}_S$ be the closure of the form $\mathfrak{t}_S$ in (5.3.1). Then the unique relation S_{F} corresponding to $\widetilde{\mathfrak{t}}_S$ via Theorem 5.1.18 is a semibounded self-adjoint extension of S with the lower bound $m(S_{\mathrm{F}}) = m(S)$. In fact, S_{F} is a self-adjoint extension of the semibounded relation $S \,\widehat{+}\, \widehat{\mathfrak{N}}_\infty(S^*)$, so that*

$$S \subset S \,\widehat{+}\, \widehat{\mathfrak{N}}_\infty(S^*) \subset S_{\mathrm{F}}, \quad \textit{where} \quad \widehat{\mathfrak{N}}_\infty(S^*) = \{0\} \times \operatorname{mul} S^*. \tag{5.3.2}$$

Moreover,

$$S_{\mathrm{F}} = (\overline{S})_{\mathrm{F}}. \tag{5.3.3}$$

Proof. By Theorem 5.1.18, the closed form $\widetilde{\mathfrak{t}}_S$ induces a unique semibounded self-adjoint relation S_{F} in $\mathfrak{H}$ such that

$$\widetilde{\mathfrak{t}}_S[f,g] = (f',g), \quad \{f,f'\} \in S_{\mathrm{F}}, \quad g \in \operatorname{dom} \widetilde{\mathfrak{t}}_S.$$

To show that S_{F} is an extension of S, let $\{f,f'\} \in S$. As $f \in \operatorname{dom} \mathfrak{t}_S$, it follows that for all $g \in \operatorname{dom} \mathfrak{t}_S$

$$\widetilde{\mathfrak{t}}_S[f,g] = \mathfrak{t}_S[f,g] = (f',g).$$

Since $\operatorname{dom} \mathfrak{t}_S = \operatorname{dom} S$ is a core of $\widetilde{\mathfrak{t}}_S$, one obtains $\{f,f'\} \in S_{\mathrm{F}}$ from Theorem 5.1.18 (iii). Hence, S_{F} is a self-adjoint extension of S with lower bound $m(S_{\mathrm{F}}) = m(S)$.

In order to verify (5.3.2) it now suffices to see that $\{0\} \times \operatorname{mul} S^* \subset S_{\mathrm{F}}$. Let $\varphi \in \operatorname{mul} S^*$ and let $g \in \operatorname{dom} S$, then clearly $\widetilde{\mathfrak{t}}_S[0,g] = 0$ and $(\varphi, g) = 0$. Therefore,

$$\widetilde{\mathfrak{t}}_S[0,g] = (\varphi,g) \quad \text{for all} \quad g \in \operatorname{dom} S.$$

Since $\operatorname{dom} S$ is a core of $\widetilde{\mathfrak{t}}_S$, it follows that $\{0,\varphi\} \in S_{\mathrm{F}}$.

To see that (5.3.3) holds, it suffices to recall from Lemma 5.1.17 that $\widetilde{\mathfrak{t}}_S = \widetilde{\mathfrak{t}}_{\overline{S}}$ holds for the closures of $\mathfrak{t}_S$ and $\mathfrak{t}_{\overline{S}}$. □

Definition 5.3.2. Let S be a semibounded relation in $\mathfrak{H}$. The semibounded self-adjoint relation S_{F} associated with the closure of the form $\mathfrak{t}_S$ in (5.3.1) is called the *Friedrichs extension* of S. The closure $\widetilde{\mathfrak{t}}_S$ of the form $\mathfrak{t}_S$ will be denoted by $\mathfrak{t}_{S_{\mathrm{F}}}$, so that $\mathfrak{t}_{S_{\mathrm{F}}} = \widetilde{\mathfrak{t}}_S$.

Let S be a semibounded relation in $\mathfrak{H}$ and let $a < m(S)$. Then $S - a$ is a nonnegative relation and it is a consequence of (5.3.1) that $\mathfrak{t}_{S-a} = \mathfrak{t}_S - a$. The translation invariance of the closures, cf. (5.1.17), leads to

$$\mathfrak{t}_{(S-a)_{\mathrm{F}}} = \widetilde{\mathfrak{t}_{S-a}} = \widetilde{\mathfrak{t}_S - a} = \widetilde{\mathfrak{t}}_S - a = \mathfrak{t}_{S_{\mathrm{F}}} - a.$$

The nonnegative self-adjoint relation $(S-a)_{\mathrm{F}}$ corresponding to the form on the left-hand side is equal to the nonnegative self-adjoint relation corresponding to the form on the right-hand side. Thus, one obtains

$$(S-a)_{\mathrm{F}} = S_{\mathrm{F}} - a, \quad a < m(S). \tag{5.3.4}$$

In other words, the Friedrichs extension is translation invariant.

By Lemma 5.3.1, the Friedrichs extension S_{F} is a semibounded self-adjoint extension of S. As a restriction of S^* the Friedrichs extensions can be characterized as follows.

Theorem 5.3.3. *Let S be a semibounded relation in $\mathfrak{H}$. The Friedrichs extension S_{F} of S admits the representation*

$$S_{\mathrm{F}} = \bigl\{ \{f, f'\} \in S^* : f \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}} \bigr\} \tag{5.3.5}$$

with $\operatorname{mul} S_{\mathrm{F}} = \operatorname{mul} S^*$. *Furthermore, if H is a self-adjoint extension of S, that is not necessarily semibounded, then*

$$\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}} \quad \Rightarrow \quad H = S_{\mathrm{F}}.$$

Proof. In order to show that S_{F} is contained in the right-hand side of (5.3.5), let $\{f, f'\} \in S_{\mathrm{F}}$. Clearly, $S \subset S_{\mathrm{F}}$ and since S_{F} is self-adjoint this implies $S_{\mathrm{F}} \subset S^*$, so that $\{f, f'\} \in S^*$. Note also that $f \in \operatorname{dom} S_{\mathrm{F}} \subset \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$. Hence, S_{F} is contained in the right-hand side of (5.3.5).

To show the opposite inclusion, let $\{f, f'\} \in S^*$ be such that $f \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$. Then there exists a sequence (f_n) in $\operatorname{dom} \mathfrak{t}_S = \operatorname{dom} S$ with

$$f_n \to_{\mathfrak{t}_{S_{\mathrm{F}}}} f.$$

Let $\{f_n, f_n'\}$ be corresponding elements in S and let $\{g, g'\} \in S$ be arbitrary. Then $\mathfrak{t}_S \subset \mathfrak{t}_{S_{\mathrm{F}}}$ and $S \subset S^*$ imply that

$$\mathfrak{t}_{S_{\mathrm{F}}}[f_n, g] = \mathfrak{t}_S[f_n, g] = (f_n', g) = (f_n, g').$$

Since $\mathfrak{t}_{S_{\rm F}}$ is a closed form, it follows that

$$\mathfrak{t}_{S_{\rm F}}[f,g] = (f,g').$$

As $\{f,f'\} \in S^*$ and $\{g,g'\} \in S$, one obtains $(f,g') = (f',g)$, so that

$$\mathfrak{t}_{S_{\rm F}}[f,g] = (f',g).$$

This identity holds for an arbitrary element $g \in \operatorname{dom} S$. Since $\operatorname{dom} S = \operatorname{dom} \mathfrak{t}_S$ is a core of the form $\mathfrak{t}_{S_{\rm F}}$ it follows from Theorem 5.1.18 (iii) that $\{f,f'\} \in S_{\rm F}$.

Now let H be any self-adjoint extension of S with $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$. Hence, if $\{f,f'\} \in H$, then $\{f,f'\} \in S^*$ and $f \in \operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$. By (5.3.5), one has $\{f,f'\} \in S_{\rm F}$. This shows $H \subset S_{\rm F}$, and since both H and $S_{\rm F}$ are self-adjoint, it follows that $H = S_{\rm F}$. □

According to Theorem 5.3.3, the inclusion $\operatorname{dom} H \subset \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$ for any self-adjoint extension H of S implies $H = S_{\rm F}$. Note that in general a self-adjoint extension of a semibounded relation is not necessarily semibounded. The situation is different when S has finite defect numbers; cf. Proposition 5.5.8.

The construction of $S_{\rm F}$ in (5.3.5) results in a description of $S_{\rm F}$ by means of approximating elements from the graph of S.

Corollary 5.3.4. *Let S be a semibounded relation in $\mathfrak{H}$. Then $S_{\rm F}$ is the set of all elements $\{f,f'\} \in S^*$ for which there exists a sequence $(\{f_n,f_n'\})$ in S such that*

$$f_n \to f \quad \textit{and} \quad (f_n', f_n) \to (f', f).$$

Proof. By Theorem 5.3.3, $S_{\rm F}$ is the set of all elements $\{f,f'\} \in S^*$ for which $f \in \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$. Hence, there exists a sequence $(\{f_n,f_n'\})$ in S such that

$$f_n \to_{\mathfrak{t}_{S_{\rm F}}} f.$$

In particular, $f_n \to f$ in $\mathfrak{H}$ and, moreover,

$$(f',f) = \mathfrak{t}_{S_{\rm F}}[f,f] = \lim_{n\to\infty} \mathfrak{t}_{S_{\rm F}}[f_n,f_n] = \lim_{n\to\infty} \mathfrak{t}_S[f_n,f_n] = \lim_{n\to\infty} (f_n',f_n).$$

Hence, $S_{\rm F}$ is contained in the relation

$$\big\{ \{f,f'\} \in S^* : \, f_n \to f \text{ and } (f_n',f_n) \to (f',f) \text{ for some } \{f_n,f_n'\} \in S \big\}.$$

Observe that this relation is symmetric since $(f_n',f_n) \in \mathbb{R}$ implies $(f',f) \in \mathbb{R}$. Thus, the self-adjoint relation $S_{\rm F}$ is contained in the symmetric relation above, and therefore they coincide. □

Since $S \subset S_{\rm F} \subset S^*$ and $\operatorname{dom} S_{\rm F} \subset \overline{\operatorname{dom} S}$ by Corollary 5.3.4 one has that

$$\operatorname{dom} S \subset \operatorname{dom} S_{\rm F} \subset \big(\overline{\operatorname{dom} S} \cap \operatorname{dom} S^* \big).$$

The next corollary shows when these inclusions are identities.

Corollary 5.3.5. *Let S be a semibounded relation in $\mathfrak{H}$. Then*

$$S \,\widehat{+}\, \widehat{\mathfrak{N}}_\infty(S^*) = S_{\mathrm{F}}, \quad \widehat{\mathfrak{N}}_\infty(S^*) = \{0\} \times \operatorname{mul} S^*, \tag{5.3.6}$$

if and only if

$$\operatorname{dom} S = \overline{\operatorname{dom} S} \cap \operatorname{dom} S^*.$$

In particular, if $\operatorname{dom} S$ *is closed, then* S_{F} *has the form* (5.3.6).

Proof. Recall from (5.3.2) that $S \,\widehat{+}\, \widehat{\mathfrak{N}}_\infty(S^*) \subset S_{\mathrm{F}}$. Note that there is equality if and only if $S \,\widehat{+}\, \widehat{\mathfrak{N}}_\infty(S^*)$ is self-adjoint. Hence, the assertion follows from Lemma 1.5.7. □

The construction of the Friedrichs extension S_{F} of a semibounded relation S via the form $\mathfrak{t}_S$ in (5.3.1) leads to an important characteristic property. First of all, recall that

$$\mathfrak{t}_{S_{\mathrm{F}}}[f,g] = (f',g), \quad \{f,f'\} \in S_{\mathrm{F}}, \quad g \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}},$$

with lower bound $m(S_{\mathrm{F}}) = m(S)$. Now assume that H is another semibounded self-adjoint extension of S. Then clearly $m(H) \le m(S)$, and according to Proposition 5.1.19, the relation H generates a closed semibounded form $\mathfrak{t}_H$ on $\mathfrak{H}$ with

$$\mathfrak{t}_H[f,g] = (f',g), \quad \{f,f'\} \in H, \quad g \in \operatorname{dom} \mathfrak{t}_H.$$

By specializing $\{f,f'\} \in S$ and $g \in \operatorname{dom} S$, it follows that

$$\mathfrak{t}_H[f,g] = (f',g) = \mathfrak{t}_S[f,g]$$

and hence $\mathfrak{t}_S \subset \mathfrak{t}_H$. By construction, $\mathfrak{t}_S \subset \widetilde{\mathfrak{t}} = \mathfrak{t}_{S_{\mathrm{F}}}$ and hence $\mathfrak{t}_{S_{\mathrm{F}}}$ is the smallest closed form extension of $\mathfrak{t}_S$; cf. Theorem 5.1.12. Therefore,

$$\mathfrak{t}_S \subset \mathfrak{t}_{S_{\mathrm{F}}} \subset \mathfrak{t}_H. \tag{5.3.7}$$

This leads to the extremality property of S_{F} stated in the next result.

Proposition 5.3.6. *Let S be a semibounded relation in $\mathfrak{H}$ and let H be a semibounded self-adjoint extension of S. Then $m(\mathfrak{t}_H) = m(H) \le m(S_{\mathrm{F}}) = m(S)$ and*

$$\mathfrak{t}_H \le \mathfrak{t}_{S_{\mathrm{F}}} \quad \textit{or} \quad H \le S_{\mathrm{F}};$$

or, equivalently, for some, and hence for all $a < m(H)$,

$$(S_{\mathrm{F}} - a)^{-1} \le (H - a)^{-1}.$$

Proof. It is a consequence of (5.3.7) and Lemma 5.2.2 (i) that $\mathfrak{t}_H \le \mathfrak{t}_{S_{\mathrm{F}}}$. The rest of the statements follow from Theorem 5.2.4 and Proposition 5.2.7. □

According to Proposition 5.3.6, the Friedrichs extension $S_{\rm F}$ is the largest semibounded self-adjoint extension of S in the sense of the ordering for forms or relations, and so it has the smallest form domain. Recall that for any semibounded self-adjoint extension H of S one has for $a < m(H) \le m(S_{\rm F})$ that

$$\mathrm{dom}\,(H-a)^{\frac{1}{2}} = \mathrm{ran}\,R(a)^{\frac{1}{2}} + \mathrm{dom}\,(S_{\rm F}-a)^{\frac{1}{2}}, \tag{5.3.8}$$

where the nonnegative operator $R(a)$ is defined by

$$R(a) = (H-a)^{-1} - (S_{\rm F}-a)^{-1} \in \mathbf{B}(\mathfrak{H}); \tag{5.3.9}$$

cf. Corollary 5.2.9. The identity (5.3.8) will now be put in a geometric context. Recall that for $a < m(H)$ the closed nonnegative form $\mathfrak{t}_H - a$ on $\mathfrak{H}$ defines the following inner product on $\mathrm{dom}\,\mathfrak{t}_H$

$$(f,g)_{\mathfrak{t}_H-a} = \mathfrak{t}_H[f,g] - a(f,g), \quad f,g \in \mathrm{dom}\,\mathfrak{t}_H = \mathrm{dom}\,(H-a)^{\frac{1}{2}}, \tag{5.3.10}$$

which makes the space $\mathfrak{H}_{\mathfrak{t}_H-a} = \mathrm{dom}\,\mathfrak{t}_H = \mathrm{dom}\,(H-a)^{\frac{1}{2}}$ complete; cf. Lemma 5.1.8. Similarly, the closed nonnegative form $\mathfrak{t}_{S_{\rm F}} - a$ on $\mathfrak{H}$ defines the following inner product on $\mathrm{dom}\,\mathfrak{t}_{S_{\rm F}}$:

$$(f,g)_{\mathfrak{t}_{S_{\rm F}}-a} = \mathfrak{t}_{S_{\rm F}}[f,g] - a(f,g), \quad f,g \in \mathrm{dom}\,\mathfrak{t}_{S_{\rm F}} = \mathrm{dom}\,(S_{\rm F}-a)^{\frac{1}{2}}, \tag{5.3.11}$$

which makes the space $\mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a} = \mathrm{dom}\,\mathfrak{t}_{S_{\rm F}} = \mathrm{dom}\,(S_{\rm F}-a)^{\frac{1}{2}}$ complete. Thus, in terms of inner product spaces one obtains

$$\mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a} \subset \mathfrak{H}_{\mathfrak{t}_H-a},$$

and by (5.3.7) the restriction of the inner product in (5.3.10) to $\mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a}$ coincides with the inner product in (5.3.11). Therefore,

$$\mathfrak{H}_{\mathfrak{t}_H-a} = \big(\mathfrak{H}_{\mathfrak{t}_H-a} \ominus_{\mathfrak{t}_H-a} \mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a}\big) \oplus_{\mathfrak{t}_H-a} \mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a}, \tag{5.3.12}$$

see Corollary 5.1.13. In terms of the spaces $\mathfrak{H}_{\mathfrak{t}_H-a}$ and $\mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a}$ the sum decomposition in (5.3.8) may be rewritten as

$$\mathfrak{H}_{\mathfrak{t}_H-a} = \mathrm{ran}\,R(a)^{\frac{1}{2}} + \mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a}. \tag{5.3.13}$$

The connection between the decompositions in (5.3.12) and (5.3.13) is discussed in the next proposition.

Proposition 5.3.7. *Let S be a semibounded relation in $\mathfrak{H}$, let $S_{\rm F}$ be its Friedrichs extension, and let H be a semibounded self-adjoint extension of S with lower bound $m(H)$. Furthermore, let $a < m(H)$ and let $R(a)$ be the nonnegative operator in (5.3.9). Then*

$$\mathfrak{H}_{\mathfrak{t}_H-a} \ominus_{\mathfrak{t}_H-a} \mathfrak{H}_{\mathfrak{t}_{S_{\rm F}}-a} = \ker\,(S^*-a) \cap \mathfrak{H}_{\mathfrak{t}_H-a} = \mathrm{ran}\,R(a)^{\frac{1}{2}}, \tag{5.3.14}$$

and, consequently, the Hilbert space $\mathfrak{H}_{\mathfrak{t}_H - a}$ *has the orthogonal decomposition*

$$\begin{aligned}\mathfrak{H}_{\mathfrak{t}_H - a} &= \big(\ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a}\big) \oplus_{\mathfrak{t}_H - a} \mathfrak{H}_{\mathfrak{t}_{S_{\mathrm{F}}} - a} \\ &= \operatorname{ran} R(a)^{\frac{1}{2}} \oplus_{\mathfrak{t}_H - a} \mathfrak{H}_{\mathfrak{t}_{S_{\mathrm{F}}} - a}.\end{aligned} \tag{5.3.15}$$

In particular, the sum decomposition in (5.3.8) *is direct for every* $a < m(H)$.

Proof. First the identity

$$\mathfrak{H}_{\mathfrak{t}_H - a} \ominus_{\mathfrak{t}_H - a} \mathfrak{H}_{\mathfrak{t}_{S_{\mathrm{F}}} - a} = \ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a} \tag{5.3.16}$$

in (5.3.14) will be shown. Recall from Lemma 5.1.17 and Theorem 5.1.12 that $\mathfrak{H}_{\mathfrak{t}_S - a}$ is a dense subspace of $\mathfrak{H}_{\mathfrak{t}_{S_{\mathrm{F}}} - a}$, and hence it suffices to verify that

$$\mathfrak{H}_{\mathfrak{t}_H - a} \ominus_{\mathfrak{t}_H - a} \mathfrak{H}_{\mathfrak{t}_S - a} = \ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a}. \tag{5.3.17}$$

Assume first that g belongs to the left-hand side of (5.3.17). Then $(f, g)_{\mathfrak{t}_H - a} = 0$ for all $f \in \operatorname{dom} S$. Hence,

$$0 = (f, g)_{\mathfrak{t}_H - a} = \mathfrak{t}_H[f, g] - a(f, g) = (f', g) - a(f, g) = (f' - af, g)$$

for all $\{f, f'\} \in S \subset H$, where in the third equality the first representation theorem was used. This implies $g \in (\operatorname{ran} (S - a))^\perp \cap \mathfrak{H}_{\mathfrak{t}_H - a} = \ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a}$. Conversely, assume that $g \in \ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a}$. Then the same reasoning as above shows that g belongs to the left-hand side of (5.3.17).

In order to prove the second equality in (5.3.14), one first shows that

$$\operatorname{ran} R(a)^{\frac{1}{2}} \subset \ker (S^* - a). \tag{5.3.18}$$

To see this, let $\{f, f'\} \in S$. Then $\{f, f'\} \in H \cap S_{\mathrm{F}}$ and hence $(H - a)^{-1}(f' - af) = f$ and $(S_{\mathrm{F}} - a)^{-1}(f' - af) = f$, which implies that for $h \in \mathfrak{H}$

$$(f' - af, R(a)h) = \big((H - a)^{-1}(f' - af) - (S_{\mathrm{F}} - a)^{-1}(f' - af), h\big) = 0.$$

Therefore,

$$\operatorname{ran} R(a) \subset (\operatorname{ran} (S - a))^\perp = \ker (S^* - a)$$

and hence also $\overline{\operatorname{ran}}\, R(a) \subset \ker (S^* - a)$. Moreover, since $R(a)$ is a nonnegative self-adjoint operator it follows from Corollary D.7 that

$$\operatorname{ran} R(a) \subset \operatorname{ran} R(a)^{\frac{1}{2}} \subset \overline{\operatorname{ran}}\, R(a)^{\frac{1}{2}} = \overline{\operatorname{ran}}\, R(a) \subset \ker (S^* - a), \tag{5.3.19}$$

which shows (5.3.18). By (5.3.8), one has $\operatorname{ran} R(a)^{\frac{1}{2}} \subset \operatorname{dom} (H - a)^{\frac{1}{2}} = \mathfrak{H}_{\mathfrak{t}_H - a}$. Together with (5.3.19) one concludes that $\operatorname{ran} R(a)^{\frac{1}{2}} \subset \ker (S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_H - a}$. From (5.3.16) it is clear that

$$\operatorname{ran} R(a)^{\frac{1}{2}} \subset \mathfrak{H}_{\mathfrak{t}_H - a} \ominus_{\mathfrak{t}_H - a} \mathfrak{H}_{\mathfrak{t}_S - a}.$$

Comparing (5.3.8) with (5.3.12), one then concludes that

$$\operatorname{ran} R(a)^{\frac{1}{2}} = \mathfrak{H}_{\mathfrak{t}_{H}-a} \ominus_{\mathfrak{t}_{H}-a} \mathfrak{H}_{\mathfrak{t}_{S}-a}.$$

Together with (5.3.16) the identities in (5.3.14) follow. Furthermore, the above reasoning shows that the sum decomposition in (5.3.8) is direct. □

The semibounded self-adjoint extensions H of S for which the subspace $\ker(S^* - a) \cap \mathfrak{H}_{\mathfrak{t}_{H}-a}$ is not a proper subset of $\ker(S^* - a)$ are of special interest. In fact, they coincide with the semibounded self-adjoint extensions H for which H and S_{F} are transversal.

Theorem 5.3.8. *Let S be a semibounded relation in $\mathfrak{H}$ and let H be a semibounded self-adjoint extension of S. Then the following statements are equivalent:*

(i) *H and S_{F} are transversal, i.e., $S^* = H \,\widehat{+}\, S_{\mathrm{F}}$;*

(ii) $\ker(S^* - a) \subset \operatorname{dom}(H - a)^{\frac{1}{2}}$ *for some, and hence for all $a < m(H)$;*

(iii) $\operatorname{dom}(H - a)^{\frac{1}{2}} = \ker(S^* - a) + \operatorname{dom}(S_{\mathrm{F}} - a)^{\frac{1}{2}}$ *for some, and hence for all $a < m(H)$;*

(iv) $\operatorname{dom} S^* \subset \operatorname{dom}(H - a)^{\frac{1}{2}}$ *for some, and hence for all $a \leq m(H)$.*

Proof. (i) ⇔ (ii) In general, the self-adjoint extensions H and S_{F} are transversal if and only if for some, and hence for all $a < m(H)$

$$\operatorname{ran} R(a) = \ker(S^* - a),$$

where $R(a) = (H - a)^{-1} - (S_{\mathrm{F}} - a)^{-1}$, which follows from Theorem 1.7.8. Since $R(a)$ is a nonnegative self-adjoint operator and since $\ker(S^* - a)$ is closed, the last statement is equivalent to

$$\operatorname{ran} R(a)^{\frac{1}{2}} = \ker(S^* - a);$$

cf. Corollary D.7. It follows from Proposition 5.3.7 that this condition is the same as

$$\ker(S^* - a) \cap \operatorname{dom}(H - a)^{\frac{1}{2}} = \ker(S^* - a).$$

(ii) ⇒ (iii) This follows immediately from the direct sum decomposition

$$\operatorname{dom}(H - a)^{\frac{1}{2}} = \bigl(\ker(S^* - a) \cap \operatorname{dom}(H - a)^{\frac{1}{2}}\bigr) + \operatorname{dom}(S_{\mathrm{F}} - a)^{\frac{1}{2}};$$

cf. (5.3.8) and Proposition 5.3.7.

(iii) ⇒ (ii) This implication is trivial.

(i) ⇒ (iv) The identity $S^* = S_{\mathrm{F}} \,\widehat{+}\, H$ shows that

$$\operatorname{dom} S^* = \operatorname{dom} S_{\mathrm{F}} + \operatorname{dom} H,$$

and note that $\operatorname{dom} H$ and $\operatorname{dom} S_{\mathrm{F}}$ are subsets of $\operatorname{dom}(H - a)^{\frac{1}{2}}$.

(iv) ⇒ (ii) This is clear. □

The following result is a consequence of Theorem 5.3.8; it describes the behavior of any semibounded self-adjoint extension H' of S in the presence of a semibounded self-adjoint extension H such that H and S_{F} are transversal.

Corollary 5.3.9. *Let S be a semibounded relation in $\mathfrak{H}$ and let H be a semibounded self-adjoint extension of S such that H and S_{F} are transversal. Then every semibounded self-adjoint extension H' of S satisfies*

$$\mathrm{dom}\,(H'-a)^{\frac{1}{2}} \subset \mathrm{dom}\,(H-a)^{\frac{1}{2}}, \quad a < \min\{m(H), m(H')\}, \tag{5.3.20}$$

in which case there exists $C > 0$ such that

$$\|(H_{\mathrm{op}}-a)^{\frac{1}{2}}\varphi\| \leq C\|((H')_{\mathrm{op}}-a)^{\frac{1}{2}}\varphi\| \tag{5.3.21}$$

for all $\varphi \in \mathrm{dom}\,(H'-a)^{\frac{1}{2}}$. Moreover, there is equality in (5.3.20) if and only if H' and S_{F} are transversal, in which case there exist $c > 0$ and $C > 0$ such that

$$c\|((H')_{\mathrm{op}}-a)^{\frac{1}{2}}\varphi\| \leq \|(H_{\mathrm{op}}-a)^{\frac{1}{2}}\varphi\| \leq C\|((H')_{\mathrm{op}}-a)^{\frac{1}{2}}\varphi\| \tag{5.3.22}$$

for all $\varphi \in \mathrm{dom}\,(H'-a)^{\frac{1}{2}} = \mathrm{dom}\,(H-a)^{\frac{1}{2}}$.

Proof. By Theorem 5.3.8, the semibounded self-adjoint extension H of S satisfies all of the equivalent conditions (i)–(iv) in Theorem 5.3.8. Let H' be another semibounded self-adjoint extension of S. Applying Proposition 5.3.7 to H' one sees that for $a < m(H')$

$$\mathrm{dom}\,(H'-a)^{\frac{1}{2}} = \big(\ker\,(S^*-a) \cap \mathfrak{H}_{\mathfrak{t}_{H'-a}}\big) + \mathrm{dom}\,(S_{\mathrm{F}}-a)^{\frac{1}{2}}.$$

Choosing $a < \min\{m(H), m(H')\}$ it follows from Theorem 5.3.8 (iii) for H that the inclusion (5.3.20) holds. The inequality (5.3.21) is a direct consequence of Proposition 1.5.11.

Assume that there is equality in (5.3.20). Then Theorem 5.3.8 (iii) implies that H' and S_{F} are transversal. Conversely, if H' and S_{F} are transversal, then it follows from Theorem 5.3.8 (iii) that there is equality in (5.3.20). If there is equality in (5.3.20), then (5.3.21) also holds for some $c > 0$ when H and H' are interchanged. Thus, the inequalities in (5.3.22) hold. □

The extreme case of equality of H and S_{F} is described in the following immediate corollary of Proposition 5.3.7 and (5.3.8).

Corollary 5.3.10. *Let S be a semibounded relation in $\mathfrak{H}$ and let H be a semibounded self-adjoint extension of S. Then $H = S_{\mathrm{F}}$ if and only if*

$$\ker\,(S^*-a) \cap \mathrm{dom}\,(H-a)^{\frac{1}{2}} = \{0\}$$

for some, and hence for all $a < m(H)$.

In the next corollary the form corresponding to a semibounded self-adjoint extension H which is transversal to S_{F} is specified.

Corollary 5.3.11. *Let S be a semibounded relation in $\mathfrak{H}$, let H be a semibounded self-adjoint extension of S such that H and $S_{\rm F}$ are transversal, and let $\mathfrak{t}_{S_{\rm F}}$ and $\mathfrak{t}_H$ be the corresponding closed semibounded forms in $\mathfrak{H}$. Then*

$$\operatorname{dom} \mathfrak{t}_H = \ker (S^* - a) \oplus_{\mathfrak{t}_H - a} \operatorname{dom} \mathfrak{t}_{S_{\rm F}}, \qquad a < m(H), \tag{5.3.23}$$

and the restriction of $\mathfrak{t}_H$ to $\mathfrak{N}_a(S^*) = \ker (S^* - a)$ is a closed form in $\mathfrak{N}_a(S^*)$ which is bounded from below by $m(H)$ and represented by a bounded self-adjoint operator $L_a \in \mathbf{B}(\mathfrak{N}_a(S^*))$. Furthermore, one has

$$\mathfrak{t}_H[f,g] - a(f,g) = \big((L_a - a)f_a, g_a\big) + \mathfrak{t}_{S_{\rm F}}[f_{\rm F}, g_{\rm F}] - a(f_{\rm F}, g_{\rm F}) \tag{5.3.24}$$

for all $f = f_a + f_{\rm F}, g = g_a + g_{\rm F} \in \operatorname{dom} \mathfrak{t}_H$, where $f_a, g_a \in \ker (S^* - a)$ and $f_{\rm F}, g_{\rm F} \in \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$.

Proof. The orthogonal decomposition (5.3.23) of $\operatorname{dom} \mathfrak{t}_H$ follows from (5.3.15) in Proposition 5.3.7 and Theorem 5.3.8 (iii). Since the restriction of the form $\mathfrak{t}_H$ to $\mathfrak{N}_a(S^*)$ is a closed form which is bounded from below by $m(H)$, it follows from Theorem 5.1.18 that there exists a semibounded self-adjoint relation L_a in $\mathfrak{N}_a(S^*)$ which represents this form. Moreover, it follows from Theorem 5.1.23 that $\mathfrak{N}_a(S^*) = \operatorname{dom} (L_a - x)^{\frac{1}{2}}$, $x \leq m(H)$, and hence $(L_a - x)^{\frac{1}{2}}$ and L_a are bounded self-adjoint operators defined on $\mathfrak{N}_a(S^*)$.

For $f, g \in \operatorname{dom} \mathfrak{t}_H$ decomposed, with respect to (5.3.23), in

$$f = f_a + f_{\rm F} \quad \text{and} \quad g = g_a + g_{\rm F},$$

where $f_a, g_a \in \ker (S^* - a)$ and $f_{\rm F}, g_{\rm F} \in \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$, one has $(f_a, g_{\rm F})_{\mathfrak{t}_H - a} = 0$ and $(f_{\rm F}, g_a)_{\mathfrak{t}_H - a} = 0$, and hence

$$\begin{aligned} \mathfrak{t}_H[f,g] - a(f,g) &= (f,g)_{\mathfrak{t}_H - a} = (f_a, g_a)_{\mathfrak{t}_H - a} + (f_{\rm F}, g_{\rm F})_{\mathfrak{t}_H - a} \\ &= \mathfrak{t}_H[f_a, g_a] - a(f_a, g_a) + \mathfrak{t}_{S_{\rm F}}[f_{\rm F}, g_{\rm F}] - a(f_{\rm F}, g_{\rm F}). \end{aligned}$$

Now (5.3.24) follows from $\mathfrak{t}_H[f_a, g_a] = (L_a f_a, g_a)$. □

5.4 Semibounded self-adjoint extensions and their lower bounds

Let S be a, not necessarily closed, semibounded relation in a Hilbert space $\mathfrak{H}$ with lower bound $m(S)$. The Friedrichs extension $S_{\rm F}$ is a self-adjoint extension of S whose lower bound $m(S_{\rm F})$ is equal to the lower bound $m(S)$ of S. If H is a semibounded self-adjoint extension of S, then necessarily $m(H) \leq m(S)$. In this section the so-called Kreĭn type extensions $S_{{\rm K},x}$ of S will be introduced. They can be viewed as generalizations of the Kreĭn–von Neumann extension of a nonnegative symmetric operator or relation. The Kreĭn type extension $S_{{\rm K},x}$ can be used to describe all semibounded self-adjoint extensions H of S whose lower bound satisfies $m(H) \in [x, m(S)]$ when $x \leq m(S)$.

Let S be a semibounded relation in $\mathfrak{H}$ with lower bound $\gamma = m(S)$. It is clear that $x \leq \gamma$ implies that $S - x \geq 0$. Hence, for $x \leq \gamma$ the relation $(S - x)^{-1}$ is nonnegative and one can define the Friedrichs extension $((S-x)^{-1})_{\mathrm{F}}$ of $(S-x)^{-1}$, which is nonnegative; cf. Definition 5.3.2.

Lemma 5.4.1. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. For $x \leq \gamma$ the relation $S_{\mathrm{K},x}$ defined by*

$$S_{\mathrm{K},x} := \big(((S - x)^{-1})_{\mathrm{F}}\big)^{-1} + x \tag{5.4.1}$$

is a semibounded self-adjoint extension of S with lower bound $m(S_{\mathrm{K},x}) = x$. Moreover, $S_{\mathrm{K},x} = (\overline{S})_{\mathrm{K},x}$ for $x \leq \gamma$ and

$$S \subset S \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*) \subset \overline{S} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*) \subset S_{\mathrm{K},x}, \quad x \leq \gamma, \tag{5.4.2}$$

while, in particular,

$$\overline{S} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*) = S_{\mathrm{K},x}, \quad x < \gamma. \tag{5.4.3}$$

Proof. Since for $x \leq \gamma$ the Friedrichs extension $((S - x)^{-1})_{\mathrm{F}}$ of the nonnegative relation $(S - x)^{-1}$ is nonnegative, it follows that

$$\big(((S - x)^{-1})_{\mathrm{F}}\big)^{-1}$$

is a nonnegative self-adjoint extension of $S - x$. Hence, $S_{\mathrm{K},x}$ defined by (5.4.1) is a self-adjoint extension of S and, clearly,

$$m(S_{\mathrm{K},x}) \geq x, \quad x \leq \gamma. \tag{5.4.4}$$

Since the closure of $(S-x)^{-1}$ is given by $(\overline{S}-x)^{-1}$, it follows from Lemma 5.3.1, with S replaced by $(S-x)^{-1}$, that

$$((\overline{S} - x)^{-1})_{\mathrm{F}} = ((S - x)^{-1})_{\mathrm{F}},$$

which leads to $S_{\mathrm{K},x} = (\overline{S})_{\mathrm{K},x}$ for $x \leq \gamma$.

Let $x \leq \gamma$ and note that the first and second inclusion in (5.4.2) are clear. It is also clear that $S \subset \overline{S} \subset S_{\mathrm{K},x}$; thus, to show the third inclusion in (5.4.2) it suffices to check that $\widehat{\mathfrak{N}}_x(S^*) \subset S_{\mathrm{K},x}$. Set $T = (S-x)^{-1}$, so that T is nonnegative and $\mathrm{mul}\, T^* = \mathfrak{N}_x(S^*)$. By Lemma 5.3.1, $\{0\} \times \mathrm{mul}\, T^* \subset T_{\mathrm{F}}$ or, equivalently,

$$\{0\} \times \mathfrak{N}_x(S^*) \subset ((S-x)^{-1})_{\mathrm{F}} \quad \text{or} \quad \mathfrak{N}_x(S^*) \times \{0\} \subset ((S-x)^{-1})_{\mathrm{F}})^{-1}.$$

Thus, $\widehat{\mathfrak{N}}_x(S^*) \subset S_{\mathrm{K},x}$ which completes the argument.

For $x < \gamma$ it follows from Proposition 1.4.6 and Lemma 1.2.2 that $\mathrm{ran}\,(\overline{S} - x)$ is closed. Hence, the relation $\overline{S} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*)$ is self-adjoint, cf. Lemma 1.5.7, and thus the equality (5.4.3) prevails.

It remains to show that $m(S_{K,x}) = x$. When $x < \gamma$ one concludes this from (5.4.3). When $x = \gamma$ observe that $S \subset S_{K,\gamma}$ implies $m(S_{K,\gamma}) \leq m(S) = \gamma$. On the other hand, from (5.4.4) it follows that $m(S_{K,\gamma}) \geq \gamma$. □

Definition 5.4.2. Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ and let $x \leq \gamma$. The semibounded self-adjoint extensions $S_{K,x}$ in (5.4.1) are called *Kreĭn type extensions* of S. In the case $\gamma \geq 0$ the nonnegative self-adjoint extension $S_{K,0}$ is called the *Kreĭn–von Neumann extension* of S.

The definition of the Kreĭn type extensions $S_{K,x}$ in (5.4.1) incorporates the lower bound $m(S) = \gamma$ of S. Note that $m(S - x) = \gamma - x$ for any $x \in \mathbb{R}$, so that $(S - x)_{K,\gamma - x}$ is well defined, and

$$(S - x)_{K,\gamma-x} = \big((((S - x) - (\gamma - x))^{-1})_F\big)^{-1} + \gamma - x, \quad x \in \mathbb{R},$$

which leads to

$$(S - x)_{K,\gamma-x} = S_{K,\gamma} - x, \quad x \in \mathbb{R}. \tag{5.4.5}$$

The identity (5.4.5) is the analog for the Kreĭn type extension $S_{K,\gamma}$ of the shift invariance property (5.3.4) of S_F. There are some more useful identities involving Kreĭn type extensions of S. First note the simple equality

$$\big(S \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_x(S^*)\big)^{-1} = S^{-1} \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_{1/x}(S^{-*}), \quad x \in \mathbb{R} \setminus \{0\}. \tag{5.4.6}$$

If $S \geq 0$ or, equivalently, $S^{-1} \geq 0$, then it follows from (5.4.6) and Lemma 5.4.1 that

$$(S_{K,x})^{-1} = (S^{-1})_{K,1/x}, \quad x < 0, \tag{5.4.7}$$

since $0 \leq \min\{m(S), m(S^{-1})\}$. In particular, one sees from (5.4.7) that

$$\big((S - \gamma)_{K,x}\big)^{-1} = \big((S - \gamma)^{-1}\big)_{K,1/x}, \quad x < 0. \tag{5.4.8}$$

Returning to the general case where S has lower bound γ, note the simple equality

$$\big(S \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_{a+x}(S^*)\big) - a = (S - a) \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_x((S - a)^*), \quad a, x \in \mathbb{R}.$$

For $a + x < \gamma$ this implies

$$S_{K,a+x} - a = (S - a)_{K,x}, \tag{5.4.9}$$

and taking $a = \gamma$ in (5.4.9) gives an analog of (5.4.5):

$$S_{K,\gamma+x} - \gamma = (S - \gamma)_{K,x}, \quad x < 0. \tag{5.4.10}$$

By Lemma 5.4.1, the Kreĭn type extensions $S_{K,x}$, $x \leq \gamma$, are semibounded self-adjoint extensions of S. As restrictions of S^* the Kreĭn type extensions can be characterized in a similar way as the Friedrichs extension S_F; cf. Theorem 5.3.3.

Theorem 5.4.3. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Then for each $x \leq \gamma$ the Kreĭn type extension $S_{K,x}$ of S has the representation*

$$S_{K,x} = \big\{\{f, f'\} \in S^* : f' - xf \in \operatorname{dom} \mathfrak{t}_{((S-x)^{-1})_F}\big\} \tag{5.4.11}$$

with $\ker (S_{K,x} - x) = \ker (S^* - x)$. *Furthermore, if H is a self-adjoint extension of S, which is not necessarily semibounded, then*

$$\operatorname{ran} (H - x) \subset \operatorname{dom} \mathfrak{t}_{((S-x)^{-1})_K} \quad \Rightarrow \quad H = S_{K,x}.$$

Proof. Let $x \leq \gamma$. Then by definition one has $(S_{K,x} - x)^{-1} = ((S - x)^{-1})_F$ and $\{f, f'\} \in S_{K,x}$ if and only if $\{f' - xf, f\} \in (S_{K,x} - x)^{-1}$. Similarly, $\{f, f'\} \in S^*$ if and only if $\{f' - xf, f\} \in (S^* - x)^{-1}$. Hence, the description (5.4.11) follows from the representation of $((S - x)^{-1})_F$ in Theorem 5.3.3 (with S now replaced by $(S - x)^{-1}$). Likewise,

$$\ker (S_{K,x} - x) = \operatorname{mul} ((S - x)^{-1})_F = \operatorname{mul} (S^* - x)^{-1} = \ker (S^* - x).$$

The last item also follows from Theorem 5.3.3. □

There is also an approximation of $S_{K,x}$ by elements in S as in Corollary 5.3.4; in particular, this gives a useful description of $\operatorname{mul} S_{K,x}$.

Corollary 5.4.4. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Then $S_{K,x}$, $x \leq \gamma$, is the set of all elements $\{f, f'\} \in S^*$ for which there exists a sequence $(\{f_n, f_n'\})$ in S such that*

$$f_n' - xf_n \to f' - xf \quad \text{and} \quad (f_n, f_n' - xf_n) \to (f, f' - xf).$$

In particular, $\operatorname{mul} S_{K,x}$ *is the set of all elements $f' \in \operatorname{mul} S^*$ for which there exists a sequence $(\{f_n, f_n'\})$ in S such that*

$$f_n' - xf_n \to f' \quad \text{and} \quad (f_n, f_n' - xf_n) \to 0.$$

As in the case of the Friedrichs extension, the Kreĭn type extension $S_{K,\gamma}$ can sometimes be explicitly given in terms of S and an eigenspace of S^*; cf. Lemma 1.5.7. The following result is the analog of Corollary 5.3.5. The special case where $\operatorname{ran} (S - \gamma)$ is closed is particularly useful.

Corollary 5.4.5. *Let S be a semibounded relation with lower bound γ. Then*

$$S_{K,\gamma} = S \,\widehat{+}\, \widehat{\mathfrak{N}}_\gamma(S^*) \tag{5.4.12}$$

if and only if

$$\operatorname{ran} (S - \gamma) = \overline{\operatorname{ran}} (S - \gamma) \cap \operatorname{ran} (S^* - \gamma).$$

In particular, if $\operatorname{ran} (S - \gamma)$ *is closed, then $S_{K,\gamma}$ has the form (5.4.12).*

The semibounded self-adjoint extensions $S_{K,x}$ with $x \leq \gamma$ become extremal extensions of S when a lower bound $x \leq \gamma$ for semibounded self-adjoint extensions of S is prescribed.

Theorem 5.4.6. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Let $x \leq \gamma$ be fixed and let H be a semibounded self-adjoint relation in $\mathfrak{H}$. Then the following equivalence holds:*

$$S \subset H \quad \textit{and} \quad x \leq m(H) \quad \Leftrightarrow \quad S_{K,x} \leq H \leq S_F. \tag{5.4.13}$$

In particular, the class of semibounded self-adjoint extensions of S preserving the lower bound of S is characterized by

$$S \subset H \quad \textit{and} \quad \gamma = m(H) \quad \Leftrightarrow \quad S_{K,\gamma} \leq H \leq S_F. \tag{5.4.14}$$

In fact, $S_{K,x} \leq H \leq S_F$, $x \leq \gamma$, implies that $S \subset (S_F \cap S_{K,x}) \subset H$.

Proof. ($\Rightarrow$) Assume that H is a semibounded self-adjoint extension of S with lower bound $m(H) \geq x$. Then clearly $S - x \subset H - x$ and here both sides are nonnegative relations. But then also $(S-x)^{-1} \subset (H-x)^{-1}$, where both sides are nonnegative relations. Applying Proposition 5.3.6 to the relation $(S-x)^{-1}$ one obtains

$$(H-x)^{-1} \leq ((S-x)^{-1})_F.$$

Since these relations are nonnegative, the antitonicity property of the inverse in Corollary 5.2.8 gives the inequality

$$\big(((S-x)^{-1})_F\big)^{-1} \leq H - x$$

or, equivalently, $S_{K,x} \leq H$. The inequality $H \leq S_F$ holds by Proposition 5.3.6.

($\Leftarrow$) Let H be a semibounded self-adjoint relation such that $S_{K,x} \leq H \leq S_F$. Then $m(S_{K,x}) \leq m(H) \leq m(S_F)$ by Lemma 5.2.5 (ii) and, since $m(S_{K,x}) = x$ and $m(S_F) = \gamma$, one concludes that H is semibounded with $x \leq m(H) \leq \gamma$.

It remains to show that H is an extension of S. With $a < x$ the assumption on H is equivalent to

$$\big((S_F - a)^{-1}h, h\big) \leq \big((H-a)^{-1}h, h\big) \leq \big((S_{K,x} - a)^{-1}h, h\big), \quad h \in \mathfrak{H}; \tag{5.4.15}$$

cf. Proposition 5.2.7. For $R(a) = (H-a)^{-1} - (S_F - a)^{-1} \in \mathbf{B}(\mathfrak{H})$ it follows from (5.4.15) that

$$0 \leq (R(a)h, h) \leq \big((S_{K,x} - a)^{-1}h, h\big) - \big((S_F - a)^{-1}h, h\big), \quad h \in \mathfrak{H}. \tag{5.4.16}$$

Now, let $\{f, f'\} \in S$ and define $h = f' - af$. Then $h \in \operatorname{ran}(S-a)$ and $\{h, f\} \in (S-a)^{-1}$, and hence

$$\{h, f\} \in (S_F - a)^{-1} \cap (S_{K,x} - a)^{-1},$$

so that $(S_F - a)^{-1}h = (S_{K,x} - a)^{-1}h$. It follows from (5.4.16) that $(R(a)h, h) = 0$, and since $R(a) \geq 0$, one concludes that $R(a)h = 0$ for all $h \in \operatorname{ran}(S-a)$.

In other words,

$$(H-a)^{-1}(f'-af) = (S_{\mathrm{F}}-a)^{-1}(f'-af) = f, \quad \{f,f'\} \in S,$$

and it follows that $\{f,f'\} \in H$. This proves the claim $S \subset H$ and completes the proof of (5.4.13). The inclusion $S_{\mathrm{F}} \cap S_{\mathrm{K},x} \subset H$ follows similarly. □

If S is nonnegative, then Theorem 5.4.6 shows that the Kreĭn–von Neumann extension

$$S_{\mathrm{K},0} = \big((S^{-1})_{\mathrm{F}}\big)^{-1} \tag{5.4.17}$$

in Definition 5.4.2 is the smallest nonnegative self-adjoint extension.

Corollary 5.4.7. *Let S be a nonnegative relation in $\mathfrak{H}$ and let H be a semibounded self-adjoint relation in $\mathfrak{H}$. Then the following equivalence holds:*

$$S \subset H \quad \textit{and} \quad m(H) \geq 0 \quad \Leftrightarrow \quad S_{\mathrm{K},0} \leq H \leq S_{\mathrm{F}}.$$

In fact, $S_{\mathrm{K},0} \leq H \leq S_{\mathrm{F}}$, implies that $S \subset (S_{\mathrm{F}} \cap S_{\mathrm{K},0}) \subset H$.

For completeness it is observed that the inequalities $S_{\mathrm{K},x} \leq H \leq S_{\mathrm{F}}$ in (5.4.13) can also be expressed by inequalities for the corresponding forms:

$$\mathfrak{t}_{S_{\mathrm{K},x}} \leq \mathfrak{t}_H \leq \mathfrak{t}_{S_{\mathrm{F}}},$$

thanks to Theorem 5.2.4. Recall from (5.3.7) that the inequality $\mathfrak{t}_H \leq \mathfrak{t}_{S_{\mathrm{F}}}$ is in fact equivalent to the inclusion $\mathfrak{t}_{S_{\mathrm{F}}} \subset \mathfrak{t}_H$.

It is clear from Lemma 5.3.1 or Theorem 5.3.3 that the Friedrichs extension S_{F} is an operator if and only if S is densely defined, in which case all self-adjoint extensions of S are operators. If S is not densely defined, then S may not be closable as an operator, in which case all self-adjoint extensions of S are multivalued. The following result shows when semibounded self-adjoint operator extensions exist.

Corollary 5.4.8. *Let S be a semibounded operator in $\mathfrak{H}$ with lower bound γ. Then the following statements are equivalent:*

(i) *S has a semibounded self-adjoint operator extension;*

(ii) *for some $x \leq \gamma$ the self-adjoint extension $S_{\mathrm{K},x}$ is an operator;*

(iii) *there exists $x \leq \gamma$ such that for any sequence $(\{f_n, f_n'\})$ in S with the properties $f_n' - xf_n \to f'$ and $(f_n, f_n' - xf_n) \to 0$ one has $f' = 0$.*

If any of these statements hold, then S is a closable operator. Furthermore, the following statements are equivalent:

(i′) *S has a bounded self-adjoint operator extension;*

(ii′) *for some $x \leq \gamma$ the self-adjoint extension $S_{\mathrm{K},x}$ belongs to $\mathbf{B}(\mathfrak{H})$;*

(iii′) *there exist $x \leq \gamma$ and $M \geq 0$ such that $\|f' - xf\|^2 \leq M(f, f' - xf)$ for all $\{f, f'\} \in S$.*

If any of these statements hold, then S is a bounded operator.

Proof. (i) $\Rightarrow$ (ii) Let H be a semibounded self-adjoint operator extension of S. Then H is densely defined and $\operatorname{mul} H = \{0\}$. If $x = m(H)$, then Theorem 5.4.6 shows that $S_{\mathrm{K},x} \leq H$. Hence, $\operatorname{mul} S_{\mathrm{K},x} \subset \operatorname{mul} H = \{0\}$ by Lemma 5.2.5 (i), and therefore $S_{\mathrm{K},x}$ is an operator extension of S.

(ii) $\Rightarrow$ (i) This is clear.

(ii) $\Leftrightarrow$ (iii) This follows from Corollary 5.4.4.

The inclusion $S \subset H$ for a self-adjoint operator H shows that S is a closable operator; this holds when one of the equivalent conditions (i)–(iii) is satisfied.

(i′) $\Rightarrow$ (iii′) Let H be a bounded self-adjoint operator which extends S and let $m(H) = x$. Then $H \in \mathbf{B}(\mathfrak{H})$ and an application of the Cauchy–Schwarz inequality for the inner product $((H-x)\cdot, \cdot)$ yields

$$\|(H-x)f\|^2 \leq \|H-x\| \, (f, (H-x)f), \quad f \in \mathfrak{H}.$$

In particular, for $f \in \operatorname{dom} S$ one obtains (iii′) with $f' = Sf$ and $M = \|H - x\|$.

(iii′) $\Rightarrow$ (ii′) Assume that (iii′) holds for some x. To show (ii′) let $\{f, f'\} \in S_{\mathrm{K},x}$. Then according to Corollary 5.4.4 there exists a sequence $(\{f_n, f_n'\})$ in S such that $f_n' - xf_n \to f' - xf$ and $(f_n, f_n' - xf_n) \to (f, f' - xf)$. By assumption $\|f_n' - xf_n\|^2 \leq M(f_n, f_n' - xf_n)$ and taking limits gives

$$\|f' - xf\|^2 \leq M(f, f' - xf) \leq M\|f\|\|f' - xf\|.$$

Thus, if $\{f, f'\} \in S_{\mathrm{K},x}$, then $\|f' - xf\| \leq M\|f\|$, which gives $\operatorname{mul} S_{\mathrm{K},x} = \{0\}$. In addition, one now sees that $S_{\mathrm{K},x} - x$ is a bounded operator, and since $S_{\mathrm{K},x}$ is self-adjoint, it follows that $S_{\mathrm{K},x} \in \mathbf{B}(\mathfrak{H})$.

(ii′) $\Rightarrow$ (i′) This is clear.

The last statement follows from any of the statements (i′), (ii′), and (iii′). □

Let S be a semibounded relation in $\mathfrak{H}$ with lower bound $m(S) = \gamma$. By means of Theorem 5.4.6 it will be shown that the mapping $x \mapsto S_{\mathrm{K},x}$, $x < \gamma$, is nondecreasing.

Corollary 5.4.9. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Let $x \leq y < \gamma$, then*

$$S_{\mathrm{K},x} \leq S_{\mathrm{K},y} \leq S_{\mathrm{K},\gamma} \leq S_{\mathrm{F}}.$$

Proof. By construction, $S_{\mathrm{K},x}$ and $S_{\mathrm{K},y}$ are semibounded self-adjoint extensions of S with lower bounds $m(S_{\mathrm{K},x}) = x$ and $m(S_{\mathrm{K},y}) = y$. Hence, $m(S_{\mathrm{K},x}) \leq m(S_{\mathrm{K},y})$ and an application of (5.4.13) in Theorem 5.4.6 gives $S_{\mathrm{K},x} \leq S_{\mathrm{K},y}$. Similarly, $m(S_{\mathrm{K},y}) \leq m(S_{\mathrm{K},\gamma}) = \gamma$ leads to $S_{\mathrm{K},y} \leq S_{\mathrm{K},\gamma}$. The inequality $S_{\mathrm{K},\gamma} \leq S_{\mathrm{F}}$ also follows from Theorem 5.4.6. □

The Friedrichs extension S_{F} and the Kreĭn type extensions $S_{\mathrm{K},x}$ can be approximated in the strong resolvent sense by the semibounded self-adjoint relations $S_{\mathrm{K},t}$ with $t \in (-\infty, \gamma)$; cf. Theorem 5.2.11.

Theorem 5.4.10. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Then the Friedrichs extension S_{F} is given by the strong resolvent limit*

$$(S_{\mathrm{F}} - \lambda)^{-1} h = \lim_{t \downarrow -\infty} (S_{\mathrm{K},t} - \lambda)^{-1} h, \quad h \in \mathfrak{H}, \tag{5.4.18}$$

where $\lambda \in \mathbb{C} \setminus [\gamma, \infty)$, and for each $x \leq \gamma$ the Kreĭn type extension $S_{\mathrm{K},x}$ is given by the strong resolvent limit

$$(S_{\mathrm{K},x} - \lambda)^{-1} h = \lim_{t \uparrow x} (S_{\mathrm{K},t} - \lambda)^{-1} h, \quad h \in \mathfrak{H}, \tag{5.4.19}$$

where $\lambda \in \mathbb{C} \setminus [x, \infty)$.

Proof. First the result in (5.4.19) will be shown. Let $x \leq \gamma$, let $\varepsilon > 0$ be arbitrary, and note that by Corollary 5.4.9

$$S_{\mathrm{K},x-\varepsilon} \leq S_{\mathrm{K},t} \leq S_{\mathrm{K},x}, \quad x - \varepsilon \leq t < x.$$

In particular, for $t \in [x-\varepsilon, x)$ the relations $S_{\mathrm{K},t}$ are bounded from below by $x - \varepsilon$. By the monotonicity of $S_{\mathrm{K},t}$ and Theorem 5.2.11, the strong resolvent limit of $S_{\mathrm{K},t}$ as $t \uparrow x$ exists for $\lambda \in \mathbb{C} \setminus [x - \varepsilon, \infty)$ and it is a semibounded self-adjoint relation S' with $x - \varepsilon \leq t \leq S_{\mathrm{K},t} \leq S'$. It will now be shown that

$$S' = S_{\mathrm{K},x}. \tag{5.4.20}$$

In fact, since $S_{\mathrm{K},t} \leq S_{\mathrm{K},x}$, there is a common upper bound and hence one has $S' \leq S_{\mathrm{K},x}$; see Corollary 5.2.12 (i). As $S \subset S_{\mathrm{K},t}$ for all $t < x$, this implies that $S \subset S'$ by Corollary 5.2.12 (ii). Thus, S' is a semibounded self-adjoint extension of S. Since $m(S_{\mathrm{K},t}) \leq m(S')$ for all $t < x$, it follows that $x \leq m(S')$ and, hence $S_{\mathrm{K},x} \leq S'$ by Theorem 5.4.6. Combining $S' \leq S_{\mathrm{K},x}$ and $S_{\mathrm{K},x} \leq S'$, it follows from Lemma 5.2.5 (iv) that (5.4.20) holds. This establishes (5.4.19) for $\lambda \in \mathbb{C} \setminus [x-\varepsilon, \infty)$. Since $\varepsilon > 0$ is arbitrary, one obtains (5.4.19).

Next, (5.4.18) will be shown. Apply the previous result (5.4.19) to the Kreĭn–von Neumann extension $((S-\gamma)^{-1})_{\mathrm{K},0}$ of the nonnegative relation $(S-\gamma)^{-1}$:

$$\big(((S-\gamma)^{-1})_{\mathrm{K},0} - \lambda\big)^{-1} h = \lim_{t \uparrow 0} \big(((S-\gamma)^{-1})_{\mathrm{K},t} - \lambda\big)^{-1} h, \quad h \in \mathfrak{H}, \tag{5.4.21}$$

where $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then it follows from (5.4.1) (with $x = 0$ and S replaced by $(S-\gamma)^{-1}$) and the translation invariance property (5.3.4) of the Friedrichs extension that

$$((S-\gamma)^{-1})_{\mathrm{K},0} = ((S-\gamma)_{\mathrm{F}})^{-1} = (S_{\mathrm{F}} - \gamma)^{-1}. \tag{5.4.22}$$

Likewise, using (5.4.8) and (5.4.10) one obtains for $t < 0$ that

$$((S-\gamma)^{-1})_{\mathrm{K},t} = ((S-\gamma)_{\mathrm{K},1/t})^{-1} = (S_{\mathrm{K},\gamma+1/t} - \gamma)^{-1}. \tag{5.4.23}$$

Substitute (5.4.22) and (5.4.23) into (5.4.21) and replace λ by $1/\lambda$:

$$\left((S_{\mathrm{F}}-\gamma)^{-1} - \frac{1}{\lambda}\right)^{-1} h = \lim_{t\uparrow 0}\left((S_{\mathrm{K},\gamma+1/t}-\gamma)^{-1} - \frac{1}{\lambda}\right)^{-1} h, \quad h \in \mathfrak{H}. \tag{5.4.24}$$

Now recall that for any relation H one has the identity

$$(H^{-1} - 1/\lambda)^{-1} = -\lambda - \lambda^2 (H-\lambda)^{-1}, \quad \lambda \neq 0,$$

see Corollary 1.1.12. Therefore, (5.4.24) yields

$$\big(S_{\mathrm{F}} - \gamma - \lambda\big)^{-1} h = \lim_{t\uparrow 0}\big(S_{\mathrm{K},\gamma+1/t} - \gamma - \lambda\big)^{-1},$$

where $\lambda \in \mathbb{C}\setminus[0,\infty)$, which is equivalent to (5.4.18). □

The next lemma and Proposition 5.4.12 below show that the convergence in (5.4.18) in Theorem 5.4.10 is uniform if the limit is a compact operator. First the case where the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$ is compact is treated. There is in general no analog of Lemma 5.4.11 for the other Kreĭn type extensions $S_{\mathrm{K},x}$, $x \neq 0$, since the eigenspace $\ker(S_{\mathrm{K},x} - x) = \ker(S^* - x)$ for the eigenvalue $x \in \sigma_{\mathrm{p}}(S_{\mathrm{K},x})$ is infinite-dimensional whenever the defect numbers of S are infinite. Hence, $S_{\mathrm{K},x}$ cannot be compact for $x \neq 0$.

Lemma 5.4.11. *Let S be a bounded nonnegative operator in $\mathfrak{H}$ and assume that the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$ is a compact operator. Then $S_{\mathrm{K},t} \in \mathbf{B}(\mathfrak{H})$ for $t < 0$ and*

$$\lim_{t\uparrow 0}\|S_{\mathrm{K},t} - S_{\mathrm{K},0}\| = 0.$$

Proof. Since $S_{\mathrm{K},0}$ is compact one has, in particular, $S_{\mathrm{K},0} \in \mathbf{B}(\mathfrak{H})$ and hence $S_{\mathrm{K},t} \in \mathbf{B}(\mathfrak{H})$ for $t < 0$; cf. Corollary 5.4.9 and Definition 5.2.3. By Theorem 5.4.10, the resolvents of $S_{\mathrm{K},t}$ converge in the strong sense to the resolvent of $S_{\mathrm{K},0}$ and since all operators belong to $\mathbf{B}(\mathfrak{H})$ it follows that $S_{\mathrm{K},t}$ converges strongly to $S_{\mathrm{K},0}$. In fact, strong resolvent convergence is equivalent to strong graph convergence by Corollary 1.9.6, and for operators in $\mathbf{B}(\mathfrak{H})$ this implies strong convergence. Now it will be shown that this convergence is uniform. Since $S_{\mathrm{K},0}$ is compact by assumption, for $\varepsilon > 0$ one can choose an orthogonal projection P_ε such that $\|S_{\mathrm{K},0}P_\varepsilon\| < \varepsilon$ and $I - P_\varepsilon$ is a finite-rank operator. Then it follows that the finite-rank operators

$$(S_{\mathrm{K},t} - S_{\mathrm{K},0})(I - P_\varepsilon) \quad \text{and} \quad (I-P_\varepsilon)(S_{\mathrm{K},t} - S_{\mathrm{K},0})P_\varepsilon \tag{5.4.25}$$

tend to zero uniformly as $t \uparrow 0$. For $t < 0$ one has $0 \le S_{\mathrm{K},t} - t \le S_{\mathrm{K},0} - t$ by Corollary 5.4.9, and hence

$$0 \le P_\varepsilon(S_{\mathrm{K},t} - t)P_\varepsilon \le P_\varepsilon(S_{\mathrm{K},0} - t)P_\varepsilon.$$

This implies

$$\begin{aligned}\|P_\varepsilon(S_{\mathrm{K},t}-S_{\mathrm{K},0})P_\varepsilon\| &\le \|P_\varepsilon(S_{\mathrm{K},t}-t)P_\varepsilon\|+\|P_\varepsilon(t-S_{\mathrm{K},0})P_\varepsilon\|\\ &\le 2\|P_\varepsilon(S_{\mathrm{K},0}-t)P_\varepsilon\|\\ &\le 2\varepsilon+2|t|,\end{aligned}$$

and now the assertion follows together with (5.4.25) and the estimate

$$\begin{aligned}\|(S_{\mathrm{K},t}-S_{\mathrm{K},0})\| \le \|(S_{\mathrm{K},t}-S_{\mathrm{K},0})(I-P_\varepsilon)\|\\ +\|P_\varepsilon(S_{\mathrm{K},t}-S_{\mathrm{K},0})P_\varepsilon\|+\|(I-P_\varepsilon)(S_{\mathrm{K},t}-S_{\mathrm{K},0})P_\varepsilon\|.\end{aligned}$$

This completes the proof. □

The counterpart of Lemma 5.4.11 for the case where the Friedrichs extension S_F has a compact resolvent is provided next.

Proposition 5.4.12. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ and assume that the resolvent $(S_\mathrm{F}-\lambda)^{-1}$ of the Friedrichs extension S_F is compact for some, and hence for all $\lambda\in\mathbb{C}\setminus[\gamma,\infty)$. Then*

$$\lim_{t\downarrow-\infty}\|(S_{\mathrm{K},t}-\lambda)^{-1}-(S_\mathrm{F}-\lambda)^{-1}\|=0.$$

Proof. It follows from the resolvent identity (see Theorem 1.2.6) that the resolvent of S_F is compact for all $\lambda\in\mathbb{C}\setminus[\gamma,\infty)$ if it is compact for some $\lambda\in\mathbb{C}\setminus[\gamma,\infty)$. Now let $x_0<\gamma$ and note that $(S-x_0)^{-1}$ is a bounded nonnegative operator. By (5.4.17) and (5.3.4) one has

$$\big((S-x_0)^{-1}\big)_{\mathrm{K},0}=\big((S-x_0)_\mathrm{F}\big)^{-1}=(S_\mathrm{F}-x_0)^{-1},$$

which is a compact operator by assumption. From Lemma 5.4.11 it follows that $((S-x_0)^{-1})_{\mathrm{K},x}$ converge uniformly to $((S-x_0)^{-1})_{\mathrm{K},0}$ when $x\uparrow 0$. This implies the assertion for $\lambda=x_0$, since

$$\begin{aligned}\lim_{x\uparrow 0}\big((S-x_0)^{-1}\big)_{\mathrm{K},x} &= \lim_{t\downarrow-\infty}\big((S-x_0)^{-1}\big)_{\mathrm{K},1/t}\\ &=\lim_{t\downarrow-\infty}\big((S-x_0)_{\mathrm{K},t}\big)^{-1}\\ &=\lim_{t\downarrow-\infty}\big(S_{\mathrm{K},t+x_0}-x_0\big)^{-1}\\ &=\lim_{t\downarrow-\infty}\big(S_{\mathrm{K},t}-x_0\big)^{-1},\end{aligned}$$

where (5.4.7) was used in the second equality and (5.4.9) was used in the third equality. The general case $\lambda\in\mathbb{C}\setminus[\gamma,\infty)$ follows with Lemma 1.11.4. □

Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ and let $x < \gamma$. Then $x \in \rho(S_{\rm F})$ is a point of regular type of S and one has that

$$S^* = S_{\rm F} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*) \quad \text{and} \quad S_{{\rm K},x} = S \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*);$$

cf. Theorem 1.7.1 and (5.4.3). Therefore, it is clear that

$$S_{{\rm K},x} \,\widehat{+}\, S_{\rm F} = S^*, \quad x < \gamma; \tag{5.4.26}$$

in other words, the extensions $S_{{\rm K},x}$ and $S_{\rm F}$ are transversal for $x < \gamma$. For $x = \gamma$ the situation is different. Now it is possible that the extensions $S_{{\rm K},\gamma}$ and $S_{\rm F}$ are transversal, but it is also possible that they are not transversal because, for instance, the extensions $S_{{\rm K},\gamma}$ and $S_{\rm F}$ may even coincide. First the case of transversality is discussed.

Corollary 5.4.13. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Then the following statements hold:*

(i) *$S_{{\rm K},\gamma}$ and $S_{\rm F}$ are transversal if and only if* $\operatorname{dom} S^* \subset \operatorname{dom}(S_{{\rm K},\gamma} - \gamma)^{\frac{1}{2}}$*;*

(ii) *$S_{{\rm K},\gamma}$ and $S_{\rm F}$ are transversal and S is bounded if and only if $S_{{\rm K},\gamma} \in \mathbf{B}(\mathfrak{H})$.*

Proof. (i) This statement follows from Theorem 5.3.8.

(ii) Assume that $S_{\rm F}$ and $S_{{\rm K},\gamma}$ are transversal and that S is a bounded operator. Then part (i) shows that

$$\operatorname{dom} S^* \subset \operatorname{dom}(S_{{\rm K},\gamma} - \gamma)^{\frac{1}{2}}. \tag{5.4.27}$$

Since S^{**} is a bounded closed operator, $\operatorname{dom} S^{**}$ is closed, and hence so is $\operatorname{dom} S^*$; see Theorem 1.3.5. Moreover, $(\operatorname{dom} S^*)^\perp = \operatorname{mul} S^{**} = \{0\}$ implies that $\operatorname{dom} S^*$ is dense in $\mathfrak{H}$, so that $\operatorname{dom} S^* = \mathfrak{H}$. Then it follows from (5.4.27) that

$$\operatorname{dom}(S_{{\rm K},\gamma} - \gamma)^{\frac{1}{2}} = \mathfrak{H}.$$

Therefore, $\operatorname{dom} S_{{\rm K},\gamma} = \mathfrak{H}$ and hence $S_{{\rm K},\gamma} \in \mathbf{B}(\mathfrak{H})$.

Conversely, assume that $S_{{\rm K},\gamma} \in \mathbf{B}(\mathfrak{H})$. Then also S is bounded. Moreover, $\operatorname{dom} S^* \subset \mathfrak{H} = \operatorname{dom}(S_{{\rm K},\gamma} - \gamma)^{\frac{1}{2}}$, which together with (i) shows that $S_{\rm F}$ and $S_{{\rm K},\gamma}$ are transversal. □

The extreme case of equality of $S_{{\rm K},\gamma}$ and $S_{\rm F}$ is described in the following corollary.

Corollary 5.4.14. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ. Then the following statements hold:*

(i) *$S_{\rm F} = S_{{\rm K},\gamma}$ if and only if* $\ker(S^* - a) \cap \operatorname{dom}(S_{{\rm K},\gamma} - a)^{\frac{1}{2}} = \{0\}$ *for some, and hence for all $a < \gamma$;*

(ii) *$S_{\rm F} = S_{{\rm K},\gamma}$ and $S_{{\rm K},\gamma} \in \mathbf{B}(\mathfrak{H})$ if and only if $\overline{S} \in \mathbf{B}(\mathfrak{H})$.*

Proof. (i) This statement follows from Corollary 5.3.10.

(ii) Assume that $S_{\mathrm{F}} = S_{\mathrm{K},\gamma}$ and $S_{\mathrm{K},\gamma} \in \mathbf{B}(\mathfrak{H})$. The assumption $S_{\mathrm{K},\gamma} \in \mathbf{B}(\mathfrak{H})$ and Corollary 5.4.13 (ii) imply that S_{F} and $S_{\mathrm{K},\gamma}$ are transversal and S is bounded. Furthermore, $S_{\mathrm{F}} = S_{\mathrm{K},\gamma}$ implies that $S^* = S_{\mathrm{F}} \widehat{+} S_{\mathrm{K},\gamma} = S_{\mathrm{K},\gamma}$, so that $\overline{S} = S_{\mathrm{K},\gamma}$ is a self-adjoint operator in $\mathbf{B}(\mathfrak{H})$.

Conversely, if $\overline{S} \in \mathbf{B}(\mathfrak{H})$, then $\overline{S}$ is the only self-adjoint extension of S and hence $S_{\mathrm{F}} = S_{\mathrm{K},\gamma} = S^{**} \in \mathbf{B}(\mathfrak{H})$. □

In the next corollary, which is a special version of Corollary 5.3.11, the form $\mathfrak{t}_{S_{\mathrm{K},x}}$ for $x < \gamma$ corresponding to the Kreĭn type extensions $S_{\mathrm{K},x}$ is expressed in terms of the Friedrichs form $\mathfrak{t}_{S_{\mathrm{F}}}$.

Corollary 5.4.15. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ, let $x < \gamma$, and let $\mathfrak{t}_{S_{\mathrm{K},x}}$ be the form corresponding to $S_{\mathrm{K},x}$. Then*

$$\operatorname{dom} \mathfrak{t}_{S_{\mathrm{K},x}} = \ker (S^* - a) \oplus_{\mathfrak{t}_{S_{\mathrm{K},x}} - a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}, \qquad a < x, \tag{5.4.28}$$

and the restriction of $\mathfrak{t}_{S_{\mathrm{K},x}}$ to $\mathfrak{N}_a(S^*) = \ker (S^* - a)$ is represented by the bounded self-adjoint operator

$$L_a = P_{\mathfrak{N}_a(S^*)}\big(x + (x-a)^2(S_{\mathrm{F}} - x)^{-1}\big)\iota_{\mathfrak{N}_a(S^*)} \in \mathbf{B}(\mathfrak{N}_a(S^*)), \tag{5.4.29}$$

where $\iota_{\mathfrak{N}_a(S^*)}$ is the canonical embedding of $\mathfrak{N}_a(S^*)$ into $\mathfrak{H}$ and $P_{\mathfrak{N}_a(S^*)}$ is the orthogonal projection onto $\mathfrak{N}_a(S^*)$. Furthermore,

$$\begin{aligned} \mathfrak{t}_{S_{\mathrm{K},x}}[f,g] - a(f,g) = (x-a)\big(\big(I + (x-a)(S_{\mathrm{F}} - x)^{-1}\big) f_a, g_a\big) \\ + \mathfrak{t}_{S_{\mathrm{F}}}[f_{\mathrm{F}}, g_{\mathrm{F}}] - a(f_{\mathrm{F}}, g_{\mathrm{F}}) \end{aligned} \tag{5.4.30}$$

holds for all $f = f_a + f_{\mathrm{F}}, g = g_a + g_{\mathrm{F}} \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{K},x}}$, where $f_a, g_a \in \ker (S^* - a)$ and $f_{\mathrm{F}}, g_{\mathrm{F}} \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$.

Proof. The decomposition (5.4.28) is clear from Corollary 5.3.11, since $S_{\mathrm{K},x}$ and S_{F} are transversal for $x < \gamma$. Next it will be shown that the representing operator for the restriction of $\mathfrak{t}_{S_{\mathrm{K},x}}$ to $\mathfrak{N}_a(S^*)$ is given by (5.4.29); then (5.3.24) in Corollary 5.3.11 also leads to (5.4.30).

In order to verify (5.4.29) consider $f_a, g_a \in \mathfrak{N}_a(S^*)$ and let

$$f_x = \big(I + (x-a)(S_{\mathrm{F}} - x)^{-1}\big) f_a.$$

Then $f_x \in \mathfrak{N}_x(S^*)$ and $f_a = (I + (a-x)(S_{\mathrm{F}} - a)^{-1}) f_x$ by Lemma 1.4.10. Moreover, since $S_{\mathrm{K},x}$ is representing the form $\mathfrak{t}_{S_{\mathrm{K},x}}$ and $f_x \in \operatorname{dom} S_{\mathrm{K},x}$ one has $\mathfrak{t}_{S_{\mathrm{K},x}}[f_x, g_a] = (x f_x, g_a)$. Using $(S_{\mathrm{F}} - a)^{-1} f_x \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ and $g_a \in \mathfrak{N}_a(S^*)$ the orthogonal decomposition (5.4.28) yields $((S_{\mathrm{F}} - a)^{-1} f_x, g_a)_{\mathfrak{t}_{S_{\mathrm{K},x}} - a} = 0$.

Now one computes

$$\begin{aligned}
\mathfrak{t}_{S_{K,x}}[f_a, g_a] &= \mathfrak{t}_{S_{K,x}}[f_x, g_a] + (a-x)\mathfrak{t}_{S_{K,x}}\big[(S_F - a)^{-1} f_x, g_a\big] \\
&= (x f_x, g_a) + (a-x)a\big((S_F - a)^{-1} f_x, g_a\big) \\
&= \big(x f_x + a(f_a - f_x), g_a\big) \\
&= \big((x-a)\big(I + (x-a)(S_F - x)^{-1}\big) f_a + a f_a, g_a\big) \\
&= \big(\big(x + (x-a)^2 (S_F - x)^{-1}\big) f_a, g_a\big),
\end{aligned}$$

which implies (5.4.29). □

Finally, the decomposition (5.4.28) in the previous corollary is used to show a similar direct sum decomposition for $a = x$.

Corollary 5.4.16. *Let S be a semibounded relation in $\mathfrak{H}$ with lower bound γ, let $x < \gamma$, and let $\mathfrak{t}_{S_{K,x}}$ be the form corresponding to $S_{K,x}$. Then*

$$\operatorname{dom} \mathfrak{t}_{S_{K,x}} = \ker (S^* - x) + \operatorname{dom} \mathfrak{t}_{S_F} \tag{5.4.31}$$

is a direct sum decomposition.

Proof. Let $a < x < \gamma$. Recall that the decomposition (5.4.28) holds since $S_{K,x}$ and S_F are transversal.

It is clear that the right-hand side of (5.4.31) is contained in the left-hand side. Observe for this that

$$\operatorname{dom} \mathfrak{t}_{S_F} \subset \operatorname{dom} \mathfrak{t}_{S_{K,x}} \quad \text{and} \quad \ker (S^* - x) \subset \operatorname{dom} S_{K,x} \subset \operatorname{dom} \mathfrak{t}_{S_{K,x}}.$$

To show that the left-hand side of (5.4.31) is contained in the right-hand side, let $f \in \operatorname{dom} \mathfrak{t}_{S_{K,x}}$. According to (5.4.28) one has $f = f_a + f_F$, with $f_a \in \ker (S^* - a)$ and $f_F \in \operatorname{dom} \mathfrak{t}_{S_F}$. Define

$$f_x = \big(I + (x-a)(S_F - x)^{-1}\big) f_a.$$

Then

$$f_x \in \ker (S^* - x) \quad \text{and} \quad f = f_x + (f_a - f_x + f_F),$$

where the last term is in $\operatorname{dom} t_{S_F}$ since $f_a - f_x \in \operatorname{dom} S_F$. Hence, the left-hand side of (5.4.31) belongs to the right-hand side. Thus, the sum decomposition (5.4.31) has been shown.

Finally, it will be shown that the sum decomposition (5.4.31) is direct. For this assume that $f_x \in \ker (S^* - x)$ is nontrivial and belongs to $\operatorname{dom} \mathfrak{t}_{S_F}$. Then $\{f_x, x f_x\} \in S^*$ implies that $\{f_x, x f_x\} \in S_F$; cf. Theorem 5.3.3. Since $x < \gamma$, this is a contradiction. □

5.5 Boundary triplets for semibounded relations

In this section semibounded self-adjoint extensions of semibounded symmetric relations are studied in the context of boundary triplets and their Weyl functions. The initial observations are general results about a closed symmetric relation S with a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, where $A_0 = \ker \Gamma_0$ is semibounded. In particular, the Friedrichs and the Kreĭn type extensions will be identified. In the remaining part of the section it will be assumed that S is a semibounded relation with a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, where $A_0 = S_{\rm F}$, and various specific properties are derived. The case where the self-adjoint extension $A_1 = \ker \Gamma_1$ is also semibounded is of specific interest. As a preparation for the main results in the following section it will be explained how the corresponding semibounded form is the first stepping stone to the notion of boundary pair.

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Assume that $A_0 = \ker \Gamma_0$ is semibounded with lower bound $\gamma_0 = m(A_0)$. Then clearly S is semibounded and $\gamma_0 \leq m(S) = \gamma$. Therefore, one may speak of the Friedrichs extension $S_{\rm F}$ so that $\gamma_0 \leq m(S_{\rm F}) = \gamma$ and the Kreĭn type extensions $S_{{\rm K},x}$ of S with $x \leq \gamma$. The corresponding Weyl function M is holomorphic on $\rho(A_0)$ and, in particular, on $\mathbb{C} \setminus [\gamma_0, \infty)$. Moreover, one has

$$M(x) = \Gamma(\widehat{\mathfrak{N}}_x(S^*)) = \Gamma(S \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*)) = \Gamma(S_{{\rm K},x}), \quad x < \gamma_0; \tag{5.5.1}$$

cf. Definition 2.3.4 and (5.4.3). By Corollary 2.3.8, the mapping $x \mapsto M(x)$ from $(-\infty, \gamma_0)$ to $\mathbf{B}(\mathcal{G})$ is nondecreasing. In particular, by Corollary 5.2.14 the limit $M(-\infty)$ exists in the strong resolvent sense,

$$\big(M(-\infty) - \lambda\big)^{-1} = \lim_{x \downarrow -\infty} \,(M(x) - \lambda)^{-1}, \tag{5.5.2}$$

and the limit $M(\gamma_0)$ exists in the strong resolvent sense,

$$\big(M(\gamma_0) - \lambda\big)^{-1} = \lim_{x \uparrow \gamma_0} \,(M(x) - \lambda)^{-1}, \tag{5.5.3}$$

where $\lambda \in \mathbb{C} \setminus [\gamma_0, \infty)$. Then $M(-\infty)$ and $M(\gamma_0)$ are self-adjoint relations in $\mathcal{G}$; cf. Theorem 5.2.11 and Corollary 5.2.14. In the following theorem the Friedrichs extension $S_{\rm F}$ and the Kreĭn type extension $S_{{\rm K},x}$ with $x \leq \gamma_0$ will be characterized by means of the limits in (5.5.2) and (5.5.3).

Theorem 5.5.1. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* and let M be the corresponding Weyl function. Assume that the self-adjoint extension $A_0 = \ker \Gamma_0$ is semibounded with $\gamma_0 = m(A_0) \leq m(S_{\rm F}) = \gamma$. Then the Friedrichs extension $S_{\rm F}$ of S is given by*

$$S_{\rm F} = \big\{ \widehat{f} \in S^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in M(-\infty) \big\} \tag{5.5.4}$$

and the Kreĭn type extension $S_{{\rm K},x}$ of S with $x \leq \gamma_0$ is given by

$$S_{{\rm K},x} = \big\{ \widehat{f} \in S^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in M(x) \big\}. \tag{5.5.5}$$

Proof. According to Theorem 5.4.10, the Friedrichs extension S_F of S is given by the strong resolvent limit

$$(S_F - \lambda)^{-1}h = \lim_{t\downarrow -\infty} (S_{K,t} - \lambda)^{-1}h, \quad h \in \mathfrak{H}, \tag{5.5.6}$$

and for each $x \le \gamma_0$ the Kreĭn type extension $S_{K,x}$ is given by the strong resolvent limit

$$(S_{K,x} - \lambda)^{-1}h = \lim_{t\uparrow x} (S_{K,t} - \lambda)^{-1}h, \quad h \in \mathfrak{H}, \tag{5.5.7}$$

where $\lambda \in \mathbb{C} \setminus [x, \infty)$. The idea behind the proof of the theorem is to connect the limit formulas in (5.5.2) and (5.5.3) with the limit formulas in (5.5.6) and (5.5.7). This in fact will be done by means of the Kreĭn formula. For this purpose observe that for $t < \gamma_0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the resolvent formula in Theorem 2.6.1 for $S_{K,t}$ reads

$$(S_{K,t} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(M(t) - M(\lambda)\big)^{-1}\gamma(\bar\lambda)^*, \tag{5.5.8}$$

due to (5.5.1). Here $(M(t)-M(\lambda))^{-1} \in \mathbf{B}(\mathcal{G})$ by Theorem 2.6.1 and Theorem 2.6.2.

First consider the Kreĭn type extension $S_{K,x}$ of S. If $x < \gamma_0$, then the formula (5.5.5) is a direct consequence of (5.5.1). To treat the case $x = \gamma_0$ let Θ_K be the self-adjoint relation in $\mathcal{G}$ which corresponds to S_{K,γ_0}, that is,

$$S_{K,\gamma_0} = \big\{ \widehat f \in S^* : \{\Gamma_0 \widehat f, \Gamma_1 \widehat f\} \in \Theta_K \big\}. \tag{5.5.9}$$

Then again by the resolvent formula in Theorem 2.6.1 one has

$$(S_{K,\gamma_0} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta_K - M(\lambda)\big)^{-1}\gamma(\bar\lambda)^* \tag{5.5.10}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here $(\Theta_K - M(\lambda))^{-1} \in \mathbf{B}(\mathcal{G})$ by Theorem 2.6.1 and Theorem 2.6.2. Subtracting (5.5.10) from (5.5.8) leads to

$$\begin{aligned}(S_{K,t} - \lambda)^{-1} &- (S_{K,\gamma_0} - \lambda)^{-1} \\ &= \gamma(\lambda)\big[\big(M(t) - M(\lambda)\big)^{-1} - \big(\Theta_K - M(\lambda)\big)^{-1}\big]\gamma(\bar\lambda)^*\end{aligned} \tag{5.5.11}$$

with $t < \gamma_0$. Now take the strong limit for $t \uparrow \gamma_0$ and apply (5.5.7). Then for each $h \in \mathfrak{H}$

$$(S_{K,t} - \lambda)^{-1}h \to (S_{K,\gamma_0} - \lambda)^{-1}h \quad \text{as} \quad t \uparrow \gamma_0,$$

which, via (5.5.11), leads to

$$\gamma(\lambda)\big[\big(M(t) - M(\lambda)\big)^{-1} - \big(\Theta_K - M(\lambda)\big)^{-1}\big]\gamma(\bar\lambda)^*h \to 0 \quad \text{as} \quad t \uparrow \gamma_0.$$

Since $\gamma(\lambda)$ maps $\mathcal{G}$ isomorphically onto $\ker(S^* - \lambda)$ and $\gamma(\bar\lambda)^* : \mathfrak{H} \to \mathcal{G}$ is surjective, see Proposition 2.3.2, it follows that for each $\varphi \in \mathcal{G}$

$$\big(M(t) - M(\lambda)\big)^{-1}\varphi \to \big(\Theta_K - M(\lambda)\big)^{-1}\varphi \quad \text{as} \quad t \uparrow \gamma_0. \tag{5.5.12}$$

Next the parameter $\Theta_{\rm K}$ will be identified with $M(\gamma_0)$. For this purpose, observe that for $\varphi \in \mathcal{G}$

$$\big\{(M(t) - M(\lambda))^{-1}\varphi, \varphi + M(\lambda)(M(t) - M(\lambda))^{-1}\varphi\big\} \in M(t). \tag{5.5.13}$$

As $t \uparrow \gamma_0$, the components on the left-hand side of (5.5.13) converge due to (5.5.12), while the bounded operators $M(t)$ converge in the strong resolvent sense, and hence in the graph sense (see Corollary 1.9.6) to the self-adjoint relation $M(\gamma_0)$. Hence, (5.5.13) implies

$$\big\{(\Theta_{\rm K} - M(\lambda))^{-1}\varphi, \varphi + M(\lambda)(\Theta_{\rm K} - M(\lambda))^{-1}\varphi\big\} \in M(\gamma_0) \tag{5.5.14}$$

for all $\varphi \in \mathcal{G}$, and thus $(\Theta_{\rm K} - M(\lambda))^{-1} \subset (M(\gamma_0) - M(\lambda))^{-1}$. Since $M(\lambda) \in \mathbf{B}(\mathcal{G})$, it follows that $\Theta_{\rm K} \subset M(\gamma_0)$, or, since both relations are self-adjoint, $\Theta_{\rm K} = M(\gamma_0)$. Now (5.5.5) for $x = \gamma_0$ follows from (5.5.9).

Next consider the Friedrichs extension $S_{\rm F}$ of S. Let $\Theta_{\rm F}$ be the self-adjoint relation in $\mathcal{G}$ which corresponds to $S_{\rm F}$, that is,

$$S_{\rm F} = \big\{ \widehat{f} \in S^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \Theta_{\rm F}\big\}.$$

Then again by the resolvent formula in Theorem 2.6.1 one has

$$(S_{\rm F} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta_{\rm F} - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* \tag{5.5.15}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. As above, $(\Theta_{\rm F} - M(\lambda))^{-1} \in \mathbf{B}(\mathcal{G})$. Subtracting (5.5.15) from (5.5.8) and using the same reasoning as above involving Theorem 5.4.10 yields

$$\big(M(t) - M(\lambda)\big)^{-1}\varphi \to \big(\Theta_{\rm F} - M(\lambda)\big)^{-1}\varphi \quad \text{as} \quad t \downarrow -\infty.$$

From the fact that $M(-\infty)$ is the strong resolvent limit, and hence the strong graph limit of $M(t)$ when $t \downarrow -\infty$, one concludes $\Theta_{\rm F} = M(-\infty)$ in the same way as in (5.5.13)–(5.5.14). This shows (5.5.4). □

The statements in the following corollary are consequences of Theorem 5.5.1 and Proposition 2.1.8.

Corollary 5.5.2. *Let S be a closed symmetric relation, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, and let M be the corresponding Weyl function. Assume that the self-adjoint extension $A_0 = \ker \Gamma_0$ is semibounded with lower bound γ_0. Then the following statements hold:*

(i) *$A_0 = S_{\rm F}$ if and only if $M(-\infty) = \{0\} \times \mathcal{G}$;*

(ii) *$A_0 \cap S_{\rm F} = S$ if and only if $M(-\infty)$ is a closed operator;*

(iii) *$A_0 \,\widehat{+}\, S_{\rm F} = S^*$ if and only if $M(-\infty) \in \mathbf{B}(\mathcal{G})$,*

and, similarly

(iv) *$A_0 = S_{{\rm K},\gamma_0}$ if and only if $M(\gamma_0) = \{0\} \times \mathcal{G}$;*

(v) *$A_0 \cap S_{{\rm K},\gamma_0} = S$ if and only if $M(\gamma_0)$ is a closed operator;*

(vi) *$A_0 \,\widehat{+}\, S_{{\rm K},\gamma_0} = S^*$ if and only if $M(\gamma_0) \in \mathbf{B}(\mathcal{G})$.*

Moreover, for $A_1 = \ker \Gamma_1$ *one has*

(vii) $A_1 = S_{\mathrm{F}}$ *if and only if* $M(-\infty) = \mathcal{G} \times \{0\}$*;*

(viii) $A_1 = S_{\mathrm{K},\gamma_0}$ *if and only if* $M(\gamma_0) = \mathcal{G} \times \{0\}$*.*

Such facts can also be stated in terms of the limit behavior of $M(-\infty)$ and $M(\gamma_0)$ via Corollary 5.2.14 applied to the Weyl function M. For this purpose recall the notations

$$\mathfrak{E}_{\gamma_0} = \left\{ \varphi \in \mathcal{G} : \lim_{x \uparrow \gamma_0} (M(x)\varphi, \varphi) < \infty \right\},$$

$$\mathfrak{E}_{-\infty} = \left\{ \varphi \in \mathcal{G} : \lim_{x \downarrow -\infty} (M(x)\varphi, \varphi) > -\infty \right\}.$$

Corollary 5.5.3. *Let* S *be a closed symmetric relation, let* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* S^**, and let* M *be the corresponding Weyl function. Assume that the self-adjoint extension* $A_0 = \ker \Gamma_0$ *is semibounded with lower bound* γ_0*. Then the following statements hold:*

(i) $A_0 = S_{\mathrm{F}}$ *if and only if* $\mathfrak{E}_{-\infty} = \{0\}$*;*

(ii) $A_0 \cap S_{\mathrm{F}} = S$ *if and only if* $\operatorname{clos} \mathfrak{E}_{-\infty} = \mathcal{G}$*;*

(iii) $A_0 \,\widehat{+}\, S_{\mathrm{F}} = S^*$ *if and only if* $\mathfrak{E}_{-\infty} = \mathcal{G}$*,*

and, similarly

(iv) $A_0 = S_{\mathrm{K},\gamma_0}$ *if and only if* $\mathfrak{E}_{\gamma_0} = \{0\}$*;*

(v) $A_0 \cap S_{\mathrm{K},\gamma_0} = S$ *if and only if* $\operatorname{clos} \mathfrak{E}_{\gamma_0} = \mathcal{G}$*;*

(vi) $A_0 \,\widehat{+}\, S_{\mathrm{K},\gamma_0} = S^*$ *if and only if* $\mathfrak{E}_{\gamma_0} = \mathcal{G}$*.*

In the context of Theorem 5.5.1, the Weyl function of the boundary triplet is holomorphic on the interval $(-\infty, \gamma_0)$. In this situation the inverse result in Theorem 4.2.4 can be formulated as follows.

Proposition 5.5.4. *Let* $\mathcal{G}$ *be a Hilbert space and let* M *be a uniformly strict* $\mathbf{B}(\mathcal{G})$*-valued Nevanlinna function, which is holomorphic on* $\mathbb{C} \setminus [\gamma_0, \infty)$ *and not holomorphic at* γ_0*. Then there exist a Hilbert space* $\mathfrak{H}$*, a closed simple symmetric operator* S *in* $\mathfrak{H}$*, and a boundary triplet* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *for* S^* *such that* A_0 *is a semibounded self-adjoint relation with lower bound* $m(A_0) = \gamma_0$ *and* M *is the corresponding Weyl function.*

Proof. Let M be a uniformly strict Nevanlinna function with values in $\mathbf{B}(\mathcal{G})$. Let $\mathfrak{H}(\mathsf{N}_M)$ be the reproducing kernel Hilbert space associated with the Nevanlinna kernel

$$\frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.$$

By Theorem 4.2.4, there exist a closed simple symmetric operator S in the reproducing kernel Hilbert space $\mathfrak{H}(\mathsf{N}_M)$ and a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^*

such that M is the corresponding Weyl function. The assumption that M is holomorphic on $(-\infty, \gamma_0)$ and the fact that S is simple imply, by Theorem 3.6.1, that $(-\infty, \gamma_0) \subset \rho(A_0)$. Moreover, since M is not holomorphic at γ_0, one has $\gamma_0 \in \sigma(A_0)$. Therefore, the self-adjoint relation A_0 is semibounded with lower bound $m(A_0) = \gamma_0$. □

The context of Theorem 5.5.1 will now be narrowed. Let S be a closed semibounded relation in $\mathfrak{H}$. Then the existence of a semibounded self-adjoint extension of S is guaranteed by the Friedrichs extension S_{F} of S. The interest in the rest of this section is in boundary triplets $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for which $A_0 = S_{\mathrm{F}}$. The following result is a consequence of Theorem 2.4.1 since for any self-adjoint extension H of S there is a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that $H = \ker \Gamma_0$.

Corollary 5.5.5. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ. Then there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that*

$$S_{\mathrm{F}} = \ker \Gamma_0.$$

The corresponding Weyl function M is holomorphic on $(-\infty, \gamma)$ and the mapping $x \mapsto M(x)$ from $(-\infty, \gamma)$ to $\mathbf{B}(\mathcal{G})$ is nondecreasing, while $M(-\infty) = \{0\} \times \mathcal{G}$.

The following result will be useful in treating the connection between semibounded self-adjoint extensions and the Weyl function.

Proposition 5.5.6. *Let S be a closed semibounded relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that $S_{\mathrm{F}} = \ker \Gamma_0$, and let M be the corresponding Weyl function. Let A_Θ be a self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathcal{G}$ and assume that $x < m(S)$. Then $M(x) \in \mathbf{B}(\mathcal{G})$ and the following equivalence holds:*

$$x \leq A_\Theta \quad \Leftrightarrow \quad M(x) \leq \Theta.$$

In particular, if A_Θ is semibounded in $\mathfrak{H}$, then Θ is semibounded in $\mathcal{G}$.

Proof. The assumption $x < m(S) = m(S_{\mathrm{F}})$ implies that $x \in \rho(S_{\mathrm{F}})$ and hence $M(x) \in \mathbf{B}(\mathcal{G})$ is clear. The formula in Theorem 2.6.1, applied to A_Θ and S_{F}, gives

$$(A_\Theta - x)^{-1} - (S_{\mathrm{F}} - x)^{-1} = \gamma(x)(\Theta - M(x))^{-1}\gamma(x)^*, \tag{5.5.16}$$

since $(S_{\mathrm{F}} - x)^{-1} \in \mathbf{B}(\mathfrak{H})$. Recall that $\gamma(x)^*$ maps $\ker(S^* - x)$ onto $\mathcal{G}$; see Proposition 2.3.2. Note that if, in addition, $x \in \rho(A_\Theta)$, then $(\Theta - M(x))^{-1} \in \mathbf{B}(\mathcal{G})$.

($\Rightarrow$) Since A_Θ is a semibounded self-adjoint extension of S, it follows from Proposition 5.3.6 that $A_\Theta \leq S_{\mathrm{F}}$. Observe that for $x \in \rho(A_\Theta)$

$$0 \leq (S_{\mathrm{F}} - x)^{-1} \leq (A_\Theta - x)^{-1},$$

where both operators belong to $\mathbf{B}(\mathfrak{H})$ since $x \in \rho(S_\mathrm{F}) \cap \rho(A_\Theta)$. Hence, it then follows from (5.5.16) that

$$(\Theta - M(x))^{-1} \geq 0 \quad \text{or, equivalently,} \quad \Theta - M(x) \geq 0.$$

Since $M(x) \in \mathbf{B}(\mathcal{G})$ it follows that $M(x) \leq \Theta$; cf. Proposition 5.2.6.

Now let $x \leq A_\Theta$. First assume that $x < m(A_\Theta)$. In this case, $x \in \rho(A_\Theta)$ and thus $M(x) \leq \Theta$. Next, assume that $x = m(A_\Theta)$ and consider an increasing sequence x_n whose limit is x. Then clearly $M(x_n) \leq \Theta$ and thus $M(x) \leq \Theta$; cf. Corollary 5.2.12 (i).

($\Leftarrow$) Assume that $M(x) \leq \Theta$. Then $\Theta - M(x) \geq 0$ by Proposition 5.2.6 and hence $(\Theta - M(x))^{-1} \geq 0$. It is straightforward to see that the right-hand side of (5.5.16) is a nonnegative relation. Thus, the relation

$$(A_\Theta - x)^{-1} - (S_\mathrm{F} - x)^{-1}$$

on the left-hand side of (5.5.16) is also nonnegative and, in fact, this relation is also self-adjoint since $(A_\Theta - x)^{-1}$ is self-adjoint and $(S_\mathrm{F} - x)^{-1} \in \mathbf{B}(\mathfrak{H})$ is self-adjoint. Therefore, one concludes with the help of Proposition 5.2.6 and $x < m(S_\mathrm{F})$ that

$$0 \leq (S_\mathrm{F} - x)^{-1} \leq (A_\Theta - x)^{-1}.$$

In particular, this shows that $0 \leq A_\Theta - x$ or $x \leq A_\Theta$. □

From Proposition 5.5.6 one sees that if the self-adjoint extension A_Θ is semibounded in $\mathfrak{H}$, then the corresponding self-adjoint relation Θ is semibounded in $\mathcal{G}$. The converse is not true in general; cf. Remark 5.6.16. However, in Proposition 5.5.8 below it will be shown that the converse holds if S has finite defect numbers or S_F has a compact resolvent. The following result is a preliminary observation.

Lemma 5.5.7. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that $S_\mathrm{F} = \ker \Gamma_0$. Let M be the corresponding Weyl function and assume that for any $C > 0$ there exists $x_1 < \gamma$ such that*

$$M(x) \leq -C, \qquad x \leq x_1. \tag{5.5.17}$$

Then for every semibounded self-adjoint relation Θ in $\mathcal{G}$ the corresponding self-adjoint extension A_Θ is semibounded from below.

Proof. Let Θ be a self-adjoint relation in $\mathcal{G}$ with lower bound ν and choose $C > 0$ in (5.5.17) such that $-C < \nu \leq \Theta$. For all $x \leq x_1$ one then has

$$0 < \nu + C \leq \Theta + C \leq \Theta - M(x),$$

which implies that $\Theta - M(x)$ is boundedly invertible for all $x \leq x_1$. From Theorem 2.6.2 one concludes $(-\infty, x_1) \in \rho(A_\Theta)$ and hence A_Θ is semibounded from below. This conclusion also follows from Proposition 5.5.6. □

Now it will be shown that the condition (5.5.17) holds when S has finite defect numbers or S_F has a compact resolvent.

Proposition 5.5.8. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that $S_F = \ker \Gamma_0$. Assume that one of the following conditions hold:*

(i) *$\mathcal{G}$ is finite-dimensional;*

(ii) *$(S_F - \lambda)^{-1}$ is compact for some, and hence for all, $\lambda \in \rho(S_F)$.*

Then for any $C > 0$ there exists $x_1 < \gamma$ such that (5.5.17) holds. In particular, if (i) *holds, then all self-adjoint extensions of S in $\mathfrak{H}$ are semibounded from below, or if* (ii) *holds and Θ is a semibounded self-adjoint relation in $\mathcal{G}$, then the self-adjoint extension A_Θ of S is semibounded from below.*

Proof. (i) As $S_F = \ker \Gamma_0$, one has $(M(x)\varphi, \varphi) \to -\infty$ for $x \to -\infty$ and all $\varphi \in \mathcal{G}$ by Corollary 5.5.3 (i). Since $\mathcal{G}$ is finite-dimensional, a compactness argument shows that there exists $x_1 < \gamma$ such that (5.5.17) holds. Every self-adjoint relation Θ in the finite-dimensional space $\mathcal{G}$ is semibounded and hence it follows from Lemma 5.5.7 that all self-adjoint extensions A_Θ are semibounded.

(ii) Recall from Proposition 5.4.12 that the resolvents of the Kreĭn type extensions $S_{K,t}$ converge uniformly to the resolvent of S_F, that is, for all $\lambda \in \mathbb{C} \setminus [\gamma, \infty)$ one has

$$\lim_{t \downarrow -\infty} \|(S_{K,t} - \lambda)^{-1} - (S_F - \lambda)^{-1}\| = 0. \tag{5.5.18}$$

In the following fix some $\lambda = x_0 < m(S_F)$ and note that, by (5.5.18), there exists $t' < x_0$ such that $x_0 \in \rho(S_{K,t})$ for all $t \leq t'$. Using (5.5.1) it follows that the resolvent of $S_{K,t}$ has the form

$$(S_{K,t} - x_0)^{-1} - (S_F - x_0)^{-1} = \gamma(x_0)\big(M(t) - M(x_0)\big)^{-1}\gamma(x_0)^*,$$

where $(M(t) - M(x_0))^{-1} \in \mathbf{B}(\mathcal{G})$ for all $t \leq t'$ by Theorem 2.6.1 and Theorem 2.6.2. Since $\gamma(x_0)$ maps $\mathcal{G}$ isomorphically to $\mathfrak{N}_{x_0}(S^*)$ and $\gamma(x_0)^*$ maps $\mathfrak{N}_{x_0}(S^*)$ isomorphically to $\mathcal{G}$, it follows together with (5.5.18) that

$$\lim_{t \downarrow -\infty} \big\|\big(M(t) - M(x_0)\big)^{-1}\big\| = 0. \tag{5.5.19}$$

This implies that for any $C > 0$ there exists $x_1 < \gamma$ such that (5.5.17) holds. In fact, otherwise there exists some $C_0 > 0$ and a sequence $s_n \to -\infty$ such that $s_n < t' < x_0$ and $M(s_n) > -C_0$. Then the estimate

$$-C_0 - \|M(x_0)\| \leq -C_0 - M(x_0) \leq M(s_n) - M(x_0) \leq 0$$

and $(M(s_n) - M(x_0))^{-1} \in \mathbf{B}(\mathcal{G})$ contradict (5.5.19). Therefore, the condition (5.5.17) is satisfied and if Θ is a semibounded self-adjoint relation in $\mathcal{G}$, then by Lemma 5.5.7 the corresponding self-adjoint extension A_Θ is semibounded. □

In the case of Corollary 5.5.5 the relationship between the Friedrichs extension S_F and the Kreĭn type extension $S_{K,\gamma}$ is described in the following corollary, which is a translation of (iv)–(vi) in Corollary 5.5.2; cf. Corollary 5.4.13 and Corollary 5.4.14.

Corollary 5.5.9. *Let S be a closed semibounded relation in $\mathfrak{H}$. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that $S_F = \ker \Gamma_0$ and let M be the corresponding Weyl function. Let $\gamma = m(S_F)$. Then the following statements hold:*

(i) *S_F and $S_{K,\gamma}$ coincide if and only if $M(\gamma) = \{0\} \times \mathcal{G}$;*

(ii) *S_F and $S_{K,\gamma}$ are disjoint if and only if $M(\gamma)$ is a closed operator;*

(iii) *S_F and $S_{K,\gamma}$ are transversal if and only if $M(\gamma) \in \mathbf{B}(\mathcal{G})$.*

In general it may not be possible to simultaneously prescribe $\ker \Gamma_0$ as the Friedrichs extension S_F and $\ker \Gamma_1$ as the Kreĭn type extension $S_{K,\gamma}$, since $\ker \Gamma_0$ and $\ker \Gamma_1$ are necessarily transversal; cf. Section 2.1. However, note that the Friedrichs extension S_F and the Kreĭn type extension $S_{K,x}$ for $x < \gamma$ are automatically transversal; cf. (5.4.26).

Proposition 5.5.10. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ. Then the following statements hold:*

(i) *For $x < \gamma$ there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that*

$$S_F = \ker \Gamma_0 \quad \text{and} \quad S_{K,x} = \ker \Gamma_1. \tag{5.5.20}$$

(ii) *If S_F and $S_{K,\gamma}$ are transversal, then there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that*

$$S_F = \ker \Gamma_0 \quad \text{and} \quad S_{K,\gamma} = \ker \Gamma_1.$$

In both cases the corresponding Weyl function satisfies $M(-\infty) = \{0\} \times \mathcal{G}$ and $M(x) = \mathcal{G} \times \{0\}$, $x \leq \gamma$. In particular, $M(t) \leq 0$ for all $t \leq x$, i.e., M belongs to the class $\mathbf{S}_{\mathcal{G}}^{-1}(-\infty, x)$ of inverse Stieltjes functions.

Proof. (i) The extensions S_F and $S_{K,x}$ for $x < \gamma$ are automatically transversal according to (5.4.26). Hence, it follows from Theorem 2.5.9 that there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that (5.5.20) holds.

(ii) Since it is assumed that S_F and $S_{K,\gamma}$ are transversal, Theorem 2.5.9 yields the statement.

The Weyl function M satisfies $M(-\infty) = \{0\} \times \mathcal{G}$ and $M(x) = \mathcal{G} \times \{0\}$ as a consequence of Corollary 5.5.2. Finally, that $M(t) \leq 0$ for all $t \leq x$ is a consequence of the monotonicity of the Weyl function M in Corollary 2.3.8. The assertion $M \in \mathbf{S}_{\mathcal{G}}^{-1}(-\infty, x)$ is immediate from the definition of the inverse Stieltjes class in Definition A.6.1. □

The following corollary is a consequence of Proposition 5.5.4 and Corollary 5.5.2.

Corollary 5.5.11. *Let $\mathcal{G}$ be a Hilbert space and let M be a uniformly strict $\mathbf{B}(\mathcal{G})$-valued Nevanlinna function, which is holomorphic on $\mathbb{C}\setminus[\gamma,\infty)$ and not holomorphic at γ. Assume, in addition, that*

$$M(-\infty) = \{0\}\times\mathcal{G} \quad \text{and} \quad M(\gamma) = \mathcal{G}\times\{0\}. \tag{5.5.21}$$

Then there exist a Hilbert space $\mathfrak{H}$, a closed simple semibounded operator S in $\mathfrak{H}$ with lower bound γ, and a boundary triplet $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ for S^* with $S_{\rm F} = \ker\Gamma_0$ and $S_{{\rm K},\gamma} = \ker\Gamma_1$, such that M is the corresponding Weyl function.

Proof. It follows from Proposition 5.5.4 that there exist a Hilbert space $\mathfrak{H}$, a closed simple symmetric operator S in $\mathfrak{H}$, and a boundary triplet $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ for S^* such that $A_0 = \ker\Gamma_0$ is a semibounded self-adjoint relation with lower bound $m(A_0)=\gamma$ and M is the corresponding Weyl function. The assumptions in (5.5.21) and Corollary 5.5.2 (i) and (viii) imply $S_{\rm F} = \ker\Gamma_0 = A_0$ and $S_{{\rm K},\gamma} = \ker\Gamma_1$. Since $m(S_{\rm F}) = m(A_0) = \gamma$, it is also clear that the symmetric operator S is semibounded with lower bound γ. □

In the next corollary a boundary triplet with the properties as in Proposition 5.5.10 (i) is exhibited.

Corollary 5.5.12. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ. Then*

$$S^* = S_{\rm F} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*), \quad x<\gamma, \tag{5.5.22}$$

is a direct sum decomposition. Let $\widehat{f} = \{f,f'\}\in S^*$ have the unique decomposition

$$\widehat{f} = \widehat{f}_{\rm F} + \widehat{f}_x,$$

with $\widehat{f}_{\rm F} = \{f_{\rm F}, f_{\rm F}'\}\in S_{\rm F}$ and $\widehat{f}_x = \{f_x, xf_x\}\in\widehat{\mathfrak{N}}_x(S^*)$. Then

$$\Gamma_0\widehat{f} = f_x \quad \text{and} \quad \Gamma_1\widehat{f} = P_{\mathfrak{N}_x(S^*)}(f_{\rm F}' - xf_{\rm F})$$

defines a boundary triplet $\{\mathfrak{N}_x(S^*),\Gamma_0,\Gamma_1\}$ for S^* such that (5.5.20) holds. For $\lambda\in\rho(S_{\rm F})$ the corresponding γ-field γ is given by

$$\gamma(\lambda) = \bigl(I + (\lambda - x)(S_{\rm F}-\lambda)^{-1}\bigr)\iota_{\mathfrak{N}_x(S^*)}, \tag{5.5.23}$$

and the corresponding Weyl function M is given by

$$M(\lambda) = \lambda - x + (\lambda - x)^2 P_{\mathfrak{N}_x(S^*)}(S_{\rm F}-\lambda)^{-1}\iota_{\mathfrak{N}_x(S^*)}. \tag{5.5.24}$$

Proof. It is clear from Theorem 1.7.1 that (5.5.22) is a direct sum decomposition. Now choose $\mu = x$ in Theorem 2.4.1 and modify the boundary triplet $\{\mathfrak{N}_x(S^*),\Gamma_0,\Gamma_1\}$ in Theorem 2.4.1 to $\{\mathfrak{N}_x(S^*),\Gamma_0,\Gamma_1 - x\Gamma_0\}$. Then $S_{\rm F} = \ker\Gamma_0$

and the corresponding γ-field and Weyl function have the form (5.5.23) and (5.5.24); cf. Theorem 2.4.1 and Corollary 2.5.5. It is easy to see that

$$S \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*) \subset \ker \Gamma_1,$$

and since $S_{\mathrm{K},x} = S \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*)$ and $\ker \Gamma_1$ are both self-adjoint, one concludes that $S_{\mathrm{K},x} = \ker \Gamma_1$, so that (5.5.20) holds. □

The following example is an illustration of Proposition 5.5.10 (i) and Corollary 5.5.12 for the case where the semibounded relation S is uniformly positive. In this situation it is convenient to have a boundary triplet for which the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$ corresponds to the boundary mapping Γ_1; cf. Chapter 8.

Example 5.5.13. Let S be a closed nonnegative symmetric relation in $\mathfrak{H}$ with lower bound $\gamma > 0$. In this case the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$ is given by $S_{\mathrm{K},0} = S \,\widehat{+}\, \widehat{\mathfrak{N}}_0(S^*)$; cf. (5.4.3). Moreover, the Friedrichs extension S_{F} and the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$ are transversal by (5.4.26). For $x = 0$ Corollary 5.5.12 shows that $\{\mathfrak{N}_0(S^*), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 \widehat{f} = f_0 \quad \text{and} \quad \Gamma_1 \widehat{f} = P_{\mathfrak{N}_0(S^*)} f_{\mathrm{F}}', \qquad \widehat{f} = \{f_{\mathrm{F}}, f_{\mathrm{F}}'\} + \{f_0, 0\},$$

is a boundary triplet for $S^* = S_{\mathrm{F}} \,\widehat{+}\, \widehat{\mathfrak{N}}_0(S^*)$ such that

$$S_{\mathrm{F}} = \ker \Gamma_0 \quad \text{and} \quad S_{\mathrm{K},0} = \ker \Gamma_1;$$

moreover, for $\lambda \in \rho(S_{\mathrm{F}})$ the corresponding γ-field γ is given by

$$\gamma(\lambda) = \bigl(I + \lambda (S_{\mathrm{F}} - \lambda)^{-1}\bigr) \iota_{\mathfrak{N}_0(S^*)},$$

and the corresponding Weyl function M is given by

$$M(\lambda) = \lambda + \lambda^2 P_{\mathfrak{N}_0(S^*)} (S_{\mathrm{F}} - \lambda)^{-1} \iota_{\mathfrak{N}_0(S^*)}.$$

Note that, in particular, $\gamma(0) = \iota_{\mathfrak{N}_0(S^*)}$ is the canonical embedding of $\mathfrak{N}_0(S^*)$ into $\mathfrak{H}$, $\gamma(0)^* = P_{\mathfrak{N}_0(S^*)}$ is the orthogonal projection onto $\mathfrak{N}_0(S^*)$, and $M(0) = 0$.

The last objective in this section is to derive an *abstract first Green identity*. For this consider a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for which $S_{\mathrm{F}} = \ker \Gamma_0$ and assume that also the self-adjoint extension corresponding to Γ_1 is semibounded. In the following the notation $S_1 = \ker \Gamma_1$ (instead of A_1) is used; this will turn out to be more convenient for the next section. As a first step rewrite the abstract Green identity (2.1.1) in the form

$$(f', g) - (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) = (f, g') - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}), \quad \widehat{f}, \widehat{g} \in S^*. \tag{5.5.25}$$

In the following theorem it will be shown that the expression on the left-hand side, and hence on the right-hand side, of (5.5.25) can be seen as a restriction of the form $\mathfrak{t}_{S_1}$.

Theorem 5.5.14. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that*

$$S_F = \ker \Gamma_0 \quad \textit{and} \quad S_1 = \ker \Gamma_1, \tag{5.5.26}$$

where S_F is the Friedrichs extension and S_1 is a semibounded self-adjoint extension of S. Moreover, let $\mathfrak{t}_{S_1}$ be the closed semibounded form corresponding to S_1. Then $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$ *and the following equality holds:*

$$(f', g) = \mathfrak{t}_{S_1}[f, g] + (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}), \quad \widehat{f}, \widehat{g} \in S^*. \tag{5.5.27}$$

Proof. By the assumption (5.5.26), the extensions S_F and S_1 are transversal and hence it follows from Theorem 5.3.8 that $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$. Moreover, every $\widehat{f}, \widehat{g} \in S^*$ can be decomposed as

$$\widehat{f} = \widehat{f}_F + \widehat{f}_1, \quad \widehat{g} = \widehat{g}_F + \widehat{g}_1, \quad \widehat{f}_F, \widehat{g}_F \in S_F, \quad \widehat{f}_1, \widehat{g}_1 \in S_1.$$

Using the conditions in (5.5.26) one sees that

$$\Gamma_0 \widehat{f}_F = \Gamma_0 \widehat{g}_F = 0 \quad \text{and} \quad \Gamma_1 \widehat{f}_1 = \Gamma_1 \widehat{g}_1 = 0, \tag{5.5.28}$$

and therefore the identity (5.5.25) can be rewritten as

$$\begin{aligned} (f', g) - (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) &= (f, g') - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}) \\ &= (f_F + f_1, g_F' + g_1') - (\Gamma_0 \widehat{f}_1, \Gamma_1 \widehat{g}_F). \end{aligned} \tag{5.5.29}$$

In order to rewrite the last term on the right-hand side of (5.5.29) observe that $\widehat{f}_1, \widehat{g}_F \in S^*$. Therefore, another application of the abstract Green identity for the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ shows that

$$\begin{aligned} (f_1', g_F) - (f_1, g_F') &= (\Gamma_1 \widehat{f}_1, \Gamma_0 \widehat{g}_F) - (\Gamma_0 \widehat{f}_1, \Gamma_1 \widehat{g}_F) \\ &= -(\Gamma_0 \widehat{f}_1, \Gamma_1 \widehat{g}_F), \end{aligned} \tag{5.5.30}$$

where (5.5.28) was used in the last equality. A combination of (5.5.29) and (5.5.30) gives

$$(f', g) - (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) = (f_F, g_F') + (f_F, g_1') + (f_1, g_1') + (f_1', g_F). \tag{5.5.31}$$

Since $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$, the last three terms on the right-hand side of (5.5.31) can be rewritten by means of Theorem 5.1.18:

$$(f_F, g_1') = \mathfrak{t}_{S_1}[f_F, g_1], \quad (f_1, g_1') = \mathfrak{t}_{S_1}[f_1, g_1], \quad (f_1', g_F) = \mathfrak{t}_{S_1}[f_1, g_F],$$

whereas the first term on the right-hand side of (5.5.31) can be rewritten by means of Theorem 5.1.18 and the inclusion $\mathfrak{t}_{S_F} \subset \mathfrak{t}_{S_1}$ as follows:

$$(f_F, g_F') = \mathfrak{t}_{S_F}[f_F, g_F] = \mathfrak{t}_{S_1}[f_F, g_F].$$

Combined with (5.5.31), the above rewriting leads to

$$\begin{aligned}(f', g) - (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) &= \mathfrak{t}_{S_1}[f_F, g_F] + \mathfrak{t}_{S_1}[f_F, g_1] + \mathfrak{t}_{S_1}[f_1, g_1] + \mathfrak{t}_{S_1}[f_1, g_F] \\ &= \mathfrak{t}_{S_1}[f_F + f_1, g_F + g_1] \\ &= \mathfrak{t}_{S_1}[f, g],\end{aligned}$$

and hence (5.5.27) has been shown. □

Let Θ be a self-adjoint relation in $\mathcal{G}$ and consider the corresponding self-adjoint extension

$$H_\Theta = \{\widehat{f} \in S^* : \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \Theta\}; \tag{5.5.32}$$

here the notation H_Θ (instead of A_Θ) is used, which is more convenient for the next section. For $\widehat{f} \in S^*$ the condition $\{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \Theta$ is equivalent to

$$\Gamma_0 \widehat{f} \in \operatorname{dom} \Theta_{\rm op} \quad \text{and} \quad P_{\rm op} \Gamma_1 \widehat{f} = \Theta_{\rm op} \Gamma_0 \widehat{f}, \tag{5.5.33}$$

where $P_{\rm op}$ denotes the orthogonal projection from $\mathcal{G}$ onto $\mathcal{G}_{\rm op} = \overline{\operatorname{dom}\,\Theta}$ and $\Theta_{\rm op}$ is the self-adjoint operator part of the self-adjoint relation Θ; cf. the end of Section 2.2. The following statement is an immediate consequence of Theorem 5.5.14.

Corollary 5.5.15. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* such that $S_F = \ker \Gamma_0$ and $S_1 = \ker \Gamma_1$, where S_F is the Friedrichs extension and S_1 is a semibounded self-adjoint extension of S. Let $\mathfrak{t}_{S_1}$ be the closed semibounded form corresponding to S_1. Assume that H_Θ is a self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathcal{G}$. Then*

$$(f', g) = \mathfrak{t}_{S_1}[f, g] + (\Theta_{\rm op} \Gamma_0 \widehat{f}, \Gamma_0 \widehat{g}), \quad \widehat{f}, \widehat{g} \in H_\Theta. \tag{5.5.34}$$

Under the assumption that H_Θ is semibounded, the left-hand side of (5.5.34) can be written as $\mathfrak{t}_{H_\Theta}[f, g]$. One may view the identity (5.5.34) as a perturbation of the form $\mathfrak{t}_{S_1}$ by means of the term $(\Theta_{\rm op} \Gamma_0 \widehat{f}, \Gamma_0 \widehat{g})$. The proper interpretation of (5.5.34) in terms of quadratic forms requires an extension of the mapping Γ_0; this procedure will be taken up in detail in Section 5.6 with the introduction of the notion of a boundary pair.

5.6 Boundary pairs and boundary triplets

In this section the notion of a boundary pair for a semibounded symmetric relation in a Hilbert space $\mathfrak{H}$ is developed. It will turn out that there is an intimate connection between boundary pairs and boundary triplets. In fact, a boundary pair helps to express the closed semibounded form associated with a semibounded self-adjoint extension in terms of the parameter provided by the boundary triplet. The concept of a boundary pair is motivated by applications occurring in the study of semibounded differential operators.

Definition 5.6.1. Let S be a closed semibounded relation in $\mathfrak{H}$ and let S_1 be a semibounded self-adjoint extension of S such that S_1 and the Friedrichs extension S_F are transversal. Let $\mathfrak{t}_{S_1}$ be the closed form associated with S_1 and let

$$\mathfrak{H}_{\mathfrak{t}_{S_1}-a} = \big(\mathrm{dom}\,\mathfrak{t}_{S_1}, (\cdot,\cdot)_{\mathfrak{t}_{S_1}-a}\big), \quad a < m(S_1),$$

be the corresponding Hilbert space. A pair $\{\mathcal{G}, \Lambda\}$ is called a *boundary pair* for S corresponding to S_1 if $\mathcal{G}$ is a Hilbert space and $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ satisfies

$$\ker \Lambda = \mathrm{dom}\,\mathfrak{t}_{S_\mathrm{F}} \quad \text{and} \quad \mathrm{ran}\,\Lambda = \mathcal{G}.$$

Let S be a closed semibounded relation in $\mathfrak{H}$, let S_F be the Friedrichs extension of S, and assume that $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to a semibounded self-adjoint extension S_1 of S. Then, by Definition 5.6.1, the semibounded self-adjoint extensions S_1 and S_F are transversal, which, by Theorem 5.3.8, is equivalent to the inclusion

$$\mathrm{dom}\,S^* \subset \mathrm{dom}\,\mathfrak{t}_{S_1} = \mathrm{dom}\,\Lambda.$$

One also has the orthogonal decomposition

$$\mathfrak{H}_{\mathfrak{t}_{S_1}-a} = \ker(S^*-a) \oplus_{\mathfrak{t}_{S_1}-a} \mathfrak{H}_{\mathfrak{t}_{S_\mathrm{F}}-a}, \quad a < m(S_1) \le m(S_\mathrm{F});$$

cf. Proposition 5.3.7 and Theorem 5.3.8. Since $\ker \Lambda = \mathrm{dom}\,\mathfrak{t}_{S_\mathrm{F}}$ and $\mathrm{ran}\,\Lambda = \mathcal{G}$, one sees that the restriction of Λ to the space $\ker(S^*-a)$ equipped with the norm $\|\cdot\|_{\mathfrak{t}_{S_1}-a}$, is a bounded mapping from $\ker(S^*-a)$ to $\mathcal{G}$ such that

$$\mathrm{ran}\,\big(\Lambda \upharpoonright \ker(S^*-a)\big) = \mathcal{G}.$$

Hence, the restriction $\Lambda \upharpoonright \ker(S^*-a)$ has a bounded everywhere defined inverse.

Boundary pairs have a useful invariance property. To see this consider a pair of semibounded self-adjoint extensions S_1 and S_2 of S which are each transversal with S_F, i.e.,

$$S^* = S_1 \,\widehat{+}\, S_\mathrm{F} = S_2 \,\widehat{+}\, S_\mathrm{F}. \tag{5.6.1}$$

Lemma 5.6.2. *Let S be a closed semibounded relation in $\mathfrak{H}$, let S_1 and S_2 be semibounded self-adjoint extensions which satisfy the transversality conditions* (5.6.1), *and assume that $a < \min\{m(S_1), m(S_2)\}$. Then $\mathrm{dom}\,\mathfrak{t}_{S_1} = \mathrm{dom}\,\mathfrak{t}_{S_2}$ and the form topologies of $\mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_2}$ coincide. Consequently, $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to S_1 if and only if $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to S_2.*

Proof. It suffices to consider the boundedness property, as the other properties of a boundary pair in Definition 5.6.1 do not depend on the choice of the transversal extensions S_1 and S_2. In fact, according to Corollary 5.3.9, the transversality conditions in (5.6.1) imply that

$$\mathrm{dom}\,(S_1-a)^{\frac{1}{2}} = \mathrm{dom}\,(S_2-a)^{\frac{1}{2}}, \quad a < \min\{m(S_1), m(S_2)\}.$$

Moreover, again by Corollary 5.3.9, this implies that

$$c_1\|((S_1)_{\rm op}-a)^{\frac{1}{2}}\varphi\|^2 \leq \|((S_2)_{\rm op}-a)^{\frac{1}{2}}\varphi\|^2 \leq c_2\|((S_1)_{\rm op}-a)^{\frac{1}{2}}\varphi\|^2$$

for all $\varphi \in \operatorname{dom}(S_2-a)^{\frac{1}{2}} = \operatorname{dom}(S_1-a)^{\frac{1}{2}}$ and for some constants $c_1, c_2 > 0$. In other words, $c_1(\mathfrak{t}_{S_1}-a) \leq \mathfrak{t}_{S_2}-a \leq c_2(\mathfrak{t}_{S_1}-a)$ for some constants $c_1, c_2 > 0$. Therefore, the form topologies of $\mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_2}$ coincide and thus $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ if and only if $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_2}-a}, \mathcal{G})$. This implies that $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to S_1 if and only if $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to S_2. □

To explore the connection between boundary pairs and boundary triplets, the notion of extension in the next definition will be important.

Definition 5.6.3. Let S be a closed semibounded relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$ and assume that $\operatorname{mul} A_0 = \operatorname{mul} S^*$. Let S_1 be a semibounded self-adjoint extension of S such that S_1 and $S_{\rm F}$ are transversal, so that, $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$. Then an operator $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ is said to be an *extension* of Γ_0 if

$$\Gamma_0 \widehat{f} = \Lambda f \quad \text{for all } \widehat{f} = \{f, f'\} \in S^*. \tag{5.6.2}$$

It follows already from the assumption $\operatorname{mul} A_0 = \operatorname{mul} S^*$ that the mapping $f \mapsto \Gamma_0 \widehat{f}$ from $\operatorname{dom} S^*$ to $\mathcal{G}$ in (5.6.2) is an operator. In fact, if $\widehat{f} = \{0, f'\} \in S^*$, then $\widehat{f} \in A_0 = \ker \Gamma_0$, so that $\Gamma_0 \widehat{f} = 0$. Note also that in the case $A_0 = S_{\rm F}$ the condition $\operatorname{mul} A_0 = \operatorname{mul} S^*$ is satisfied by Theorem 5.3.3.

Next the notion of a compatible boundary pair will be introduced.

Definition 5.6.4. Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with

$$A_0 = \ker \Gamma_0 \quad \text{and} \quad A_1 = \ker \Gamma_1.$$

Let S_1 be a semibounded self-adjoint extension of S such that S_1 and $S_{\rm F}$ are transversal and let $\{\mathcal{G}, \Lambda\}$ be a boundary pair for S corresponding to S_1. Then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}, \Lambda\}$ are said to be *compatible* if Λ is an extension of Γ_0 and the self-adjoint relations A_1 and S_1 coincide.

The next lemma provides a sufficient condition for an extension Λ of Γ_0 such that $\{\mathcal{G}, \Lambda\}$ is a boundary pair, or compatible boundary pair, for S corresponding to S_1. In the special case where the defect numbers of S are finite this condition is automatically satisfied, which makes the lemma useful in applications to Sturm–Liouville operators in Chapter 6.

Lemma 5.6.5. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$. Let S_1 be a semibounded self-adjoint extension of S such that S_1 and $S_{\rm F}$ are transversal or, equivalently, $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$, and let $a < m(S_1)$. Then the following statements hold:*

(i) *If* $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ *is an extension of* Γ_0, *then* $\operatorname{ran}\Lambda = \mathcal{G}$ *and*

$$\operatorname{dom}\mathfrak{t}_{S_F} \subset \ker\Lambda.$$

(ii) *If* $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ *is an extension of* Γ_0 *and*

$$\operatorname{dom}\mathfrak{t}_{S_F} = \ker\Lambda, \tag{5.6.3}$$

then $A_0 = \ker\Gamma_0 = S_F$ *and* $\{\mathcal{G}, \Lambda\}$ *is a boundary pair for* S *corresponding to* S_1. *If, in addition,* $A_1 = S_1$, *then* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *and* $\{\mathcal{G}, \Lambda\}$ *are compatible.*

In particular, if $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ *is an extension of* Γ_0 *and the defect numbers of* S *are finite, then* $A_0 = \ker\Gamma_0 = S_F$ *and* $\{\mathcal{G}, \Lambda\}$ *is a boundary pair for* S *corresponding to* S_1. *If, in this case, also* $A_1 = S_1$, *then* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *and* $\{\mathcal{G}, \Lambda\}$ *are compatible.*

Proof. (i) Let $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ be an extension of Γ_0. It follows from the extension property (5.6.2) that $\operatorname{ran}\Lambda = \mathcal{G}$ and that

$$\operatorname{dom} S \subset \operatorname{dom} A_0 \subset \ker\Lambda. \tag{5.6.4}$$

Since $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$, it is clear that $\ker\Lambda$ is closed in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ and, by (5.6.4), the closure of $\operatorname{dom} S$ with respect to the inner product $(\cdot,\cdot)_{\mathfrak{t}_{S_1}-a}$ is contained in $\ker\Lambda$. On the other hand, $\operatorname{dom} S \subset \operatorname{dom} S_F \subset \operatorname{dom}\mathfrak{t}_{S_F}$ and the inner product on $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ restricted to $\mathfrak{H}_{\mathfrak{t}_{S_F}-a}$ coincides with the inner product $(\cdot,\cdot)_{\mathfrak{t}_{S_F}-a}$ in $\mathfrak{H}_{\mathfrak{t}_{S_F}-a}$ (see the discussion below (5.3.10) and (5.3.11)). As $\operatorname{dom} S$ is a core of $\mathfrak{t}_{S_F}$, it follows from Corollary 5.1.15 that the closure of $\operatorname{dom} S$ with respect to the inner product $(\cdot,\cdot)_{\mathfrak{t}_{S_F}-a}$ coincides with $\operatorname{dom}\mathfrak{t}_{S_F}$. Thus, one concludes $\operatorname{dom}\mathfrak{t}_{S_F} \subset \ker\Lambda$.

(ii) Assume that $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ is an extension of Γ_0 and that (5.6.3) holds. Then

$$A_0 = \bigl\{\widehat{f} \in S^* : \widehat{f} \in \ker\Gamma_0\bigr\} \subset \bigl\{\widehat{f} \in S^* : f \in \ker\Lambda\bigr\}. \tag{5.6.5}$$

Since $\ker\Lambda = \operatorname{dom}\mathfrak{t}_{S_F}$, it follows from Theorem 5.3.3 that the right-hand side of (5.6.5) concides with S_F. Thus, $A_0 \subset S_F$ and since both relations are self-adjoint, it follows that $A_0 = S_F$. Moreover, one has $\operatorname{ran}\Lambda = \mathcal{G}$ by (i) and hence $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S^* corresponding to S_1. It follows directly from Definition 5.6.4 that the additional assumption $A_1 = S_1$ yields compatibility of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}, \Lambda\}$.

For the last statement assume that the defect numbers of S are finite and let $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$ be an extension of Γ_0. Then $\operatorname{ran}\Lambda = \mathcal{G}$ and $\operatorname{dom}\mathfrak{t}_{S_F} \subset \ker\Lambda$ by (i). Since S_1 and S_F are transversal, one has the orthogonal decomposition

$$\mathfrak{H}_{\mathfrak{t}_{S_1}-a} = \ker(S^* - a) \oplus_{\mathfrak{t}_{S_1}-a} \mathfrak{H}_{\mathfrak{t}_{S_F}-a}$$

for $a < m(S_1)$ by Proposition 5.3.7 and Theorem 5.3.8. Then

$$\dim\operatorname{ran}\Lambda = \dim\mathcal{G} = \dim\ker(S^* - a) < \infty$$

together with $\operatorname{dom}\mathfrak{t}_{S_F} \subset \ker\Lambda$ implies $\operatorname{dom}\mathfrak{t}_{S_F} = \ker\Lambda$. Now the assertions follow from (ii). □

If the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and the boundary pair $\{\mathcal{G}, \Lambda\}$ are compatible, then automatically $\ker \Gamma_0 = S_{\mathrm{F}}$. In the next theorem it will be shown that for a boundary triplet for S^* such that $\ker \Gamma_0 = S_{\mathrm{F}}$ and $\ker \Gamma_1 = S_1$ is semibounded, there exists a compatible boundary pair $\{\mathcal{G}, \Lambda\}$ for S corresponding to S_1.

Theorem 5.6.6. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Assume that*

$$\ker \Gamma_0 = S_{\mathrm{F}} \quad \textit{and} \quad \ker \Gamma_1 = S_1,$$

where S_{F} is the Friedrichs extension and S_1 is a semibounded self-adjoint extension of S. Then, with $a < m(S_1)$ fixed, the mapping

$$\Lambda_0 = \left\{ \{f, \Gamma_0 \widehat{f}\} : \widehat{f} \in S^* \right\} \subset \mathfrak{H}_{\mathfrak{t}_{S_1} - a} \times \mathcal{G}, \tag{5.6.6}$$

is (the graph of) a densely defined bounded operator. Its unique bounded extension Λ to all of $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$ induces a boundary pair $\{\mathcal{G}, \Lambda\}$ for S corresponding to S_1 which is compatible with the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$.

Proof. The assumption that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^* implies that S_{F} and S_1 are transversal extensions of S. By Theorem 5.3.8, this is equivalent to $\mathrm{dom}\, S^* \subset \mathrm{dom}\, \mathfrak{t}_{S_1}$, and hence $\mathrm{dom}\, \Lambda_0 = \mathrm{dom}\, S^*$ is contained in $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$, i.e., the relation Λ_0 in (5.6.6) is well defined from $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$ to $\mathcal{G}$.

In order to see that Λ_0 is an operator, assume that $\{f, \Gamma_0 \widehat{f}\} \in \Lambda_0$ with $\widehat{f} \in S^*$ satisfying $f = 0$. Hence, $\widehat{f} = \{0, f'\} \in S^*$, which by Theorem 5.3.3 shows that $\widehat{f} \in S_{\mathrm{F}}$. Now the identity $S_{\mathrm{F}} = \ker \Gamma_0$ implies that $\Gamma_0 \widehat{f} = 0$. Hence, $\mathrm{mul}\, \Lambda_0 = \{0\}$, that is, Λ_0 in (5.6.6) is an operator. Furthermore, it is clear that $S_{\mathrm{F}} = \ker \Gamma_0$ yields $\ker \Lambda_0 = \mathrm{dom}\, S_{\mathrm{F}}$.

Next it will be shown that the operator

$$\Lambda_0 : \mathfrak{H}_{\mathfrak{t}_{S_1} - a} \supset \mathrm{dom}\, S^* \to \mathcal{G}, \qquad f \mapsto \Lambda_0 f = \Gamma_0 \widehat{f}, \tag{5.6.7}$$

is bounded. By assumption, $a < m(S_1) \le m(S_{\mathrm{F}})$, and hence one has the decomposition $S^* = S_{\mathrm{F}} \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_a(S^*)$; see Theorem 1.7.1. Let $\widehat{f} = \{f, f'\} \in S^*$ and decompose it accordingly,

$$\widehat{f} = \widehat{f}_{\mathrm{F}} + \widehat{f}_a, \quad \widehat{f}_{\mathrm{F}} \in S_{\mathrm{F}}, \ \widehat{f}_a = \{f_a, a f_a\} \in \widehat{\mathfrak{N}}_a(S^*).$$

From $S_{\mathrm{F}} = \ker \Gamma_0$ and the fact that $\Gamma_0 : S^* \to \mathcal{G}$ is bounded (with respect to the graph norm; cf. Proposition 2.1.2) it follows that

$$\|\Lambda_0 f\|^2 = \|\Gamma_0 \widehat{f}\|^2 = \|\Gamma_0 \widehat{f}_a\|^2 \le M(\|f_a\|^2 + a^2 \|f_a\|^2) \le M' \|f_a\|^2$$

holds for some constants $M, M' > 0$. Then is clear from (5.1.9) that there exists $M'' > 0$ such that

$$\|\Lambda_0 f\|^2 \le M'' \|f_a\|^2_{\mathfrak{t}_{S_1} - a} \le M'' \big(\|f_{\mathrm{F}}\|^2_{\mathfrak{t}_{S_1} - a} + \|f_a\|^2_{\mathfrak{t}_{S_1} - a} \big) = M'' \|f\|^2_{\mathfrak{t}_{S_1} - a},$$

where $f_F \in \operatorname{dom} S_F \subset \operatorname{dom} \mathfrak{t}_{S_F}$; here in the last equality one uses the orthogonal decomposition

$$\operatorname{dom} \mathfrak{t}_{S_1} = \ker (S^* - a) \oplus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_F}$$

with respect to the inner product $(\cdot,\cdot)_{\mathfrak{t}_{S_1}-a}$ in the Hilbert space $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$, see Proposition 5.3.7 and Theorem 5.3.8. This shows that the operator Λ_0 in (5.6.7) is bounded.

By Theorem 5.1.18 (ii), $\operatorname{dom} S_1$ is a core of $\mathfrak{t}_{S_1}$ and hence $\operatorname{dom} S_1$ is dense in the Hilbert space $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ by Corollary 5.1.15. As $\operatorname{dom} S_1 \subset \operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$, it follows that $\operatorname{dom} S^*$ is also a dense subspace of $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$. Therefore, the bounded operator Λ_0 in (5.6.7) is densely defined, and hence Λ_0 admits a unique extension Λ by continuity to all of $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$.

Since the restriction of the inner product in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ to $\mathfrak{H}_{\mathfrak{t}_{S_F}-a}$ coincides with the inner product in $\mathfrak{H}_{\mathfrak{t}_{S_F}-a}$ (see the discussion below (5.3.10) and (5.3.11)), the closure of $\operatorname{dom} S_F = \ker \Lambda_0$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ is $\operatorname{dom} \mathfrak{t}_{S_F}$. This implies $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_F}$. Finally, by definition $\operatorname{ran} \Lambda = \operatorname{ran} \Lambda_0 = \mathcal{G}$, and hence $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S corresponding to S_1. It is also clear that the boundary pair $\{\mathcal{G}, \Lambda\}$ and the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are compatible. □

In the next corollary it is shown that in the context of Theorem 5.6.6 the continuity of $\Lambda : \mathfrak{H}_{\mathfrak{t}_{S_1}-a} \to \mathcal{G}$ makes it possible to extend the identity

$$(f', g) = \mathfrak{t}_{S_1}[f, g] + (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}), \quad \widehat{f}, \widehat{g} \in S^*, \tag{5.6.8}$$

in Theorem 5.5.14 to $\widehat{f} \in S^*$ and $g \in \operatorname{dom} \mathfrak{t}_{S_1}$.

Corollary 5.6.7. *Let S be a closed semibounded relation in $\mathfrak{H}$, and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}, \Lambda\}$ be the boundary triplet and boundary pair in Theorem 5.6.6, respectively. Then the following equality holds:*

$$(f', g) = \mathfrak{t}_{S_1}[f, g] + (\Gamma_1 \widehat{f}, \Lambda g), \quad \widehat{f} \in S^*, \ g \in \operatorname{dom} \mathfrak{t}_{S_1}.$$

Proof. As $\operatorname{dom} S^*$ is a dense subspace of $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$, there exist $g_n \in \operatorname{dom} S^*$ and $\widehat{g}_n = \{g_n, g_n'\} \in S^*$ such that $g_n \to g$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$, and hence $\Gamma_0 \widehat{g}_n = \Lambda g_n \to \Lambda g$ in $\mathcal{G}$. Furthermore, $g_n \to g$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ also implies $g_n \to g \in \mathfrak{H}$ and $\mathfrak{t}_{S_1}[f, g_n] \to \mathfrak{t}_{S_1}[f, g]$ for $\widehat{f} = \{f, f'\} \in S^*$. By (5.6.8), the identity

$$(f', g_n) = \mathfrak{t}_{S_1}[f, g_n] + (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}_n) = \mathfrak{t}_{S_1}[f, g_n] + (\Gamma_1 \widehat{f}, \Lambda g_n)$$

holds for $\widehat{f} = \{f, f'\}$ and $\widehat{g}_n = \{g_n, g_n'\} \in S^*$. Now the assertion follows by taking limits. □

For the sake of completeness the existence of boundary pairs is stated in the following corollary as an addendum to Definition 5.6.1.

Corollary 5.6.8. *Let S be a closed semibounded relation in $\mathfrak{H}$. Then there exist a semibounded self-adjoint extension S_1 of S such that S_1 and $S_{\rm F}$ are transversal and a mapping $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_1}-a}, \mathcal{G})$, $a < m(S_1)$, such that $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S.*

Proof. By Proposition 5.5.10, there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* such that $S_{\rm F} = \ker \Gamma_0$ and $S_{{\rm K},x} = \ker \Gamma_1$ for $x < m(S) = m(S_{\rm F})$. Now the statement follows with $S_1 = S_{{\rm K},x}$ and $a < m(S_1) = x$ from Theorem 5.6.6. □

Example 5.6.9. Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ, fix $x < \gamma$, and consider the boundary triplet $\{\mathfrak{N}_x(S^*), \Gamma_0, \Gamma_1\}$ for S^* in Corollary 5.5.12 with

$$\Gamma_0 \widehat{f} = f_x, \qquad \widehat{f} = \{f_{\rm F}, f_{\rm F}'\} + \{f_x, x f_x\} \in S_{\rm F} \,\widehat{+}\, \widehat{\mathfrak{N}}_x(S^*).$$

Then $S_{\rm F} = \ker \Gamma_0$ and $S_{{\rm K},x} = \ker \Gamma_1$ and for $a < x$ one has the direct sum decomposition

$$\mathfrak{H}_{\mathfrak{t}_{S_{{\rm K},x}}-a} = \operatorname{dom} \mathfrak{t}_{S_{{\rm K},x}} = \operatorname{dom} \mathfrak{t}_{S_{\rm F}} + \mathfrak{N}_x(S^*), \quad a < x < \gamma; \tag{5.6.9}$$

cf. Corollary 5.4.16. Then the mapping

$$\Lambda f = f_x, \qquad f = f_{\rm F} + f_x \in \operatorname{dom} \mathfrak{t}_{S_{\rm F}} + \mathfrak{N}_x(S^*),$$

belongs to $\mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_{{\rm K},x}}-a}, \mathfrak{N}_x(S^*))$. In fact, let $f \in \mathfrak{H}_{\mathfrak{t}_{S_{{\rm K},x}}-a}$ have the decomposition $f = f_{\rm F} + f_x$ as in (5.6.9), and define $f_a = (I + (a-x)(S_{\rm F} - a)^{-1}) f_x$. Then $f = g_{\rm F} + f_a$, where $g_{\rm F} = f_x - f_a + f_{\rm F} \in \operatorname{dom} \mathfrak{t}_{S_{\rm F}}$ and $f_a \in \mathfrak{N}_a(S^*)$. Now observe that $f_x = (I + (x-a)(S_{\rm F} - x)^{-1}) f_a$, so that Proposition 1.4.6 leads to the estimate

$$\|f_x\| \leq \frac{\gamma - a}{\gamma - x} \|f_a\|.$$

Recall from (5.1.9) (with $\mathfrak{t} = \mathfrak{t}_{S_{{\rm K},x}}$, $\gamma = x$, $\varphi = f_a$) and (5.4.28) that

$$(x-a)\|f_a\|^2 \leq \|f_a\|^2_{\mathfrak{t}_{S_{{\rm K},x}}-a} \leq \|f_a\|^2_{\mathfrak{t}_{S_{{\rm K},x}}-a} + \|g_{\rm F}\|^2_{\mathfrak{t}_{S_{{\rm K},x}}-a} = \|f\|^2_{\mathfrak{t}_{S_{{\rm K},x}}-a},$$

which proves that $\Lambda \in \mathbf{B}(\mathfrak{H}_{\mathfrak{t}_{S_{{\rm K},x}}-a}, \mathfrak{N}_x(S^*))$. Thus, Λ extends Γ_0 in the sense of Definition 5.6.3. It is clear that $\operatorname{dom} \mathfrak{t}_{S_{\rm F}} = \ker \Lambda$, and hence Lemma 5.6.5 (ii) implies that $\{\mathfrak{N}_x(S^*), \Lambda\}$ is a boundary pair for S corresponding to $S_{{\rm K},x}$ which is compatible with the boundary triplet $\{\mathfrak{N}_x(S^*), \Gamma_0, \Gamma_1\}$ in Corollary 5.5.12.

Theorem 5.6.6 admits a converse. If S is a semibounded relation and $\{\mathcal{G}, \Lambda\}$ is a boundary pair for S in the sense of Definition 5.6.1, then there exists a compatible boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^*. The construction of the mapping $\Gamma_0 : S^* \to \mathcal{G}$ is inspired by Lemma 5.6.5 and the construction of $\Gamma_1 : S^* \to \mathcal{G}$ is inspired by the first Green formula in Theorem 5.5.14.

Theorem 5.6.10. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let S_1 be a semibounded self-adjoint extension of S such that S_1 and S_F are transversal. Let $\{\mathcal{G}, \Lambda\}$ be a boundary pair for S corresponding to S_1. Then*

$$\Gamma_0 = \big\{\{\widehat{f}, \Lambda f\} : \widehat{f} \in S^*\big\} \tag{5.6.10}$$

is (the graph of) a linear operator from S^* to $\mathcal{G}$ and there exists a unique linear operator $\Gamma_1 : S^* \to \mathcal{G}$ such that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ defines a boundary triplet for S^* which is compatible with the boundary pair $\{\mathcal{G}, \Lambda\}$ for S corresponding to S_1.

Proof. The relations S_1 and S_F are semibounded self-adjoint extensions of S and hence $m(S_1) \leq m(S_F) = m(S)$. There are the following decompositions of the relation S^*:

$$S^* = S_F \,\widehat{+}\, \widehat{\mathfrak{N}}_a(S^*), \quad a < m(S_F), \tag{5.6.11}$$

and, likewise,

$$S^* = S_1 \,\widehat{+}\, \widehat{\mathfrak{N}}_a(S^*), \quad a < m(S_1); \tag{5.6.12}$$

cf. Theorem 1.7.1. Recall that $\mathfrak{t}_{S_F} \subset \mathfrak{t}_{S_1}$ and, since S_1 and S_F are transversal, there is the orthogonal decomposition

$$\operatorname{dom} \mathfrak{t}_{S_1} = \ker(S^* - a) \oplus_{\mathfrak{t}_{S_1} - a} \operatorname{dom} \mathfrak{t}_{S_F}, \quad a < m(S_1), \tag{5.6.13}$$

of the Hilbert space $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$. Moreover, in this case one also has $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$; cf. Proposition 5.3.7 and Theorem 5.3.8. The proof will be given in a number of steps. The mapping Γ_0 is considered in Step 1. Step 2 and Step 3 are preparations for the construction of Γ_1 in Step 4. In the remaining steps the various properties of Γ_1 are established.

Step 1. This step concerns the properties of Γ_0 in (5.6.10). Since $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$ one sees from Definition 5.6.1 that the relation Γ_0 is well defined. It is clear that Γ_0 is the graph of an operator,

$$\Gamma_0 : S^* \to \mathcal{G}, \qquad \widehat{f} \mapsto \Gamma_0 \widehat{f} = \Lambda f, \tag{5.6.14}$$

and that $\{0\} \times \operatorname{mul} S^* \subset \ker \Gamma_0$. Furthermore,

$$S_F = \big\{\widehat{f} \in S^* : f \in \operatorname{dom} \mathfrak{t}_{S_F}\big\} = \big\{\widehat{f} \in S^* : f \in \ker \Lambda\big\} = \ker \Gamma_0, \tag{5.6.15}$$

where the first equality holds by Theorem 5.3.3, the second equality is due to $\operatorname{dom} \mathfrak{t}_{S_F} = \ker \Lambda$, and the third equality follows from (5.6.14).

Since $\operatorname{ran} \Lambda = \mathcal{G}$ and $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_F}$, it follows from (5.6.13) that Λ maps $\ker(S^* - a)$ bijectively onto $\mathcal{G}$. Therefore,

$$\Gamma_0 \text{ is a bijection between } \widehat{\mathfrak{N}}_a(S^*) \text{ and } \mathcal{G}, \tag{5.6.16}$$

and, in particular,

$$\operatorname{ran} \Gamma_0 = \mathcal{G}. \tag{5.6.17}$$

Step 2. Now it will be shown that the identity

$$\mathfrak{t}_{S_1}[f, g_\mathrm{F}] = (f', g_\mathrm{F}), \quad \widehat{f} \in S^*, \, g_\mathrm{F} \in \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}, \tag{5.6.18}$$

holds. For this, assume that $\widehat{f}$ is decomposed as

$$\widehat{f} = \widehat{f}_\mathrm{F} + \widehat{h}_a, \qquad \widehat{f}_\mathrm{F} \in S_\mathrm{F}, \; \widehat{h}_a \in \widehat{\mathfrak{N}}_a(S^*); \tag{5.6.19}$$

cf. (5.6.11). Recall that $\operatorname{dom} S^* \subset \operatorname{dom} \mathfrak{t}_{S_1}$ and observe that with (5.6.19) one gets

$$\mathfrak{t}_{S_1}[f, g_\mathrm{F}] = \mathfrak{t}_{S_1}[f_\mathrm{F} + h_a, g_\mathrm{F}] = \mathfrak{t}_{S_\mathrm{F}}[f_\mathrm{F}, g_\mathrm{F}] + \mathfrak{t}_{S_1}[h_a, g_\mathrm{F}]. \tag{5.6.20}$$

The orthogonal decomposition in (5.6.13) gives

$$0 = (h_a, g_\mathrm{F})_{\mathfrak{t}_{S_1} - a} = \mathfrak{t}_{S_1}[h_a, g_\mathrm{F}] - a(h_a, g_\mathrm{F}).$$

Hence, (5.6.20) leads to the identity

$$\mathfrak{t}_{S_1}[f, g_\mathrm{F}] = \mathfrak{t}_{S_\mathrm{F}}[f_\mathrm{F}, g_\mathrm{F}] + a(h_a, g_\mathrm{F}) = (f'_\mathrm{F}, g_\mathrm{F}) + a(h_a, g_\mathrm{F}),$$

which shows (5.6.18).

Step 3. Next it will be shown that

$$\mathfrak{t}_{S_1}[f, g] - (f', g) = (f_a, g_a)_{\mathfrak{t}_{S_1} - a}, \quad \widehat{f}, \widehat{g} \in S^*, \tag{5.6.21}$$

where $\widehat{f}$ and $\widehat{g}$ are decomposed as

$$\widehat{f} = \widehat{f}_1 + \widehat{f}_a, \qquad \widehat{f}_1 \in S_1, \; \widehat{f}_a \in \widehat{\mathfrak{N}}_a(S^*), \tag{5.6.22}$$

and

$$\widehat{g} = \widehat{g}_\mathrm{F} + \widehat{g}_a, \qquad \widehat{g}_\mathrm{F} \in S_\mathrm{F}, \; \widehat{g}_a \in \widehat{\mathfrak{N}}_a(S^*); \tag{5.6.23}$$

cf. (5.6.12) and (5.6.11). For this note first that with (5.6.22) the identity (5.6.18) in Step 2 gives

$$\mathfrak{t}_{S_1}[f_1 + f_a, g_\mathrm{F}] = (f'_1 + a f_a, g_\mathrm{F}). \tag{5.6.24}$$

Furthermore, note that with g_a from (5.6.23) one has

$$\mathfrak{t}_{S_1}[f_1, g_a] = (f'_1, g_a) \tag{5.6.25}$$

due to $\widehat{f}_1 \in S_1$ and $g_a \in \mathfrak{N}_a(S^*) \subset \operatorname{dom} \mathfrak{t}_{S_1}$; cf. Theorem 5.1.18. A combination of (5.6.24) and (5.6.25) leads to

$$\begin{aligned}
\mathfrak{t}_{S_1}[f, g] - (f', g) &= \mathfrak{t}_{S_1}[f_1 + f_a, g_\mathrm{F} + g_a] - (f'_1 + a f_a, g_\mathrm{F} + g_a) \\
&= \mathfrak{t}_{S_1}[f_1 + f_a, g_a] - (f'_1 + a f_a, g_a) \\
&= \mathfrak{t}_{S_1}[f_a, g_a] - a(f_a, g_a),
\end{aligned}$$

which gives (5.6.21).

Step 4. In this step the operator $\Gamma_1 : S^* \to \mathcal{G}$ will be constructed. For this purpose fix $\widehat{f} \in S^*$ and consider the linear relation

$$\Phi_{\widehat{f}} = \bigl\{\{\Gamma_0\,\widehat{g}, (g, f') - \mathfrak{t}_{S_1}[g, f]\} : \widehat{g} \in S^* \bigr\}. \tag{5.6.26}$$

It follows from $\operatorname{ran}\Gamma_0 = \mathcal{G}$ in (5.6.17) that $\operatorname{dom}\Phi_{\widehat{f}} = \mathcal{G}$. Next it will be shown that $\Phi_{\widehat{f}}$ is the graph of a bounded linear functional. If $\widehat{f}$ and $\widehat{g}$ are decomposed as in (5.6.22) and (5.6.23), then it follows from (5.6.21) in Step 3 that

$$\bigl|(g, f') - \mathfrak{t}_{S_1}[g, f]\bigr| = \bigl|\mathfrak{t}_{S_1}[f, g] - (f', g)\bigr| = |(f_a, g_a)_{\mathfrak{t}_{S_1}-a}| \leq \|f_a\|_{\mathfrak{t}_{S_1}-a}\|g_a\|_{\mathfrak{t}_{S_1}-a}.$$

Recall that the restriction of Λ to $\ker(S^* - a)$ has a bounded inverse (with respect to the norm $\|\cdot\|_{\mathfrak{t}_{S_1}-a}$ on $\ker(S^* - a)$). Therefore, by (5.6.10) and (5.6.15),

$$\|g_a\|_{\mathfrak{t}_{S_1}-a} \leq C\|\Lambda g_a\| = C\|\Gamma_0\widehat{g}_a\| = C\|\Gamma_0\widehat{g}\| \tag{5.6.27}$$

for some constant $C > 0$ and, as a consequence,

$$\bigl|(g, f') - \mathfrak{t}_{S_1}[g, f]\bigr| \leq C\|f_a\|_{\mathfrak{t}_{S_1}-a}\|\Gamma_0\widehat{g}\|, \qquad \widehat{g} \in S^*.$$

This implies that the relation $\Phi_{\widehat{f}}$ in (5.6.26) is the graph of an everywhere defined bounded functional. Hence, by the Riesz representation theorem, there exists a unique $\varphi_{\widehat{f}} \in \mathcal{G}$ such that

$$\Phi_{\widehat{f}}(\Gamma_0\widehat{g}) = \bigl(\Gamma_0\widehat{g}, \varphi_{\widehat{f}}\bigr), \quad \widehat{g} \in S^*.$$

Define the mapping Γ_1 by

$$\Gamma_1 : S^* \to \mathcal{G}, \qquad \widehat{f} \mapsto \Gamma_1\widehat{f} := \varphi_{\widehat{f}}. \tag{5.6.28}$$

By construction, Γ_1 is linear and it follows from (5.6.21) and (5.6.26) that

$$(\Gamma_1\widehat{f}, \Gamma_0\widehat{g}) = (f', g) - \mathfrak{t}_{S_1}[f, g] = -(f_a, g_a)_{\mathfrak{t}_{S_1}-a} \tag{5.6.29}$$

for all $\widehat{f} \in S^*$ and $\widehat{g} \in S^*$ decomposed in the forms (5.6.22) and (5.6.23), respectively.

Step 5. It will be shown that the operator Γ_1, constructed in Step 4, satisfies

$$S_1 = \ker\Gamma_1. \tag{5.6.30}$$

To show that $S_1 \subset \ker\Gamma_1$, assume that $\widehat{f} \in S_1$. Then (5.6.29) in Step 4 implies that $(\Gamma_1\widehat{f}, \Gamma_0\widehat{g}) = 0$ for all $\widehat{g} \in S^*$, and since Γ_0 is surjective (see (5.6.17)), one concludes that $\Gamma_1\widehat{f} = 0$. Thus, $S_1 \subset \ker\Gamma_1$. To show the reverse inclusion, assume that $\widehat{f} = \{f, f'\} \in \ker\Gamma_1$. Then it follows from (5.6.29) that

$$\mathfrak{t}_{S_1}[f, g] = (f', g) \quad \text{for all} \quad g \in \operatorname{dom} S_1 \subset \operatorname{dom} S^*.$$

Since $\operatorname{dom} S_1$ is a core of $\mathfrak{t}_{S_1}$, it is a consequence of the first representation theorem (Theorem 5.1.18) that $\widehat{f} = \{f, f'\} \in S_1$. Thus, $\ker\Gamma_1 \subset S_1$, and so (5.6.30) has been proved.

Step 6. Next it will be shown that the operator Γ_1, constructed in Step 4, satisfies

$$\operatorname{ran}\Gamma_1 = \mathcal{G}. \tag{5.6.31}$$

For this purpose note first that $\overline{\operatorname{ran}}\,\Gamma_1 = \mathcal{G}$. In fact, if $\overline{\operatorname{ran}}\,\Gamma_1 \neq \mathcal{G}$, then in view of (5.6.16) there exists $\widehat{g}_a \in \widehat{\mathfrak{N}}_a(S^*)$ such that $\Gamma_0\widehat{g}_a \neq 0$ and

$$(\Gamma_1\widehat{f}, \Gamma_0\widehat{g}_a) = 0 \quad \text{for all} \quad \widehat{f} \in S^*. \tag{5.6.32}$$

Now apply (5.6.29) with $\widehat{f} = \widehat{f}_1 + \widehat{f}_a \in S^*$, $\widehat{f}_1 \in S_1$, $\widehat{f}_a \in \widehat{\mathfrak{N}}_a(S^*)$, and $\widehat{g}_a \in \widehat{\mathfrak{N}}_a(S^*)$. Then (5.6.32) implies

$$(f_a, g_a)_{\mathfrak{t}_{S_1}-a} = 0$$

for all $f_a \in \mathfrak{N}_a(S^*)$. Therefore, $g_a = 0$ and hence $\widehat{g}_a = 0$ and $\Gamma_0\widehat{g}_a = 0$, which is a contradiction. Thus, $\overline{\operatorname{ran}}\,\Gamma_1 = \mathcal{G}$.

To conclude (5.6.31), it suffices to show that $\operatorname{ran}\Gamma_1$ is closed. For this consider the restriction Γ_1' to $\widehat{\mathfrak{N}}_a(S^*)$. It follows from $S^* = S_1 \,\widehat{+}\, \widehat{\mathfrak{N}}_a(S^*)$ and (5.6.30) that Γ_1' is injective and that

$$\operatorname{ran}\Gamma_1' = \operatorname{ran}\Gamma_1. \tag{5.6.33}$$

With the inner product $(\cdot,\cdot)_{\mathfrak{t}_{S_1}-a}$ the space $\widehat{\mathfrak{N}}_a(S^*)$ is a closed subspace of the Hilbert space $\mathfrak{H}_{\mathfrak{t}_{S_1}-a} \times \mathfrak{H}_{\mathfrak{t}_{S_1}-a}$ (see (5.6.13)). Since

$$\left|(\Gamma_1'\widehat{f}_a, \Gamma_0\widehat{g})\right| = |(f_a, g_a)_{\mathfrak{t}_{S_1}-a}| \leq C\|f_a\|_{\mathfrak{t}_{S_1}-a}\|\Gamma_0\widehat{g}\|$$

by (5.6.29) and (5.6.27), it follows from

$$\|\Gamma_1'\widehat{f}_a\| = \sup_{\|\Gamma_0\widehat{g}\|=1} \left|(\Gamma_1\widehat{f}_a, \Gamma_0\widehat{g})\right| \leq C\|f_a\|_{\mathfrak{t}_{S_1}-a}$$

that the operator Γ_1' is bounded in the topology of $\mathfrak{H}_{\mathfrak{t}_{S_1}-a} \times \mathfrak{H}_{\mathfrak{t}_{S_1}-a}$. Hence, Γ_1' is closed and the same is true for the inverse operator

$$(\Gamma_1')^{-1} : \mathcal{G} \supset \operatorname{ran}\Gamma_1' \to \widehat{\mathfrak{N}}_a(S^*).$$

Assume that $(\Gamma_1')^{-1}$ is unbounded. Then there exists a sequence $(\widehat{g}_n)$ in $\widehat{\mathfrak{N}}_a(S^*)$ such that $\|g_n\|_{\mathfrak{t}_{S_1}-a} = 1$ and $\Gamma_1'\widehat{g}_n \to 0$ in $\mathcal{G}$. From (5.6.29) and the definition of Γ_0 one obtains

$$1 = (g_n, g_n)_{\mathfrak{t}_{S_1}-a} = -(\Gamma_1'\widehat{g}_n, \Gamma_0\widehat{g}_n) = -(\Gamma_1'\widehat{g}_n, \Lambda g_n) \leq \|\Gamma_1'\widehat{g}_n\|\|\Lambda g_n\|,$$

and as $\Lambda : \mathfrak{H}_{\mathfrak{t}_{S_1}-a} \to \mathcal{G}$ is bounded this yields

$$1 \leq C'\|\Gamma_1'\widehat{g}_n\|\|g_n\|_{\mathfrak{t}_{S_1}-a} = C'\|\Gamma_1'\widehat{g}_n\| \to 0;$$

a contradiction. Hence, the operator $(\Gamma_1')^{-1}$ is bounded. As $(\Gamma_1')^{-1}$ is closed, it follows that $\operatorname{ran}\Gamma_1' = \operatorname{dom}(\Gamma_1')^{-1}$ is closed, which together with (5.6.33) and $\overline{\operatorname{ran}}\,\Gamma_1 = \mathcal{G}$ shows (5.6.31).

Step 7. First it will be verified that the mappings Γ_0 and Γ_1 form a boundary triplet for S^*. Observe that (5.6.29) implies the Green identity

$$(f',g)-(f,g')=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g}),\quad \widehat{f},\widehat{g}\in S^*.$$

It remains to show that

$$\operatorname{ran}\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}=\mathcal{G}\times\mathcal{G}. \tag{5.6.34}$$

For this, let $\varphi,\varphi'\in\mathcal{G}$. From $\operatorname{ran}\Gamma_0=\mathcal{G}$ in (5.6.17) and $\operatorname{ran}\Gamma_1=\mathcal{G}$ in (5.6.31) it is clear that there exist $\widehat{h},\widehat{k}\in S^*$ such that $\Gamma_0\widehat{h}=\varphi$ and $\Gamma_1\widehat{k}=\varphi'$. It follows from the transversality $S^*=S_{\mathrm{F}}\,\widehat{+}\,S_1$ that

$$\widehat{h}=\widehat{h}_{\mathrm{F}}+\widehat{h}_1\quad\text{and}\quad \widehat{k}=\widehat{k}_{\mathrm{F}}+\widehat{k}_1,\quad \widehat{h}_{\mathrm{F}},\widehat{k}_{\mathrm{F}}\in S_{\mathrm{F}},\,\widehat{h}_1,\widehat{k}_1\in S_1.$$

Define $\widehat{f}:=\widehat{h}_1+\widehat{k}_{\mathrm{F}}\in S^*$. Making use of the facts that $\ker\Gamma_0=S_{\mathrm{F}}$ in (5.6.15) and $\ker\Gamma_1=S_1$ in (5.6.30), one obtains

$$\begin{aligned}\Gamma_0\widehat{f}&=\Gamma_0\widehat{h}_1=\Gamma_0\widehat{h}=\varphi,\\ \Gamma_1\widehat{f}&=\Gamma_1\widehat{k}_{\mathrm{F}}=\Gamma_1\widehat{k}=\varphi',\end{aligned}$$

which shows (5.6.34). Therefore, $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is a boundary triplet for S^*.

Since $\ker\Gamma_0=S_{\mathrm{F}}$ and $\ker\Gamma_1=S_1$, and since Λ is an extension of Γ_0, see (5.6.10), one concludes that the boundary triplet $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ and the boundary pair $\{\mathcal{G},\Lambda\}$ are compatible; see Definition 5.6.4.

It remains to check that Γ_1 constructed in (5.6.28)–(5.6.29) is uniquely determined. Note that the mapping Γ_0 and the kernel S_1 of Γ_1 are uniquely determined as the boundary triplet is required to be compatible with the boundary pair $\{\mathcal{G},\Lambda\}$. Under these circumstances the action of Γ_1 is uniquely determined by formula (5.5.27) in Theorem 5.5.14. □

The following result gives a connection via a boundary pair $\{\mathcal{G},\Lambda\}$ between closed semibounded forms $\mathfrak{t}_H$ corresponding to semibounded self-adjoint extensions H of S such that $S_1\le H\le S_{\mathrm{F}}$ and closed nonnegative forms ω in $\mathcal{G}$. A similar result also involving boundary triplets follows later.

Theorem 5.6.11. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let S_1 be a semibounded self-adjoint extension of S such that S_1 and S_{F} are transversal. Let $\{\mathcal{G},\Lambda\}$ be a boundary pair for S corresponding to S_1. Then the following statements hold:*

(i) *If H is a semibounded self-adjoint extension of S such that $S_1\le H$, then there exists a closed nonnegative form ω in $\mathcal{G}$ defined on $\operatorname{dom}\omega=\Lambda(\operatorname{dom}\mathfrak{t}_H)$ such that*

$$\mathfrak{t}_H[f,g]=\mathfrak{t}_{S_1}[f,g]+\omega[\Lambda f,\Lambda g],\qquad f,g\in\operatorname{dom}\mathfrak{t}_H. \tag{5.6.35}$$

Moreover, the space $\Lambda(\operatorname{dom}H)$ is a core of the form ω.

(ii) *If ω is a closed nonnegative form in $\mathcal{G}$, then*

$$\begin{aligned} \mathfrak{t}[f,g] &= \mathfrak{t}_{S_1}[f,g] + \omega[\Lambda f, \Lambda g], \\ \operatorname{dom} \mathfrak{t} &= \bigl\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom} \omega \bigr\}, \end{aligned} \tag{5.6.36}$$

is a closed semibounded form in $\mathfrak{H}$ and the corresponding self-adjoint relation H is a semibounded self-adjoint extension of S which satisfies $S_1 \le H$.

The formulas (5.6.35) and (5.6.36) establish a one-to-one correspondence between all closed nonnegative forms ω in $\mathcal{G}$ and all semibounded self-adjoint extensions H of S satisfying the inequalities $S_1 \le H \le S_{\mathrm{F}}$.

Proof. (i) Let H be a semibounded self-adjoint extension of S and let $\mathfrak{t}_H$ be the corresponding closed semibounded form. By assumption, $S_1 \le H$ or, equivalently, $\mathfrak{t}_{S_1} \le \mathfrak{t}_H$; cf. Theorem 5.2.4. Hence, $\operatorname{dom} \mathfrak{t}_H \subset \operatorname{dom} \mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_1}[f] \le \mathfrak{t}_H[f]$ for all $f \in \operatorname{dom} \mathfrak{t}_H$. Recall that $\mathfrak{t}_{S_{\mathrm{F}}}$, as the closure of $\mathfrak{t}_S$, satisfies $\mathfrak{t}_{S_{\mathrm{F}}} \subset \mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_{\mathrm{F}}} \subset \mathfrak{t}_H$. Since $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$, one concludes that the form

$$\omega[\Lambda f, \Lambda g] := \mathfrak{t}_H[f,g] - \mathfrak{t}_{S_1}[f,g], \quad \operatorname{dom} \omega = \Lambda(\operatorname{dom} \mathfrak{t}_H), \quad f,g \in \operatorname{dom} \mathfrak{t}_H, \tag{5.6.37}$$

is well defined and nonnegative in the Hilbert space $\mathcal{G}$. To see that it is well defined, just note that for $f,g \in \operatorname{dom} \mathfrak{t}_H$ the Cauchy–Schwarz inequality shows

$$\bigl|\mathfrak{t}_H[f,g] - \mathfrak{t}_{S_1}[f,g]\bigr| \le \bigl|\mathfrak{t}_H[f,f] - \mathfrak{t}_{S_1}[f,f]\bigr|^{\frac{1}{2}} \bigl|\mathfrak{t}_H[g,g] - \mathfrak{t}_{S_1}[g,g]\bigr|^{\frac{1}{2}},$$

and hence $\mathfrak{t}_H[f,g] - \mathfrak{t}_{S_1}[f,g]$ in (5.6.37) vanishes when either f or g belongs to $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$.

Next it will be shown that the form ω is closed in $\mathcal{G}$. To this end consider a sequence (φ_n) in $\operatorname{dom} \omega$ and assume that $\varphi_n \to_\omega \varphi$ for some $\varphi \in \mathcal{G}$, that is, (φ_n) is a sequence in $\operatorname{dom} \omega = \Lambda(\operatorname{dom} \mathfrak{t}_H)$, such that

$$\varphi_n \to \varphi \in \mathcal{G} \qquad \text{and} \qquad \omega[\varphi_n - \varphi_m] \to 0. \tag{5.6.38}$$

Since $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ and

$$\operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}} \subset \operatorname{dom} \mathfrak{t}_H \subset \operatorname{dom} \mathfrak{t}_{S_1} = \bigl(\operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}\bigr) \oplus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$$

for $a < m(S_1)$, there exists a sequence (f_n) in $\operatorname{dom} \mathfrak{t}_H \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ such that $\Lambda f_n = \varphi_n$. Moreover, since $\operatorname{ran} \Lambda = \mathcal{G}$, there exists $f \in \operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ such that $\Lambda f = \varphi$; see Proposition 5.3.7. Since the restriction of Λ to the space $\operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ has a bounded inverse (see the discussion following Definition 5.6.1), it follows that $f_n \to f$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$. In particular, $f_n \to f$ in $\mathfrak{H}$ and $\mathfrak{t}_{S_1}[f_n - f_m] \to 0$. Then (5.6.37) and (5.6.38) imply

$$\begin{aligned} \mathfrak{t}_H[f_n - f_m] &= \mathfrak{t}_{S_1}[f_n - f_m] + \omega[\Lambda f_n - \Lambda f_m] \\ &= \mathfrak{t}_{S_1}[f_n - f_m] + \omega[\varphi_n - \varphi_m] \to 0, \end{aligned}$$

and as $\mathfrak{t}_H$ is closed one concludes $f \in \operatorname{dom} \mathfrak{t}_H$ and $\mathfrak{t}_H[f_n - f] \to 0$. This implies $\varphi = \Lambda f \in \operatorname{dom} \omega$. Furthermore, as $\mathfrak{t}_{S_1}$ is closed, also $\mathfrak{t}_{S_1}[f_n - f] \to 0$, and hence

$$\omega[\varphi_n - \varphi] = \omega[\Lambda f_n - \Lambda f] = \mathfrak{t}_H[f_n - f] - \mathfrak{t}_{S_1}[f_n - f] \to 0,$$

so that ω is a closed form in $\mathcal{G}$. It is clear that the definition of ω in (5.6.37) implies the representation of $\mathfrak{t}_H$ in (i).

It remains to show that $\Lambda(\operatorname{dom} H)$ is a core of ω. For this let $\varphi \in \operatorname{dom} \omega$ and choose $f \in \operatorname{dom} \mathfrak{t}_H$ such that $\varphi = \Lambda f$. As $\operatorname{dom} H$ is a core of $\mathfrak{t}_H$, there exists a sequence (f_n) in $\operatorname{dom} H$ such that $f_n \to f$ in $\mathfrak{H}$ and $\mathfrak{t}_H[f_n - f] \to 0$. Then $0 \leq (\mathfrak{t}_{S_1} - a)[f_n - f] \leq (\mathfrak{t}_H - a)[f_n - f] \to 0$ and, in particular, one has $f_n \to f$ in $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$. Setting $\varphi_n := \Lambda f_n$ one has $\varphi_n \subset \Lambda(\operatorname{dom} H)$ and using the fact that Λ is bounded one concludes that

$$\varphi_n = \Lambda f_n \to \Lambda f = \varphi$$

and

$$\omega[\varphi_n - \varphi] = \omega[\Lambda f_n - \Lambda f] = \mathfrak{t}_H[f_n - f] - \mathfrak{t}_{S_1}[f_n - f] \to 0.$$

This shows that $\Lambda(\operatorname{dom} H)$ is a core of ω.

(ii) Assume that ω is a closed nonnegative form in $\mathcal{G}$. Then it is clear that the form

$$\mathfrak{t}[f, g] = \mathfrak{t}_{S_1}[f, g] + \omega[\Lambda f, \Lambda g] \tag{5.6.39}$$

defined on $\operatorname{dom} \mathfrak{t} = \Lambda^{-1}(\operatorname{dom} \omega) \subset \operatorname{dom} \mathfrak{t}_{S_1}$ is semibounded and $\mathfrak{t}[f] \geq \mathfrak{t}_{S_1}[f]$ holds for all $f \in \operatorname{dom} \mathfrak{t}$. To verify that $\mathfrak{t}$ is closed consider a sequence (f_n) in $\operatorname{dom} \mathfrak{t}$ such that $f_n \to_{\mathfrak{t}} f$ for some $f \in \mathfrak{H}$, that is, $f_n \to f$ in $\mathfrak{H}$ and $\mathfrak{t}[f_n - f_m] \to 0$. Since the forms $\mathfrak{t}_{S_1} - a$, $a < m(S_1)$, and ω are nonnegative, it follows from (5.6.39) and $(\mathfrak{t} - a)[f_n - f_m] \to 0$ that $0 \leq (\mathfrak{t}_{S_1} - a)[f_n - f_m] \to 0$ and $\omega[\Lambda f_n - \Lambda f_m] \to 0$. As $\mathfrak{t}_{S_1}$ is a closed form in $\mathfrak{H}$, one concludes that $f \in \operatorname{dom} \mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_1}[f_n - f] \to 0$. This shows that f_n converges to f in $\mathfrak{H}_{\mathfrak{t}_{S_1} - a}$ and as Λ is bounded one has $\Lambda f_n \to \Lambda f$ in $\mathcal{G}$. Moreover, since $\omega[\Lambda f_n - \Lambda f_m] \to 0$ and ω is closed in $\mathcal{G}$, one concludes that $\Lambda f \in \operatorname{dom} \omega$ and $\omega[\Lambda f_n - \Lambda f] \to 0$. Hence, $f \in \operatorname{dom} \mathfrak{t} = \Lambda^{-1}(\operatorname{dom} \omega)$ and $\mathfrak{t}[f_n - f] \to 0$, and $\mathfrak{t}$ is a closed form in $\mathfrak{H}$.

Let H be the semibounded self-adjoint relation associated with $\mathfrak{t}$ via the first representation theorem; see Theorem 5.1.18. Since $\operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}} = \ker \Lambda$, it follows from (5.6.36) that $\mathfrak{t}_{S_{\mathrm{F}}} \subset \mathfrak{t}$. Hence, $\mathfrak{t}_{S_1} \leq \mathfrak{t} \leq \mathfrak{t}_{S_{\mathrm{F}}}$ or, equivalently, $S_1 \leq H \leq S_{\mathrm{F}}$; see Theorem 5.2.4. One concludes from Theorem 5.4.6 (or its proof) that H is a self-adjoint extension of S. This completes the proof of (ii).

The indicated one-to-one correspondence is clear from (i) and (ii) by the uniqueness of the representing semibounded self-adjoint relation associated with a closed semibounded form. □

A combination of Theorem 5.6.11 with Theorem 5.4.6 leads to the following observations. Recall that the Kreĭn type extensions $S_{\mathrm{K},x}$ and S_{F} are transversal when $x < \gamma = m(S) = m(S_F)$ (see (5.4.26)) and that in the nonnegative case $\gamma \geq 0$ the Kreĭn–von Neumann extension is given by $S_{K,0}$; cf. Definition 5.4.2.

Corollary 5.6.12. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ.*

(i) *For $x < \gamma$ and $S_1 = S_{K,x}$ the formulas (5.6.35) and (5.6.36) establish a one-to-one correspondence between all closed nonnegative forms ω in $\mathcal{G}$ and all self-adjoint extensions H of S satisfying $m(H) \geq x$.*

(ii) *If $S_{K,\gamma}$ and S_F are transversal, then with $S_1 = S_{K,\gamma}$ the formulas (5.6.35) and (5.6.36) establish a one-to-one correspondence between all closed nonnegative forms ω in $\mathcal{G}$ and all self-adjoint extensions H of S satisfying $m(H) = \gamma$.*

(iii) *If $\gamma > 0$, then with the Kreĭn–von Neumann extension $S_1 = S_{K,0}$ the formulas (5.6.35) and (5.6.36) establish a one-to-one correspondence between all closed nonnegative forms ω in $\mathcal{G}$ and all nonnegative self-adjoint extensions H of S. If $\gamma = 0$ the same is true if the Kreĭn–von Neumann extension $S_1 = S_{K,0}$ and S_F are transversal.*

Theorem 5.6.11 is a first step towards a full description of all semibounded self-adjoint extensions and their associated forms. The following result is a continuation of Theorem 5.5.14 for semibounded self-adjoint extensions (see Corollary 5.5.15) and an extension of the first part of Theorem 5.6.11, in which also the boundary conditions of the extensions and the corresponding forms are connected.

Theorem 5.6.13. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let S_1 be a semibounded self-adjoint extension of S such that S_1 and S_F are transversal. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* and let $\{\mathcal{G}, \Lambda\}$ be a compatible boundary pair for S corresponding to S_1. Assume that H_Θ is a semibounded self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathcal{G}$ as in (5.5.32)–(5.5.33). Then Θ is semibounded in $\mathcal{G}$ and the corresponding closed semibounded form ω_Θ in $\mathcal{G}$ and the closed semibounded form $\mathfrak{t}_{H_\Theta}$ corresponding to H_Θ are related by*

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[f,g] &= \mathfrak{t}_{S_1}[f,g] + \omega_\Theta[\Lambda f, \Lambda g], \\ \operatorname{dom} \mathfrak{t}_{H_\Theta} &= \big\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom} \omega_\Theta \big\}. \end{aligned} \tag{5.6.40}$$

Proof. The proof of the theorem will rely on the results in Theorem 5.5.14 and Corollary 5.5.15, where H_Θ is now taken to be semibounded. In the first two steps of the proof the equality between the forms in (5.6.40) will be verified. In the last step the domain characterization in (5.6.40) will be shown.

Step 1. First recall from Corollary 5.5.15 the formula (5.5.34):

$$(f', g) = \mathfrak{t}_{S_1}[f,g] + (\Theta_{\mathrm{op}} \Gamma_0 \widehat{f}, \Gamma_0 \widehat{g}), \quad \widehat{f}, \widehat{g} \in H_\Theta. \tag{5.6.41}$$

Since H_Θ is assumed to be semibounded, it follows from Theorem 5.1.18 that $(f', g) = \mathfrak{t}_{H_\Theta}[f,g]$. As the boundary pair $\{\mathcal{G}, \Lambda\}$ is compatible with the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, the mapping Λ is an extension of Γ_0. Hence, (5.6.41) may now be rewritten as

$$\mathfrak{t}_{H_\Theta}[f,g] = \mathfrak{t}_{S_1}[f,g] + (\Theta_{\rm op}\Lambda f, \Lambda g), \quad f,g \in \operatorname{dom} H_\Theta. \tag{5.6.42}$$

Step 2. In this step it is shown that the formula (5.6.42) can be extended to the form domain of $\mathfrak{t}_{H_\Theta}$ as in (5.6.40). First observe that by Lemma 5.6.5 one has $A_0 = S_{\rm F}$. Moreover, since H_Θ is a semibounded extension of S, it follows from Proposition 5.5.6 that Θ is semibounded from below. Hence, (5.6.42) can be written as

$$\mathfrak{t}_{H_\Theta}[f,g] = \mathfrak{t}_{S_1}[f,g] + \omega_\Theta[\Lambda f, \Lambda g], \quad f,g \in \operatorname{dom} H_\Theta, \tag{5.6.43}$$

where ω_Θ is the closed semibounded form corresponding to Θ in $\mathcal{G}$. It follows from Corollary 5.3.9 that

$$\operatorname{dom}(H_\Theta - a)^{\frac{1}{2}} \subset \operatorname{dom}(S_1 - a)^{\frac{1}{2}} \tag{5.6.44}$$

and hence there is a constant $C > 0$ such that

$$\|((S_1)_{\rm op} - a)^{\frac{1}{2}}\varphi\| \le C\|((H_\Theta)_{\rm op} - a)^{\frac{1}{2}}\varphi\| \tag{5.6.45}$$

for all $\varphi \in \operatorname{dom}(H_\Theta - a)^{\frac{1}{2}}$.

Now let $f \in \operatorname{dom}\mathfrak{t}_{H_\Theta}$. As $\operatorname{dom} H_\Theta$ is a core of $\mathfrak{t}_{H_\Theta}$, there exists a sequence (f_n) in $\operatorname{dom} H_\Theta$ such that $f_n \to f$ in $\mathfrak{H}$ and $\mathfrak{t}_{H_\Theta}[f_n - f] \to 0$. By (5.6.44)–(5.6.45) it follows that $f \in \operatorname{dom}\mathfrak{t}_{S_1}$ and $\mathfrak{t}_{S_1}[f_n - f] \to 0$, so that $f_n \to f$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$. Since Λ is bounded, this shows that $\Lambda f_n \to \Lambda f$ in $\mathcal{G}$. Furthermore, from (5.6.43) one sees that

$$\omega_\Theta[\Lambda f_n - \Lambda f_m] = \mathfrak{t}_{H_\Theta}[f_n - f_m] - \mathfrak{t}_{S_1}[f_n - f_m] \to 0.$$

Since ω_Θ is closed, one obtains

$$\Lambda f \in \operatorname{dom}\omega_\Theta \quad \text{and} \quad \omega_\Theta[\Lambda f_n - \Lambda f] \to 0.$$

Therefore, the following inclusion has been shown

$$\operatorname{dom}\mathfrak{t}_{H_\Theta} \subset \bigl\{f \in \operatorname{dom}\mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom}\omega_\Theta\bigr\}. \tag{5.6.46}$$

Let $f,g \in \operatorname{dom}\mathfrak{t}_{H_\Theta}$ and choose $(f_n),(g_n)$ in $\operatorname{dom} H_\Theta$ as above. Then one has $\mathfrak{t}_{H_\Theta}[f_n,g_n] \to \mathfrak{t}_{H_\Theta}[f,g]$, $\mathfrak{t}_{S_1}[f_n,g_n] \to \mathfrak{t}_{S_1}[f,g]$, and $\omega_\Theta[\Lambda f_n, \Lambda g_n] \to \omega_\Theta[\Lambda f, \Lambda g]$ as $n \to \infty$ by Lemma 5.1.8, and hence (5.6.43) extends to

$$\mathfrak{t}_{H_\Theta}[f,g] = \mathfrak{t}_{S_1}[f,g] + \omega_\Theta[\Lambda f, \Lambda g], \quad f,g \in \operatorname{dom}\mathfrak{t}_{H_\Theta}. \tag{5.6.47}$$

Step 3. To complete the proof of the theorem the equality between the domains in (5.6.40) must be verified. Due to (5.6.46) it suffices to show that

$$\bigl\{f \in \operatorname{dom}\mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom}\omega_\Theta\bigr\} \subset \operatorname{dom}\mathfrak{t}_{H_\Theta}.$$

Let $f \in \operatorname{dom} \mathfrak{t}_{S_1}$ and assume that $\varphi = \Lambda f \in \operatorname{dom} \omega_\Theta$. Using the orthogonal decomposition

$$\operatorname{dom} \mathfrak{t}_{S_1} = \bigl(\operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}\bigr) \oplus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}, \quad a < m(S_1), \tag{5.6.48}$$

write f in the form $f = h + k$, where $h \in \operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}$ and $k \in \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}$. Then $k \in \operatorname{dom} \mathfrak{t}_{H_\Theta}$, and since $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}$, one has $\varphi = \Lambda h$. It remains to show that $h \in \operatorname{dom} \mathfrak{t}_{H_\Theta}$.

Recall that $\operatorname{dom} \Theta$ is a core of ω_Θ. Hence, there exists a sequence (φ_n) in $\operatorname{dom} \Theta$ such that $\varphi_n \to_{\omega_\Theta} \varphi$, that is,

$$\varphi_n \to \varphi \in \mathcal{G} \qquad \text{and} \qquad \omega_\Theta[\varphi_n - \varphi_m] \to 0.$$

Note that $\varphi_n \in \operatorname{dom} \Theta$ means $\{\varphi_n, \varphi_n'\} \in \Theta$ for some $\varphi_n' \in \mathcal{G}$ and there exists $\{f_n, f_n'\} \in H_\Theta$ such that $\Gamma\{f_n, f_n'\} = \{\varphi_n, \varphi_n'\}$. Hence, $\Lambda f_n = \Gamma_0\{f_n, f_n'\} = \varphi_n$, where $f_n \in \operatorname{dom} H_\Theta \subset \operatorname{dom} \mathfrak{t}_{H_\Theta} \subset \operatorname{dom} \mathfrak{t}_{S_1}$. Using (5.6.48), one can write f_n in the form

$$f_n = h_n + k_n, \quad h_n \in \operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}, \ k_n \in \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}.$$

From $\ker \Lambda = \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}$ it is clear that $\varphi_n = \Lambda f_n = \Lambda h_n$. Since the restriction of Λ to $\operatorname{dom} \mathfrak{t}_{S_1} \ominus_{\mathfrak{t}_{S_1}-a} \operatorname{dom} \mathfrak{t}_{S_\mathrm{F}}$ has a bounded inverse it follows from $\varphi_n \to \varphi$ in $\mathcal{G}$ that $h_n \to h$ in $\mathfrak{H}_{\mathfrak{t}_{S_1}-a}$. In particular, $h_n \to h$ in $\mathfrak{H}$ and $\mathfrak{t}_{S_1}[h_n - h_m] \to 0$. Then it follows from (5.6.47) that

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[h_n - h_m] &= \mathfrak{t}_{S_1}[h_n - h_m] + \omega_\Theta[\Lambda h_n - \Lambda h_m] \\ &= \mathfrak{t}_{S_1}[h_n - h_m] + \omega_\Theta[\varphi_n - \varphi_m] \to 0, \end{aligned}$$

and as $\mathfrak{t}_{H_\Theta}$ is closed, one concludes that $h \in \operatorname{dom} \mathfrak{t}_{H_\Theta}$. □

One may apply the second representation theorem (Theorem 5.1.23) to the closed form ω_Θ in Theorem 5.6.13. Assume that $\mu \le m(\Theta)$, then it follows that

$$\begin{aligned} \omega_\Theta[\Lambda f, \Lambda g] &= \bigl((\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda f, (\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda g\bigr) + \mu\,(\Lambda f, \Lambda g)\,, \\ \operatorname{dom} \omega_\Theta &= \operatorname{dom}\,(\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}}. \end{aligned}$$

Hence, one obtains the following result; cf. Corollary 5.5.15.

Corollary 5.6.14. *Let the assumptions be as in Theorem 5.6.13 and let $\mu \le m(\Theta)$. Then the closed semibounded form $\mathfrak{t}_{H_\Theta}$ corresponding to H_Θ is given by*

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[f, g] &= \mathfrak{t}_{S_1}[f, g] + \bigl((\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda f, (\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda g\bigr) + \mu\,(\Lambda f, \Lambda g)\,, \\ \operatorname{dom} \mathfrak{t}_{H_\Theta} &= \bigl\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom}\,(\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \bigr\}. \end{aligned}$$

Furthermore, if $\Theta_{\mathrm{op}} \in \mathbf{B}(\mathcal{G}_{\mathrm{op}})$, then

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[f, g] &= \mathfrak{t}_{S_1}[f, g] + \bigl(\Theta_{\mathrm{op}} \Lambda f, \Lambda g\bigr), \\ \operatorname{dom} \mathfrak{t}_{H_\Theta} &= \bigl\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom} \Theta_{\mathrm{op}} \bigr\}, \end{aligned} \tag{5.6.49}$$

and in the special case $\Theta \in \mathbf{B}(\mathcal{G})$

$$\mathfrak{t}_{H_\Theta}[f, g] = \mathfrak{t}_{S_1}[f, g] + \bigl(\Theta \Lambda f, \Lambda g\bigr), \qquad \operatorname{dom} \mathfrak{t}_{H_\Theta} = \operatorname{dom} \mathfrak{t}_{S_1}.$$

Example 5.6.15. Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound γ, fix $x < \gamma$, and consider the boundary triplet $\{\mathfrak{N}_x(S^*), \Gamma_0, \Gamma_1\}$ for S^* in Corollary 5.5.12 and the corresponding compatible boundary pair $\{\mathfrak{N}_x(S^*), \Lambda\}$ in Example 5.6.9. Assume that H_Θ is a semibounded self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathfrak{N}_x(S^*)$ as in (5.5.32)–(5.5.33). Then Θ is semibounded in $\mathfrak{N}_x(S^*)$ and the corresponding closed semibounded form ω_Θ in $\mathfrak{N}_x(S^*)$ and the closed semibounded form $\mathfrak{t}_{H_\Theta}$ corresponding to H_Θ are related by

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[f,g] &= \mathfrak{t}_{S_{\mathrm{K},x}}[f,g] + \omega_\Theta[f_x, g_x], \\ \operatorname{dom} \mathfrak{t}_{H_\Theta} &= \bigl\{ f = f_{\mathrm{F}} + f_x \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}} + \mathfrak{N}_x(S^*) : f_x \in \operatorname{dom} \omega_\Theta \bigr\}. \end{aligned}$$

Let H_Θ be a semibounded self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathcal{G}$ as in (5.5.32)–(5.5.33). The first boundary condition in (5.5.33) is the *essential boundary condition* given by $\Gamma_0 \widehat{f} \in \operatorname{dom} \Theta_{\mathrm{op}}$. Since H_Θ is now assumed to be semibounded, it follows from $f \in \operatorname{dom} S^* \subset \operatorname{dom} \Lambda$ that this condition can be written as

$$\Lambda f = \Gamma_0 \widehat{f} \in \operatorname{dom} \Theta_{\mathrm{op}} \subset \operatorname{dom} (\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}}, \quad \mu \leq m(\Theta),$$

which implies that $f \in \operatorname{dom} \mathfrak{t}_{H_\Theta}$. The second boundary condition in (5.5.33) is the *natural boundary condition* given by $P_{\mathrm{op}} \Gamma_1 \widehat{f} = \Theta_{\mathrm{op}} \Gamma_0 \widehat{f}$. It is subsumed in the additive term in the structure of the form $\mathfrak{t}_{H_\Theta}$:

$$\bigl((\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda f, (\Theta_{\mathrm{op}} - \mu)^{\frac{1}{2}} \Lambda g \bigr) + \mu \, (\Lambda f, \Lambda g)\,;$$

cf. Corollary 5.5.15, which in case of a bounded operator part Θ_{op} simplifies to

$$\bigl(\Theta_{\mathrm{op}} \Lambda f, \Lambda g \bigr);$$

cf. (5.6.49). In particular, the elements in $\operatorname{dom} H_\Theta$ satisfy an essential boundary condition if and only if $\operatorname{dom} \Theta \neq \mathcal{G}$, that is, $\Theta \notin \mathbf{B}(\mathcal{G})$. Note that the extreme case $\operatorname{dom} \Theta = \{0\}$ corresponds to $\Lambda f = 0$ and $\Gamma_0 \widehat{f} = 0$, i.e., $f \in \operatorname{dom} \mathfrak{t}_{S_{\mathrm{F}}}$ and $\widehat{f} \in S_{\mathrm{F}}$.

Remark 5.6.16. In Theorem 5.6.11 a one-to-one correspondence between the closed nonnegative forms ω in $\mathcal{G}$ and the semibounded self-adjoint extensions H of S in $\mathfrak{H}$ satisfying $S_1 \leq H \leq S_{\mathrm{F}}$ is established. For closed semibounded forms ω in $\mathcal{G}$ the situation is different: Although Theorem 5.6.13 shows that for each semibounded self-adjoint extension $H = H_\Theta$ of S there exists a closed semibounded form $\omega = \omega_\Theta$ in $\mathcal{G}$ such that

$$\mathfrak{t}_H[f,g] = \mathfrak{t}_{S_1}[f,g] + \omega[\Lambda f, \Lambda g], \tag{5.6.50}$$

one can also see that for an arbitrary closed semibounded form ω in $\mathcal{G}$ the right-hand side in (5.6.50) is not necessarily bounded from below. However, if, e.g., ω is a symmetric form with $\operatorname{dom} \omega = \mathcal{G}$ such that for some $a \geq 0$ and $b \in [0,1)$

$$|\omega[\Lambda f]| \leq a \|f\|^2 + b \mathfrak{t}_{S_1}[f], \qquad f \in \operatorname{dom} \mathfrak{t}_{S_1},$$

then Theorem 5.1.16 shows that $\mathfrak{t}_H$ in (5.6.50) is a closed semibounded form with $\operatorname{dom}\mathfrak{t}_H = \operatorname{dom}\mathfrak{t}_{S_1}$ in $\mathfrak{H}$. In particular, in this situation the corresponding self-adjoint extension H of S is semibounded.

Recall from Proposition 5.5.8 that in the case of finite defect numbers or in the case that S_F has a compact resolvent the implication

$$\Theta \text{ semibounded in } \mathcal{G} \quad \Rightarrow \quad H_\Theta \text{ semibounded in } \mathfrak{H}$$

holds. The following corollary supplements Theorem 5.6.13 and can be seen as an extension and completion of the second part of Theorem 5.6.11. When the defect numbers are not finite or the resolvent of S_F is not compact there is in general no analog of the second part of Theorem 5.6.11.

Corollary 5.6.17. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let S_1 be a semibounded self-adjoint extension of S such that S_1 and S_F are transversal. Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^*, let $\{\mathcal{G},\Lambda\}$ be a compatible boundary pair for S corresponding to S_1 and assume, in addition, that one of the following conditions hold:*

- (i) *the defect numbers of S are finite;*
- (ii) *$(S_\mathrm{F}-\lambda)^{-1}$ is a compact operator for some $\lambda \in \rho(S_\mathrm{F})$.*

Let Θ be a semibounded self-adjoint relation in $\mathcal{G}$ and let H_Θ be the corresponding self-adjoint extension of S as in (5.5.32)–(5.5.33). Then H_Θ is semibounded and the closed semibounded forms $\mathfrak{t}_{H_\Theta}$ and ω_Θ are related by (5.6.40).

The following corollary complements Corollary 5.6.12 (iii). If the symmetric relation S is positive a natural choice for S_1 is the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$. A possible explicit choice for the boundary triplet can be found in Example 5.5.13.

Corollary 5.6.18. *Let S be a closed semibounded relation in $\mathfrak{H}$ with lower bound $\gamma > 0$, let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^*, and let $\{\mathcal{G},\Lambda\}$ be a compatible boundary pair for S corresponding to the Kreĭn–von Neumann extension $S_{\mathrm{K},0}$. Then the formula*

$$\begin{aligned} \mathfrak{t}_{H_\Theta}[f,g] &= \mathfrak{t}_{S_{\mathrm{K},0}}[f,g] + \bigl(\Theta_{\mathrm{op}}^{\frac12}\Lambda f, \Theta_{\mathrm{op}}^{\frac12}\Lambda g\bigr), \\ \operatorname{dom}\mathfrak{t}_{H_\Theta} &= \bigl\{ f \in \operatorname{dom}\mathfrak{t}_{S_{\mathrm{K},0}} : \Lambda f \in \operatorname{dom}\Theta_{\mathrm{op}}^{\frac12} \bigr\}, \end{aligned} \tag{5.6.51}$$

establishes a one-to-one correspondence between all closed nonnegative forms $\mathfrak{t}_{H_\Theta}$ corresponding to nonnegative self-adjoint extensions H_Θ of S in $\mathfrak{H}$ and all closed nonnegative forms ω_Θ corresponding to nonnegative self-adjoint relations Θ in $\mathcal{G}$.

Proof. By assumption, one has $S_{\mathrm{K},0} = \ker\Gamma_1$ and hence the Weyl function M corresponding to $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ satisfies $M(0)=0$ by Corollary 5.5.2 (viii). Assume that H_Θ is a nonnegative self-adjoint extension of S with corresponding closed nonnegative form $\mathfrak{t}_{H_\Theta}$. Since $\gamma > 0$, Proposition 5.5.6 with $x = 0$ shows that the self-adjoint relation Θ in $\mathcal{G}$ is nonnegative. Formula (5.6.51) follows from Theorem 5.6.13 and Corollary 5.6.14 with $\mu = 0$. Conversely, if Θ is a nonnegative self-adjoint relation in $\mathcal{G}$, then Theorem 5.6.11 (ii) shows that H_Θ is a nonnegative self-adjoint extension of S and (5.6.51) holds. □

In the next corollary the ordering of semibounded self-adjoint extensions is translated in the ordering of the corresponding parameters.

Corollary 5.6.19. *Let S be a closed semibounded relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Assume that*

$$\ker \Gamma_0 = S_{\mathrm{F}} \quad \textit{and} \quad \ker \Gamma_1 = S_1,$$

where S_{F} is the Friedrichs extension and S_1 is a semibounded self-adjoint extension of S. Let H_{Θ_1} and H_{Θ_2} be semibounded self-adjoint extensions of S corresponding to the semibounded self-adjoint relations Θ_1 and Θ_2. Then

$$H_{\Theta_1} \leq H_{\Theta_2} \quad \Leftrightarrow \quad \Theta_1 \leq \Theta_2. \tag{5.6.52}$$

In particular, $S_1 \leq H_{\Theta_2} \;\Leftrightarrow\; 0 \leq \Theta_2$.

Proof. Let $\{\mathcal{G}, \Lambda\}$ be a compatible boundary pair for S corresponding to S_1 as in Theorem 5.6.6. Then according to Theorem 5.6.13 one has the following identities

$$\begin{aligned} \mathfrak{t}_{H_{\Theta_1}}[f,g] &= \mathfrak{t}_{S_1}[f,g] + \omega_{\Theta_1}[\Lambda f, \Lambda g], \\ \operatorname{dom} \mathfrak{t}_{H_{\Theta_1}} &= \bigl\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom} \omega_{\Theta_1} \bigr\}, \end{aligned} \tag{5.6.53}$$

and

$$\begin{aligned} \mathfrak{t}_{H_{\Theta_2}}[f,g] &= \mathfrak{t}_{S_1}[f,g] + \omega_{\Theta_2}[\Lambda f, \Lambda g], \\ \operatorname{dom} \mathfrak{t}_{H_{\Theta_2}} &= \bigl\{ f \in \operatorname{dom} \mathfrak{t}_{S_1} : \Lambda f \in \operatorname{dom} \omega_{\Theta_2} \bigr\}. \end{aligned} \tag{5.6.54}$$

Recall from Theorem 5.2.4 that $H_{\Theta_1} \leq H_{\Theta_2}$ if and only if $\mathfrak{t}_{H_{\Theta_1}} \leq \mathfrak{t}_{H_{\Theta_2}}$. This last statement means by definition that

$$\operatorname{dom} \mathfrak{t}_{H_{\Theta_2}} \subset \operatorname{dom} \mathfrak{t}_{H_{\Theta_1}} \quad \text{and} \quad \mathfrak{t}_{H_{\Theta_1}}[f] \leq \mathfrak{t}_{H_{\Theta_2}}[f], \qquad f \in \operatorname{dom} \mathfrak{t}_{H_{\Theta_2}}, \tag{5.6.55}$$

which, via (5.6.53) and (5.6.54), is equivalent to

$$\operatorname{dom} \mathfrak{t}_{H_{\Theta_2}} \subset \operatorname{dom} \mathfrak{t}_{H_{\Theta_1}} \quad \text{and} \quad \omega_{\Theta_1}[\Lambda f] \leq \omega_{\Theta_2}[\Lambda f], \qquad f \in \operatorname{dom} \mathfrak{t}_{H_{\Theta_2}}. \tag{5.6.56}$$

Assume now that $H_{\Theta_1} \leq H_{\Theta_2}$, i.e., that (5.6.56) (and (5.6.55)) holds. First it will be shown that $\operatorname{dom} \mathfrak{t}_{H_{\Theta_2}} \subset \operatorname{dom} \mathfrak{t}_{H_{\Theta_1}}$ implies that

$$\operatorname{dom} \Theta_2 \subset \operatorname{dom} \omega_{\Theta_1}. \tag{5.6.57}$$

To see this, let $\varphi \in \operatorname{dom}\Theta_2$. Then $\{\varphi, \varphi'\} \in \Theta_2$ for some $\varphi' \in \mathcal{G}$. Now choose $\{f, f'\} \in H_{\Theta_2} \subset S^*$ with the property $\Gamma\{f, f'\} = \{\varphi, \varphi'\}$. Then it follows that $\Lambda f = \Gamma_0\{f, f'\} = \varphi$. Furthermore, since $f \in \operatorname{dom} S^* \subset \operatorname{dom}\mathfrak{t}_{S_1}$ and

$$\varphi = \Lambda f \in \operatorname{dom}\Theta_2 \subset \operatorname{dom}\omega_{\Theta_2},$$

it follows from $\operatorname{dom}\mathfrak{t}_{H_{\Theta_2}} \subset \operatorname{dom}\mathfrak{t}_{H_{\Theta_1}}$ that $\varphi = \Lambda f \in \operatorname{dom}\omega_{\Theta_1}$. Hence, (5.6.57) has been shown. Next observe that due to the previous reasoning the inequality in (5.6.56) gives

$$\omega_{\Theta_1}[\varphi] \le \omega_{\Theta_2}[\varphi], \quad \varphi \in \operatorname{dom}\Theta_2. \tag{5.6.58}$$

Denote the restriction of the form ω_{Θ_2} to $\operatorname{dom}\Theta_2$ by $\mathring{\omega}_{\Theta_2}$. Then the inclusion (5.6.57) and the inequality (5.6.58) can be written as

$$\omega_{\Theta_1} \le \mathring{\omega}_{\Theta_2}, \tag{5.6.59}$$

and, since $\operatorname{dom}\Theta_2$ is a core of ω_{Θ_2}, it follows from (5.6.59) and Lemma 5.2.2 (v) that

$$\omega_{\Theta_1} \le \omega_{\Theta_2} \quad \text{or, equivalently,} \quad \Theta_1 \le \Theta_2.$$

Hence, $H_{\Theta_1} \le H_{\Theta_2}$ implies that $\Theta_1 \le \Theta_2$.

For the converse statement assume $\Theta_1 \le \Theta_2$ or, equivalently, $\omega_{\Theta_1} \le \omega_{\Theta_2}$, i.e.,

$$\operatorname{dom}\omega_{\Theta_2} \subset \operatorname{dom}\omega_{\Theta_1} \quad \text{and} \quad \omega_{\Theta_1}[\varphi] \le \omega_{\Theta_2}[\varphi], \qquad \varphi \in \operatorname{dom}\omega_{\Theta_2}. \tag{5.6.60}$$

It will be shown that (5.6.56) holds. Let $f \in \operatorname{dom}\mathfrak{t}_{H_{\Theta_2}}$, so that $f \in \operatorname{dom}\mathfrak{t}_{S_1}$ and $\Lambda f \in \operatorname{dom}\omega_{\Theta_2}$. Then it follows from (5.6.60) that also $\Lambda f \in \operatorname{dom}\omega_{\Theta_1}$. Hence, one sees that $\operatorname{dom}\mathfrak{t}_{H_{\Theta_2}} \subset \operatorname{dom}\mathfrak{t}_{H_{\Theta_1}}$. Furthermore, if $f \in \operatorname{dom}\mathfrak{t}_{H_{\Theta_2}}$, then it follows directly from (5.6.60) that $\omega_{\Theta_1}[\Lambda f] \le \omega_{\Theta_2}[\Lambda f]$. Thus, (5.6.56) holds and one concludes that $H_1 \le H_2$.

Finally, note that for the choice $\Theta_1 = 0$ one has $H_{\Theta_1} = \ker\Gamma_1 = S_1$ and hence the equivalence (5.6.52) takes the form $S_1 \le H_{\Theta_2} \Leftrightarrow 0 \le \Theta_2$. □

If S is a semibounded relation in $\mathfrak{H}$ with lower bound γ and one chooses H_{Θ_1} to be the Kreĭn type extension $S_{\mathrm{K},x}$ for some $x \le \gamma$ in the previous corollary, then the next statement follows from (5.5.1).

Corollary 5.6.20. *Let the assumptions be as in Corollary 5.6.19 and let H_Θ be a semibounded self-adjoint extension of S corresponding to the self-adjoint relation Θ in $\mathcal{G}$ as in (5.5.32)–(5.5.33). Then for any $x \le m(S)$*

$$S_{\mathrm{K},x} \le H_\Theta \quad \Leftrightarrow \quad M(x) \le \Theta.$$