Chapter 3

Spectra, Simple Operators, and Weyl Functions

In this chapter the spectrum of a self-adjoint operator or relation will be completely characterized in terms of the analytic behavior and the limit properties of the Weyl function. In order to be able to treat the different parts of the spectrum, a short introduction to finite Borel measures on $\mathbb{R}$ and the corresponding Borel transforms will be given in Section 3.1 and Section 3.2. The notions and some properties of the absolutely continuous, singular continuous, pure point, and other spectral subsets of a self-adjoint relation are recalled in Section 3.3. Moreover, the concepts of simplicity (or complete non-self-adjointness) and local simplicity of symmetric operators and relations will be explained in detail in Section 3.4. For a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ with corresponding Weyl function M, the spectrum of the self-adjoint extension $A_0 = \ker \Gamma_0$ is then characterized. An analytic description for the point spectrum of A_0 in terms of M is given in Section 3.5, the rest of the spectrum and its different parts, namely absolutely continuous, singular, and continuous spectrum are studied in Section 3.6 under the additional condition that the underlying symmetric relation S is simple or locally simple. The limit properties of the Weyl function are also connected with defect elements belonging to the domain or range of A_0. This is discussed in Section 3.7. Finally, it is shown with the help of tranformation properties of boundary triplets and Weyl functions in Section 3.8 how the earlier results in this chapter extend to a description of the spectrum of an arbitrary self-adjoint extension A_Θ.

3.1 Analytic descriptions of minimal supports of Borel measures

A Borel measure on $\mathbb{R}$ can be decomposed with respect to the Lebesgue measure into an absolutely continuous measure and a singular measure. The minimal supports of the measure and its parts can be described by means of the derivative of

the measure. The present interest is in an analytic description of these minimal supports in terms of the Borel transform. For the convenience of the reader, a brief review on Borel measures on $\mathbb{R}$ and some properties of their Borel transforms are recalled.

In the following let μ be a regular Borel measure on $\mathbb{R}$ and denote the Lebesgue measure on $\mathbb{R}$ by m. Recall that any Borel measure on $\mathbb{R}$ which is finite on compact sets is automatically regular. Associated with the regular Borel measure μ is the nondecreasing, left-continuous function

$$
\nu_\mu(x) = \begin{cases} \mu([0,x)), & x > 0, \\ 0, & x = 0, \\ -\mu([x,0)), & x < 0, \end{cases} \tag{3.1.1}
$$

on $\mathbb{R}$. Observe that ν_μ is bounded if and only if μ is a finite measure, that the derivative ν'_μ of the nondecreasing function ν_μ exists m-almost everywhere, and that

$$
\mu([x,y)) = \nu_\mu(y) - \nu_\mu(x), \qquad x < y. \tag{3.1.2}
$$

It is important to note that via (3.1.2) the function ν_μ induces a Lebesgue-Stieltjes measure on $\mathbb{R}$, which is a complete measure that coincides with the completion of μ. In the following it is often more convenient to work with this completion, which will also be denoted by μ, and the corresponding μ-measurable subsets of $\mathbb{R}$.

The regular Borel measure μ has a *Lebesgue decomposition* with respect to the Lebesgue measure m:

$$
\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}},
$$

where the measure μ_{ac} is absolutely continuous and the measure μ_{s} is singular, each with respect to the Lebesgue measure. The singular measure μ_{s} is further decomposed into the singular continuous part μ_{sc} and the pure point part μ_{p}, so that

$$
\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{p}}.
$$

The corresponding nondecreasing, left-continuous functions $\nu_{\mu_{\mathrm{ac}}}$, $\nu_{\mu_{\mathrm{sc}}}$, and $\nu_{\mu_{\mathrm{p}}}$ defined via (3.1.1), are absolutely continuous, continuous with $\nu'_{\mu_{\mathrm{sc}}} = 0$ m-almost everywhere, and a step function, respectively, and

$$
\nu_\mu = \nu_{\mu_{\mathrm{ac}}} + \nu_{\mu_{\mathrm{sc}}} + \nu_{\mu_{\mathrm{p}}}.
$$

Furthermore,

$$
\mu_{\mathrm{ac}}(\mathfrak{B}) = \int_{\mathfrak{B}} \nu'_\mu(x)\, dm(x) \tag{3.1.3}
$$

for all Borel sets $\mathfrak{B}$, and hence the derivative ν'_μ coincides with the Radon–Nikodým derivative of μ_{ac} m-almost everywhere.

For $x \in \mathbb{R}$ the derivative $\mu'(x)$ of the Borel measure μ with respect to the Lebesgue measure m is defined by

$$\mu'(x) = \lim_{m(I_x) \downarrow 0} \left\{ \frac{\mu(I_x)}{m(I_x)} : I_x \text{ an interval containing } x \right\}, \tag{3.1.4}$$

whenever the limit exists and takes values in $[0, \infty]$. It can be shown that the sets

$$\mathfrak{E}_0 = \left\{ x \in \mathbb{R} : \mu'(x) \text{ exists finitely} \right\} \tag{3.1.5}$$

and

$$\mathfrak{E} = \left\{ x \in \mathbb{R} : \mu'(x) \text{ exists finitely or infinitely} \right\} \tag{3.1.6}$$

are Borel sets, and for the set $\mathbb{R} \setminus \mathfrak{E}_0$ on which the derivative μ' does not exist finitely one has that

$$m(\mathbb{R} \setminus \mathfrak{E}_0) = 0, \tag{3.1.7}$$

while for the set $\mathbb{R} \setminus \mathfrak{E}$ on which the derivative μ' does not exist finitely or infinitely one has that

$$m(\mathbb{R} \setminus \mathfrak{E}) = 0 \quad \text{and} \quad \mu(\mathbb{R} \setminus \mathfrak{E}) = 0; \tag{3.1.8}$$

note that $\mathbb{R} \setminus \mathfrak{E} \subset \mathbb{R} \setminus \mathfrak{E}_0$. Recall also that the derivative ν'_μ of the function ν_μ in (3.1.2) and the derivative μ' in (3.1.4) of the measure μ coincide m-almost everywhere.

A μ-measurable set $\mathfrak{S} \subset \mathbb{R}$ is called a *support* of μ if $\mu(\mathbb{R} \setminus \mathfrak{S}) = 0$. In particular, this implies that $\mu(\mathfrak{A}) = \mu(\mathfrak{A} \cap \mathfrak{S})$ for all μ-measurable sets $\mathfrak{A} \subset \mathbb{R}$. A support $\mathfrak{S} \subset \mathbb{R}$ of μ is called *minimal* if for subsets $\mathfrak{S}_0 \subset \mathfrak{S}$ that are μ-measurable and m-measurable, $\mu(\mathfrak{S}_0) = 0$ implies $m(\mathfrak{S}_0) = 0$. A minimal support is not uniquely defined. The next auxiliary lemma provides some useful properties of minimal supports.

Lemma 3.1.1. *Let μ be a Borel measure on $\mathbb{R}$ and let $\mathfrak{S}, \mathfrak{S}' \subset \mathbb{R}$ be sets that are measurable with respect to μ and m.*

(i) *If $\mathfrak{S}$ and $\mathfrak{S}'$ are minimal supports for μ, then the symmetric difference $\mathfrak{S} \Delta \mathfrak{S}'$ satisfies $\mu(\mathfrak{S} \Delta \mathfrak{S}') = 0$ and $m(\mathfrak{S} \Delta \mathfrak{S}') = 0$.*

(ii) *If $\mathfrak{S}$ is a minimal support for μ while $\mu(\mathfrak{S} \setminus \mathfrak{S}') = 0$ and $m(\mathfrak{S}' \setminus \mathfrak{S}) = 0$, then $\mathfrak{S}'$ is a minimal support of μ. In particular, if $\mathfrak{S}$ is a minimal support for μ and $\mathfrak{S} \subset \mathfrak{S}'$ is such that $m(\mathfrak{S}' \setminus \mathfrak{S}) = 0$, then $\mathfrak{S}'$ is a minimal support of μ.*

Proof. (i) Since $\mathfrak{S} \Delta \mathfrak{S}' \subset ((\mathbb{R} \setminus \mathfrak{S}) \cup (\mathbb{R} \setminus \mathfrak{S}'))$ and both $\mathfrak{S}$ and $\mathfrak{S}'$ are supports for μ, one has

$$\mu(\mathfrak{S} \Delta \mathfrak{S}') \leq \mu(\mathbb{R} \setminus \mathfrak{S}) + \mu(\mathbb{R} \setminus \mathfrak{S}') = 0.$$

In particular, $\mu(\mathfrak{S} \setminus \mathfrak{S}') = 0$. Now $\mathfrak{S} \setminus \mathfrak{S}' \subset \mathfrak{S}$ is μ-measurable and m-measurable, and since $\mathfrak{S}$ is a minimal support, it follows that $m(\mathfrak{S} \setminus \mathfrak{S}') = 0$. A similar argument shows that $m(\mathfrak{S}' \setminus \mathfrak{S}) = 0$. Hence, $m(\mathfrak{S} \Delta \mathfrak{S}') = 0$.

(ii) From $\mathbb{R} \setminus \mathfrak{S}' = ((\mathbb{R} \setminus \mathfrak{S}) \cup (\mathfrak{S} \setminus \mathfrak{S}')) \setminus (\mathfrak{S}' \setminus \mathfrak{S})$ one concludes that

$$\mu(\mathbb{R} \setminus \mathfrak{S}') \leq \mu(\mathbb{R} \setminus \mathfrak{S}) + \mu(\mathfrak{S} \setminus \mathfrak{S}').$$

Since $\mathfrak{S}$ is a support of μ and it is assumed that $\mu(\mathfrak{S} \setminus \mathfrak{S}') = 0$, it follows that $\mu(\mathbb{R} \setminus \mathfrak{S}') = 0$. Hence, $\mathfrak{S}'$ is a support of μ.

To prove that $\mathfrak{S}'$ is a minimal support for μ, let $\mathfrak{S}_0 \subset \mathfrak{S}'$ be μ-measurable and m-measurable, and assume that $m(\mathfrak{S}_0) > 0$. Since

$$\mathfrak{S}_0 = (\mathfrak{S}_0 \cap \mathfrak{S}) \cup \big(\mathfrak{S}_0 \cap (\mathfrak{S}' \setminus \mathfrak{S})\big) \tag{3.1.9}$$

and $m(\mathfrak{S}' \setminus \mathfrak{S}) = 0$ by assumption, it follows that $m(\mathfrak{S}_0 \cap \mathfrak{S}) = m(\mathfrak{S}_0) > 0$. As $\mathfrak{S}$ is a minimal support for μ, this implies $\mu(\mathfrak{S}_0 \cap \mathfrak{S}) > 0$. Therefore, (3.1.9) leads to

$$\mu(\mathfrak{S}_0) = \mu(\mathfrak{S}_0 \cap \mathfrak{S}) + \mu\big(\mathfrak{S}_0 \cap (\mathfrak{S}' \setminus \mathfrak{S})\big) \geq \mu(\mathfrak{S}_0 \cap \mathfrak{S}) > 0.$$

Thus, $\mathfrak{S}'$ is a minimal support for μ. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Minimal supports for the parts of the spectrum in the Lebesgue decomposition can be expressed in terms of the behavior of the derivative μ'; cf. [335, Lemma 4] (see also [676, 682]).

Theorem 3.1.2. *Let μ be a regular Borel measure on $\mathbb{R}$. Then the following sets*

  (i) $\{x \in \mathfrak{E} : 0 < \mu'(x) \leq \infty\}$;

  (ii) $\{x \in \mathfrak{E} : 0 < \mu'(x) < \infty\}$;

  (iii) $\{x \in \mathfrak{E} : \mu'(x) = \infty\}$;

  (iv) $\{x \in \mathfrak{E} : \mu'(x) = \infty,\ \mu(\{x\}) = 0\}$;

  (v) $\{x \in \mathfrak{E} : \mu'(x) = \infty,\ \mu(\{x\}) > 0\}$,

are minimal supports for μ, μ_{ac}, μ_{s}, μ_{sc}, and μ_{p}, respectively.

For practical reasons the attention is now restricted to finite Borel measures on $\mathbb{R}$. The properties of such measures are reflected by the boundary behavior of their so-called Borel transform in a sense to be made precise; cf. Appendix A.

Definition 3.1.3. *Let μ be a finite Borel measure on $\mathbb{R}$. Then the Borel transform F of μ is the function F defined by*

$$F(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \lambda}\, d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1.10}$$

If for some $x \in \mathbb{R}$ the limit $\lim_{y \downarrow 0} F(x + iy)$ exists and takes values in $[0, \infty]$, it will be denoted by $F(x + i0)$. The set of points in $\mathbb{R}$ where the limit of the imaginary part of F exists and takes values in $[0, \infty]$ is denoted by

$$\mathfrak{F} = \big\{x \in \mathbb{R} : \operatorname{Im} F(x + i0) \text{ exists finitely or infinitely}\big\}. \tag{3.1.11}$$

It follows from the integral representation (3.1.10) that

$$y \operatorname{Re} F(x + iy) = \int_{\mathbb{R}} \frac{(t - x)y}{(t - x)^2 + y^2} \, d\mu(t),$$

$$y \operatorname{Im} F(x + iy) = \int_{\mathbb{R}} \frac{y^2}{(t - x)^2 + y^2} \, d\mu(t),$$

and hence, by dominated convergence,

$$\lim_{y \downarrow 0} y \operatorname{Re} F(x + iy) = 0 \quad \text{and} \quad \lim_{y \downarrow 0} y \operatorname{Im} F(x + iy) = \mu(\{x\}) \qquad (3.1.12)$$

for all $x \in \mathbb{R}$; cf. Lemma A.2.6. In particular,

$$\lim_{y \downarrow 0} y \, F(x + iy) = \lim_{y \downarrow 0} iy \operatorname{Im} F(x + iy) \qquad (3.1.13)$$

for all $x \in \mathbb{R}$. Note also that the Borel transform F is a Nevanlinna function (see Definition A.2.3) and $\mu(\mathbb{R}) = \sup_{y>0} y \operatorname{Im} F(iy)$. Conversely, every Nevanlinna function F with

$$\sup_{y>0} y \operatorname{Im} F(iy) < \infty \quad \text{and} \quad \lim_{y \to \infty} F(iy) = 0$$

is the Borel transform of a finite Borel measure μ as in (3.1.10); cf. Proposition A.5.3.

An important observation concerning the boundary values $\operatorname{Im} F(x + i0)$ is contained in the following theorem, which is formulated in terms of the *symmetric derivative*

$$(D\mu)(x) = \lim_{\epsilon \downarrow 0} \frac{\mu((x - \epsilon, x + \epsilon))}{2\epsilon} \qquad (3.1.14)$$

of μ. Here the limit is assumed to take values in $[0, \infty]$. Note that if for some $x \in \mathbb{R}$ the derivative $\mu'(x)$ in (3.1.4) exists with values in $[0, \infty]$, then the same is true for the symmetric derivative $(D\mu)(x)$.

Theorem 3.1.4. *Let μ be a finite Borel measure on $\mathbb{R}$, let F be its Borel transform, and let $x \in \mathbb{R}$. If the symmetric derivative $(D\mu)(x)$ exists with values in $[0, \infty]$, then also $\operatorname{Im} F(x + i0)$ exists with values in $[0, \infty]$ and*

$$\operatorname{Im} F(x + i0) = \pi (D\mu)(x) \quad (\in [0, \infty]). \qquad (3.1.15)$$

In particular, the following statements hold:

(i) *$\operatorname{Im} F(x + i0)$ and $(D\mu)(x)$ exist simultaneously finitely m-almost everywhere and (3.1.15) holds;*

(ii) *$\operatorname{Im} F(x + i0)$ and $(D\mu)(x)$ exist simultaneously finitely or infinitely μ-almost everywhere and m-almost everywhere and (3.1.15) holds.*

Proof. Assume first that the symmetric derivative $(D\mu)(x)$ exists in $[0, \infty)$ for some $x \in \mathbb{R}$ and choose $c_-, c_+ \in \mathbb{R}$ with $c_- < (D\mu)(x) < c_+$. From the definition

(3.1.14) it follows that there exists $\delta > 0$ such that

$$2c_-\epsilon \leq \mu(I_\epsilon) \leq 2c_+\epsilon, \qquad I_\epsilon := (x - \epsilon, x + \epsilon), \tag{3.1.16}$$

holds for all $\epsilon \in (0, \delta]$. In the following set $K_y(s) := \frac{y}{s^2 + y^2}$ for $y > 0$ and $s \in \mathbb{R}$. Then one has

$$\begin{aligned}
\operatorname{Im} F(x + iy) &= \int_{\mathbb{R}} \frac{y}{(x - t)^2 + y^2} \, d\mu(t) \\
&= \int_{\mathbb{R}} K_y(x - t) \, d\mu(t) \\
&= \int_{I_\delta} K_y(x - t) \, d\mu(t) + \int_{\mathbb{R} \setminus I_\delta} K_y(x - t) \, d\mu(t)
\end{aligned} \tag{3.1.17}$$

for $y > 0$. First one estimates the second term on the right-hand side in (3.1.17). Since $t \in \mathbb{R} \setminus I_\delta$, one has $|t - x| \geq \delta$, so that $0 \leq K_y(t - x) \leq K_y(\delta)$. Then it is clear that

$$0 \leq \int_{\mathbb{R} \setminus I_\delta} K_y(x - t) \, d\mu(t) \leq K_y(\delta)\mu(\mathbb{R}) \to 0 \tag{3.1.18}$$

for $y \downarrow 0$. In order to estimate the first integral on the right-hand side in (3.1.17) one uses the identity

$$\int_{I_\delta} K_y(t - x) \, d\mu(t) = \mu(I_\delta)K_y(\delta) - \int_0^\delta K_y'(\epsilon)\mu(I_\epsilon) \, d\epsilon. \tag{3.1.19}$$

To prove (3.1.19), observe that

$$\begin{aligned}
\int_0^\delta K_y'(\epsilon)\mu(I_\epsilon) \, d\epsilon &= \int_0^\delta \int_{x-\epsilon}^{x+\epsilon} K_y'(\epsilon) \, d\mu(t) \, d\epsilon \\
&= \int_{x-\delta}^x \int_{x-t}^\delta K_y'(\epsilon) \, d\epsilon \, d\mu(t) + \int_x^{x+\delta} \int_{t-x}^\delta K_y'(\epsilon) \, d\epsilon \, d\mu(t) \\
&= \mu(I_\delta)K_y(\delta) - \int_{x-\delta}^{x+\delta} K_y(t - x) \, d\mu(t),
\end{aligned}$$

where Fubini's theorem on the triangle in the (t, ϵ)-plane given by $\epsilon = t - x$, $\epsilon = x - t$, with $0 \leq \epsilon \leq \delta$, was used. Now integration by parts, the fact that (3.1.16), $-K_y'(\epsilon) \geq 0$ for $\epsilon, y > 0$, and (3.1.19) give the estimate

$$\begin{aligned}
2c_- \arctan(\delta/y) &= 2c_- \int_0^\delta K_y(\epsilon) \, d\epsilon \\
&= 2c_-\delta K_y(\delta) + 2c_- \int_0^\delta (-\epsilon K_y'(\epsilon)) \, d\epsilon \\
&\leq \mu(I_\delta)K_y(\delta) - \int_0^\delta K_y'(\epsilon)\mu(I_\epsilon) \, d\epsilon \\
&= \int_{I_\delta} K_y(t - x) \, d\mu(t).
\end{aligned}$$

In the same way one verifies the estimate

$$\int_{I_\delta} K_y(t-x)\,d\mu(t) \leq 2c_+ \arctan(\delta/y).$$

It follows that

$$\pi c_- \leq \liminf_{y\downarrow 0} \int_{I_\delta} K_y(t-x)\,d\mu(t) \leq \limsup_{y\downarrow 0} \int_{I_\delta} K_y(t-x)\,d\mu(t) \leq \pi c_+.$$

Now (3.1.18) and (3.1.17) imply

$$\pi c_- \leq \liminf_{y\downarrow 0} \operatorname{Im} F(x+iy) \leq \limsup_{y\downarrow 0} \operatorname{Im} F(x+iy) \leq \pi c_+.$$

Letting $c_- \uparrow (D\mu)(x)$ and $c_+ \downarrow (D\mu)(x)$, one obtains

$$\lim_{y\downarrow 0} \operatorname{Im} F(x+iy) = \pi (D\mu)(x).$$

Next the case where the symmetric derivative $(D\mu)(x)$ exists and equals ∞ for some $x \in \mathbb{R}$ is discussed. In this situation the above reasoning leads to

$$\pi c_- \leq \liminf_{y\downarrow 0} \operatorname{Im} F(x+iy)$$

for all $c_- > 0$. This yields $\lim_{y\downarrow 0} \operatorname{Im} F(x+iy) = \infty$.

It remains to show assertions (i) and (ii). Recall that if $\mu'(x)$ exists at some point $x \in \mathbb{R}$, then so does the symmetric derivative $(D\mu)(x)$ and

$$\mu'(x) = (D\mu)(x),$$

with equality in $[0,\infty]$. For (ii) the above reasoning implies that the set $\mathfrak{E}$ in (3.1.6) is contained in the set $\mathfrak{F}$ in (3.1.11) and hence $\mu(\mathbb{R} \setminus \mathfrak{F}) = 0$ and $m(\mathbb{R} \setminus \mathfrak{F}) = 0$ by (3.1.8). Assertion (i) follows in the same way from (3.1.5) and (3.1.7). □

It follows from Theorem 3.1.4 and (3.1.12) that Theorem 3.1.2 has a counterpart expressing minimal supports in terms of the Borel transform of μ.

Theorem 3.1.5. *Let μ be a finite Borel measure and let F be its Borel transform. Then the sets*

(i) $\{x \in \mathfrak{F} : 0 < \operatorname{Im} F(x+i0) \leq \infty\}$;

(ii) $\{x \in \mathfrak{F} : 0 < \operatorname{Im} F(x+i0) < \infty\}$;

(iii) $\{x \in \mathfrak{F} : \operatorname{Im} F(x+i0) = \infty\}$;

(iv) $\{x \in \mathfrak{F} : \operatorname{Im} F(x+i0) = \infty,\ \lim_{y\downarrow 0} y \operatorname{Im} F(x+iy) = 0\}$;

(v) $\{x \in \mathfrak{F} : \operatorname{Im} F(x+i0) = \infty,\ \lim_{y\downarrow 0} y \operatorname{Im} F(x+iy) > 0\}$,

are minimal supports for μ, μ_{ac}, μ_{s}, μ_{sc}, and μ_{p}, respectively.

Proof. Only statement (i) will be proved. The proofs of the other statements are similar. Let

$$\mathfrak{M} = \{x \in \mathfrak{E} : 0 < \mu'(x) \leq \infty\},$$

and note that $\mathfrak{M}$ is a Borel set. Recall that, by Theorem 3.1.2 (i), $\mathfrak{M}$ is a minimal support for μ. Now introduce the set

$$\mathfrak{M}' = \{x \in \mathfrak{F} : 0 < \operatorname{Im} F(x + i0) \leq \infty\},$$

which is also a Borel set, as $\operatorname{Im} F(x + iy)$, $y > 0$, and hence $\operatorname{Im} F(x + i0)$ are Borel measurable functions in x. Then Theorem 3.1.4 shows that $\mathfrak{M} \subset \mathfrak{M}'$ and furthermore one has

$$\mathfrak{M}' \setminus \mathfrak{M} \subset \mathbb{R} \setminus \mathfrak{E}.$$

Since $m(\mathbb{R} \setminus \mathfrak{E}) = 0$ according to (3.1.8), it follows that $m(\mathfrak{M}' \setminus \mathfrak{M}) = 0$ and as $\mathfrak{M} \subset \mathfrak{M}'$, and $\mathfrak{M}$ is a minimal support for μ, one concludes from Lemma 3.1.1 (ii) that $\mathfrak{M}'$ is a minimal support for μ. $\qquad\square$

Most of the results in this section have been stated in the context of finite Borel measures on $\mathbb{R}$ and their Borel transforms. They will be applied to study the spectrum of self-adjoint relations and operators in Section 3.6. However, it is also useful for later references to have similar results in the more general context of scalar Nevanlinna functions and the corresponding spectral functions; cf. Chapter 6 and Chapter 7. Let N be a scalar Nevanlinna function of the form

$$N(\lambda) = \alpha + \beta\lambda + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\tau(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.1.20}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and τ is a Borel measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \frac{1}{t^2 + 1} d\tau(t) < \infty; \tag{3.1.21}$$

cf. Theorem A.2.5. Then the last condition implies that μ defined by

$$d\mu(t) = \frac{d\tau(t)}{t^2 + 1} \tag{3.1.22}$$

is a finite Borel measure on $\mathbb{R}$. Let F be the Borel transform of μ:

$$F(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \lambda} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1.23}$$

The connection between N and F is given in the following lemma.

Lemma 3.1.6. *The Nevanlinna function N in* (3.1.20) *and the Borel transform F in* (3.1.23) *are connected by*

$$N(\lambda) = a + b\lambda + (\lambda^2 + 1)F(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.1.24}$$

where $a, b \in \mathbb{R}$. If $x \in \mathbb{R}$, then the limits $\operatorname{Im} N(x + i0)$ and $\operatorname{Im} F(x + i0)$ exist simultaneously with values in $[0, \infty]$, and in that case

$$\operatorname{Im} N(x + i0) = (x^2 + 1)\operatorname{Im} F(x + i0) \quad (\in [0, \infty]). \tag{3.1.25}$$

Moreover, for each $x \in \mathbb{R}$,

$$\lim_{y \downarrow 0} y \operatorname{Re} N(x + iy) = 0 \tag{3.1.26}$$

and

$$\lim_{y \downarrow 0} y \operatorname{Im} N(x + iy) = (x^2 + 1) \lim_{y \downarrow 0} y \operatorname{Im} F(x + iy). \tag{3.1.27}$$

Proof. It is an immediate consequence of the integral representation (3.1.20) that N can be rewritten as

$$N(\lambda) = \alpha + \lambda \left(\beta + \int_{\mathbb{R}} d\mu(t) \right) + (\lambda^2 + 1) \int_{\mathbb{R}} \frac{1}{t - \lambda} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

cf. Theorem A.2.4. This leads to (3.1.24). Note that for $\lambda = x + iy$ one has

$$N(x + iy) = a + b(x + iy) + ((x + iy)^2 + 1)F(x + iy),$$

whence

$$\operatorname{Im} N(x + iy) = by + (x^2 + 1 - y^2)\operatorname{Im} F(x + iy) + 2xy \operatorname{Re} F(x + iy).$$

Now observe that for each $x \in \mathbb{R}$ one has $\lim_{y \downarrow 0} y \operatorname{Re} F(x + iy) = 0$ by (3.1.12). Together with the previous identity this proves the assertion in (3.1.25). Furthermore, now one sees (3.1.27) directly; cf. (3.1.12). Finally, note that

$$\operatorname{Re} N(x + iy) = a + bx + (x^2 + 1 - y^2)\operatorname{Re} F(x + iy) - 2xy \operatorname{Im} F(x + iy),$$

which together with (3.1.12) leads to the identity (3.1.26).   □

The next corollary deals with the existence of the limit $\lim_{\epsilon \downarrow 0} N(x + i\epsilon)$ for any scalar Nevanlinna function N.

Corollary 3.1.7. *Let N be a scalar Nevanlinna function. Then the limit $N(x + i0)$ exists finitely m-almost everywhere.*

Proof. It is clear from (3.1.25) and Theorem 3.1.4 that $\lim_{\epsilon \downarrow 0} \operatorname{Im} N(x + i\epsilon)$ exists finitely m-almost everywhere. Hence, it suffices to show that

$$\lim_{\epsilon \downarrow 0} \operatorname{Re} N(x + i\epsilon) \tag{3.1.28}$$

exists finitely m-almost everywhere. Denote by $\sqrt{\cdot}$ the branch of the square root fixed by $\operatorname{Im} \sqrt{\lambda} > 0$ for $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, \infty)$. Then it is easy to

see that $\operatorname{Im}\sqrt{N(\lambda)} \geq 0$ and $\operatorname{Im}\left(i\sqrt{N(\lambda)}\right) \geq 0$ for $\lambda \in \mathbb{C}^+$ and hence $\lambda \mapsto \sqrt{N(\lambda)}$ and $\lambda \mapsto i\sqrt{N(\lambda)}$ are scalar Nevanlinna functions when they are extended to $\mathbb{C}^-$ by symmetry. It follows from (3.1.25) and Theorem 3.1.4 that the limits

$$\lim_{\epsilon \downarrow 0} \operatorname{Im}\sqrt{N(x+i\epsilon)} \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \operatorname{Re}\sqrt{N(x+i\epsilon)} = \lim_{\epsilon \downarrow 0} \operatorname{Im}\left(i\sqrt{N(x+i\epsilon)}\right)$$

exist finitely m-almost everywhere. Since

$$\operatorname{Re}N(x+i\epsilon) = \left(\operatorname{Re}\sqrt{N(x+i\epsilon)}\right)^2 - \left(\operatorname{Im}\sqrt{N(x+i\epsilon)}\right)^2$$

it follows that the limit in (3.1.28) exists finitely m-almost everywhere. □

Let τ be the Borel measure on $\mathbb{R}$ in (3.1.20) which satisfies the condition (3.1.21). It has the Lebesgue decomposition

$$\tau = \tau_{\mathrm{ac}} + \tau_{\mathrm{s}}, \quad \tau_{\mathrm{s}} = \tau_{\mathrm{sc}} + \tau_{\mathrm{p}},$$

where τ_{ac} is absolutely continuous, τ_{s} is singular, τ_{sc} is singular continuous, and τ_{p} is pure point. In the next corollary, which is a consequence of Theorem 3.1.5, (3.1.22), and (3.1.25), minimal supports for these measures are expressed in terms of the boundary behavior of N.

Corollary 3.1.8. *Let N be a Nevanlinna function with the integral representation* (3.1.20). *Then the sets*

(i) $\{x \in \mathfrak{F} : 0 < \operatorname{Im}N(x+i0) \leq \infty\}$;

(ii) $\{x \in \mathfrak{F} : 0 < \operatorname{Im}N(x+i0) < \infty\}$;

(iii) $\{x \in \mathfrak{F} : \operatorname{Im}N(x+i0) = \infty\}$;

(iv) $\{x \in \mathfrak{F} : \operatorname{Im}N(x+i0) = \infty, \lim_{y \downarrow 0} y\operatorname{Im}N(x+iy) = 0\}$;

(v) $\{x \in \mathfrak{F} : \operatorname{Im}N(x+i0) = \infty, \lim_{y \downarrow 0} y\operatorname{Im}N(x+iy) > 0\}$,

are minimal supports for τ, τ_{ac}, τ_{s}, τ_{sc}, and τ_{p}, respectively.

3.2 Growth points of finite Borel measures

Let μ be a finite Borel measure on $\mathbb{R}$. In this section the set of its *growth points* $\sigma(\mu)$, defined by

$$\sigma(\mu) = \left\{x \in \mathbb{R} : \mu\left((x-\epsilon, x+\epsilon)\right) > 0 \text{ for all } \epsilon > 0\right\}, \qquad (3.2.1)$$

is studied. The growth points $\sigma(\mu)$ and the growth points $\sigma(\mu_{\mathrm{ac}})$, $\sigma(\mu_{\mathrm{s}})$, and $\sigma(\mu_{\mathrm{sc}})$ of the absolutely continuous, singular, and singular continuous part of μ will be located by means of the minimal supports expressed in terms of the Borel transform of μ.

There is an intimate connection between the set of growth points $\sigma(\mu)$ and supports for μ.

Lemma 3.2.1. *Let μ be a finite Borel measure on $\mathbb{R}$. Then the following statements hold:*

(i) *If $\mathfrak{S} \subset \mathbb{R}$ is a support of μ, then $\sigma(\mu) \subset \overline{\mathfrak{S}}$.*

(ii) *The set $\sigma(\mu)$ is closed and it is a support of μ.*

Proof. (i) Let $\mathfrak{S}$ be a support of μ, so that $\mu(\mathbb{R} \setminus \mathfrak{S}) = 0$. Assume that $x \in \sigma(\mu)$, so that for any $\epsilon > 0$ one has $\mu((x - \epsilon, x + \epsilon)) > 0$. Since $\mathfrak{S}$ is a support of μ, it follows that

$$0 < \mu\big((x - \epsilon, x + \epsilon)\big) = \mu\big((x - \epsilon, x + \epsilon) \cap \mathfrak{S}\big),$$

which implies that for any $\epsilon > 0$ the set $(x - \epsilon, x + \epsilon) \cap \mathfrak{S}$ is nonempty. Hence, there exists a sequence $x_n \in (x - 1/n, x + 1/n) \cap \mathfrak{S}$ converging to x from inside $\mathfrak{S}$. This shows that $\sigma(\mu) \subset \overline{\mathfrak{S}}$.

(ii) In order to show that $\sigma(\mu)$ is closed, let $x_n \in \sigma(\mu)$ converge to $x \in \mathbb{R}$. Assume that $x \notin \sigma(\mu)$. Then there is $\epsilon > 0$ such that $\mu((x - \epsilon, x + \epsilon)) = 0$. For this ϵ there exist $n_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ with $(x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0) \subset (x - \epsilon, x + \epsilon)$, and hence

$$\mu\big((x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0)\big) \leq \mu\big((x - \epsilon, x + \epsilon)\big) = 0,$$

a contradiction, since $x_{n_0} \in \sigma(\mu)$. Therefore, $x \in \sigma(\mu)$ and $\sigma(\mu)$ is closed.

Next it will be verified that $\sigma(\mu)$ is a support for μ. For each $x \in \mathbb{R} \setminus \sigma(\mu)$ there is $\epsilon_x > 0$ such that $\mu((x - \epsilon_x, x + \epsilon_x)) = 0$. Since the set $\sigma(\mu)$ is closed, it follows that the open intervals $(x - \epsilon_x, x + \epsilon_x)$, $x \in \mathbb{R} \setminus \sigma(\mu)$, form an open cover for $\mathbb{R} \setminus \sigma(\mu)$. Then there is a countable subcover of open intervals I_n with $\mu(I_n) = 0$ for $\mathbb{R} \setminus \sigma(\mu)$. It follows that

$$\mu(\mathbb{R} \setminus \sigma(\mu)) \leq \sum_n \mu(I_n) = 0$$

and hence $\mu(\mathbb{R} \setminus \sigma(\mu)) = 0$, that is, $\sigma(\mu)$ is a support for μ. $\qquad\square$

For completeness it is noted that in general the set $\sigma(\mu)$ is not a minimal support of μ. Observe also that, by Lemma 3.2.1, the set of growth points $\sigma(\mu)$ has the following minimality property: each closed support $\mathfrak{S} \subset \mathbb{R}$ of μ satisfies $\sigma(\mu) \subset \mathfrak{S}$. Therefore, one has the next corollary.

Corollary 3.2.2. *Let μ be a finite Borel measure on $\mathbb{R}$. Then $\sigma(\mu)$ is the smallest closed support of μ.*

The set of growth points of μ will now be described by means of the Borel transform of μ.

Theorem 3.2.3. *Let μ be a finite Borel measure on $\mathbb{R}$ and let F be its Borel transform. Then*

$$\sigma(\mu) = \overline{\big\{ x \in \mathbb{R} : 0 < \liminf_{y \downarrow 0} \operatorname{Im} F(x + iy) \big\}}.$$

Proof. With the notation

$$\mathfrak{N} = \big\{ x \in \mathbb{R} : 0 < \liminf_{y \downarrow 0} \operatorname{Im} F(x + iy) \big\}$$

it will be proved that $\sigma(\mu) = \overline{\mathfrak{N}}$. Recall first that, by Theorem 3.1.5 (i), the set

$$\mathfrak{M}' = \big\{ x \in \mathfrak{F} : 0 < \operatorname{Im} F(x + i0) \leq \infty \big\}$$

is a (minimal) support for μ. Since $\mathfrak{M}' \subset \mathfrak{N}$, it follows that $\mathfrak{N}$ is also a support for μ. Hence, Lemma 3.2.1 (i) yields $\sigma(\mu) \subset \overline{\mathfrak{N}}$. For the inclusion $\overline{\mathfrak{N}} \subset \sigma(\mu)$ it suffices to show $\mathfrak{N} \subset \sigma(\mu)$, since $\sigma(\mu)$ is closed; cf. Lemma 3.2.1 (ii). Assume that $x \notin \sigma(\mu)$. Then there exists $\epsilon > 0$ such that $\mu((x - \epsilon, x + \epsilon)) = 0$ and it follows from

$$\operatorname{Im} F(x + iy) = \int_{\mathbb{R} \setminus (x - \epsilon, x + \epsilon)} \frac{y}{(t - x)^2 + y^2} \, d\mu(t)$$

that $\operatorname{Im} F(x + i0) = 0$. This implies $x \notin \mathfrak{N}$ and hence $\mathfrak{N} \subset \sigma(\mu)$. $\square$

Analogous to Theorem 3.2.3 there are also results for the parts of the finite Borel measure μ on $\mathbb{R}$ in its Lebesgue decomposition. In order to describe these results one needs the following notions of closure.

Definition 3.2.4. Let $\mathfrak{B} \subset \mathbb{R}$ be a Borel set. The *absolutely continuous closure* (or *essential closure*) of $\mathfrak{B}$ is defined by

$$\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B}) := \big\{ x \in \mathbb{R} : m\big((x - \epsilon, x + \epsilon) \cap \mathfrak{B}\big) > 0 \text{ for all } \epsilon > 0 \big\}.$$

The *continuous closure* of $\mathfrak{B}$ is defined by

$$\operatorname{clos}_{\mathrm{c}}(\mathfrak{B}) := \big\{ x \in \mathbb{R} : (x - \epsilon, x + \epsilon) \cap \mathfrak{B} \text{ is not countable for all } \epsilon > 0 \big\}.$$

In general, $\mathfrak{B}$ is not a subset of $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$ since, e.g., isolated points in $\mathfrak{B}$ are not contained in $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$. Moreover, if $\mathfrak{B} \subset \mathfrak{B}'$ and $m(\mathfrak{B}' \setminus \mathfrak{B}) = 0$, then $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B}) = \operatorname{clos}_{\mathrm{ac}}(\mathfrak{B}')$.

Lemma 3.2.5. *Let $\mathfrak{B} \subset \mathbb{R}$ be a Borel set. Then the sets $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$ and $\operatorname{clos}_{\mathrm{c}}(\mathfrak{B})$ are both closed and*

$$\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B}) \subset \operatorname{clos}_{\mathrm{c}}(\mathfrak{B}) \subset \overline{\mathfrak{B}}. \tag{3.2.2}$$

Moreover, the following statements hold:

(i) $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B}) = \emptyset$ *if and only if* $m(\mathfrak{B}) = 0$;

(ii) $\operatorname{clos}_{\mathrm{c}}(\mathfrak{B}) = \emptyset$ *if and only if* $\mathfrak{B}$ *is countable.*

Proof. First it will be shown that for any Borel set $\mathfrak{B} \subset \mathbb{R}$ both sets $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$ and $\operatorname{clos}_{\mathrm{c}}(\mathfrak{B})$ are closed.

In order to show that $\operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$ is closed, let $x_n \in \operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$ converge to $x \in \mathbb{R}$. Assume that $x \notin \operatorname{clos}_{\mathrm{ac}}(\mathfrak{B})$. Then there is $\epsilon > 0$ such that

$$m\big((x - \epsilon, x + \epsilon) \cap \mathfrak{B}\big) = 0. \tag{3.2.3}$$

For this ϵ there is $n_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ with $(x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0) \subset (x - \epsilon, x + \epsilon)$. One then concludes from (3.2.3) that $m((x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0) \cap \mathfrak{B}) = 0$, a contradiction as $x_{n_0} \in \mathrm{clos}_{\mathrm{ac}}(\mathfrak{B})$. Therefore, $x \in \mathrm{clos}_{\mathrm{ac}}(\mathfrak{B})$ and $\mathrm{clos}_{\mathrm{ac}}(\mathfrak{B})$ is closed.

To show that $\mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$ is closed, let $x_n \in \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$ converge to $x \in \mathbb{R}$. Assume that $x \notin \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$. Then there is $\epsilon > 0$ such that the set $(x - \epsilon, x + \epsilon) \cap \mathfrak{B}$ is countable. For this ϵ there exist $n_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ with

$$(x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0) \subset (x - \epsilon_0, x + \epsilon_0),$$

so that

$$\big((x_{n_0} - \epsilon_0, x_{n_0} + \epsilon_0) \cap \mathfrak{B}\big) \subset \big((x - \epsilon, x + \epsilon) \cap \mathfrak{B}\big)$$

is countable, a contradiction, as $x_{n_0} \in \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$. Therefore, $x \in \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$ and $\mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$ is closed.

To see the first inclusion in (3.2.2) assume that $x \in \mathrm{clos}_{\mathrm{ac}}(\mathfrak{B})$. Then one has $m((x - \epsilon, x + \epsilon) \cap \mathfrak{B}) > 0$ for all $\epsilon > 0$ and hence for all $\epsilon > 0$ the set $(x - \epsilon, x + \epsilon) \cap \mathfrak{B}$ is not countable. This implies $\mathrm{clos}_{\mathrm{ac}}(\mathfrak{B}) \subset \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$. Likewise, to see the second inclusion assume that $x \in \mathrm{clos}_{\mathrm{c}}(\mathfrak{B})$ and that $x \notin \overline{\mathfrak{B}}$. Then there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap \mathfrak{B} = \emptyset$, a contradiction. Hence, $\mathrm{clos}_{\mathrm{c}}(\mathfrak{B}) \subset \overline{\mathfrak{B}}$.

(i) ($\Rightarrow$) Assume that $\mathrm{clos}_{\mathrm{ac}}(\mathfrak{B}) = \emptyset$. This implies that for all $x \in \mathbb{R}$ there exists $\epsilon_x > 0$ such that $m((x - \epsilon_x, x + \epsilon_x) \cap \mathfrak{B}) = 0$. First assume that $\mathfrak{B}$ is compact. Then for all $x \in \mathfrak{B}$ the open sets $(x - \epsilon_x, x + \epsilon_x)$ form an open cover for $\mathfrak{B}$. Therefore, there exists a finite subcover $(x_i - \epsilon_i, x_i + \epsilon_i)$ of $\mathfrak{B}$ such that

$$\mathfrak{B} \subset \bigcup_{i=1}^{n} (x_i - \epsilon_i, x_i + \epsilon_i) \cap \mathfrak{B},$$

and hence

$$m(\mathfrak{B}) \leq \sum_{i=1}^{n} m\big((x_i - \epsilon_i, x_i + \epsilon_i) \cap \mathfrak{B}\big) = 0.$$

For arbitrary Borel sets $\mathfrak{B}$ the (inner) regularity of the Lebesgue measure implies $m(\mathfrak{B}) = 0$.

($\Leftarrow$) If $m(\mathfrak{B}) = 0$, then $m((x - \epsilon, x + \epsilon) \cap \mathfrak{B}) = 0$ for all $x \in \mathbb{R}$ and all $\epsilon > 0$. Therefore, $\mathrm{clos}_{\mathrm{ac}}(\mathfrak{B}) = \emptyset$.

(ii) ($\Rightarrow$) Assume that $\mathrm{clos}_{\mathrm{c}}(\mathfrak{B}) = \emptyset$. This implies that for all $x \in \mathbb{R}$ there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \cap \mathfrak{B}$ is countable; in particular, this holds for all rational x_i. The countable many open sets $(x_i - \epsilon_{x_i}, x_i + \epsilon_{x_i})$ form an open cover for $\mathfrak{B}$ and this implies that $\mathfrak{B}$ is countable.

($\Leftarrow$) If $\mathfrak{B}$ is countable, then $(x - \epsilon, x + \epsilon) \cap \mathfrak{B}$ is countable for all $x \in \mathbb{R}$ and all $\epsilon > 0$. Therefore, $\mathrm{clos}_{\mathrm{c}}(\mathfrak{B}) = \emptyset$. $\square$

Here is the promised treatment of the absolutely continuous, singular, and singular continuous parts of the Borel measure μ.

Theorem 3.2.6. *Let μ be a finite Borel measure on $\mathbb{R}$ and let F be its Borel transform. Then the following statements hold:*

(i) $\sigma(\mu_{\mathrm{ac}}) = \mathrm{clos}_{\mathrm{ac}}(\{x \in \mathfrak{F} : 0 < \mathrm{Im}\, F(x + i0) < \infty\})$;

(ii) $\sigma(\mu_{\mathrm{s}}) \subset \overline{\{x \in \mathfrak{F} : \mathrm{Im}\, F(x + i0) = \infty\}}$;

(iii) $\sigma(\mu_{\mathrm{sc}}) \subset \mathrm{clos}_{\mathrm{c}}(\{x \in \mathfrak{F} : \mathrm{Im}\, F(x + i0) = \infty,\ \lim_{y \downarrow 0} y F(x + iy) = 0\})$.

Proof. (i) Let
$$\mathfrak{M}'_{\mathrm{ac}} := \{x \in \mathfrak{F} : 0 < \mathrm{Im}\, F(x + i0) < \infty\}$$

and note that $\mathfrak{M}'_{\mathrm{ac}}$ is a Borel set. It is claimed that

$$\sigma(\mu_{\mathrm{ac}}) = \mathrm{clos}_{\mathrm{ac}}(\mathfrak{M}'_{\mathrm{ac}}). \qquad (3.2.4)$$

To verify the inclusion ($\subset$) in (3.2.4), assume that $x \notin \mathrm{clos}_{\mathrm{ac}}(\mathfrak{M}'_{\mathrm{ac}})$. Then there exists $\epsilon > 0$ such that

$$m\big((x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{ac}}\big) = 0.$$

As μ_{ac} is absolutely continuous with respect to the Lebesgue measure m, also

$$\mu_{\mathrm{ac}}\big((x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{ac}}\big) = 0. \qquad (3.2.5)$$

Furthermore, by Theorem 3.1.5 (ii), the set $\mathfrak{M}'_{\mathrm{ac}}$ is a minimal support for μ_{ac} and, in particular, $\mu_{\mathrm{ac}}(\mathbb{R} \setminus \mathfrak{M}'_{\mathrm{ac}}) = 0$. Hence,

$$\mu_{\mathrm{ac}}\big((x - \epsilon, x + \epsilon) \setminus \mathfrak{M}'_{\mathrm{ac}}\big) = 0 \qquad (3.2.6)$$

and from (3.2.5)–(3.2.6) one obtains $\mu_{\mathrm{ac}}((x - \epsilon, x + \epsilon)) = 0$. Hence, $x \notin \sigma(\mu_{\mathrm{ac}})$. Thus, the inclusion ($\subset$) in (3.2.4) has been shown.

For the converse inclusion ($\supset$), let $x \notin \sigma(\mu_{\mathrm{ac}})$. Then there exists $\epsilon > 0$ such that

$$0 = \mu_{\mathrm{ac}}\big((x - \epsilon, x + \epsilon)\big) = \int_{(x-\epsilon, x+\epsilon)} (D\mu)(t)\, dm(t),$$

where in the last equality the Radon–Nikodým theorem was used; cf. (3.1.3) and note that $\nu'_\mu = \mu' = D\mu$ m-almost everywhere. Due to Theorem 3.1.4 and the fact that $\mathrm{Im}\, F(t + i0) \geq 0$ for all $t \in \mathfrak{F}$, one concludes that

$$0 = \frac{1}{\pi} \int_{(x-\epsilon, x+\epsilon)} \mathrm{Im}\, F(t + i0)\, dm(t)$$
$$= \frac{1}{\pi} \int_{(x-\epsilon, x+\epsilon) \cap \mathfrak{M}'_{\mathrm{ac}}} \mathrm{Im}\, F(t + i0)\, dm(t).$$

This implies $m((x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{ac}}) = 0$ since $\mathrm{Im}\, F(t + i0)$ is positive on $\mathfrak{M}'_{\mathrm{ac}}$. Hence, $x \notin \mathrm{clos}_{\mathrm{ac}}(\mathfrak{M}'_{\mathrm{ac}})$. Thus, the inclusion ($\supset$) in (3.2.4) has been shown. Therefore, the equality (3.2.4) has been established, which gives the assertion (i).

(ii) According to Theorem 3.1.5 (iii) the set $\{x \in \mathfrak{F} : \operatorname{Im} F(x + i0) = \infty\}$ is a minimal support for the singular part μ_s of μ. Since $\sigma(\mu_s)$ is contained in the closure of this set by Lemma 3.2.1 (i), the assertion follows.

(iii) By Theorem 3.1.5 (iv) and (3.1.13), the Borel set

$$\mathfrak{M}'_{\mathrm{sc}} := \big\{x \in \mathfrak{F} : \operatorname{Im} F(x + i0) = \infty, \lim_{y \downarrow 0} y F(x + iy) = 0\big\}$$

is a minimal support for μ_{sc} and hence, in particular, $\mu_{\mathrm{sc}}(\mathbb{R} \setminus \mathfrak{M}'_{\mathrm{sc}}) = 0$. Let $\operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})$ be the continuous closure of $\mathfrak{M}'_{\mathrm{sc}}$, which is a Borel set, as it is closed; cf. Lemma 3.2.5. It will be shown that $\operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})$ is a support for μ_{sc}, that is,

$$\mu_{\mathrm{sc}}\big(\mathbb{R} \setminus \operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})\big) = 0, \tag{3.2.7}$$

since this implies that $\sigma(\mu_{\mathrm{sc}}) \subset \operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})$; cf. Lemma 3.2.1 (i) and Lemma 3.2.5.

In fact, for $x \in \mathbb{R} \setminus \operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})$ by definition there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{sc}}$ is countable; thus $\mu_{\mathrm{sc}}((x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{sc}}) = 0$, as μ_{sc} is continuous. Consequently,

$$\mu_{\mathrm{sc}}((x - \epsilon, x + \epsilon)) \leq \mu_{\mathrm{sc}}\big((x - \epsilon, x + \epsilon) \cap \mathfrak{M}'_{\mathrm{sc}}\big) + \mu_{\mathrm{sc}}(\mathbb{R} \setminus \mathfrak{M}'_{\mathrm{sc}}) = 0.$$

This yields $\mu_{\mathrm{sc}}(K) = 0$ for each compact set $K \subset \mathbb{R} \setminus \operatorname{clos}_{\mathrm{c}}(\mathfrak{M}'_{\mathrm{sc}})$ and hence, by the (inner) regularity of the finite measure μ_{sc}, (3.2.7) follows. $\square$

3.3 Spectra of self-adjoint relations

The spectrum of a self-adjoint relation or operator in a Hilbert space will be studied in terms of its spectral measure. In particular, a division of the spectrum into absolutely continuous and singular spectra will be introduced based on the Lebesgue decomposition of a finite Borel measure; cf. Section 3.1.

Let A be a self-adjoint relation in the Hilbert space $\mathfrak{H}$. Then $\sigma(A) \subset \mathbb{R}$ by Theorem 1.5.5 and $\sigma_{\mathrm{r}}(A) = \emptyset$, and hence $\sigma(A) = \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A)$; cf. Proposition 1.4.4. The spectral measure $E(\cdot)$ of A satisfies

$$(A - \lambda)^{-1} = \int_{\mathbb{R}} \frac{1}{t - \lambda} \, dE(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

cf. (1.5.6). First the parts $\sigma_{\mathrm{p}}(A)$ and $\sigma_{\mathrm{c}}(A)$ of the spectrum $\sigma(A)$ will be characterized in terms of the spectral measure $E(\cdot)$. These results will play an important role in the further development; cf. Section 3.5 and Section 3.6. The facts in Proposition 3.3.1 are immediate consequences of the orthogonal decomposition

$$\mathfrak{H} = \mathfrak{H}_{\mathrm{op}} \oplus \mathfrak{H}_{\mathrm{mul}}, \quad A = A_{\mathrm{op}} \,\widehat{\oplus}\, A_{\mathrm{mul}}, \tag{3.3.1}$$

where $\mathfrak{H}_{\mathrm{op}} = \overline{\operatorname{dom} A}$ and $\mathfrak{H}_{\mathrm{mul}} = \operatorname{mul} A$, of the self-adjoint relation A (see Theorem 1.5.1) and the properties of the spectral measure of A_{op}.

Proposition 3.3.1. *Let A be a self-adjoint relation in $\mathfrak{H}$ with spectral measure $E(\cdot)$. Then the following statements hold:*

(i) $\lambda \in \rho(A) \cap \mathbb{R}$ *if and only if* $E((\lambda - \epsilon, \lambda + \epsilon)) = 0$ *for some* $\epsilon > 0$;

(ii) $\lambda \in \sigma_{\mathrm{p}}(A)$ *if and only if* $E(\{\lambda\}) \neq 0$, *in which case* $\mathfrak{N}_\lambda(A) = \operatorname{ran} E(\{\lambda\})$ *and*

$$\widehat{\mathfrak{N}}_\lambda(A) = \{\{E(\{\lambda\})h, \lambda E(\{\lambda\})h\} : h \in \mathfrak{H}\};$$

(iii) $\lambda \in \sigma_{\mathrm{c}}(A)$ *if and only if* $E(\{\lambda\}) = 0$ *and* $E((\lambda - \epsilon, \lambda + \epsilon)) \neq 0$ *for all* $\epsilon > 0$.

A further subdivision of the spectrum will be introduced analogous to the Lebesgue decomposition of a finite Borel measure on $\mathbb{R}$; cf. Section 3.1. This requires another description of the spectrum via the introduction of a collection of finite Borel measures induced by the spectral function. Let A be a self-adjoint relation in $\mathfrak{H}$ with spectral measure $E(\cdot)$. For each $h \in \mathfrak{H}$, define μ_h by

$$\mu_h = (E(\cdot)h, h) = (E_{\mathrm{op}}(\cdot)P_{\mathrm{op}}h, P_{\mathrm{op}}h), \tag{3.3.2}$$

so that μ_h is a regular Borel measure on $\mathbb{R}$. Note that $\mu_h = 0$ for $h \in \mathfrak{H}_{\mathrm{mul}}$. The set of growth points $\sigma(\mu_h)$ of μ_h is given by

$$\sigma(\mu_h) = \{x \in \mathbb{R} : \mu_h((x - \epsilon, x + \epsilon)) > 0 \text{ for all } \epsilon > 0\}.$$

It will be shown that the spectrum of A is made up of the growth points of μ_h for a dense set of elements $h \in \mathfrak{H}$. Furthermore, the statement in the following proposition is in a local sense, namely, it concerns the spectrum of A relative to an open interval $\Delta \subset \mathbb{R}$; cf. Definition 3.4.9.

Proposition 3.3.2. *Let A be a self-adjoint relation in $\mathfrak{H}$, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that $\mathcal{D}_\Delta$ is a subset of the closed subspace $E(\Delta)\mathfrak{H}$ such that*

$$\overline{\operatorname{span}}\, \mathcal{D}_\Delta = E(\Delta)\mathfrak{H}.$$

Then the following identities hold:

$$\overline{\sigma(A) \cap \Delta} = \overline{\bigcup_{h \in E(\Delta)\mathfrak{H}} \sigma(\mu_h)} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_h)}. \tag{3.3.3}$$

Proof. First it will be shown that

$$\overline{\sigma(A) \cap \Delta} \supset \overline{\bigcup_{h \in E(\Delta)\mathfrak{H}} \sigma(\mu_h)} \supset \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_h)}. \tag{3.3.4}$$

For this purpose assume that $x \notin \overline{\sigma(A) \cap \Delta}$. Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap \Delta$ contains no spectrum of A. By Proposition 3.3.1 (i), this yields

$$E\big((x - \epsilon, x + \epsilon) \cap \Delta\big) = 0$$

and for $h \in E(\Delta)\mathfrak{H}$ one obtains

$$
\begin{aligned}
\mu_h\big((x - \epsilon, x + \epsilon)\big) &= \big(E\big((x - \epsilon, x + \epsilon)\big)h, h\big) \\
&= \big(E\big((x - \epsilon, x + \epsilon)\big)E(\Delta)h, h\big) \\
&= \big(E\big((x - \epsilon, x + \epsilon) \cap \Delta\big)h, h\big) \\
&= 0.
\end{aligned}
$$

Therefore, $(x - \epsilon, x + \epsilon) \cap \sigma(\mu_h) = \emptyset$ for all $h \in E(\Delta)\mathfrak{H}$, and thus

$$
x \notin \overline{\bigcup_{h \in E(\Delta)\mathfrak{H}} \sigma(\mu_h)}.
$$

Hence, the inclusions (3.3.4) follow. Next it will be shown that

$$
\overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_h)} \supset \overline{\sigma(A) \cap \Delta},
$$

which, together with (3.3.4), yields (3.3.3). For this purpose, assume that

$$
x \notin \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_h)}.
$$

Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus \sigma(\mu_h)$ for all $h \in \mathcal{D}_\Delta$, that is,

$$
\big\| E\big((x - \epsilon, x + \epsilon)\big)h \big\|^2 = \mu_h\big((x - \epsilon, x + \epsilon)\big) = 0 \tag{3.3.5}
$$

for all $h \in \mathcal{D}_\Delta$, and hence for all $h \in \operatorname{span} \mathcal{D}_\Delta$. Since by assumption $\operatorname{span} \mathcal{D}_\Delta$ is dense in $E(\Delta)\mathfrak{H}$, it follows that (3.3.5) holds for all $h \in E(\Delta)\mathfrak{H}$ and hence again by Proposition 3.3.1 (i),

$$
E\big((x - \epsilon, x + \epsilon) \cap \Delta\big)h = E\big((x - \epsilon, x + \epsilon)\big)E(\Delta)h = 0
$$

for all $h \in \mathfrak{H}$. This shows that $(x - \epsilon, x + \epsilon) \cap \Delta$ does not contain spectrum of A, in particular, $x \notin \overline{\sigma(A) \cap \Delta}$. $\qquad\square$

The collection of Borel measures μ_h, $h \in \mathfrak{H}$, as defined in (3.3.2), is now used to introduce a number of subspaces of $\mathfrak{H}$.

Definition 3.3.3. Let A be a self-adjoint relation in $\mathfrak{H}$. The *pure point subspace*, the *absolutely continuous subspace*, and the *singular continuous subspace* corresponding to A_{op} are defined by

$$
\begin{aligned}
\mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}}) &= \big\{ h \in \mathfrak{H} : \mu_h \text{ is pure point} \big\}, \\
\mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}}) &= \big\{ h \in \mathfrak{H} : \mu_h \text{ is absolutely continuous} \big\}, \\
\mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}}) &= \big\{ h \in \mathfrak{H} : \mu_h \text{ is singular continuous} \big\},
\end{aligned}
$$

respectively.

In conjunction with the orthogonal decomposition (3.3.1), these subspaces span the original Hilbert space and lead to invariant parts of the self-adjoint relation, see, e.g., [649, Theorem VII.4] or [691, Proposition 9.3].

Theorem 3.3.4. *Let A be a self-adjoint relation in $\mathfrak{H}$. Then $\mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}})$, $\mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}})$, and $\mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}})$ are mutually orthogonal closed subspaces of $\mathfrak{H}$ and*

$$\mathfrak{H} = \mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}}) \oplus \mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}}) \oplus \mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}}) \oplus \mathfrak{H}_{\mathrm{mul}}.$$

Each of the Hilbert spaces $\mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}})$, $\mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}})$, and $\mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}})$ is invariant for the operator A_{op}, and the restrictions

$$A_{\mathrm{op}}^{\mathrm{p}} = A_{\mathrm{op}} \upharpoonright \mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}}),$$
$$A_{\mathrm{op}}^{\mathrm{ac}} = A_{\mathrm{op}} \upharpoonright \mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}}),$$
$$A_{\mathrm{op}}^{\mathrm{sc}} = A_{\mathrm{op}} \upharpoonright \mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}}),$$

are self-adjoint operators in $\mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}})$, $\mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}})$, and $\mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}})$, respectively.

By means of these subspaces one defines, in analogy with the case of finite Borel measures, the *singular subspace* and the *continuous subspace* corresponding to A_{op} by

$$\mathfrak{H}_{\mathrm{s}}(A_{\mathrm{op}}) = \mathfrak{H}_{\mathrm{p}}(A_{\mathrm{op}}) \oplus \mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}}) \quad \text{and} \quad \mathfrak{H}_{\mathrm{c}}(A_{\mathrm{op}}) = \mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}}) \oplus \mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}}),$$

respectively. The restrictions of A_{op} to these subspaces are denoted by $A_{\mathrm{op}}^{\mathrm{s}}$ and $A_{\mathrm{op}}^{\mathrm{c}}$, respectively, and it follows that

$$A_{\mathrm{op}}^{\mathrm{s}} = A_{\mathrm{op}}^{\mathrm{p}} \,\widehat{\oplus}\, A_{\mathrm{op}}^{\mathrm{sc}} \quad \text{and} \quad A_{\mathrm{op}}^{\mathrm{c}} = A_{\mathrm{op}}^{\mathrm{ac}} \,\widehat{\oplus}\, A_{\mathrm{op}}^{\mathrm{sc}}.$$

Definition 3.3.5. *Let A be a self-adjoint relation in $\mathfrak{H}$. The absolutely continuous spectrum $\sigma_{\mathrm{ac}}(A)$, the singular continuous spectrum $\sigma_{\mathrm{sc}}(A)$, and the singular spectrum $\sigma_{\mathrm{s}}(A)$ of A are defined by*

$$\sigma_{\mathrm{ac}}(A) = \sigma\big(A_{\mathrm{op}}^{\mathrm{ac}}\big), \quad \sigma_{\mathrm{sc}}(A) = \sigma\big(A_{\mathrm{op}}^{\mathrm{sc}}\big), \quad \text{and} \quad \sigma_{\mathrm{s}}(A) = \sigma\big(A_{\mathrm{op}}^{\mathrm{s}}\big),$$

respectively.

Note that for the pure point part $A_{\mathrm{op}}^{\mathrm{p}}$ one only has $\overline{\sigma_{\mathrm{p}}(A)} = \sigma(A_{\mathrm{op}}^{\mathrm{p}})$. The spectral measures of the self-adjoint operators $A_{\mathrm{op}}^{\mathrm{ac}}$, $A_{\mathrm{op}}^{\mathrm{sc}}$, and $A_{\mathrm{op}}^{\mathrm{s}}$ in the Hilbert spaces $\mathfrak{H}_{\mathrm{ac}}(A_{\mathrm{op}})$, $\mathfrak{H}_{\mathrm{sc}}(A_{\mathrm{op}})$, and $\mathfrak{H}_{\mathrm{s}}(A_{\mathrm{op}})$, are given by the corresponding restrictions of the spectral measure $E(\cdot)$ of A. These spectral measures will be denoted by $E_{\mathrm{ac}}(\cdot)$, $E_{\mathrm{sc}}(\cdot)$, and $E_{\mathrm{s}}(\cdot)$, respectively.

The following corollary relates the absolutely continuous, singular continuous, and singular spectrum of A in an open interval Δ with the growth points of the absolutely continuous, singular continuous, and singular parts of the measures μ_h.

Corollary 3.3.6. *Let A be a self-adjoint relation in $\mathfrak{H}$, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that $\mathcal{D}_\Delta$ is a subset of the closed subspace $E(\Delta)\mathfrak{H}$ such that*

$$\overline{\mathrm{span}}\,\mathcal{D}_\Delta = E(\Delta)\mathfrak{H}.$$

Denote by $\mu_{h,\mathrm{ac}}$, $\mu_{h,\mathrm{sc}}$, and $\mu_{h,\mathrm{s}}$ the absolutely continuous, singular continuous, and singular part in the Lebesgue decomposition of the Borel measure μ_h in (3.3.2). Then the following identity holds:

$$\overline{\sigma_i(A) \cap \Delta} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{h,i})}, \qquad i = \mathrm{ac},\ \mathrm{sc},\ \mathrm{s}.$$

Proof. Observe first that the absolutely continuous, singular continuous, and singular part of the Borel measure μ_h, $h \in \mathfrak{H}$, are given by

$$\mu_{h,\mathrm{ac}} = \mu_{P_{\mathrm{ac}}h}, \quad \mu_{h,\mathrm{sc}} = \mu_{P_{\mathrm{sc}}h}, \quad \text{and} \quad \mu_{h,\mathrm{s}} = \mu_{P_{\mathrm{s}}h}, \qquad (3.3.6)$$

respectively, where P_i denote the orthogonal projections onto the corresponding Hilbert spaces $\mathfrak{H}_i(A_{\mathrm{op}})$, $i = \mathrm{ac}$, sc, s. This follows from the uniqueness of the Lebesgue decomposition and Theorem 3.3.4. If $\mu_{h_i}^i = (E_i(\cdot)h_i, h_i)$, $h_i \in \mathfrak{H}_i(A_{\mathrm{op}})$, is the Borel measure defined with the help of the spectral measures $E_i(\cdot)$ of A_{op}^i, $i = \mathrm{ac}$, sc, s, then Definition 3.3.5, Proposition 3.3.2 and (3.3.6) yield

$$\overline{\sigma_i(A) \cap \Delta} = \overline{\bigcup_{h_i \in P_i \mathcal{D}_\Delta} \sigma(\mu_{h_i}^i)} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{P_i h})} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{h,i})}$$

for $i = \mathrm{ac}$, sc, s. Here it was also used that the linear span of the set $P_i \mathcal{D}_\Delta$ is dense in $E_i(\Delta)\mathfrak{H}_i(A_{\mathrm{op}}) = P_i E(\Delta)\mathfrak{H}$. $\square$

Example 3.3.7. Let μ be a Borel measure on $\mathbb{R}$ and consider the maximal multiplication operator by the independent variable in $L_\mu^2(\mathbb{R})$, given by

$$(Af)(t) = tf(t), \quad \mathrm{dom}\, A = \{f \in L_\mu^2(\mathbb{R}) : t \mapsto tf(t) \in L_\mu^2(\mathbb{R})\}.$$

The operator A is self-adjoint in $L_\mu^2(\mathbb{R})$ and for every Borel set $\mathfrak{B} \subset \mathbb{R}$ the spectral measure of A is given by

$$E(\mathfrak{B})h = \chi_\mathfrak{B} h, \qquad h \in L_\mu^2(\mathbb{R}),$$

where $\chi_\mathfrak{B}$ denotes the characteristic function of $\mathfrak{B}$. For $h \in L_\mu^2(\mathbb{R})$ the Borel measure in (3.3.2) satisfies

$$\mu_h(\mathfrak{B}) = (E(\mathfrak{B})h, h)_{L_\mu^2(\mathbb{R})} = \int_\mathfrak{B} |h(t)|^2 \, d\mu(t)$$

for all Borel sets $\mathfrak{B} \subset \mathbb{R}$. It is not difficult to check that $\sigma(A) = \sigma(\mu)$. Furthermore, the Lebesgue decomposition $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$, where $\mu_{\mathrm{s}} = \mu_{\mathrm{sc}} + \mu_{\mathrm{p}}$, gives rise to the orthogonal decompositions

$$L_\mu^2(\mathbb{R}) = L_{\mu_{\mathrm{ac}}}^2(\mathbb{R}) \oplus L_{\mu_{\mathrm{s}}}^2(\mathbb{R}) \quad \text{and} \quad L_{\mu_{\mathrm{s}}}^2(\mathbb{R}) = L_{\mu_{\mathrm{sc}}}^2(\mathbb{R}) \oplus L_{\mu_{\mathrm{p}}}^2(\mathbb{R}).$$

For the spectral subspaces of A in Definition 3.3.3 one has $\mathfrak{H}_i(A) = L_{\mu_i}^2(\mathbb{R})$, $i = \mathrm{ac}$, sc, s, and this implies

$$\sigma_{\mathrm{ac}}(A) = \sigma(\mu_{\mathrm{ac}}), \quad \sigma_{\mathrm{sc}}(A) = \sigma(\mu_{\mathrm{sc}}), \quad \text{and} \quad \sigma_{\mathrm{s}}(A) = \sigma(\mu_{\mathrm{s}}).$$

3.4 Simple symmetric operators

It will be shown that any closed symmetric relation in a Hilbert space can be decomposed into the orthogonal componentwise sum of a closed simple, i.e., completely non-self-adjoint, symmetric operator, and a self-adjoint relation. Criteria for the absence of the self-adjoint relation in this decomposition will be given, and a local version of simplicity will be studied.

First some attention is paid to the notions of invariance and reduction. These notions appeared already in the self-adjoint case in the previous section when subdividing the spectrum, and are also important in the description of self-adjoint extensions of symmetric relations. Let S be a closed symmetric relation in the Hilbert space $\mathfrak{H}$. Decompose $\mathfrak{H}$ as $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$, let P' and P'' be the orthogonal projections onto $\mathfrak{H}'$ and $\mathfrak{H}''$, and define

$$\widehat{P}'\{f, g\} = \{P'f, P'g\} \quad \text{and} \quad \widehat{P}''\{f, g\} = \{P''f, P''g\}, \quad f, g \in \mathfrak{H}.$$

The closed symmetric relation S gives rise to the restrictions

$$S' = S \cap (\mathfrak{H}')^2 \subset \widehat{P}'S \quad \text{and} \quad S'' = S \cap (\mathfrak{H}'')^2 \subset \widehat{P}''S, \tag{3.4.1}$$

which are closed symmetric relations and

$$S' \mathbin{\widehat{\oplus}} S'' \subset S. \tag{3.4.2}$$

In order to describe when S' and S'' span S the following notions are useful. The subspaces $\mathfrak{H}'$ and $\mathfrak{H}''$ are called *invariant* under the symmetric relation S if $S' = \widehat{P}'S$ or $S'' = \widehat{P}''S$, respectively. Clearly, the spaces $\mathfrak{H}'$ or $\mathfrak{H}''$ are invariant under S if

$$\widehat{P}'S \subset S \quad \text{or} \quad \widehat{P}''S \subset S,$$

respectively. In the next lemma it turns out that $\mathfrak{H}'$ is invariant under S if and only if $\mathfrak{H}''$ is invariant under S; in which case S' and S'' can be orthogonally split off from S, i.e., $S = S' \mathbin{\widehat{\oplus}} S''$.

Lemma 3.4.1. *Let S be a closed symmetric relation in $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ and let S' and S'' be as in* (3.4.1). *Then the following statements hold:*

(i) *$S' = \widehat{P}'S$ or, equivalently, $S'' = \widehat{P}''S$ implies that $S = S' \mathbin{\widehat{\oplus}} S''$.*

(ii) *If S' is self-adjoint in $\mathfrak{H}'$ or S'' is self-adjoint in $\mathfrak{H}''$, then $S' = \widehat{P}'S$ and $S'' = \widehat{P}''S$.*

Assume, in addition, that S is self-adjoint. Then

(iii) *$S' = \widehat{P}'S$ or, equivalently, $S'' = \widehat{P}''S$ implies that S' and S'' are self-adjoint in $\mathfrak{H}'$ and $\mathfrak{H}''$, respectively.*

Proof. (i) Assume that $S' = \widehat{P}'S$. Since $S' \,\widehat{\oplus}\, S'' \subset S$ by (3.4.2), it suffices to show that $S \subset S' \,\widehat{\oplus}\, S''$. Let $\{f, f'\} \in S$ and decompose $\{f, f'\}$ with respect to $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ as

$$\{f, f'\} = \{h, h'\} + \{k, k'\}, \quad h, h' \in \mathfrak{H}', \quad k, k' \in \mathfrak{H}''.$$

Then $\{h, h'\} \in \widehat{P}'S = S' \subset S$ and therefore $\{k, k'\} \in S \cap (\mathfrak{H}'')^2 = S''$. Hence, $S = S' \,\widehat{\oplus}\, S''$, which implies that $S'' = \widehat{P}''S$.

(ii) Assume that S' is self-adjoint in $\mathfrak{H}'$. To show that $\widehat{P}'S \subset S'$, let $\{f, f'\} \in S$ and consider $\{P'f, P'f'\} \in \widehat{P}'S$. Since S is symmetric it follows for all $\{h, h'\} \in S' \subset S$ that

$$(P'f', h)_{\mathfrak{H}'} - (P'f, h')_{\mathfrak{H}'} = (f', h)_{\mathfrak{H}} - (f, h')_{\mathfrak{H}} = 0.$$

The assumption that S' is self-adjoint in $\mathfrak{H}'$ implies $\{P'f, P'f'\} \in S'$. Therefore, $\widehat{P}'S \subset S'$. This implies $S' = \widehat{P}'S$ and (i) yields $S'' = \widehat{P}''S$.

(iii) According to (i), either of the conditions $S' = \widehat{P}'S$ or $S'' = \widehat{P}''S$ implies that $S = S' \,\widehat{\oplus}\, S''$. Since S is self-adjoint, this shows that S' is self-adjoint in $\mathfrak{H}'$ and that S'' is self-adjoint in $\mathfrak{H}''$. $\qquad\square$

Before introducing the notion of simplicity in Definition 3.4.3 below, the following lemma on symmetric and self-adjoint extensions of symmetric relations that contain a self-adjoint part is discussed.

Lemma 3.4.2. *Let S be a closed symmetric relation in $\mathfrak{H}$ whose defect numbers are not necessarily equal and assume that there are orthogonal decompositions*

$$\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}'', \quad S = S' \,\widehat{\oplus}\, S'', \tag{3.4.3}$$

such that S' is closed and symmetric in $\mathfrak{H}'$ and S'' is self-adjoint in $\mathfrak{H}''$. Then every closed symmetric (self-adjoint) extension A of S in $\mathfrak{H}$ admits the decomposition

$$A = A' \,\widehat{\oplus}\, S'',$$

where A' is a closed symmetric (self-adjoint) extension of S' in $\mathfrak{H}'$.

Proof. Observe that the inclusion $S \subset A$ and the decomposition (3.4.3) imply that

$$S'' = S \cap (\mathfrak{H}'')^2 \subset A \cap (\mathfrak{H}'')^2.$$

Therefore, the assumption that S'' is self-adjoint in $\mathfrak{H}''$ shows that the closed symmetric relation $A \cap (\mathfrak{H}'')^2$ is actually self-adjoint in $\mathfrak{H}''$ and that $S'' = A \cap (\mathfrak{H}'')^2$. Hence, by Lemma 3.4.1 (i)–(ii) the relation A decomposes as $A = A' \,\widehat{\oplus}\, S''$, where $A' = A \cap (\mathfrak{H}')^2$ is a symmetric extension of S' in $\mathfrak{H}'$. Therefore,

$$S' \,\widehat{\oplus}\, S'' \subset A' \,\widehat{\oplus}\, S''.$$

If A is self-adjoint in $\mathfrak{H}$, then Lemma 3.4.1 (iii) implies that $A' = A \cap (\mathfrak{H}')^2$ is a self-adjoint extension of S' in $\mathfrak{H}'$. This completes the proof. $\qquad\square$

The notion of simplicity or complete non-self-adjointness is defined next.

Definition 3.4.3. Let S be a closed symmetric relation in $\mathfrak{H}$ whose defect numbers are not necessarily equal. Then S is *simple* if there is no orthogonal decomposition

$$S = S' \,\widehat{\oplus}\, S'', \quad \text{where} \quad \mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}'', \tag{3.4.4}$$

such that $\mathfrak{H}'' \neq \{0\}$ and S'' is self-adjoint in $\mathfrak{H}''$.

Every closed symmetric relation S in $\mathfrak{H}$ has the orthogonal componentwise decomposition $S = S_{\mathrm{op}} \,\widehat{\oplus}\, S_{\mathrm{mul}}$, where S_{mul} is a purely multivalued self-adjoint relation in the closed subspace $\mathfrak{H}_{\mathrm{mul}} = \mathrm{mul}\, S$; cf. Theorem 1.4.11. Hence, a closed simple symmetric relation is necessarily an operator. A similar argument shows that a closed simple symmetric relation does not have any eigenvalues; cf. Lemma 3.4.7.

Any closed symmetric relation S in $\mathfrak{H}$ has a decomposition as in (3.4.4), where S' is simple in $\mathfrak{H}'$ and S'' is self-adjoint in $\mathfrak{H}''$. To see this, define the closed subspace $\mathfrak{R} \subset \mathfrak{H}$ by

$$\mathfrak{R} := \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \mathrm{ran}\,(S - \lambda), \tag{3.4.5}$$

and the closed subspace $\mathfrak{K} = \mathfrak{R}^\perp$, so that

$$\mathfrak{K} = \overline{\mathrm{span}}\,\big\{ \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \big\}, \quad \mathfrak{N}_\lambda(S^*) = \ker\,(S^* - \lambda). \tag{3.4.6}$$

It follows from Lemma 1.6.11 that the set $\mathbb{C} \setminus \mathbb{R}$ in (3.4.6), and hence in the intersection in (3.4.5), can be replaced by any subset of $\mathbb{C} \setminus \mathbb{R}$ which has an accumulation point in $\mathbb{C}^+$ and an accumulation point in $\mathbb{C}^-$.

Theorem 3.4.4. *Let S be a closed symmetric relation in $\mathfrak{H}$ whose defect numbers are not necessarily equal. Let $\mathfrak{H}$ be decomposed as $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{R}$, where the closed subspaces $\mathfrak{K}$ and $\mathfrak{R}$ are defined as in (3.4.5) and (3.4.6), and denote*

$$S' = S \cap \mathfrak{K}^2 \quad and \quad S'' = S \cap \mathfrak{R}^2. \tag{3.4.7}$$

Then the relation S admits the orthogonal decomposition

$$S = S' \,\widehat{\oplus}\, S'', \tag{3.4.8}$$

where S' is a closed simple symmetric operator in $\mathfrak{K}$ and S'' is a self-adjoint relation in $\mathfrak{R}$.

Proof. Step 1. First it will be shown that $\mathfrak{R}$ satisfies the following invariance property: for any $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$

$$(S - \lambda_0)^{-1}\mathfrak{R} \subset \mathfrak{R}. \tag{3.4.9}$$

To see this, let $h \in \mathfrak{R}$ and $h' = (S - \lambda_0)^{-1}h$. Hence, $\{h', h + \lambda_0 h'\} \in S$ and thus

$$(h + \lambda_0 h', f_{\bar\lambda}) = (h', \bar\lambda f_{\bar\lambda}), \quad \{f_{\bar\lambda}, \bar\lambda f_{\bar\lambda}\} \in S^*,\ \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Since $h \in \operatorname{ran}(S - \lambda) = (\ker(S^* - \bar{\lambda}))^{\perp}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $f_{\bar{\lambda}} \in \ker(S^* - \bar{\lambda})$, this implies

$$0 = (h, f_{\bar{\lambda}}) = (\lambda - \lambda_0)(h', f_{\bar{\lambda}}),$$

that is, $h' \perp f_{\bar{\lambda}}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \lambda_0$. Hence, $h' \in \operatorname{ran}(S - \lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \lambda_0$, and it follows from Lemma 1.6.11 that

$$h' \in \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{R}, \, \lambda \neq \lambda_0} \operatorname{ran}(S - \lambda) = \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(S - \lambda) = \mathfrak{R},$$

which proves the inclusion in (3.4.9).

Step 2. Next it will be shown that the relation $S \cap \mathfrak{R}^2$ is self-adjoint. Fix some $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and define the relation S'' first by

$$S'' = \left\{ \{(S - \lambda_0)^{-1}h, (I + \lambda_0(S - \lambda_0)^{-1})h\} : h \in \mathfrak{R} \right\}. \tag{3.4.10}$$

It follows from (3.4.5) that $\mathfrak{R} \subset \operatorname{ran}(S - \lambda_0)$, and hence Lemma 1.1.8 implies $S'' \subset S$, so that in particular S'' is symmetric in $\mathfrak{H}$. It follows from (3.4.9) that $S'' \subset \mathfrak{R}^2$. Therefore, $S'' \subset S \cap \mathfrak{R}^2$. Next $S \cap \mathfrak{R}^2 \subset S''$ will be verified. Let $\{f, f'\} \in S \cap \mathfrak{R}^2$, so that by Lemma 1.1.8

$$\{f, f'\} = \{(S - \lambda_0)^{-1}h, (I + \lambda_0(S - \lambda_0)^{-1})h\}$$

for some $h \in \operatorname{ran}(S - \lambda_0)$. Since $\{f, f'\} \in \mathfrak{R}^2$, it follows that

$$(S - \lambda_0)^{-1}h \in \mathfrak{R} \quad \text{and} \quad (I + \lambda_0(S - \lambda_0)^{-1})h \in \mathfrak{R}.$$

Therefore, $h \in \mathfrak{R}$ and hence $\{f, f'\} \in S''$, so that $S \cap \mathfrak{R}^2 \subset S''$. This leads to the equality $S'' = S \cap \mathfrak{R}^2$ in (3.4.7); in particular, S'' in (3.4.10) does not depend on the choice of $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

From $S'' \subset S$ it follows that S'' is symmetric and from (3.4.10) one obtains that $\operatorname{ran}(S'' - \lambda_0) = \mathfrak{R}$. Since S'' is independent of the choice of λ_0, it follows that $\operatorname{ran}(S'' - \lambda) = \mathfrak{R}$ holds for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence, $S'' = S \cap \mathfrak{R}^2$ is a self-adjoint relation in $\mathfrak{R}$ by Theorem 1.5.5. Now Lemma 3.4.1 (i)–(ii) imply (3.4.8).

Step 3. In order to show that $S' = S \cap \mathfrak{K}^2$ is simple in the Hilbert space $\mathfrak{K}$, assume that there is an orthogonal decomposition $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ and a corresponding orthogonal decomposition $S' = S_1 \,\widehat{\oplus}\, S_2$ such that S_2 is self-adjoint in $\mathfrak{K}_2$. Then $\operatorname{ran}(S_2 - \lambda) = \mathfrak{K}_2$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and thus

$$\mathfrak{K}_2 = \operatorname{ran}(S_2 - \lambda) \subset \operatorname{ran}(S' - \lambda) \subset \operatorname{ran}(S - \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

According to (3.4.5), this implies $\mathfrak{K}_2 \subset \mathfrak{R}$ while $\mathfrak{K}_2 \subset \mathfrak{K} = \mathfrak{R}^{\perp}$. Thus, $\mathfrak{K}_2 = \{0\}$, so that S' is simple. $\qquad\square$

Corollary 3.4.5. *Let S be a closed symmetric relation in $\mathfrak{H}$. Then S is simple if and only if*

$$\mathfrak{H} = \overline{\operatorname{span}}\left\{ \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}. \tag{3.4.11}$$

The set $\mathbb{C} \setminus \mathbb{R}$ on the right-hand side can be replaced by any set $\mathfrak{U}$ which has an accumulation point in $\mathbb{C}^+$ and in $\mathbb{C}^-$.

Proof. It follows from Theorem 3.4.4 and the definition of $\mathfrak{K}$ in (3.4.6) that the equality (3.4.11) holds if and only if S is simple. The last assertion in the corollary follows from Lemma 1.6.11.   □

Corollary 3.4.6. *Let S be a closed symmetric relation in $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{H} = \overline{\operatorname{span}} \{ \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \} \oplus \operatorname{mul} S$;

(ii) S_{op} *is a closed simple symmetric operator in $\mathfrak{H}_{\mathrm{op}} = \mathfrak{H} \ominus \operatorname{mul} S$.*

The set $\mathbb{C} \setminus \mathbb{R}$ on the right-hand side in (i) *can be replaced by any set $\mathfrak{U}$ which has an accumulation point in $\mathbb{C}^+$ and in $\mathbb{C}^-$.*

Proof. (i) $\Rightarrow$ (ii) The assumption implies in the context of Theorem 3.4.4 that $\mathfrak{R} = \operatorname{mul} S$, so that

$$S'' = S \cap \mathfrak{R}^2 = \{0\} \times \operatorname{mul} S,$$

which is a self-adjoint relation in $\operatorname{mul} S$. Hence, by the decomposition $S = S' \,\widehat{\oplus}\, S''$ in Theorem 3.4.4 it follows that $S' = S_{\mathrm{op}}$ in $\mathfrak{H}_{\mathrm{op}} = \mathfrak{H} \ominus \operatorname{mul} S$.

(ii) $\Rightarrow$ (i) Recall that $\mathfrak{H} = \mathfrak{H}_{\mathrm{op}} \oplus \mathfrak{H}_{\mathrm{mul}}$. By Corollary 3.4.5 one has

$$\mathfrak{H}_{\mathrm{op}} = \overline{\operatorname{span}} \{ \mathfrak{N}_\lambda(S_{\mathrm{op}}^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \}.$$

From the decomposition $S = S_{\mathrm{op}} \,\widehat{\oplus}\, (\{0\} \times \operatorname{mul} S)$ and Proposition 1.3.13 one concludes that $S^* = S_{\mathrm{op}}^* \,\widehat{\oplus}\, (\{0\} \times \operatorname{mul} S)$. Hence, $\mathfrak{N}_\lambda(S^*) = \mathfrak{N}_\lambda(S_{\mathrm{op}}^*)$, which yields (i).   □

Lemma 3.4.7. *Let S be a closed simple symmetric relation in $\mathfrak{H}$. Then S is an operator and it has no eigenvalues.*

Proof. Indeed, it follows from Definition 3.4.3 that also $S - x$ and $(S - x)^{-1}$, $x \in \mathbb{R}$, are closed simple symmetric relations in $\mathfrak{H}$. In particular, $(S - x)^{-1}$ is an operator; cf. the discussion following Definition 3.4.3. This implies $\ker (S - x) = \{0\}$ for all $x \in \mathbb{R}$ and hence S has no eigenvalues.   □

In certain situations the assertion in Lemma 3.4.7 has a converse.

Proposition 3.4.8. *Let S be a closed symmetric relation in $\mathfrak{H}$ and assume that there exists a self-adjoint extension A of S in $\mathfrak{H}$ such that $\sigma(A) = \sigma_{\mathrm{p}}(A)$. If $\sigma_{\mathrm{p}}(S) = \emptyset$, then the operator part S_{op} of S is a closed simple symmetric operator in the Hilbert space $\mathfrak{H}_{\mathrm{op}} = (\operatorname{mul} S)^\perp$.*

Proof. By Lemma 3.4.2 and Theorem 1.4.11, it suffices to consider the case where S is a closed symmetric operator and A is a self-adjoint extension of S. Now assume

that S is not simple, so that by Theorem 3.4.4 there are nontrivial decompositions $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ and $S = S' \,\widehat{\oplus}\, S''$ with S' closed, simple and symmetric in $\mathfrak{H}'$, and S'' self-adjoint in $\mathfrak{H}''$. Then A decomposes accordingly as $A = A' \,\widehat{\oplus}\, S''$ with A' self-adjoint in $\mathfrak{H}'$ by Lemma 3.4.2. Now $\sigma(A) = \sigma_{\mathrm{p}}(A)$ implies that S'' and thus S has a nontrivial point spectrum, which gives a contradiction. $\qquad\square$

The notion of simplicity of a closed symmetric relation S in $\mathfrak{H}$ will now be specified in a local sense. This will be done relative to a Borel set $\Delta \subset \mathbb{R}$ and by means of a self-adjoint extension A of S and its spectral measure $E(\cdot)$. Then $\mathfrak{H}$ admits the orthogonal decomposition

$$\mathfrak{H} = E(\Delta)\mathfrak{H} \oplus (I - E(\Delta))\mathfrak{H},$$

which leads to the orthogonal componentwise decomposition of A into self-adjoint components:

$$A = \left[A \cap \left(E(\Delta)\mathfrak{H}\right)^2\right] \,\widehat{\oplus}\, \left[A \cap \left((I - E(\Delta))\mathfrak{H}\right)^2\right].$$

Note that $A \cap (E(\Delta)\mathfrak{H})^2$ is a self-adjoint operator in $E(\Delta)\mathfrak{H}$ which coincides with $A_{\mathrm{op}} \upharpoonright E_{\mathrm{op}}(\Delta)\mathfrak{H}_{\mathrm{op}}$; cf. Section 1.5.

Definition 3.4.9. Let S be a closed symmetric relation in $\mathfrak{H}$ and let A be a self-adjoint extension of S with spectral measure $E(\cdot)$. Let $\Delta \subset \mathbb{R}$ be a Borel set. Then S is said to be *simple with respect to* $\Delta \subset \mathbb{R}$ and the self-adjoint extension A if

$$E(\Delta)\mathfrak{H} = \overline{\mathrm{span}}\left\{E(\Delta)k : k \in \mathfrak{N}_\lambda(S^*),\ \lambda \in \mathbb{C} \setminus \mathbb{R}\right\}. \tag{3.4.12}$$

In the next proposition this local notion and some of its consequences are discussed.

Proposition 3.4.10. *Let S be a closed symmetric relation in $\mathfrak{H}$ and let A be a self-adjoint extension of S with spectral measure $E(\cdot)$. Assume that S is simple with respect to the Borel set $\Delta \subset \mathbb{R}$ and the self-adjoint extension A. Then the following statements hold:*

(i) *For every Borel set $\Delta' \subset \Delta$ one has*

$$E(\Delta')\mathfrak{H} = \overline{\mathrm{span}}\left\{E(\Delta')k : k \in \mathfrak{N}_\lambda(S^*),\ \lambda \in \mathbb{C} \setminus \mathbb{R}\right\}. \tag{3.4.13}$$

(ii) *There is no point spectrum of S in Δ:*

$$\Delta \cap \sigma_{\mathrm{p}}(S) = \emptyset.$$

(iii) *If $\mathfrak{U}$ is a subset of $\rho(A)$ with an accumulation point in each connected component of $\rho(A)$, then*

$$E(\Delta)\mathfrak{H} = \overline{\mathrm{span}}\left\{E(\Delta)k : k \in \mathfrak{N}_\lambda(S^*),\ \lambda \in \mathfrak{U}\right\}. \tag{3.4.14}$$

Proof. (i) First note that the inclusion ($\supset$) in (3.4.13) holds. To see the converse inclusion, let $f \in E(\Delta')\mathfrak{H}$. As $\Delta' \subset \Delta$, one has

$$E(\Delta')\mathfrak{H} \subset E(\Delta)\mathfrak{H},$$

and hence $f \in E(\Delta)\mathfrak{H}$. By assumption, the identity (3.4.12) holds and so, in the linear span of

$$\big\{ E(\Delta)k : k \in \mathfrak{N}_\lambda(S^*), \, \lambda \in \mathbb{C} \setminus \mathbb{R} \big\}$$

there exists a sequence (f_n), that converges to f. Then $(E(\Delta')f_n)$ is a sequence in the linear span of

$$\big\{ E(\Delta')k : k \in \mathfrak{N}_\lambda(S^*), \, \lambda \in \mathbb{C} \setminus \mathbb{R} \big\}$$

which converges to $E(\Delta')f = f$. This shows the inclusion ($\subset$) in (3.4.13).

(ii) Assume that $\{f, xf\} \in S$ for some $x \in \Delta$. Since $S \subset A$, it follows that $f \in E(\Delta)\mathfrak{H}$. Observe that for $k \in \mathfrak{N}_\lambda(S^*)$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has $\{k, \lambda k\} \in S^*$ and hence $(\lambda k, f) = (k, xf)$. As $x \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, it follows that $(k, f) = 0$. Further, since $f \in E(\Delta)\mathfrak{H}$, one concludes that

$$0 = (k, f) = (k, E(\Delta)f) = (E(\Delta)k, f)$$

for all $k \in \mathfrak{N}_\lambda(S^*)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence, (3.4.12) implies that $f \in E(\Delta)\mathfrak{H}$ is orthogonal to $E(\Delta)\mathfrak{H}$, which shows that $f = 0$. Thus, S does not possess any eigenvalues in Δ.

(iii) The inclusion ($\supset$) in (3.4.14) is clear. In order to prove the identity, fix $\mu \in \mathcal{U}$ and recall from Lemma 1.4.10 that the operator $I + (\lambda - \mu)(A - \lambda)^{-1}$ maps $\mathfrak{N}_\mu(S^*)$ bijectively onto $\mathfrak{N}_\lambda(S^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It suffices to verify that the vectors $E(\Delta)k$, $k \in \mathfrak{N}_\lambda(S^*)$, $\lambda \in \mathcal{U}$, span a dense set in $E(\Delta)\mathfrak{H}$. Suppose that $E(\Delta)f$ is orthogonal to this set, that is,

$$0 = \big(E(\Delta)(I + (\lambda - \mu)(A - \lambda)^{-1})g_\mu, E(\Delta)f \big) \qquad (3.4.15)$$

for all $g_\mu \in \mathfrak{N}_\mu(S^*)$ and all $\lambda \in \mathcal{U}$. Since for each $g_\mu \in \mathfrak{N}_\mu(S^*)$ the function

$$\lambda \mapsto \big(E(\Delta)(I + (\lambda - \mu)(A - \lambda)^{-1})g_\mu, E(\Delta)f \big)$$

is analytic on $\rho(A)$, it follows from (3.4.15) and the assumption that $\mathcal{U}$ has an accumulation point in each connected component of $\rho(A)$ that this function is identically equal to zero. Hence, $(E(\Delta)k, E(\Delta)f) = 0$ for all $k \in \mathfrak{N}_\lambda(S^*)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now (3.4.12) yields $E(\Delta)f = 0$ and (iii) follows. $\square$

The connection with the global notion of simplicity is given in the following corollary.

Corollary 3.4.11. *Let S be a closed symmetric relation in $\mathfrak{H}$ and let A be a self-adjoint extension of S with spectral measure $E(\cdot)$. Then S is simple if and only if S is simple with respect to any Borel set $\Delta \subset \mathbb{R}$ and the self-adjoint extension A.*

Proof. Assume that S is simple. Then (3.4.11) holds and hence (3.4.12) holds with $\Delta = \mathbb{R}$. Then Proposition 3.4.10 (i) implies that S is simple with respect to any Borel set $\Delta \subset \mathbb{R}$ and the self-adjoint extension A. Conversely, if (3.4.12) holds for any Borel set $\Delta \subset \mathbb{R}$, then (3.4.12) also holds for $\Delta = \mathbb{R}$, and hence reduces to (3.4.11), that is, S is simple. $\square$

In the following lemma the eigenspace of A corresponding to an eigenvalue x is described in the case where x is not an eigenvalue of S. In particular, this observation leads to a characterization of local simplicity if the Borel set $\Delta \subset \mathbb{R}$ in Definition 3.4.9 is a singleton; cf. Corollary 3.4.13.

Lemma 3.4.12. *Let S be a closed symmetric relation in $\mathfrak{H}$, let A be a self-adjoint extension of S with spectral measure $E(\cdot)$, and let $x \in \mathbb{R}$. Then*

$$E(\{x\})\mathfrak{H} = E(\{x\})\mathfrak{N}_\lambda(S^*) \tag{3.4.16}$$

for some, and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, if and only if $x \notin \sigma_p(S)$.

Proof. Assume first that (3.4.16) holds for some fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Assume that $\{f, xf\} \in S$, which implies $\{f, xf\} \in A$ and hence $f \in E(\{x\})\mathfrak{H}$. Moreover, one has $(xf, k_\lambda) = (f, \lambda k_\lambda)$ for all $k_\lambda \in \mathfrak{N}_\lambda(S^*)$ as $\{k_\lambda, \lambda k_\lambda\} \in S^*$. It follows that

$$\big(xf, E(\{x\})k_\lambda\big) = (xf, k_\lambda) = (f, \lambda k_\lambda) = \big(f, \lambda E(\{x\})k_\lambda\big)$$

and hence $(f, E(\{x\})k_\lambda) = 0$ for all $k_\lambda \in \mathfrak{N}_\lambda(S^*)$. Now (3.4.16) and $f \in E(\{x\})\mathfrak{H}$ yield $f = 0$, which implies $x \notin \sigma_p(S)$.

Conversely, assume that $x \notin \sigma_p(S)$ and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The inclusion ($\supset$) in (3.4.16) is clear and since both subspaces in (3.4.16) are closed, it suffices to verify that $E(\{x\})\mathfrak{N}_\lambda(S^*)$ is dense in $E(\{x\})\mathfrak{H}$. Suppose that there exists $f \in E(\{x\})\mathfrak{H}$ such that

$$\big(f, E(\{x\})k_\lambda\big) = 0, \qquad k_\lambda \in \mathfrak{N}_\lambda(S^*).$$

As $f \in E(\{x\})\mathfrak{H}$, this implies $(f, k_\lambda) = 0$ and hence $f \in \operatorname{ran}(S - \bar\lambda)$. Choose $\{g, g'\} \in S$ such that $g' - \bar\lambda g = f$. Then

$$g = (S - \bar\lambda)^{-1}f = (A - \bar\lambda)^{-1}f = \frac{1}{x - \bar\lambda}f$$

and

$$g' = f + \bar\lambda g = f + \frac{\bar\lambda}{x - \bar\lambda}f = \frac{x}{x - \bar\lambda}f,$$

and it follows that $\{f, xf\} \in S$. Since $x \notin \sigma_p(S)$ by assumption this yields $f = 0$. Hence, $E(\{x\})\mathfrak{N}_\lambda(S^*)$ is dense in $E(\{x\})\mathfrak{H}$ and therefore (3.4.16) holds. $\square$

The above lemma together with Proposition 3.4.10 (ii) implies that S is simple with respect to a point $x \in \mathbb{R}$ if and only if x is not an eigenvalue of S.

Corollary 3.4.13. *Let S be a closed symmetric relation in $\mathfrak{H}$, let A be a self-adjoint extension of S with spectral measure $E(\cdot)$, and let $x \in \mathbb{R}$. Then*

$$E(\{x\})\mathfrak{H} = \overline{\operatorname{span}}\big\{E(\{x\})k : k \in \mathfrak{N}_\lambda(S^*),\, \lambda \in \mathbb{C} \setminus \mathbb{R}\big\}$$

holds if and only if $x \notin \sigma_p(S)$.

3.5   Eigenvalues and eigenspaces

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$, and corresponding γ-field γ and Weyl function M. The purpose of the present section is to characterize eigenvalues and the associated eigenspaces of the self-adjoint relation A_0 by means of the corresponding Weyl function M.

Recall that the Weyl function M can be expressed in terms of the γ-field and the resolvent of the self-adjoint relation A_0; cf. Proposition 2.3.6 (v). In particular, for $\lambda = x + iy$, $y > 0$, and $\lambda_0 \in \rho(A_0)$ one has

$$M(x + iy) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^* \big[(x + iy - \operatorname{Re} \lambda_0)$$
$$+ (x + iy - \lambda_0)(x + iy - \bar{\lambda}_0)\big(A_0 - (x + iy)\big)^{-1}\big] \gamma(\lambda_0). \tag{3.5.1}$$

This formula will be used to study the behavior of the Weyl function M at a point $x \in \mathbb{R}$. In the next proposition it turns out that the strong limit of $iyM(x + iy)$, $y \downarrow 0$, is closely connected with the eigenspace of A_0 at x. Here the spectral measure E of A_0 is not used explicitly in the assertion; the orthogonal projection onto the eigenspace $\mathfrak{N}_x(A_0) = \ker(A_0 - x)$ is denoted by $P_{\mathfrak{N}_x(A_0)}$ instead of $E(\{x\})$.

Proposition 3.5.1. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, let M and γ be the corresponding Weyl function and γ-field, and let $x \in \mathbb{R}$. Then for each $\lambda_0 \in \rho(A_0)$ and all $\varphi \in \mathcal{G}$ one has*

$$\lim_{y \downarrow 0} iyM(x + iy)\varphi = -|x - \lambda_0|^2 \gamma(\lambda_0)^* P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0)\varphi. \tag{3.5.2}$$

Proof. For $x \in \mathbb{R}$ and $\lambda_0 \in \rho(A_0)$, it follows from (3.5.1) that

$$iyM(x + iy) = iy \operatorname{Re} M(\lambda_0) + iy\gamma(\lambda_0)^*(x + iy - \operatorname{Re} \lambda_0)\gamma(\lambda_0)$$
$$+ iy\,\gamma(\lambda_0)^*(x + iy - \lambda_0)(x + iy - \bar{\lambda}_0)\big(A_0 - (x + iy)\big)^{-1}\gamma(\lambda_0).$$

As the first and second terms on the right-hand side tend to 0 as $y \downarrow 0$, one obtains

$$\lim_{y \downarrow 0} iyM(x + iy)\varphi = |x - \lambda_0|^2 \gamma(\lambda_0)^* \big[\lim_{y \downarrow 0} iy\big(A_0 - (x + iy)\big)^{-1}\big]\gamma(\lambda_0)\varphi \tag{3.5.3}$$

for all $\varphi \in \mathcal{G}$. Since $x \in \mathbb{R}$ is fixed and $y \downarrow 0$, one has that

$$\frac{iy}{t - (x + iy)} \to -\mathbf{1}_x(t), \quad t \in \mathbb{R},$$

where the approximating functions are uniformly bounded by 1. The spectral calculus for the self-adjoint relation A_0 in Lemma 1.5.3 yields

$$\lim_{y \downarrow 0} iy\big(A_0 - (x + iy)\big)^{-1}\gamma(\lambda_0)\varphi = -P_{\mathfrak{N}_x(A_0)}\gamma(\lambda_0)\varphi, \quad \varphi \in \mathcal{G}.$$

Now the assertion follows from (3.5.3). $\qquad\square$

Definition 3.5.2. Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M be the corresponding Weyl function. For $x \in \mathbb{R}$ the operator $\mathcal{R}_x : \mathcal{G} \to \mathcal{G}$ is defined as the strong limit

$$\mathcal{R}_x \varphi = \lim_{y \downarrow 0} iy M(x + iy)\varphi, \qquad \varphi \in \mathcal{G}.$$

Observe that $\mathcal{R}_x$ in Definition 3.5.2 is a well-defined operator in $\mathbf{B}(\mathcal{G})$; indeed, this is clear from the identity (3.5.2). It also follows from (3.5.2) that $\mathcal{R}_x = 0$ when $x \in \rho(A_0) \cap \mathbb{R}$.

Remark 3.5.3. If $x \in \mathbb{R}$ is an isolated singularity of the function M, then x is a pole of first order of M; cf. Corollary 2.3.9. Moreover, in a sufficiently small punctured disc $B_x \setminus \{x\}$ centered at x such that M is holomorphic in $\overline{B}_x \setminus \{x\}$, one has a norm convergent Laurent series expansion of the form

$$M(\lambda) = \frac{M_{-1}}{\lambda - x} + \sum_{k=0}^{\infty} M_k (\lambda - x)^k, \quad M_{-1}, M_0, M_1, \ldots \in \mathbf{B}(\mathcal{G}).$$

It follows that $\mathcal{R}_x$ coincides with the residue of M at x, i.e.,

$$\mathcal{R}_x = \frac{1}{2\pi i} \int_{\mathcal{C}} M(\lambda) \, d\lambda = M_{-1},$$

where $\mathcal{C}$ denotes the boundary of B_x.

In the following let $x \in \mathbb{R}$ and recall that the corresponding eigenspaces of S and A_0 are given by

$$\widehat{\mathfrak{N}}_x(S) = \{\{f, xf\} : f \in \mathfrak{N}_x(S)\}, \quad \mathfrak{N}_x(S) = \ker(S - x),$$

and

$$\widehat{\mathfrak{N}}_x(A_0) = \{\{f, xf\} : f \in \mathfrak{N}_x(A_0)\}, \quad \mathfrak{N}_x(A_0) = \ker(A_0 - x).$$

The main interest will be in the closed linear subspace $\widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S)$, which is the orthogonal complement of $\widehat{\mathfrak{N}}_x(S)$ in $\widehat{\mathfrak{N}}_x(A_0)$. Similarly, the orthogonal complement of $\mathfrak{N}_x(S)$ in $\mathfrak{N}_x(A_0)$ is denoted by $\mathfrak{N}_x(A_0) \ominus \mathfrak{N}_x(S)$.

Lemma 3.5.4. *Let $\lambda_0 \in \rho(A_0)$, let $x \in \mathbb{R}$, and let P_x be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_x(A_0) \ominus \mathfrak{N}_x(S)$. Then the operator $\mathcal{R}_x$ has the representation*

$$\mathcal{R}_x \varphi = (\lambda_0 - x)\Gamma_1 \{P_x \gamma(\lambda_0)\varphi, x P_x \gamma(\lambda_0)\varphi\}, \quad \varphi \in \mathcal{G}. \qquad (3.5.4)$$

Proof. First, recall from Corollary 2.3.3 that for $x \in \mathbb{R}$ and $\{h, xh\} \in A_0$ one has

$$\Gamma_1 \{h, xh\} = (x - \overline{\lambda}_0)\gamma(\lambda_0)^* h, \quad \lambda \in \rho(A_0).$$

Now let $\varphi \in \mathcal{G}$ and consider

$$h = (\lambda_0 - x) P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0)\varphi \in \ker(A_0 - x).$$

According to Proposition 3.5.1 and Definition 3.5.2,

$$
\begin{aligned}
\mathcal{R}_x \varphi &= -|x - \lambda_0|^2 \gamma(\lambda_0)^* P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0) \varphi \\
&= (x - \bar{\lambda}_0) \gamma(\lambda_0)^* h \\
&= \Gamma_1 \{h, xh\} \\
&= (\lambda_0 - x) \Gamma_1 \{ P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0) \varphi, x P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0) \varphi \}.
\end{aligned}
\tag{3.5.5}
$$

Now observe that for $\varphi \in \mathcal{G}$

$$
P_{\mathfrak{N}_x(A_0)} \gamma(\lambda_0) \varphi = P_x \gamma(\lambda_0) \varphi + P_{\mathfrak{N}_x(S)} \gamma(\lambda_0) \varphi.
$$

Since $\{P_{\mathfrak{N}_x(S)} \gamma(\lambda_0) \varphi, x P_{\mathfrak{N}_x(S)} \gamma(\lambda_0) \varphi\} \in S$ and $S = \ker \Gamma_0 \cap \ker \Gamma_1$ by Proposition 2.1.2 (ii), it follows that

$$
\Gamma_1 \{ P_{\mathfrak{N}_x(S)} \gamma(\lambda_0) \varphi, x P_{\mathfrak{N}_x(S)} \gamma(\lambda_0) \varphi \} = 0
$$

and hence (3.5.5) leads to (3.5.4). □

In the following theorem the eigenvalue $x \in \mathbb{R}$ and the corresponding eigenspace of A_0 are characterized by means of the Weyl function M and the operator $\mathcal{R}_x$. Later it will be shown how to distinguish between isolated and embedded eigenvalues of A_0; cf. Theorem 3.6.1.

Theorem 3.5.5. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, let M and γ be the corresponding Weyl function and γ-field, and let $x \in \mathbb{R}$. Then the mapping*

$$
\tau : \widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S) \to \overline{\mathrm{ran}}\, \mathcal{R}_x, \quad \widehat{f} \mapsto \Gamma_1 \widehat{f},
\tag{3.5.6}
$$

is an isomorphism. In particular,

$$
x \in \sigma_{\mathrm{p}}(A_0) \text{ and } \widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S) \neq \{0\} \quad \Leftrightarrow \quad \mathcal{R}_x \neq 0.
$$

Proof. Let $x \in \mathbb{R}$ and define $\mathfrak{K}_x = \widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S)$. The mapping $\Gamma_1 : S^* \to \mathcal{G}$ is continuous and, in particular, its restriction to $\mathfrak{K}_x \subset S^*$ is continuous. The proof consists of three steps. In Step 1 it will be shown that the restriction of Γ_1 to $\mathfrak{K}_x$ is injective and in Step 2 it will be shown that it has closed range. Then it follows from Step 3 that τ in (3.5.6) is an isomorphism.

Step 1. The restriction of the mapping Γ_1 to $\mathfrak{K}_x$ is injective. Indeed, let $\widehat{f} \in \mathfrak{K}_x$ with $\Gamma_1 \widehat{f} = 0$. The assumption $\widehat{f} \in \mathfrak{K}_x$ implies that $\widehat{f} \in A_0$ and hence $\Gamma_0 \widehat{f} = 0$. Therefore, $\widehat{f} \in \ker \Gamma_0 \cap \ker \Gamma_1 = S$. Since $\widehat{f} = \{f, xf\} \in \widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S)$, this implies $\widehat{f} = 0$.

Step 2. The range of the restriction of Γ_1 to $\mathfrak{K}_x$ is closed. In fact, let (φ_n) be a sequence in $\mathrm{ran}\,(\Gamma_1 \upharpoonright \mathfrak{K}_x)$ such that $\varphi_n \to \varphi \in \mathcal{G}$. Then there exists a sequence

$(\widehat{f}_n)$ in $\mathfrak{K}_x$ such that $\Gamma_1 \widehat{f}_n = \varphi_n$ and as $\widehat{f}_n \in A_0$ one has $\Gamma_0 \widehat{f}_n = 0$. Therefore, $\Gamma \widehat{f}_n = \{0, \varphi_n\} \to \{0, \varphi\}$. Recall from Proposition 2.1.2 that the restriction of Γ to $S^* \ominus S$ is an isomorphism onto $\mathcal{G} \times \mathcal{G}$. It follows that $\widehat{f}_n$ converge to some element $\widehat{f}$, which belongs to the closed subspace $\mathfrak{K}_x$. This yields $\Gamma_1 \widehat{f} = \varphi$ and hence $\operatorname{ran}(\Gamma_1 \upharpoonright \mathfrak{K}_x)$ is closed.

Step 3. The linear space

$$\big\{\{P_x \gamma(\lambda_0)\varphi, x P_x \gamma(\lambda_0)\varphi\} : \varphi \in \mathcal{G}\big\}$$

is dense in the Hilbert space $\mathfrak{K}_x = \widehat{\mathfrak{N}}_x(A_0) \ominus \widehat{\mathfrak{N}}_x(S)$. To see this, let $\widehat{f} \in \mathfrak{K}_x$ be orthogonal to all $\{P_x \gamma(\lambda_0)\varphi, x P_x \gamma(\lambda_0)\varphi\}$, $\varphi \in \mathcal{G}$. Then, since $\widehat{f} = \{f, xf\}$, Corollary 2.3.3 shows that for all $\varphi \in \mathcal{G}$ one has

$$\begin{aligned}
0 &= \big(\widehat{f}, \{P_x \gamma(\lambda_0)\varphi, x P_x \gamma(\lambda_0)\varphi\}\big) \\
&= (f, P_x \gamma(\lambda_0)\varphi) + (xf, x P_x \gamma(\lambda_0)\varphi) \\
&= (1 + x^2)(f, \gamma(\lambda_0)\varphi) \\
&= (1 + x^2)(\gamma(\lambda_0)^* f, \varphi) \\
&= (1 + x^2)(x - \bar{\lambda}_0)^{-1}(\Gamma_1 \widehat{f}, \varphi),
\end{aligned}$$

so that $\Gamma_1 \widehat{f} = 0$, and hence $\widehat{f} = 0$ by Step 1.

Step 4. The mapping in (3.5.6) is an isomorphism. To see this, observe that

$$\operatorname{ran} \mathcal{R}_x \subset \operatorname{ran}(\Gamma_1 \upharpoonright \mathfrak{K}_x) \subset \overline{\operatorname{ran}} \, \mathcal{R}_x. \tag{3.5.7}$$

The first inclusion in (3.5.7) follows directly from (3.5.4). From the same identity one also sees that

$$\Gamma_1 \big\{P_x \gamma(\lambda_0)\varphi, x P_x \gamma(\lambda_0)\varphi\big\} = \frac{1}{\lambda_0 - x} \, \mathcal{R}_x \varphi \in \operatorname{ran} \mathcal{R}_x \subset \overline{\operatorname{ran}} \, \mathcal{R}_x.$$

Hence, the second inclusion in (3.5.7) follows from Step 3 and the boundedness of Γ_1. It is clear from (3.5.7) and Step 2 that

$$\operatorname{ran}(\Gamma_1 \upharpoonright \mathfrak{K}_x) = \overline{\operatorname{ran}} \, \mathcal{R}_x,$$

and hence, due to Step 1, the mapping in (3.5.6) is an isomorphism. □

The statement of Theorem 3.5.5 can be simplified if x is not an eigenvalue of the symmetric relation S, that is, S satisfies a local simplicity condition at $x \in \mathbb{R}$; cf. Corollary 3.4.13.

Corollary 3.5.6. *Assume that x is not an eigenvalue of the closed symmetric relation S in Theorem 3.5.5. Then*

$$x \in \sigma_{\mathrm{p}}(A_0) \quad \Leftrightarrow \quad \mathcal{R}_x \neq 0.$$

Now the behavior of M at ∞ will be considered and the multivalued part of A_0 will be described. First recall that the self-adjoint relation A_0 is decomposed into the orthogonal sum

$$A_0 = A_{0,\mathrm{op}} \,\widehat{\oplus}\, A_{0,\mathrm{mul}}, \tag{3.5.8}$$

where $A_{0,\mathrm{op}}$ is a self-adjoint operator in the Hilbert space

$$\mathfrak{H}_{\mathrm{op}} = (\mathrm{mul}\, A_0)^{\perp} = \overline{\mathrm{dom}}\, A_0 \tag{3.5.9}$$

and $A_{0,\mathrm{mul}}$ is the purely multivalued self-adjoint relation in $\mathfrak{H}_{\mathrm{mul}} = \mathrm{mul}\, A_0$. Then the resolvent of A_0 has the form

$$(A_0 - \lambda)^{-1} = \begin{pmatrix} (A_{0,\mathrm{op}} - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda \in \rho(A_0), \tag{3.5.10}$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_{\mathrm{op}} \oplus \mathfrak{H}_{\mathrm{mul}}$; cf. (1.5.1).

The representation (3.5.1) of M in terms of A_0 gives for $\lambda_0 \in \rho(A_0)$ and $x = 0$ that

$$\begin{aligned}
M(iy) &= \operatorname{Re} M(\lambda_0) \\
&\quad + \gamma(\lambda_0)^* \big[iy - \operatorname{Re} \lambda_0 + (iy - \lambda_0)(iy - \bar{\lambda}_0)(A_0 - iy)^{-1} \big] \gamma(\lambda_0).
\end{aligned} \tag{3.5.11}$$

In order to use this formula for large y decompose the term $\gamma(\lambda_0)^* \gamma(\lambda_0)$ as

$$\gamma(\lambda_0)^* \gamma(\lambda_0) = \gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0) + \gamma(\lambda_0)^* \iota_{\mathrm{op}} P_{\mathrm{op}} \gamma(\lambda_0),$$

where P_{op} denotes the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{\mathrm{op}}$, ι_{op} is the canonical embedding of $\mathfrak{H}_{\mathrm{op}}$ into $\mathfrak{H}$, and $I - P_{\mathrm{op}}$ is viewed as an orthogonal projection in $\mathfrak{H}$. From the representation of the resolvent of A_0 in terms of the resolvent of $A_{0,\mathrm{op}}$ in (3.5.10) it follows that (3.5.11) may be rewritten as

$$\begin{aligned}
M(iy) &= \operatorname{Re} M(\lambda_0) + (iy - \operatorname{Re}\lambda_0)\, \gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0) \\
&\quad + \gamma(\lambda_0)^* \iota_{\mathrm{op}} \big[iy - \operatorname{Re}\lambda_0 + (iy - \lambda_0)(iy - \bar{\lambda}_0)(A_{0,\mathrm{op}} - iy)^{-1} \big] P_{\mathrm{op}} \gamma(\lambda_0)
\end{aligned} \tag{3.5.12}$$

for all $y > 0$. This formula will be used to study the behavior of M at ∞. It turns out that the strong limit $\frac{1}{iy} M(iy)$, $y \to +\infty$, is closely connected with the multivalued part of A_0.

Proposition 3.5.7. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M and γ be the corresponding Weyl function and γ-field. Then for each $\lambda_0 \in \rho(A_0)$ and $\varphi \in \mathcal{G}$ one has*

$$\lim_{y \to +\infty} \frac{1}{iy} M(iy)\varphi = \gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0)\varphi. \tag{3.5.13}$$

Proof. It follows from (3.5.12) with $\lambda_0 \in \rho(A_0)$ that

$$\frac{1}{iy}M(iy) = \frac{1}{iy}\mathrm{Re}\,M(\lambda_0) + \frac{iy - \mathrm{Re}\,\lambda_0}{iy}\gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0)$$

$$+ \frac{1}{iy}\gamma(\lambda_0)^*\iota_{\mathrm{op}}\big[iy - \mathrm{Re}\,\lambda_0 + (iy - \lambda_0)(iy - \bar{\lambda}_0)(A_{0,\mathrm{op}} - iy)^{-1}\big]P_{\mathrm{op}}\gamma(\lambda_0).$$

It suffices to show that the first and the third term on the right-hand side converge to 0 strongly. This is obvious for the first term on the right-hand side. For the third term note that for $y \to +\infty$ one has

$$\frac{iy - \mathrm{Re}\,\lambda_0}{iy} + \frac{(iy - \lambda_0)(iy - \bar{\lambda}_0)}{iy}\frac{1}{t - iy} \to 0, \qquad t \in \mathbb{R},$$

and hence the spectral calculus for $A_{0,\mathrm{op}}$ shows that for $y \to +\infty$

$$\frac{1}{iy}\gamma(\lambda_0)^*\iota_{\mathrm{op}}\big[iy - \mathrm{Re}\,\lambda_0 + (iy - \lambda_0)(iy - \bar{\lambda}_0)(A_{0,\mathrm{op}} - iy)^{-1}\big]P_{\mathrm{op}}\gamma(\lambda_0)\varphi$$

tends to zero for all $\varphi \in \mathcal{G}$; cf. Lemma 1.5.3. This leads to (3.5.13). $\qquad\square$

Definition 3.5.8. Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker\Gamma_0$, and let M be the corresponding Weyl function. The operator $\mathcal{R}_\infty : \mathcal{G} \to \mathcal{G}$ is defined as the strong limit

$$\mathcal{R}_\infty\varphi = \lim_{y \to +\infty}\frac{1}{iy}M(iy)\varphi, \qquad \varphi \in \mathcal{G}.$$

It follows from Proposition 3.5.7 that $\mathcal{R}_\infty \in \mathbf{B}(\mathcal{G})$. For the following properties of $\mathcal{R}_\infty$ recall the notations

$$\widehat{\mathfrak{N}}_\infty(S) = \big\{\{0, f\} : f \in \mathfrak{N}_\infty(S)\big\}, \quad \mathfrak{N}_\infty(S) = \mathrm{mul}\,S,$$

and

$$\widehat{\mathfrak{N}}_\infty(A_0) = \big\{\{0, f\} : f \in \mathfrak{N}_\infty(A_0)\big\}, \quad \mathfrak{N}_\infty(A_0) = \mathrm{mul}\,A_0.$$

The next lemma can be viewed as a variant of Lemma 3.5.4 for $x = \infty$. Here the main interest is in the closed subspace $\widehat{\mathfrak{N}}_\infty(A_0) \ominus \widehat{\mathfrak{N}}_\infty(S)$, that is, the orthogonal complement of $\widehat{\mathfrak{N}}_\infty(S)$ in $\widehat{\mathfrak{N}}_\infty(A_0)$.

Lemma 3.5.9. *Let $\lambda_0 \in \rho(A_0)$ and let P_∞ be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_\infty(A_0) \ominus \mathfrak{N}_\infty(S)$. Then the operator $\mathcal{R}_\infty$ has the representation*

$$\mathcal{R}_\infty\varphi = \Gamma_1\big\{0, P_\infty\gamma(\lambda_0)\varphi\big\}, \qquad \varphi \in \mathcal{G}. \tag{3.5.14}$$

Proof. First recall from Corollary 2.3.3 that for $\{0, h'\} \in A_0$ one has

$$\Gamma_1\{0, h'\} = \gamma(\lambda_0)^*h', \qquad \lambda_0 \in \rho(A_0).$$

Now let $\varphi \in \mathcal{G}$ and consider $h' = (I - P_{\mathrm{op}})\gamma(\lambda_0)\varphi \in \mathrm{mul}\, A_0$. According to Proposition 3.5.7,

$$\begin{aligned}
\mathcal{R}_\infty \varphi &= \gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0)\varphi = \gamma(\lambda_0)^* h' = \Gamma_1\{0, h'\} \\
&= \Gamma_1\{0, (I - P_{\mathrm{op}})\gamma(\lambda_0)\varphi\}.
\end{aligned} \tag{3.5.15}$$

Now observe that for $\varphi \in \mathcal{G}$

$$(I - P_{\mathrm{op}})\gamma(\lambda_0)\varphi = P_\infty \gamma(\lambda_0)\varphi + P_{\mathfrak{N}_\infty(S)}\gamma(\lambda_0)\varphi.$$

Since $\{0, P_{\mathfrak{N}_\infty(S)}\gamma(\lambda_0)\varphi\} \in S$ and $S = \ker \Gamma_0 \cap \ker \Gamma_1$ by Proposition 2.1.2 (ii), it follows that

$$\Gamma_1\{0, P_{\mathfrak{N}_\infty(S)}\gamma(\lambda_0)\varphi\} = 0$$

and hence (3.5.15) leads to (3.5.14). □

In the next theorem the multivalued part of A_0 is characterized by means of the Weyl function M and the operator $\mathcal{R}_\infty$.

Theorem 3.5.10. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M and γ be the corresponding Weyl function and γ-field. Then the mapping*

$$\tau : \widehat{\mathfrak{N}}_\infty(A_0) \ominus \widehat{\mathfrak{N}}_\infty(S) \to \overline{\mathrm{ran}}\, \mathcal{R}_\infty, \quad \widehat{f} \mapsto \Gamma_1 \widehat{f}, \tag{3.5.16}$$

is an isomorphism. In particular,

$$\mathrm{mul}\, A_0 \ominus \mathrm{mul}\, S \neq \{0\} \quad \Leftrightarrow \quad \mathcal{R}_\infty \neq 0.$$

Proof. The proof follows a strategy similar to the one used in the proof of Theorem 3.5.5. To simplify notation, set

$$\mathfrak{K}_\infty := \widehat{\mathfrak{N}}_\infty(A_0) \ominus \widehat{\mathfrak{N}}_\infty(S) = \big\{\{0, f'\} : f' \in \mathrm{mul}\, A_0 \ominus \mathrm{mul}\, S\big\}.$$

The mapping $\Gamma_1 : S^* \to \mathcal{G}$ is continuous and, in particular, its restriction to $\mathfrak{K}_\infty \subset S^*$ is continuous.

Step 1. The restriction of the mapping Γ_1 to $\mathfrak{K}_\infty$ is injective. Indeed, let $\widehat{f} \in \mathfrak{K}_\infty$ with $\Gamma_1 \widehat{f} = 0$. The assumption $\widehat{f} \in \mathfrak{K}_\infty$ implies that $\widehat{f} \in A_0$ and hence $\Gamma_0 \widehat{f} = 0$. Therefore, $\widehat{f} \in \ker \Gamma_0 \cap \ker \Gamma_1 = S$. Since $\widehat{f} = \{0, f'\} \in \widehat{\mathfrak{N}}_\infty(A_0) \ominus \widehat{\mathfrak{N}}_\infty(S)$, this implies $\widehat{f} = 0$.

Step 2. The range of the restriction of Γ_1 to $\mathfrak{K}_\infty$ is closed. In fact, let (φ_n) be a sequence in $\mathrm{ran}\,(\Gamma_1 \upharpoonright \mathfrak{K}_\infty)$ such that $\varphi_n \to \varphi \in \mathcal{G}$. Then there exist $(\widehat{f}_n)$ in $\mathfrak{K}_\infty$ such that $\Gamma_1 \widehat{f}_n = \varphi_n$ and as $\widehat{f}_n \in A_0$ one has $\Gamma_0 \widehat{f}_n = 0$. Thus, $\Gamma \widehat{f}_n = \{0, \varphi_n\} \to \{0, \varphi\}$, and since the restriction of Γ to $S^* \ominus S$ is an isomorphism onto $\mathcal{G} \times \mathcal{G}$, it follows that $\widehat{f}_n$ converge to some element $\widehat{f}$, which belongs to the closed subspace $\mathfrak{K}_\infty$. Therefore, $\Gamma_1 \widehat{f} = \varphi$ and hence $\mathrm{ran}\,(\Gamma_1 \upharpoonright \mathfrak{K}_\infty)$ is closed.

Step 3. The linear space

$$\big\{\{0, P_\infty \gamma(\lambda_0)\varphi\} : \varphi \in \mathcal{G}\big\}$$

is dense in the Hilbert space $\mathfrak{K}_\infty = \widehat{\mathfrak{N}}_\infty(A_0) \ominus \widehat{\mathfrak{N}}_\infty(S)$. To see this, let $\widehat{f} \in \mathfrak{K}_\infty$ be orthogonal to all elements $\{0, P_\infty \gamma(\lambda_0)\varphi\}$, $\varphi \in \mathcal{G}$. Then it follows from $\widehat{f} = \{0, f'\}$ and Corollary 2.3.3 that for all $\varphi \in \mathcal{G}$ one has

$$0 = \big(\widehat{f}, \{0, P_\infty \gamma(\lambda_0)\varphi\}\big) = (f', P_\infty \gamma(\lambda_0)\varphi) = (\gamma(\lambda_0)^* f', \varphi) = (\Gamma_1 \widehat{f}, \varphi),$$

so that $\Gamma_1 \widehat{f} = 0$, and hence $\widehat{f} = 0$ by Step 1.

Step 4. The mapping in (3.5.16) is an isomorphism. To see this, observe that

$$\operatorname{ran} \mathcal{R}_\infty \subset \operatorname{ran}(\Gamma_1 \upharpoonright \mathfrak{K}_\infty) \subset \overline{\operatorname{ran}} \, \mathcal{R}_\infty. \tag{3.5.17}$$

The first inclusion in (3.5.17) follows from (3.5.14). From the same identity one also sees that

$$\Gamma_1\{0, P_\infty \gamma(\lambda_0)\varphi\} = \mathcal{R}_\infty \varphi \in \operatorname{ran} \mathcal{R}_\infty \subset \overline{\operatorname{ran}} \, \mathcal{R}_\infty.$$

Hence, the second inclusion in (3.5.17) follows from Step 3 and the boundedness of Γ_1. It is clear from (3.5.17) and Step 2 that

$$\operatorname{ran}(\Gamma_1 \upharpoonright \mathfrak{K}_\infty) = \overline{\operatorname{ran}} \, \mathcal{R}_\infty,$$

and hence, due to Step 1, the mapping in (3.5.16) is an isomorphism. $\qquad\square$

Corollary 3.5.11. *Assume that the closed symmetric relation S in Theorem 3.5.10 is an operator. Then A_0 is an operator if and only if $\mathcal{R}_\infty = 0$.*

An equivalent statement is that A_0 is an operator if and only if for all $\varphi \in \mathcal{G}$

$$\lim_{y \to +\infty} \frac{1}{iy} M(iy)\varphi = 0. \tag{3.5.18}$$

3.6 Spectra and local minimality

As in Section 3.5, let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$, and corresponding γ-field γ and Weyl function M. The spectrum of the self-adjoint extension A_0 and its division into absolutely continuous and singular spectra (cf. Section 3.3) will now be discussed in detail in terms of the boundary behavior of M. For this purpose it is assumed that S either is simple or satisfies a local simplicity condition with respect to an open interval $\Delta \subset \mathbb{R}$ and the self-adjoint extension A_0; see Definition 3.4.9 for the notion of local simplicity.

The following theorem describes the point spectrum and the continuous spectrum of A_0 in terms of the boundary behavior of the Weyl function M; cf. Proposition 3.3.1.

Theorem 3.6.1. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with $A_0 = \ker \Gamma_0$, let M and γ be the corresponding Weyl function and γ-field, and let $\mathcal{R}_x = \lim_{y\downarrow 0} iyM(x+iy)$, $x \in \mathbb{R}$, be the operator in Definition 3.5.2. Let $\Delta \subset \mathbb{R}$ be an open interval and assume that the condition*

$$E(\Delta)\mathfrak{H} = \overline{\operatorname{span}} \left\{ E(\Delta)\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R}, \varphi \in \mathcal{G} \right\} \tag{3.6.1}$$

is satisfied, where $E(\cdot)$ is the spectral measure of A_0. Then the following statements hold for each $x \in \Delta$:

- (i) *$x \in \rho(A_0)$ if and only if M can be continued analytically to x;*
- (ii) *$x \in \sigma_c(A_0)$ if and only if $\mathcal{R}_x = 0$ and M cannot be continued analytically to x;*
- (iii) *x is an eigenvalue of A_0 if and only if $\mathcal{R}_x \neq 0$;*
- (iv) *x is an isolated eigenvalue of A_0 if and only if x is a pole (of first order) of M; in this case $\mathcal{R}_x$ is the residue of M at x.*

Proof. (i) Recall first that by Proposition 2.3.6 (iii) or (v) the function $\lambda \mapsto M(\lambda)$ is holomorphic on $\rho(A_0)$, which proves the implication ($\Rightarrow$). In order to verify the other implication assume that M can be continued analytically to some $x \in \Delta$. Then there exists an open neighborhood $\mathcal{O}$ of x in $\mathbb{C}$ with $\mathcal{O} \cap \mathbb{R} \subset \Delta$ to which M can be continued analytically. Choose $a, b \in \mathbb{R}$ with $x \in (a, b)$, $[a, b] \subset \mathcal{O}$, and $a, b \notin \sigma_p(A_0)$. The spectral projection $E((a,b))$ of A_0 corresponding to the interval (a, b) is given by Stone's formula (1.5.7)

$$E((a,b)) = \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_a^b \left((A_0 - (t+i\delta))^{-1} - (A_0 - (t-i\delta))^{-1} \right) dt,$$

where the integral on the right-hand side is understood in the strong sense. For $\nu \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in \mathcal{G}$ this implies

$$
\begin{aligned}
\|E((a,b))\gamma(\nu)\varphi\|^2 &= \left(\gamma(\nu)^* E((a,b))\gamma(\nu)\varphi, \varphi \right) \\
&= \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_a^b \left(\left(\gamma(\nu)^*(A_0 - (t+i\delta))^{-1}\gamma(\nu)\varphi, \varphi \right) \right. \\
&\qquad\qquad\qquad \left. - \left(\gamma(\nu)^*(A_0 - (t-i\delta))^{-1}\gamma(\nu)\varphi, \varphi \right) \right) dt
\end{aligned}
\tag{3.6.2}
$$

and the identities

$$
\begin{aligned}
&\gamma(\nu)^* \left(A_0 - (t \pm i\delta) \right)^{-1} \gamma(\nu) \\
&\quad = \frac{M(t \pm i\delta)}{|t \pm i\delta - \nu|^2} + \frac{M(\bar{\nu})}{(\bar{\nu} - (t \pm i\delta))(\bar{\nu} - \nu)} + \frac{M(\nu)}{(\nu - (t \pm i\delta))(\nu - \bar{\nu})}
\end{aligned}
$$

from Proposition 2.3.6 (vi) (with $\lambda = t \pm i\delta$ and $\bar{\mu} = \nu$) together with the holomorphy of M in $\mathcal{O}$ yield that the integral on the right-hand side of (3.6.2) is zero.

Hence, $E((a,b))\gamma(\nu)\varphi = 0$ for all $\nu \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in \mathcal{G}$. On the other hand, since $(a,b) \subset \Delta$, the assumption (3.6.1) and Proposition 3.4.10 (i) yield

$$E((a,b))\mathfrak{H} = \overline{\text{span}}\left\{E((a,b))\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R}, \varphi \in \mathcal{G}\right\},$$

and hence one concludes from $E((a,b))\gamma(\nu)\varphi = 0$ for $\nu \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in \mathcal{G}$ that $E((a,b)) = 0$. In particular, $x \in \rho(A_0)$ by Proposition 3.3.1 (i).

(ii)–(iii) According to Proposition 3.4.10 (ii), the condition (3.6.1) implies that S does not have eigenvalues in Δ. Hence, items (ii) and (iii) follow immediately from item (i) and Corollary 3.5.6.

(iv) Assume that $x \in \Delta$ is an isolated eigenvalue of A_0. Then by Proposition 2.3.6 (iii) or (v) there exists an open neighborhood $\mathcal{O}$ of x such that M is holomorphic on $\mathcal{O} \setminus \{x\}$. Since $x \notin \sigma_{\mathrm{p}}(S)$ by Proposition 3.4.10 (ii), it follows from Corollary 3.5.6 that there exists $\varphi \in \mathcal{G}$ such that

$$\mathcal{R}_x\varphi = \lim_{y \downarrow 0} iyM(x+iy)\varphi \neq 0. \tag{3.6.3}$$

This implies that M has a pole at x, which is of first order; cf. Corollary 2.3.9. By Remark 3.5.3 the residue of M at x is given by $\mathcal{R}_x$. Conversely, if M has a pole (of first order) at x, then (3.6.3) holds for some $\varphi \in \mathcal{G}$. Thus, x is an eigenvalue of A_0 by Corollary 3.5.6 and from item (i) it follows that there exists an open neighborhood $\mathcal{O}$ of x in $\mathbb{C}$ such that $\mathcal{O} \setminus \{\lambda\} \subset \rho(A_0)$. Hence, x is an isolated point in the spectrum of A_0. $\qquad\square$

Under the condition that S is simple the spectrum of A_0 can be described completely in terms of the Weyl function M.

Corollary 3.6.2. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M and γ be the corresponding Weyl function and γ-field. Assume that S is simple. Then the assertions* (i)–(iv) *in Theorem 3.6.1 hold for all $x \in \mathbb{R}$.*

To describe the absolutely continuous, singular, and singular continuous parts of the spectrum of A_0 in terms of the boundary behavior of the Weyl function M, some preliminary lemmas are needed.

Lemma 3.6.3. *Let $\lambda_0 \in \rho(A_0)$, $x \in \mathbb{R}$, and $\varphi \in \mathcal{G}$. Then the (possibly improper) limits*

$$\text{Im}\,(M(x+i0)\varphi, \varphi) \quad \text{and} \quad \text{Im}\,\big((A_0 - (x+i0))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\big)$$

exist simultaneously, and they satisfy

$$\text{Im}\,(M(x+i0)\varphi, \varphi) = |x - \lambda_0|^2 \,\text{Im}\,\big((A_0 - (x+i0))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\big).$$

Proof. It is no restriction to assume that $x \neq \lambda_0$, as otherwise $\lambda \mapsto M(\lambda)$ and $\lambda \mapsto (A_0 - \lambda)^{-1}$ are both holomorphic at $x = \lambda_0 \in \rho(A_0) \cap \mathbb{R}$, so that the above limits are zero and the identities hold.

For $x \neq \lambda_0$ it follows from (3.5.1) that

$$\operatorname{Im}(M(x+iy)\varphi, \varphi) = y\|\gamma(\lambda_0)\varphi\|^2$$
$$+ (|x - \lambda_0|^2 - y^2)\operatorname{Im}\left((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right)$$
$$+ 2(x - \operatorname{Re}\lambda_0)\, y \operatorname{Re}\left((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right).$$

The first term on the right-hand side clearly goes to 0 as $y \downarrow 0$. For the third term on the right-hand side, observe that for $y \downarrow 0$ one has

$$y \operatorname{Re}\left(\frac{1}{t - (x+iy)}\right) = \frac{y(t - x)}{(t - x)^2 + y^2} \to 0, \quad t \in \mathbb{R},$$

and since the approximating functions are uniformly bounded, the spectral calculus for A_0 (see Lemma 1.5.3) yields

$$\lim_{y \downarrow 0} y \operatorname{Re}\left((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right) = 0.$$

Hence, also the third term on the right-hand side goes to 0 as $y \downarrow 0$. Furthermore, $|x - \lambda_0|^2 - y^2 \to |x - \lambda_0|^2 > 0$ as $y \downarrow 0$. Therefore, $\operatorname{Im}(M(x+iy)\varphi, \varphi)$ converges as $y \downarrow 0$ if and only if

$$\operatorname{Im}\left((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right)$$

converges as $y \downarrow 0$. In addition, it is clear that the identity in the lemma for the limits is satisfied. $\square$

Recall that the self-adjoint extension A_0 generates a collection of finite Borel measures on $\mathbb{R}$: for each $h \in \mathfrak{H}$ the finite Borel measure μ_h in (3.3.2) is defined by $\mu_h = (E(\cdot)h, h)$, where E is the spectral measure of A_0. Now the interest is in the Borel transform F_h of $\mu_h = (E(\cdot)h, h)$, that is

$$F_h(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \lambda}\, d(E(t)h, h), \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

cf. Definition 3.1.3. In particular, if $\lambda = x + iy$, where $x \in \mathbb{R}$ and $y > 0$, then one has

$$\operatorname{Im} F_h(x+iy) = \operatorname{Im}\left((A_0 - (x+iy))^{-1}h, h\right) \qquad (3.6.4)$$

and

$$yF_h(x+iy) = y\left((A_0 - (x+iy))^{-1}h, h\right). \qquad (3.6.5)$$

By means of Lemma 3.6.3 the boundary values of the Borel transform F_h for a class of elements $h \in \mathfrak{H}$ are expressed in terms of the boundary values of the Weyl function M.

Lemma 3.6.4. *Let $\Delta \subset \mathbb{R}$ be an open interval and let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Then for elements of the form $h = E(\Delta)\gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, the following statements hold:*

(i) *If $x \in \Delta$, then the (possibly improper) limits*

$$\operatorname{Im} F_h(x + i0) \quad and \quad \operatorname{Im}(M(x + i0)\varphi, \varphi)$$

exist simultaneously, and

$$\operatorname{Im} F_h(x + i0) = |x - \lambda_0|^{-2}\operatorname{Im}(M(x + i0)\varphi, \varphi).$$

(ii) *If $x \notin \overline{\Delta}$, then $\operatorname{Im} F_h(x + i0) = |x - \lambda_0|^{-2}\operatorname{Im}(M(x + i0)\varphi, \varphi) = 0$.*

Proof. It follows from (3.6.4) that for all $h \in \mathfrak{H}$ the (possibly improper) limits $\operatorname{Im} F_h(x + i0)$ and $\operatorname{Im}((A_0 - (x + i0))^{-1}h, h)$ exist simultaneously and coincide. For the choice $h = \gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, it follows from Lemma 3.6.3 that the (possibly improper) limits $\operatorname{Im} F_h(x + i0)$ and $\operatorname{Im}(M(x + i0)\varphi, \varphi)$ exist simultaneously, and

$$\operatorname{Im} F_h(x + i0) = \operatorname{Im}\left((A_0 - (x + i0))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right)$$
$$= |x - \lambda_0|^{-2}\operatorname{Im}(M(x + i0)\varphi, \varphi).$$

If $h = E(\Delta)\gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, then for $x \in \Delta$ the spectral calculus implies

$$\operatorname{Im} F_h(x + i0) = \operatorname{Im}\left((A_0 - (x + i0))^{-1}E(\Delta)\gamma(\lambda_0)\varphi, E(\Delta)\gamma(\lambda_0)\varphi\right)$$
$$= \operatorname{Im}\left((A_0 - (x + i0))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\right)$$
$$= |x - \lambda_0|^{-2}\operatorname{Im}(M(x + i0)\varphi, \varphi),$$

while for $x \notin \overline{\Delta}$ it follows that

$$\operatorname{Im} F_h(x + i0) = \operatorname{Im}\left((A_0 - (x + i0))^{-1}E(\Delta)\gamma(\lambda_0)\varphi, E(\Delta)\gamma(\lambda_0)\varphi\right)$$
$$= 0.$$

This shows the assertions in (i) and (ii). $\square$

Now the absolutely continuous spectrum, the singular spectrum, and the singular continuous spectrum (cf. Section 3.3) of A_0 can be described in terms of the boundary behavior of the Weyl function M, still under the assumption of local simplicity. The results are essentially consequences of Theorem 3.2.3, Theorem 3.2.6, and Corollary 3.3.6.

Theorem 3.6.5. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with let $A_0 = \ker \Gamma_0$, and let M and γ be the corresponding Weyl function and γ-field. Let $\Delta \subset \mathbb{R}$ be an open interval and assume that the condition*

$$E(\Delta)\mathfrak{H} = \overline{\operatorname{span}}\left\{E(\Delta)\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R}, \varphi \in \mathcal{G}\right\} \tag{3.6.6}$$

is satisfied, where $E(\cdot)$ is the spectral measure of A_0. Then the absolutely contin-uous spectrum of A_0 in Δ is given by

$$\overline{\sigma_{\mathrm{ac}}(A_0) \cap \Delta} = \overline{\bigcup_{\varphi \in \mathcal{G}} \mathrm{clos}_{\mathrm{ac}}\big(\{x \in \Delta : 0 < \mathrm{Im}\,(M(x+i0)\varphi, \varphi) < \infty\}\big)}. \qquad (3.6.7)$$

If S is simple, then (3.6.7) holds for every open interval Δ, including $\Delta = \mathbb{R}$.

Proof. By assumption, the span of the set

$$\mathcal{D}_\Delta := \big\{E(\Delta)\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R},\ \varphi \in \mathcal{G}\big\}$$

is dense in $E(\Delta)\mathfrak{H}$ and hence Corollary 3.3.6 implies the identity

$$\overline{\sigma_{\mathrm{ac}}(A_0) \cap \Delta} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{h,\mathrm{ac}})}.$$

According to Theorem 3.2.6 (i) (where the set $\mathfrak{F}$ was replaced by $\mathbb{R}$),

$$\sigma(\mu_{h,\mathrm{ac}}) = \mathrm{clos}_{\mathrm{ac}}\big(\{x \in \mathbb{R} : 0 < \mathrm{Im}\,F_h(x+i0) < \infty\}\big),$$

which for $h = E(\Delta)\gamma(\nu)\varphi \in \mathcal{D}_\Delta$ is equivalent to

$$\sigma(\mu_{h,\mathrm{ac}}) = \mathrm{clos}_{\mathrm{ac}}\big(\{x \in \Delta : 0 < \mathrm{Im}\,(M(x+i0)\varphi, \varphi) < \infty\big\}$$

by Lemma 3.6.4. This yields (3.6.7). $\square$

The next corollary gives a necessary and sufficient condition for the absence of absolutely continuous spectrum.

Corollary 3.6.6. *Let A_0 and M be as in Theorem 3.6.5 and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.6.6) is satisfied. Then*

$$\sigma_{\mathrm{ac}}(A_0) \cap \Delta = \emptyset$$

if and only if for all $\varphi \in \mathcal{G}$ and for almost all $x \in \Delta$

$$\mathrm{Im}\,(M(x+i0)\varphi, \varphi) = 0.$$

If S is simple, then the assertion holds for every open interval Δ, including $\Delta = \mathbb{R}$.

Proof. Since $\mathrm{clos}_{\mathrm{ac}}(\mathfrak{B}) = \emptyset$ if and only if $m(\mathfrak{B}) = 0$ for any Borel set $\mathfrak{B} \subset \mathbb{R}$ by Lemma 3.2.5 (i), it is clear that for $\varphi \in \mathcal{G}$

$$\mathrm{clos}_{\mathrm{ac}}\big(\{x \in \Delta : 0 < \mathrm{Im}\,(M(x+i0)\varphi, \varphi) < \infty\}\big) = \emptyset \qquad (3.6.8)$$

if and only if

$$m\big(\{x \in \Delta : 0 < \mathrm{Im}\,(M(x+i0)\varphi, \varphi) < \infty\}\big) = 0. \qquad (3.6.9)$$

Assume first that $\sigma_{\mathrm{ac}}(A_0) \cap \Delta = \emptyset$. Then (3.6.7) yields (3.6.8) for all $\varphi \in \mathcal{G}$, and hence (3.6.9) holds for all $\varphi \in \mathcal{G}$. Moreover, for $h = \gamma(\lambda_0)\varphi$, $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, and $x \in \mathbb{R}$

one has

$$\mathrm{Im}\,(M(x+i0)\varphi,\varphi) = |x-\lambda_0|^2 \mathrm{Im}\,F_h(x+i0)$$

by Lemma 3.6.4 (with $\Delta = \mathbb{R}$), and according to Theorem 3.1.4 (i) this limit exists and is finite for m-almost all $x \in \mathbb{R}$. Hence, (3.6.9) implies $\mathrm{Im}\,(M(x+i0)\varphi,\varphi) = 0$ for all $\varphi \in \mathcal{G}$ and m-almost all $x \in \Delta$. For the converse implication assume that $\mathrm{Im}\,(M(x+i0)\varphi,\varphi) = 0$ for all $\varphi \in \mathcal{G}$ and for m-almost all $x \in \Delta$. Then (3.6.9) and hence also (3.6.8) hold for all $\varphi \in \mathcal{G}$. Thus, (3.6.7) yields $\sigma_{\mathrm{ac}}(A_0) \cap \Delta = \emptyset$. $\square$

The next lemma is of similar nature as Lemma 3.6.4. Here the limits exist for all $x \in \mathbb{R}$ by (3.1.12)–(3.1.13) and Proposition 3.5.1.

Lemma 3.6.7. *Let* $\Delta \subset \mathbb{R}$ *be an open interval and let* $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. *Then for elements of the form* $h = E(\Delta)\gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, *one has*

$$\lim_{y\downarrow 0} yF_h(x+iy) = \begin{cases} |x-\lambda_0|^{-2}\,\lim_{y\downarrow 0} y(M(x+iy)\varphi,\varphi), & x \in \Delta, \\ 0, & x \notin \overline{\Delta}. \end{cases}$$

Proof. For $h = \gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, it follows from (3.6.5) and (3.5.1) (cf. (3.5.3) in the proof of Proposition 3.5.1) that

$$\lim_{y\downarrow 0} yF_h(x+iy) = \lim_{y\downarrow 0} y\big((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\big)$$
$$= |x-\lambda_0|^{-2}\lim_{y\downarrow 0} y(M(x+iy)\varphi,\varphi)$$

for all $x \in \mathbb{R}$. If $h = E(\Delta)\gamma(\lambda_0)\varphi$, $\varphi \in \mathcal{G}$, then for $x \in \Delta$ the spectral calculus shows that

$$\lim_{y\downarrow 0} yF_h(x+iy) = \lim_{y\downarrow 0} y\big((A_0 - (x+iy))^{-1}E(\Delta)\gamma(\lambda_0)\varphi, E(\Delta)\gamma(\lambda_0)\varphi\big)$$
$$= \lim_{y\downarrow 0} y\big((A_0 - (x+iy))^{-1}\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi\big)$$
$$= |x-\lambda_0|^{-2}\lim_{y\downarrow 0} y(M(x+iy)\varphi,\varphi),$$

while for $x \notin \overline{\Delta}$ one has

$$\lim_{y\downarrow 0} yF_h(x+iy) = \lim_{y\downarrow 0} y\big((A_0 - (x+iy))^{-1}E(\Delta)\gamma(\lambda_0)\varphi, E(\Delta)\gamma(\lambda_0)\varphi\big)$$
$$= 0.$$

This completes the proof. $\square$

Next some inclusions for the singular and singular continuous spectra of A_0 will be shown.

Theorem 3.6.8. *Let* S *be a closed symmetric relation in* $\mathfrak{H}$, *let* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* S^* *with* $A_0 = \ker \Gamma_0$, *and let* M *and* γ *be the corresponding Weyl function and* γ-*field. Let* $\Delta \subset \mathbb{R}$ *be an open interval and assume that*

the condition

$$E(\Delta)\mathfrak{H} = \overline{\mathrm{span}}\left\{E(\Delta)\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R}, \varphi \in \mathcal{G}\right\} \qquad (3.6.10)$$

is satisfied, where $E(\cdot)$ is the spectral measure of A_0. Then the following statements hold:

(i) *The singular spectrum of A_0 in Δ satisfies*

$$\left(\sigma_{\mathrm{s}}(A_0) \cap \Delta\right) \subset \overline{\bigcup_{\varphi \in \mathcal{G}} \left\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty\right\}}.$$

(ii) *The singular continuous spectrum of A_0 in Δ, i.e., $\sigma_{\mathrm{sc}}(A_0) \cap \Delta$, is contained in the set*

$$\overline{\bigcup_{\varphi \in \mathcal{G}} \mathrm{clos}_{\mathrm{c}}\left(\left\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty, \lim_{y \downarrow 0} y(M(x+iy)\varphi, \varphi) = 0\right\}\right)}.$$

If S is simple, then (i) *and* (ii) *hold for every open interval Δ, including $\Delta = \mathbb{R}$.*

Proof. By assumption, the span of the set

$$\mathcal{D}_\Delta := \left\{E(\Delta)\gamma(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R},\ \varphi \in \mathcal{G}\right\}$$

is dense in $E(\Delta)\mathfrak{H}$.

(i) Recall that by Corollary 3.3.6 one has

$$\overline{\sigma_{\mathrm{s}}(A_0) \cap \Delta} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{h,\mathrm{s}})} \qquad (3.6.11)$$

and according to Theorem 3.2.6 (ii) (with $\mathfrak{F}$ replaced by $\mathbb{R}$)

$$\sigma(\mu_{h,\mathrm{s}}) \subset \overline{\left\{x \in \mathbb{R} : \mathrm{Im}\, F_h(x+i0) = \infty\right\}}.$$

For $h = E(\Delta)\gamma(\nu)\varphi \in \mathcal{D}_\Delta$ this gives, via Lemma 3.6.4,

$$\sigma(\mu_{h,\mathrm{s}}) \subset \overline{\left\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty\right\}}.$$

Hence, the set $\overline{\sigma_{\mathrm{s}}(A_0) \cap \Delta}$ in (3.6.11) is contained in

$$\overline{\bigcup_{h \in \mathcal{D}_\Delta} \overline{\left\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty\right\}}}$$

$$= \overline{\bigcup_{h \in \mathcal{D}_\Delta} \left\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty\right\}},$$

which yields the assertion in (i).

(ii) Likewise, Corollary 3.3.6 implies

$$\overline{\sigma_{\mathrm{sc}}(A_0) \cap \Delta} = \overline{\bigcup_{h \in \mathcal{D}_\Delta} \sigma(\mu_{h,\mathrm{sc}})}. \qquad (3.6.12)$$

By Theorem 3.2.6 (iii) (again with $\mathfrak{F}$ replaced by $\mathbb{R}$),

$$\sigma(\mu_{h,\mathrm{sc}}) \subset \mathrm{clos}_{\mathrm{c}}\big(\big\{x \in \mathbb{R} : \mathrm{Im}\, F_h(x+i0) = \infty,\ \lim_{y\downarrow 0} y F_h(x+iy) = 0\big\}\big),$$

and for $h = E(\Delta)\gamma(\nu)\varphi \in \mathcal{D}_\Delta$ this gives, via Lemma 3.6.4 and Lemma 3.6.7, that $\sigma(\mu_{h,\mathrm{sc}})$ is contained in

$$\mathrm{clos}_{\mathrm{c}}\big(\big\{x \in \Delta : \mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty,\ \lim_{y\downarrow 0} y(M(x+iy)\varphi, \varphi) = 0\big\}\big).$$

Hence, the assertion follows from (3.6.12). $\qquad\square$

An immediate corollary of the previous theorem and Lemma 3.2.5 (ii) is a sufficient condition for the absence of the singular continuous spectrum in terms of the limit behavior of the function M.

Corollary 3.6.9. *Let A_0 and M be as in Theorem 3.6.8 and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.6.10) is satisfied. Assume that for each $\varphi \in \mathcal{G}$ there exist at most countably many $x \in \Delta$ such that*

$$\mathrm{Im}\,(M(x+iy)\varphi, \varphi) \to \infty \quad \text{and} \quad y(M(x+iy)\varphi, \varphi) \to 0 \quad \text{as} \quad y \downarrow 0.$$

Then

$$\sigma_{\mathrm{sc}}(A_0) \cap \Delta = \emptyset.$$

If S is simple, then the assertion holds for every open interval Δ, including $\Delta = \mathbb{R}$.

As a further corollary of the theorems of this section sufficient conditions are provided for the spectrum of A_0 to be purely absolutely continuous or purely singularly continuous, respectively, in some set.

Corollary 3.6.10. *Let A_0 and M be as in Theorem 3.6.5 or Theorem 3.6.8 and let $\Delta \subset \mathbb{R}$ be an open interval such that the condition (3.6.6) or (3.6.10) is satisfied. Assume that for all $\varphi \in \mathcal{G}$ and all $x \in \Delta$*

$$\lim_{y\downarrow 0} y M(x+iy)\varphi = 0. \tag{3.6.13}$$

Then the following statements hold:

(i) *If for each $\varphi \in \mathcal{G}$ there exist at most countably many $x \in \Delta$ such that $\mathrm{Im}\,(M(x+i0)\varphi, \varphi) = \infty$, then $\sigma(A_0) \cap \Delta = \sigma_{\mathrm{ac}}(A_0) \cap \Delta$.*

(ii) *If $\mathrm{Im}\,(M(x+i0)\varphi, \varphi) = 0$ holds for all $\varphi \in \mathcal{G}$ and almost all $x \in \Delta$, then $\sigma(A_0) \cap \Delta = \sigma_{\mathrm{sc}}(A_0) \cap \Delta$.*

If S is simple and Δ is an open interval such that (3.6.13) holds for all $\varphi \in \mathcal{G}$ and all $x \in \Delta$, then (i) and (ii) are satisfied.

Proof. Note first that the assumption (3.6.13) yields $\sigma_{\mathrm{p}}(A_0) \cap \Delta = \emptyset$; this follows immediately from Corollary 3.5.6 and the fact that the condition (3.6.6) or (3.6.10) implies $\sigma_{\mathrm{p}}(S) \cap \Delta = \emptyset$; cf. Proposition 3.4.10 (ii). The assumption in (i) and Corollary 3.6.9 imply $\sigma_{\mathrm{sc}}(A_0) \cap \Delta = \emptyset$ and hence $\sigma(A_0) \cap \Delta = \sigma_{\mathrm{ac}}(A_0) \cap \Delta$. Similarly, the assumption in (ii) and Corollary 3.6.6 imply $\sigma_{\mathrm{ac}}(A_0) \cap \Delta = \emptyset$ and hence $\sigma(A_0) \cap \Delta = \sigma_{\mathrm{sc}}(A_0) \cap \Delta$ follows. $\qquad\square$

3.7 Limit properties of Weyl functions

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M be the corresponding Weyl function. The aim of this section is to relate limit properties of the imaginary part $\operatorname{Im} M$ of the Weyl function with defect elements in $\operatorname{dom} A_0$ and $\operatorname{dom} |A_0|^{\frac{1}{2}}$, and $\operatorname{ran}(A_0 - x)$ and $\operatorname{ran} |A_0 - x|^{\frac{1}{2}}$, $x \in \mathbb{R}$, respectively, where

$$A_0 = A_{0,\mathrm{op}} \,\widehat{\oplus}\, A_{0,\mathrm{mul}} \quad \text{and} \quad |A_0| = |A_{0,\mathrm{op}}| \,\widehat{\oplus}\, A_{0,\mathrm{mul}} \qquad (3.7.1)$$

with respect to the usual decomposition $\mathfrak{H} = \mathfrak{H}_{\mathrm{op}} \oplus \mathfrak{H}_{\mathrm{mul}}$. This also leads to necessary and sufficient conditions for S to be a densely defined operator in terms of the Weyl function.

The first result connects limit properties of the Weyl function at ∞ with elements in $\operatorname{dom} A_0 \cap \ker(S^* - \lambda)$ and $\operatorname{dom} |A_0|^{\frac{1}{2}} \cap \ker(S^* - \lambda)$ for $\lambda \in \rho(A_0)$. Although the decomposition $S^* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*)$ is direct for all $\lambda \in \rho(A_0)$, it may happen that $\operatorname{dom} A_0 \cap \ker(S^* - \lambda) \neq \{0\}$ if S^* is multivalued. In fact, if $f \in \operatorname{dom} A_0 \cap \ker(S^* - \lambda)$, $f \neq 0$, then $\{f, f'\} \in A_0$ for some f' and hence $\{0, f' - \lambda f\} \in S^*$. Since $\lambda \in \rho(A_0)$, this yields $\operatorname{mul} S^* \neq \{0\}$.

The representation (3.5.12) of the Weyl function M in terms of the extension $A_0 = \ker \Gamma_0$ will now be used; cf. (3.5.8)–(3.5.9). For simplicity one takes $\lambda_0 = i$ in (3.5.12), which leads to the representation

$$\begin{aligned}
M(iy) = \operatorname{Re} M(i) &+ iy\, \gamma(i)^*(I - P_{\mathrm{op}})\gamma(i) \\
&+ \gamma(i)^* \iota_{\mathrm{op}}\big[iy + (1 - y^2)(A_{0,\mathrm{op}} - iy)^{-1}\big] P_{\mathrm{op}}\gamma(i)
\end{aligned} \qquad (3.7.2)$$

for all $y > 0$. The spectral calculus for the self-adjoint operator $A_{0,\mathrm{op}}$ applied to (3.7.2) shows that for $\varphi \in \mathcal{G}$ and $y > 0$

$$\begin{aligned}
\operatorname{Im}(M(iy)\varphi, \varphi) = y\|(I - P_{\mathrm{op}})\gamma(i)\varphi\|^2 \\
+ y \int_{\mathbb{R}} \frac{t^2 + 1}{t^2 + y^2}\, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big).
\end{aligned} \qquad (3.7.3)$$

Proposition 3.7.1. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$, and let M and γ be the corresponding Weyl function and γ-field. Then the following statements hold for $\varphi \in \mathcal{G}$:*

(i) $\gamma(\lambda)\varphi \in \operatorname{dom} A_0$ *for some, and hence for all* $\lambda \in \rho(A_0)$ *if and only if*

$$\lim_{y \to +\infty} y \operatorname{Im}(M(iy)\varphi, \varphi) < \infty;$$

(ii) $\gamma(\lambda)\varphi \in \operatorname{dom} |A_0|^{\frac{1}{2}}$ *for some, and hence for all* $\lambda \in \rho(A_0)$ *if and only if*

$$\int_1^\infty \frac{\operatorname{Im}(M(iy)\varphi, \varphi)}{y} \, dy < \infty. \tag{3.7.4}$$

Proof. (i) It suffices to prove the assertion for $\lambda = i$, since by Proposition 2.3.2 (ii)

$$\gamma(\lambda) = \big(I + (\lambda - i)(A_0 - \lambda)^{-1}\big)\gamma(i),$$
$$\gamma(i) = \big(I + (i - \lambda)(A_0 - i)^{-1}\big)\gamma(\lambda) \tag{3.7.5}$$

for $\lambda \in \rho(A_0)$ and hence $\gamma(i)\varphi \in \operatorname{dom} A_0$ if and only if $\gamma(\lambda)\varphi \in \operatorname{dom} A_0$. Note first that (3.7.3) yields

$$y\operatorname{Im}(M(iy)\varphi, \varphi) = y^2\|(I - P_{\mathrm{op}})\gamma(i)\varphi\|^2$$
$$+ \int_{\mathbb{R}} \frac{y^2(t^2 + 1)}{t^2 + y^2} \, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big). \tag{3.7.6}$$

It is clear that the left-hand side of (3.7.6) has a finite limit for $y \to +\infty$ if and only if $(I - P_{\mathrm{op}})\gamma(i)\varphi = 0$ and

$$\int_{\mathbb{R}} t^2 \, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big) < \infty,$$

which follows from the monotone convergence theorem. In other words, the left-hand side of (3.7.6) has a finite limit for $y \to +\infty$ if and only if $\gamma(i)\varphi \in \operatorname{dom} A_0$.

(ii) As in the proof of (i), it suffices to verify the assertion for $\lambda = i$. In fact, if $\gamma(i)\varphi \in \operatorname{dom} |A_0|^{\frac{1}{2}}$, then $\gamma(i)\varphi = (|A_0|^{\frac{1}{2}} - \mu)^{-1}g$ for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ and $g \in \mathfrak{H}$. The first identity in (3.7.5) and the functional calculus for the self-adjoint operator $A_{0,\mathrm{op}}$ or self-adjoint relation A_0 (see Section 1.5) show

$$\gamma(\lambda)\varphi = \big(I + (\lambda - i)(A_0 - \lambda)^{-1}\big)(|A_0|^{\frac{1}{2}} - \mu)^{-1}g$$
$$= (|A_0|^{\frac{1}{2}} - \mu)^{-1}\big(I + (\lambda - i)(A_0 - \lambda)^{-1}\big)g \in \operatorname{dom} |A_0|^{\frac{1}{2}}.$$

The same argument and the second identity in (3.7.5) show that $\gamma(\lambda)\varphi \in \operatorname{dom} |A_0|^{\frac{1}{2}}$ implies $\gamma(i)\varphi \in \operatorname{dom} |A_0|^{\frac{1}{2}}$.

It follows from (3.7.3) that

$$\frac{\operatorname{Im}(M(iy)\varphi, \varphi)}{y} = \|(I - P_{\mathrm{op}})\gamma(i)\varphi\|^2$$
$$+ \int_{\mathbb{R}} \frac{t^2 + 1}{t^2 + y^2} \, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big).$$

Hence, (3.7.4) holds if and only if $(I - P_{\mathrm{op}})\gamma(i)\varphi = 0$ and

$$\int_1^\infty \left(\int_{\mathbb{R}} \frac{t^2 + 1}{t^2 + y^2} \, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big) \right) dy < \infty.$$

Change the order of integration in the last integral, note that

$$(t^2 + 1) \int_1^\infty \frac{1}{t^2 + y^2} \, dy = (t^2 + 1)\frac{1}{|t|} \left(\frac{\pi}{2} - \arctan\frac{1}{|t|} \right), \qquad t \neq 0,$$

and observe that for large $|t|$ one has

$$(t^2 + 1) \frac{1}{|t|} \left(\frac{\pi}{2} - \arctan\frac{1}{|t|} \right) \sim |t|$$

and that on compact subsets of $\mathbb{R}$ the function

$$t \mapsto (t^2 + 1) \frac{1}{|t|} \left(\frac{\pi}{2} - \arctan\frac{1}{|t|} \right)$$

is bounded. Hence, (3.7.4) holds if and only if $(I - P_{\mathrm{op}})\gamma(i)\varphi = 0$ and

$$\int_{\mathbb{R}} |t| \, d\big(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(i)\varphi, P_{\mathrm{op}}\gamma(i)\varphi\big) < \infty.$$

In other words, (3.7.4) holds if and only if $\gamma(i)\varphi \in \operatorname{dom}|A_0|^{\frac{1}{2}}$.   □

The following result is essentially a consequence of Proposition 3.7.1 (i).

Corollary 3.7.2. *Let S, A_0, and M be as in Proposition 3.7.1. Then $\operatorname{dom} S$ is dense in $\overline{\operatorname{dom}} A_0$ if and only if*

$$\lim_{y \to +\infty} y \operatorname{Im}(M(iy)\varphi, \varphi) = \infty \quad \textit{for all} \quad \varphi \in \mathcal{G}, \ \varphi \neq 0.$$

Proof. Let $\lambda \in \rho(A_0)$ and note that $f \in (\operatorname{dom} S)^\perp$ if and only if for all $\{h, h'\} \in S$

$$0 = (f, h) = \big(f, (A_0 - \bar\lambda)^{-1}(h' - \bar\lambda h)\big) = \big((A_0 - \lambda)^{-1}f, h' - \bar\lambda h\big).$$

Hence, $f \in (\operatorname{dom} S)^\perp$ if and only if $(A_0 - \lambda)^{-1}f \in \ker(S^* - \lambda) = \operatorname{ran}\gamma(\lambda)$. Furthermore, $(A_0 - \lambda)^{-1}f \neq 0$ if and only if $f \notin \operatorname{mul} A_0 = (\overline{\operatorname{dom}} A_0)^\perp$.

Now assume that $\operatorname{dom} S$ is not dense in $\overline{\operatorname{dom}} A_0$. Then there exists a nontrivial $f \in \overline{\operatorname{dom}} A_0$ such that $f \in (\operatorname{dom} S)^\perp$, and hence

$$(A_0 - \lambda)^{-1}f \in \ker(S^* - \lambda).$$

Since $f \in \overline{\operatorname{dom}} A_0$ it follows that $(A_0 - \lambda)^{-1}f = \gamma(\lambda)\varphi$ for a nontrivial $\varphi \in \mathcal{G}$. This means $\gamma(\lambda)\varphi \in \operatorname{dom} A_0$, and hence

$$\lim_{y \to +\infty} y \operatorname{Im}(M(iy)\varphi, \varphi) < \infty \tag{3.7.7}$$

by Proposition 3.7.1 (i). Conversely, if (3.7.7) holds for some nontrivial $\varphi \in \mathcal{G}$, then by Proposition 3.7.1 (i) it follows that $\gamma(\lambda)\varphi \in \operatorname{dom} A_0$. Hence, there exists a nontrivial $f \in \overline{\operatorname{dom}} A_0$ such that $\gamma(\lambda)\varphi = (A_0 - \lambda)^{-1}f$. Therefore, one sees that $f \in (\operatorname{dom} S)^\perp$ and hence $\operatorname{dom} S$ is not dense in $\overline{\operatorname{dom}} A_0$. $\square$

Corollary 3.7.3. *Let S, A_0, and M be as in Proposition 3.7.1. Then S is a densely defined operator if and only if the following conditions hold:*

(i) $\displaystyle\lim_{y \to +\infty} \frac{1}{iy}(M(iy)\varphi, \varphi) = 0$ *for all $\varphi \in \mathcal{G}$;*

(ii) $\lim_{y \to +\infty} y \operatorname{Im}(M(iy)\varphi, \varphi) = \infty$ *for all $\varphi \in \mathcal{G}$, $\varphi \neq 0$.*

In this case, S^* is an operator and all intermediate extensions of S are operators.

Proof. Note that Proposition 3.5.7 and the fact that $\gamma(\lambda_0)^*(I - P_{\mathrm{op}})\gamma(\lambda_0)$ in (3.5.13) is a nonnegative operator in $\mathcal{G}$ show that condition (i) is equivalent to the condition

$$\lim_{y \to +\infty} \frac{1}{iy}M(iy)\varphi = 0, \qquad \varphi \in \mathcal{G}.$$

By (3.5.18), this condition is necessary and sufficient for A_0 to be an operator, which is the case if and only if $\overline{\operatorname{dom}} A_0 = \mathfrak{H}$. Moreover, according to Corollary 3.7.2, the condition (ii) is necessary and sufficient for the equality $\overline{\operatorname{dom}} S = \overline{\operatorname{dom}} A_0$ to hold. Therefore, $\overline{\operatorname{dom}} S = \mathfrak{H}$ if and only if conditions (i) and (ii) hold. $\square$

In the next result, which is parallel to Proposition 3.7.1, the limit properties of the Weyl function at $x \in \mathbb{R}$ will be connected with elements in

$$\ker(S^* - \lambda) \cap \operatorname{ran}(A_0 - x) \quad \text{and} \quad \ker(S^* - \lambda) \cap \operatorname{ran}|A_0 - x|^{\frac{1}{2}}.$$

For this reason the representation (3.5.1) expressing the Weyl function M in terms of the self-adjoint relation $A_0 = \ker \Gamma_0$ will be used. For simplicity one takes $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $\operatorname{Re} \lambda_0 = x$ in (3.5.1), which leads to

$$M(x + iy) = \operatorname{Re} M(\lambda_0) \\ + \gamma(\lambda_0)^* \big[iy + (|\operatorname{Im} \lambda_0|^2 - y^2)(A_0 - (x + iy))^{-1}\big]\gamma(\lambda_0). \tag{3.7.8}$$

It follows by means of the spectral calculus applied to (3.7.8) that for $x \in \mathbb{R}$ and $\varphi \in \mathcal{G}$ one has

$$\frac{\operatorname{Im}(M(x + iy)\varphi, \varphi)}{y} = \|\gamma(\lambda_0)\varphi\|^2 \\ + (|\operatorname{Im} \lambda_0|^2 - y^2)\int_\mathbb{R} \frac{1}{(t - x)^2 + y^2}\, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi). \tag{3.7.9}$$

Proposition 3.7.4. *Let S be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, let $A_0 = \ker \Gamma_0$ be decomposed as in (3.7.1), and let M and γ be the corresponding Weyl function and γ-field. Then the following statements hold for $x \in \mathbb{R}$ and $\varphi \in \mathcal{G}$:*

(i) $\gamma(\lambda)\varphi \in \operatorname{ran}(A_0 - x)$ *for some, and hence for all* $\lambda \in \rho(A_0)$ *if and only if*

$$\lim_{y \downarrow 0} \frac{\operatorname{Im}(M(x+iy)\varphi, \varphi)}{y} < \infty; \tag{3.7.10}$$

(ii) $P_{\mathrm{op}}\gamma(\lambda)\varphi \in \operatorname{ran}|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ *for some, and hence for all* $\lambda \in \rho(A_0)$ *if and only if*

$$\int_0^1 \frac{\operatorname{Im}(M(x+iy)\varphi, \varphi)}{y} \, dy < \infty. \tag{3.7.11}$$

Proof. (i) It will first be shown that for $\lambda, \lambda_0 \in \rho(A_0)$ one has $\gamma(\lambda)\varphi \in \operatorname{ran}(A_0 - x)$ if and only if $\gamma(\lambda_0)\varphi \in \operatorname{ran}(A_0 - x)$. Assume that $\gamma(\lambda_0)\varphi \in \operatorname{ran}(A_0 - x)$. Then there is $\{f, f'\} \in A_0$ such that $\gamma(\lambda_0)\varphi = f' - xf$. As

$$\{f' - xf, (A_0 - \lambda)^{-1}(f' - xf)\} \in (A_0 - \lambda)^{-1},$$

it follows that

$$\{(A_0 - \lambda)^{-1}(f' - xf), f' - xf + (\lambda - x)(A_0 - \lambda)^{-1}(f' - xf)\} \in A_0 - x.$$

Hence, $f' - xf + (\lambda - x)(A_0 - \lambda)^{-1}(f' - xf) \in \operatorname{ran}(A_0 - x)$ and

$$(A_0 - \lambda)^{-1}(f' - xf) \in \operatorname{ran}(A_0 - x).$$

From the identity $\gamma(\lambda) = (I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma(\lambda_0)$, established in Proposition 2.3.2 (ii), one finds that

$$\gamma(\lambda)\varphi = f' - xf + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}(f' - xf) \in \operatorname{ran}(A_0 - x).$$

Thus, $\gamma(\lambda_0)\varphi \in \operatorname{ran}(A_0 - x)$ implies that $\gamma(\lambda)\varphi \in \operatorname{ran}(A_0 - x)$. Since λ_0 and λ in the above argument can be interchanged, it is clear that $\gamma(\lambda)\varphi \in \operatorname{ran}(A_0 - x)$ if and only if $\gamma(\lambda_0)\varphi \in \operatorname{ran}(A_0 - x)$.

To verify the remaining assertion in (i) with $\lambda = \lambda_0$, note first that the limit as $y \downarrow 0$ in (3.7.10) is finite if and only if the limit of the integral in the second term in (3.7.9) is finite. An application of the monotone convergence theorem shows that the limit as $y \downarrow 0$ of the integral in the second term in (3.7.9) is finite if and only if

$$\int_{\mathbb{R}} \frac{1}{(t-x)^2} \, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi) < \infty,$$

that is, if and only if

$$\int_{\mathbb{R}} \frac{1}{(t-x)^2} \, d(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(\lambda_0)\varphi, P_{\mathrm{op}}\gamma(\lambda_0)\varphi) < \infty,$$

where the definition of the spectral measure $E(\cdot)$ of A_0 via the spectral measure $E_{\mathrm{op}}(\cdot)$ of $A_{0,\mathrm{op}}$ was used. Therefore, the limit as $y \downarrow 0$ in (3.7.10) is finite if and

only if $P_{\mathrm{op}}\gamma(\lambda_0)\varphi \in \mathrm{dom}\,(A_{0,\mathrm{op}} - x)^{-1} = \mathrm{ran}\,(A_{0,\mathrm{op}} - x)$, that is, if and only if $\gamma(\lambda_0)\varphi \in \mathrm{ran}\,(A_0 - x)$.

(ii) As in (i), it will first be shown that $P_{\mathrm{op}}\gamma(\lambda)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ if and only if $P_{\mathrm{op}}\gamma(\lambda_0)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ for $\lambda, \lambda_0 \in \rho(A_0)$. Assume that

$$P_{\mathrm{op}}\gamma(\lambda_0)\varphi = |A_{0,\mathrm{op}} - x|^{\frac{1}{2}} f$$

for some $f \in \mathrm{dom}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$. It follows from the functional calculus for unbounded self-adjoint operators that

$$\overline{(A_{0,\mathrm{op}} - \lambda)^{-1}|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}} = \overline{|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}(A_{0,\mathrm{op}} - \lambda)^{-1}}$$
$$= |A_{0,\mathrm{op}} - x|^{\frac{1}{2}}(A_{0,\mathrm{op}} - \lambda)^{-1}$$

and hence, since $\gamma(\lambda) = (I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma(\lambda_0)$, one has that

$$P_{\mathrm{op}}\gamma(\lambda)\varphi = P_{\mathrm{op}}\gamma(\lambda_0)\varphi + (\lambda - \lambda_0)(A_{0,\mathrm{op}} - \lambda)^{-1}P_{\mathrm{op}}\gamma(\lambda_0)\varphi$$
$$= |A_{0,\mathrm{op}} - x|^{\frac{1}{2}} f + (\lambda - \lambda_0)(A_{0,\mathrm{op}} - \lambda)^{-1}|A_{0,\mathrm{op}} - x|^{\frac{1}{2}} f$$
$$= |A_{0,\mathrm{op}} - x|^{\frac{1}{2}} f + (\lambda - \lambda_0)|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}(A_{0,\mathrm{op}} - \lambda)^{-1} f,$$

that is, $P_{\mathrm{op}}\gamma(\lambda)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$. Thus, $P_{\mathrm{op}}\gamma(\lambda_0)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ implies $P_{\mathrm{op}}\gamma(\lambda)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$. Since λ_0 and λ in the above argument can be interchanged, it is clear that $P_{\mathrm{op}}\gamma(\lambda_0)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ holds if and only if $P_{\mathrm{op}}\gamma(\lambda)\varphi \in \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}}$ holds.

To verify the remaining assertion in (ii), it is convenient to fix $\lambda = \lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $|\mathrm{Im}\,\lambda_0| > 1$. One then concludes from (3.7.9) that the integral in (3.7.11) converges if and only if the integral

$$\int_0^1 \left(\int_{\mathbb{R}} \frac{1}{(t - x)^2 + y^2}\, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi) \right) dy$$

converges. Changing the order of integration in the last integral and observing that

$$\int_0^1 \frac{1}{(t - x)^2 + y^2}\, dy = \frac{1}{|t - x|} \arctan \frac{1}{|t - x|}, \qquad t \neq x,$$

one sees that the integral in (3.7.11) converges if and only if

$$\int_{\mathbb{R}} \frac{1}{|t - x|} \arctan \frac{1}{|t - x|}\, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi) < \infty. \tag{3.7.12}$$

Since the integrand in (3.7.12) is bounded on $\mathbb{R} \setminus (x - 1, x + 1)$, it follows that the integral in (3.7.11) converges if and only

$$\int_{x-1}^{x+1} \frac{1}{|t - x|}\, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi) < \infty,$$

which is equivalent to

$$\int_{\mathbb{R}} \frac{1}{|t-x|} \, d(E(t)\gamma(\lambda_0)\varphi, \gamma(\lambda_0)\varphi) < \infty$$

and to

$$\int_{\mathbb{R}} \frac{1}{|t-x|} \, d(E_{\mathrm{op}}(t)P_{\mathrm{op}}\gamma(\lambda_0)\varphi, P_{\mathrm{op}}\gamma(\lambda_0)\varphi) < \infty.$$

Therefore, (3.7.11) holds if and only if

$$P_{\mathrm{op}}\gamma(\lambda_0)\varphi \in \mathrm{dom}\,|A_{0,\mathrm{op}} - x|^{-\frac{1}{2}} = \mathrm{ran}\,|A_{0,\mathrm{op}} - x|^{\frac{1}{2}},$$

that is, if and only if $\gamma(\lambda_0)\varphi \in \mathrm{ran}\,|A_0 - x|^{\frac{1}{2}}$. $\qquad\square$

3.8 Spectra and local minimality for self-adjoint extensions

In this section the results on eigenvalues, eigenspaces, continuous, absolutely continuous and singular continuous spectra from Section 3.5 and Section 3.6 will be explicitly formulated for arbitrary self-adjoint extensions of a symmetric relation.

Let S be a closed symmetric relation in $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with γ-field γ and Weyl function M. Consider a self-adjoint extension

$$A_\Theta = \{\widehat{f} \in S^* : \Gamma\widehat{f} \in \Theta\} = \ker\left(\Gamma_1 - \Theta\Gamma_0\right) \qquad (3.8.1)$$

of S in $\mathfrak{H}$, where $\Theta = \Theta^*$ is a self-adjoint relation in $\mathcal{G}$. Recall from Corollary 1.10.9 that there exist operators $\mathcal{A}, \mathcal{B} \in \mathbf{B}(\mathcal{G})$ with the properties

$$\mathcal{A}^*\mathcal{B} = \mathcal{B}^*\mathcal{A}, \quad \mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*, \quad \mathcal{A}^*\mathcal{A} + \mathcal{B}^*\mathcal{B} = I = \mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*,$$

such that

$$\Theta = \{\{\mathcal{A}\varphi, \mathcal{B}\varphi\} : \varphi \in \mathcal{G}\} = \{\{\psi, \psi'\} \in \mathcal{G}^2 : \mathcal{A}^*\psi' = \mathcal{B}^*\psi\}.$$

According to Section 2.2, the self-adjoint extensions A_Θ in (3.8.1) can also be written in the form

$$A_\Theta = \{\widehat{f} \in S^* : \mathcal{A}^*\Gamma_1\widehat{f} = \mathcal{B}^*\Gamma_0\widehat{f}\}.$$

In order to describe the spectrum of A_Θ consider the boundary triplet $\{\mathcal{G}, \Gamma_0', \Gamma_1'\}$, where

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = \begin{pmatrix} \mathcal{B}^* & -\mathcal{A}^* \\ \mathcal{A}^* & \mathcal{B}^* \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}; \qquad (3.8.2)$$

cf. Corollary 2.5.11. Then one has

$$A_\Theta = \ker\Gamma_0', \qquad (3.8.3)$$

and the corresponding Weyl function and γ-field will be denoted by M_Θ and γ_Θ. For $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ they are given by

$$M_\Theta(\lambda) = \left(\mathcal{A}^* + \mathcal{B}^* M(\lambda)\right)\left(\mathcal{B}^* - \mathcal{A}^* M(\lambda)\right)^{-1} \tag{3.8.4}$$

and

$$\gamma_\Theta(\lambda) = \gamma(\lambda)\left(\mathcal{B}^* - \mathcal{A}^* M(\lambda)\right)^{-1},$$

respectively; cf. (2.5.17) and (2.5.18). From (3.8.3) it is clear that the spectrum of A_Θ can be described by means of the Weyl function M_Θ. Therefore, the earlier results expressing the spectrum of A_0 in terms of the Weyl function M (and the γ-field γ) can now be simply translated to the present context. The main results will be listed below; it is left to the reader to formulate analogs of the results in Section 3.7 in the present setting.

First the analogs of Theorem 3.5.5 and Theorem 3.5.10 will be described. For this purpose define the operators $\mathcal{R}_x^\Theta$, $x \in \mathbb{R}$, and $\mathcal{R}_\infty^\Theta$ similar to Definition 3.5.2 and Definition 3.5.8:

$$\mathcal{R}_x^\Theta \varphi = \lim_{y \downarrow 0} iy M_\Theta(x + iy)\varphi, \qquad \varphi \in \mathcal{G},$$

and

$$\mathcal{R}_\infty^\Theta \varphi = \lim_{y \to +\infty} \frac{1}{iy} M_\Theta(iy)\varphi, \qquad \varphi \in \mathcal{G}.$$

As in Section 3.5, one has that $\mathcal{R}_x^\Theta, \mathcal{R}_\infty^\Theta \in \mathbf{B}(\mathcal{G})$. In terms of the boundary triplet $\{\mathcal{G}, \Gamma_0', \Gamma_1'\}$ in (3.8.2) and the corresponding Weyl function M_Θ in (3.8.4), Theorem 3.5.5 and Corollary 3.5.6 read as follows.

Corollary 3.8.1. *Let S, A_Θ, and M_Θ be as above and let $x \in \mathbb{R}$. Then the mapping*

$$\tau : \widehat{\mathfrak{N}}_x(A_\Theta) \ominus \widehat{\mathfrak{N}}_x(S) \to \overline{\mathrm{ran}}\,\mathcal{R}_x^\Theta, \quad \widehat{f} \mapsto \mathcal{A}^*\Gamma_0\widehat{f} + \mathcal{B}^*\Gamma_1\widehat{f},$$

is an isomorphism. In particular,

$$x \in \sigma_\mathrm{p}(A_\Theta) \text{ and } \widehat{\mathfrak{N}}_x(A_\Theta) \ominus \widehat{\mathfrak{N}}_x(S) \neq \{0\} \quad \Leftrightarrow \quad \mathcal{R}_x^\Theta \neq 0,$$

and if $x \notin \sigma_\mathrm{p}(S)$, then $x \in \sigma_\mathrm{p}(A_\Theta)$ if and only if $\mathcal{R}_x^\Theta \neq 0$.

Similarly, Theorem 3.5.10 and Corollary 3.5.11 take the following form.

Corollary 3.8.2. *Let S, A_Θ, and M_Θ be as above. Then the mapping*

$$\tau : \widehat{\mathfrak{N}}_\infty(A_\Theta) \ominus \widehat{\mathfrak{N}}_\infty(S) \to \overline{\mathrm{ran}}\,\mathcal{R}_\infty^\Theta, \quad \widehat{f} \mapsto \mathcal{A}^*\Gamma_0\widehat{f} + \mathcal{B}^*\Gamma_1\widehat{f},$$

is an isomorphism. In particular,

$$\mathrm{mul}\,A_\Theta \ominus \mathrm{mul}\,S \neq \{0\} \quad \Leftrightarrow \quad \mathcal{R}_\infty^\Theta \neq 0,$$

and if $\mathrm{mul}\,S = \{0\}$, then A_Θ is an operator if and only if $\mathcal{R}_\infty^\Theta = 0$.

For the next results the local simplicity condition appearing in many of the results in Section 3.6 has to be reformulated with respect to A_Θ. According to Definition 3.4.9, the closed symmetric relation S is simple with respect to $\Delta \subset \mathbb{R}$ and the self-adjoint extension A_Θ if

$$E_\Theta(\Delta)\mathfrak{H} = \overline{\mathrm{span}}\,\{E_\Theta(\Delta)\gamma_\Theta(\nu)\varphi : \nu \in \mathbb{C} \setminus \mathbb{R}, \varphi \in \mathcal{G}\}, \qquad (3.8.5)$$

where $E_\Theta(\cdot)$ is the spectral measure of A_Θ.

Then Theorem 3.6.1 yields the following statement.

Corollary 3.8.3. *Let S, A_Θ, and M_Θ be as above, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that the local simplicity condition (3.8.5) is satisfied. Then the following statements hold for each $x \in \Delta$:*

(i) *$x \in \rho(A_\Theta)$ if and only if M_Θ can be continued analytically to x;*

(ii) *$x \in \sigma_c(A_\Theta)$ if and only if $\mathcal{R}_x^\Theta = 0$ and M_Θ cannot be continued analytically to x;*

(iii) *x is an eigenvalue of A_Θ if and only if $\mathcal{R}_x^\Theta \neq 0$;*

(iv) *x is an isolated eigenvalue of A_Θ if and only if x is a pole (of first order) of M_Θ; in this case $\mathcal{R}_x^\Theta$ is the residue of M_Θ at x.*

If S is simple, then the statements (i)–(iv) hold for all $x \in \mathbb{R}$.

Finally, the corresponding results for the absolutely continuous, singular, and singular continuous spectra will be formulated; it is left to the reader to state the analogs of Corollaries 3.6.6, 3.6.9, and 3.6.10.

In the present situation Theorem 3.6.5 reads as follows.

Corollary 3.8.4. *Let S, A_Θ, and M_Θ be as above, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that the local simplicity condition (3.8.5) is satisfied. Then the absolutely continuous spectrum of A_Θ in Δ is given by*

$$\overline{\sigma_{\mathrm{ac}}(A_\Theta) \cap \Delta} = \overline{\bigcup_{\varphi \in \mathcal{G}} \mathrm{clos}_{\mathrm{ac}}\big(\{x \in \Delta : 0 < \mathrm{Im}\,(M_\Theta(x+i0)\varphi, \varphi) < \infty\}\big)}. \qquad (3.8.6)$$

If S is simple, then (3.8.6) holds for every open interval Δ, including $\Delta = \mathbb{R}$.

For the singular and singular continuous spectra one obtains the following version of Theorem 3.6.8.

Corollary 3.8.5. *Let S, A_Θ, and M_Θ be as above, let $\Delta \subset \mathbb{R}$ be an open interval, and assume that the local simplicity condition (3.8.5) is satisfied. Then the following statements hold:*

(i) *The singular spectrum of A_Θ in Δ satisfies*

$$(\sigma_s(A_\Theta) \cap \Delta) \subset \overline{\bigcup_{\varphi \in \mathcal{G}} \{x \in \Delta : \mathrm{Im}\,(M_\Theta(x+i0)\varphi, \varphi) = \infty\}}.$$

(ii) *The singular continuous spectrum of A_Θ in Δ, $\sigma_{sc}(A_\Theta) \cap \Delta$, is contained in the set*

$$\bigcup_{\varphi \in \mathcal{G}} \text{clos}_c\big(\{x \in \Delta : \text{Im}\,(M_\Theta(x+i0)\varphi, \varphi) = \infty, \lim_{y \downarrow 0} y(M_\Theta(x+iy)\varphi, \varphi) = 0\}\big).$$

If S is simple, then (i) *and* (ii) *hold for every open interval Δ, including $\Delta = \mathbb{R}$.*

Finally, the special case where the self-adjoint relation Θ in (3.8.1) is a bounded self-adjoint operator will be briefly discussed. In this situation there is a more natural choice of the transformed boundary triplet $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$ above. In fact, if S is a closed symmetric relation, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^* with γ-field γ and Weyl function M, and $\Theta \in \mathbf{B}(\mathcal{G})$ is self-adjoint, then, by Corollary 2.5.7, the mappings

$$\Gamma'_0 = \Gamma_1 - \Theta\Gamma_0 \quad \text{and} \quad \Gamma'_1 = -\Gamma_0$$

lead to a boundary triplet $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$ for S^* such that

$$\ker \Gamma'_0 = \ker (\Gamma_1 - \Theta\Gamma_0) = A_\Theta.$$

For $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ the corresponding γ-field γ_Θ and the Weyl function M_Θ are given by

$$\gamma_\Theta(\lambda) = -\gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1} \quad \text{and} \quad M_\Theta(\lambda) = \big(\Theta - M(\lambda)\big)^{-1}, \qquad (3.8.7)$$

respectively. Then the above results in Corollaries 3.8.1–3.8.5 remain valid with the function M_Θ in (3.8.7) and the mapping $\widehat{f} \mapsto \mathcal{A}^*\Gamma_0\widehat{f} + \mathcal{B}^*\Gamma_1\widehat{f}$ in Corollary 3.8.1 and Corollary 3.8.2 replaced by $\widehat{f} \mapsto -\Gamma_0\widehat{f}$.