Introduction

In this monograph the theory of boundary triplets and their Weyl functions is developed and applied to the analysis of boundary value problems for differential equations and general operators in Hilbert spaces. Concrete illustrations by means of weighted Sturm–Liouville differential operators, canonical systems of differential equations, and multidimensional Schrödinger operators are provided. The abstract notions of boundary triplets and Weyl functions have their roots in the theory of ordinary differential operators; they appear in a slightly different context also in the treatment of partial differential operators.

Before describing the contents of the monograph it may be helpful to explain the ideas in this text by means of the following simple Sturm–Liouville differential expression

$$L = -\frac{d^2}{dx^2} + V, \tag{1}$$

where it is assumed that the potential V is a real measurable function. The context in which this differential expression will be placed serves as an example as well as a motivation. The first step is to associate with L some differential operators in a suitable Hilbert space. Assume, e.g., that (1) is given on the positive half-line $\mathbb{R}^+ = (0, \infty)$ and assume for simplicity that the real function V is bounded. Define the linear space $\mathfrak{D}_{\max}$ by

$$\mathfrak{D}_{\max} = \big\{ f \in L^2(\mathbb{R}^+) : f, f' \text{ absolutely continuous}, \, Lf \in L^2(\mathbb{R}^+) \big\}$$

and define the *minimal operator* S associated with L by

$$Sf = -f'' + Vf, \qquad \operatorname{dom} S = \big\{ f \in \mathfrak{D}_{\max} : f(0) = f'(0) = 0 \big\}.$$

Then S is a closed densely defined symmetric operator $L^2(\mathbb{R}^+)$; in fact, it is the closure of (the graph of) the restriction of S to $C_0^\infty(\mathbb{R}^+)$. It can be shown that the adjoint operator S^* is given by

$$S^* f = -f'' + Vf, \qquad \operatorname{dom} S^* = \mathfrak{D}_{\max},$$

which is usually called the *maximal operator* associated with L. Roughly speaking, S is a two-dimensional restriction of S^* by means of the boundary conditions

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$f(0) = 0$ and $f'(0) = 0$. Note that the maximal domain $\mathfrak{D}_{\max}$ coincides with the second-order Sobolev space $H^2(\mathbb{R}^+)$.

The notion of *boundary triplet* will now be explained in the present situation. For this consider $f, g \in \operatorname{dom} S^*$ and observe that integration by parts leads to

$$\begin{aligned}(S^* f, g)_{L^2(\mathbb{R}^+)} - (f, S^* g)_{L^2(\mathbb{R}^+)} &= -f'(x)\overline{g(x)}\Big|_0^\infty + f(x)\overline{g'(x)}\Big|_0^\infty \\ &= f'(0)\overline{g(0)} - f(0)\overline{g'(0)},\end{aligned}$$

where it was used that the products $f'\bar{g}$ and $f\bar{g}'$ vanish at ∞. Inspired by the above identity, define boundary mappings

$$\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathbb{C}, \quad f \mapsto \Gamma_0 f := f(0) \quad \text{and} \quad f \mapsto \Gamma_1 f := f'(0), \tag{2}$$

so that for all $f, g \in \operatorname{dom} S^*$ one has

$$(S^* f, g)_{L^2(\mathbb{R}^+)} - (f, S^* g)_{L^2(\mathbb{R}^+)} = (\Gamma_1 f, \Gamma_0 g)_{\mathbb{C}} - (\Gamma_0 f, \Gamma_1 g)_{\mathbb{C}}, \tag{3}$$

which is the so-called *abstract Green identity* in the definition of a boundary triplet; note that on the right-hand side of (3) the scalar product in the (boundary) Hilbert space $\mathbb{C}$ is used. This abstract Green identity is the key feature in the notion of a boundary triplet and it is primarily responsible for the succesful functioning of the whole theory. Note also that the combined boundary mapping

$$(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^* \to \mathbb{C}^2$$

is surjective, which is understood as a maximality condition in the sense that the image space of the boundary maps is not unnecessarily large. Observe that one has $\operatorname{dom} S = \ker \Gamma_0 \cap \ker \Gamma_1$. The operator realizations A of the Sturm–Liouville differential expression L which are intermediate extensions, that is, $S \subset A \subset S^*$, can be described by boundary conditions expressed via the boundary maps. More precisely, for $\tau \in \mathbb{C} \cup \{\infty\}$ the operator A_τ is defined by

$$A_\tau f = S^* f, \qquad \operatorname{dom} A_\tau = \ker(\Gamma_1 - \tau \Gamma_0), \tag{4}$$

which in a more explicit form reads

$$A_\tau f = -f'' + Vf, \qquad \operatorname{dom} A_\tau = \big\{ f \in \mathfrak{D}_{\max} : f'(0) = \tau f(0) \big\};$$

the case $\tau = \infty$ is understood as the boundary condition $\ker \Gamma_0$, that is,

$$A_\infty f = -f'' + Vf, \qquad \operatorname{dom} A_\infty = \big\{ f \in \mathfrak{D}_{\max} : f(0) = 0 \big\}. \tag{5}$$

In the definition (4) the quantity τ plays the role of a boundary parameter that links the boundary values $\Gamma_0 f = f(0)$ and $\Gamma_1 f = f'(0)$ of the functions $f \in \operatorname{dom} S^*$, which determine the Dirichlet and Neumann boundary conditions, respectively. The properties of the boundary parameter are directly connected with the properties of the corresponding operator A_τ; in particular, the realization A_τ is self-adjoint in $L^2(\mathbb{R}^+)$ if and only if $\tau \in \mathbb{R} \cup \{\infty\}$.

The next main goal is to motivate and illustrate the definition of the *Weyl function* as an analytic object corresponding to a boundary triplet, which is indispensable in the spectral theory of the intermediate extensions. For this, let $\lambda \in \mathbb{C}$ and consider first the unique solutions φ_λ and ψ_λ of the boundary value problems

$$\begin{aligned} -\varphi_\lambda'' + V\varphi_\lambda &= \lambda\varphi_\lambda, \qquad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0, \\ -\psi_\lambda'' + V\psi_\lambda &= \lambda\psi_\lambda, \qquad \psi_\lambda(0) = 0, \quad \psi_\lambda'(0) = 1, \end{aligned} \tag{6}$$

and note that in general $\varphi_\lambda, \psi_\lambda \notin L^2(\mathbb{R}^+)$. It was shown by H. Weyl more than a century ago that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists $m(\lambda) \in \mathbb{C}$ such that

$$x \mapsto f_\lambda(x) = \varphi_\lambda(x) + m(\lambda)\psi_\lambda(x) \in L^2(\mathbb{R}^+), \tag{7}$$

and it turned out that the function $m : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ is holomorphic and has a positive imaginary part in the upper half-plane $\mathbb{C}^+$. This function and its interplay with spectral theory were later studied extensively by E.C. Titchmarsh; hence the frequently used terminology Titchmarsh–Weyl m-function. It plays a key role in the spectral analysis of Sturm–Liouville differential operators. E.g., the (real) poles of m coincide with the isolated eigenvalues of the self-adjoint Dirichlet operator A_∞ in (5) and the absolutely continuous spectrum of A_∞ is, roughly speaking, given by those $\lambda \in \mathbb{R}$ for which $\operatorname{Im} m(\lambda + i0) > 0$. In a similar way one can also characterize the continuous spectrum, the embedded eigenvalues, and exclude singular continuous spectrum of A_∞.

Observe that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the function $x \mapsto f_\lambda(x)$ in (7) belongs to $\operatorname{dom} S^* = \mathfrak{D}_{\max}$ and that, in fact, $-f_\lambda'' + Vf_\lambda = \lambda f_\lambda$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$; in other words, $f_\lambda \in \ker(S^* - \lambda)$. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for S^* with the boundary mappings defined in (2). From the choice of φ_λ and ψ_λ in (6) it is clear that

$$m(\lambda)\Gamma_0 f_\lambda = m(\lambda)f_\lambda(0) = m(\lambda) = \Gamma_1 f_\lambda, \quad f_\lambda \in \ker(S^* - \lambda). \tag{8}$$

In the general theory this identity is used as the definition of the *Weyl function* corresponding to a boundary triplet. In other words, the Weyl function corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is defined as the function m that satisfies (8) for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (and even for the possibly larger set of λ belonging to the resolvent set of the self-adjoint Dirichlet operator A_∞) and hence coincides with the Titchmarsh–Weyl m-function introduced via (7). Here the Weyl function maps Dirichlet boundary values of L^2-solutions of the equation $-f_\lambda'' + Vf_\lambda = \lambda f_\lambda$ onto the corresponding Neumann boundary values and therefore $m(\lambda)$ acts formally like a *Dirichlet-to-Neumann map*. Besides the Weyl function, one associates to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ the so-called γ-*field* as the mapping $\gamma(\lambda) : \mathbb{C} \to L^2(\mathbb{R}^+)$ that assigns to a prescribed boundary value $c \in \mathbb{C}$ the solution $h_\lambda \in \operatorname{dom} S^*$ of the boundary value problem

$$-h_\lambda'' + Vh_\lambda = \lambda h_\lambda, \qquad \Gamma_0 h_\lambda = h_\lambda(0) = c.$$

Since $\gamma(\lambda)c = h_\lambda = cf_\lambda$, it is clear that $m(\lambda) = \Gamma_1\gamma(\lambda)$. Moreover, one can show with the help of the abstract Green identity that the adjoint $\gamma(\lambda)^* : L^2(\mathbb{R}^+) \to \mathbb{C}$ is given by $\gamma(\lambda)^* = \Gamma_1(A_\infty - \bar{\lambda})^{-1}$. The Weyl function and γ-field associated to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ appear in the perturbation term in Kreĭn's formula

$$(A_\tau - \lambda)^{-1} = (A_\infty - \lambda)^{-1} + \gamma(\lambda)(\tau - m(\lambda))^{-1}\gamma(\bar{\lambda})^*,$$

where, for simplicity, it is assumed that A_τ is a self-adjoint realization of L as in (4) corresponding to some boundary parameter $\tau \in \mathbb{R}$ and $\lambda \in \rho(A_\tau) \cap \rho(A_\infty)$. Kreĭn's formula in this particular case provides a description of the resolvent difference of A_τ and the fixed self-adjoint extension A_∞. It is important to note that $\gamma(\lambda)$ and $\gamma(\bar{\lambda})^*$ in the perturbation term provide a link between the original Hilbert space $L^2(\mathbb{R}^+)$ and the boundary space $\mathbb{C}$, but do not affect the resolvents of A_∞ and A_τ. Therefore, if $\lambda \in \rho(A_0)$, then the singularities of the resolvent $\lambda \mapsto (A_\tau - \lambda)^{-1}$ are reflected in the singularities of the term $\lambda \mapsto (\tau - m(\lambda))^{-1}$ and vice versa. In fact, the function $\lambda \mapsto (\tau - m(\lambda))^{-1}$ is connected with the spectrum of A_τ in the same way as the function $\lambda \mapsto m(\lambda)$ is connected with the spectrum of A_∞.

There is another efficient technique to associate differential operators with the differential expression L, which is based on the *sesquilinear form* $\mathfrak{t}$ corresponding to L,

$$\mathfrak{t}[f, g] = (f', g')_{L^2(\mathbb{R}^+)} + (Vf, g)_{L^2(\mathbb{R}^+)}, \tag{9}$$

defined on, e.g.,

$$\mathfrak{D} = \bigl\{ f \in L^2(\mathbb{R}^+) : f \text{ absolutely continuous, } f' \in L^2(\mathbb{R}^+) \bigr\}, \tag{10}$$

and the first representation theorem for sesquilinear forms. In fact, one verifies that $\mathfrak{t}$ in (9)–(10) is a densely defined closed semibounded form in $L^2(\mathbb{R}^+)$, and hence there exists a uniquely determined self-adjoint operator S_1 with $\operatorname{dom} S_1 \subset \mathfrak{D}$ such that

$$(S_1 f, g)_{L^2(\mathbb{R}^+)} = \mathfrak{t}[f, g], \qquad f \in \operatorname{dom} S_1,\ g \in \operatorname{dom} \mathfrak{D}. \tag{11}$$

Note that here the form domain $\mathfrak{D}$ coincides with the first-order Sobolev space $H^1(\mathbb{R}^+)$. It can be shown that the self-adjoint operator S_1 is actually an extension of the minimal operator S. Instead of the domain $\mathfrak{D}$ in (10) one may consider the sesquilinear form $\mathfrak{t}$ on the smaller domain $\mathfrak{D}_0 = \{f \in \mathfrak{D} : f(0) = 0\}$, which also leads to a densely defined closed semibounded form in $L^2(\mathbb{R}^+)$. Again, via the first representation theorem, there is a corresponding self-adjoint operator S_0 with $\operatorname{dom} S_0 \subset \mathfrak{D}_0$ determined by

$$(S_0 f, g)_{L^2(\mathbb{R}^+)} = \mathfrak{t}[f, g], \qquad f \in \operatorname{dom} S_0,\ g \in \operatorname{dom} \mathfrak{D}_0. \tag{12}$$

One verifies that the self-adjoint operator S_1 in (11) coincides with the self-adjoint realization of L determined by the boundary condition $\ker \Gamma_1$ and that the self-adjoint operator S_0 in (12) coincides with the self-adjoint realization of L determined by the boundary condition $\ker \Gamma_0$ in (4), that is, S_1 corresponds to the boundary parameter $\tau = 0$ and S_0 is the Dirichlet operator corresponding to the boundary parameter $\tau = \infty$. Furthermore, in the situation discussed here the self-adjoint operator S_0 in (12) is the *Friedrichs extension* of the minimal (or preminimal) operator associated to L.

The concept of boundary triplet is supplemented by the notion of *boundary pair*, which is inspired by the form approach indicated above. More precisely, in the present situation it turns out that $\{\mathcal{G}, \Lambda\}$, where $\mathcal{G} = \mathbb{C}$ and

$$\Lambda : \mathfrak{D} \to \mathbb{C}, \qquad f \mapsto \Lambda f := f(0), \tag{13}$$

is a boundary pair for the minimal operator S (corresponding to S_1). For this, one has to ensure that the mapping Λ defined on the form domain of S_1 is continuous with respect to the Hilbert space topology generated by the closed form $\mathfrak{t}$ on $\mathfrak{D}$, and that $\ker \Lambda$ coincides with the form domain corresponding to the Friedrichs extension of S. Note also that in the present situation the mapping Λ in (13) is an extension of the boundary mapping $\Gamma_0 : \operatorname{dom} S^* \to \mathbb{C}$ to the form domain $\mathfrak{D}$. With the help of the boundary pair $\{\mathbb{C}, \Lambda\}$ one can parametrize all densely defined closed semibounded forms corresponding to semibounded self-adjoint extensions of S via

$$\mathfrak{t}_\tau[f, g] = \mathfrak{t}[f, g] + (\tau \Lambda f, \Lambda g)_{\mathbb{C}}, \qquad f, g \in \mathfrak{D}, \tag{14}$$

where $\tau \in \mathbb{R} \cup \{\infty\}$, and the case $\tau = \infty$ corresponds to the boundary condition $\Lambda f = 0$ in $\mathfrak{D}_0$. The boundary pair and the boundary triplet are connected via the first Green identity

$$(S^* f, g)_{L^2(\mathbb{R}^+)} = \mathfrak{t}[f, g] + (\Gamma_1 f, \Lambda g)_{\mathbb{C}}, \quad f \in \operatorname{dom} S^*,\ g \in \mathfrak{D}.$$

The first Green identity makes it possible to identify the closed semibounded forms in (14) with the corresponding self-adjoint operator realizations A_τ of L described via boundary conditions in (4). For $f \in \operatorname{dom} A_\tau$ and $g \in \mathfrak{D}$, the first Green identity reduces to

$$(A_\tau f, g)_{L^2(\mathbb{R}^+)} = \mathfrak{t}[f, g] + (\tau \Gamma_0 f, \Lambda g)_{\mathbb{C}} = \mathfrak{t}[f, g] + (\tau \Lambda f, \Lambda g)_{\mathbb{C}},$$

and the expression $(\tau \Lambda f, \Lambda g)_{\mathbb{C}}$ on the right-hand side can also be interpreted as a sesquilinear form in the boundary space $\mathbb{C}$. In this sense the theory of boundary pairs for semibounded symmetric operators complements the theory of boundary triplets in a natural way: it provides a description of the closed semibounded forms corresponding to semibounded self-adjoint extensions of the minimal operator S.

Methods to treat Sturm–Liouville problems such as the one discussed above go back to H. Weyl [758, 759, 760], whose papers on this topic appeared in 1910/1911; see also [761]. The interpretation of a Sturm–Liouville expression as an operator in a Hilbert space can already be found in the 1932 book of M.H. Stone [724]. In this monograph Stone gave an abstract treatment of operators in a Hilbert space including the work of J. von Neumann [610, 611] from 1929 and 1932, who had also introduced the extension theory of densely defined symmetric operators and found the formulas which carry his name: self-adjoint extensions correspond to unitary mappings between the defect spaces. The von Neumann formulas are abstract, since they are formulated in terms of the defect spaces of the symmetric operator, and they needed to be related to concrete boundary value problems. With this in mind another approach involving abstract boundary conditions was developed by J.W. Calkin [187] in his 1937 Harvard doctoral dissertation, which was written under the direction of Stone, who suggested the topic. Calkin was also advised by von Neumann. Calkin's work on boundary value problems did not receive the attention it might have deserved. It seems that he never returned to it; his later mathematical work was related to World War II and the Manhattan project in Los Alamos.

Another way to deal with the self-adjoint extensions of a symmetric operator is via Kreĭn's resolvent formula. The early background of this formula can be found in the idea of perturbation of self-adjoint operators. Kreĭn's formula describes the resolvent of a self-adjoint extension in terms of the resolvent of a fixed self-adjoint extension and a perturbation term which involves a so-called Q-function and a parameter describing the self-adjoint extension. The Q-function uniquely determines the underlying symmetry and the fixed self-adjoint extension, up to unitary equivalence, and thus reflects their spectral properties. The original Kreĭn formula for equal finite defect numbers goes back to M.G. Kreĭn [491, 492] in the middle of the 1940s; only in 1965 it was finally established for the case of equal infinite defect numbers by S.N. Saakyan [679]. In fact, the self-adjoint extensions were allowed to be in a Hilbert space which contains the original Hilbert space as a closed subspace. This type of extension appeared after 1940 in papers by M.G. Kreĭn and M.A. Naĭmark [605, 606, 607]. Later A.V. Štraus in the 1950s and 1960s described such exit space extensions in the framework of the von Neumann formulas via holomorphic contractions between the defect spaces [731]. The Q-function in Kreĭn's formula can be seen as an abstract analog of the Titchmarsh–Weyl function in the above Sturm–Liouville example; it was extensively studied in the 1960s and 1970s by M.G. Kreĭn and H. Langer [497]–[504], also in the context of Pontryagin spaces.

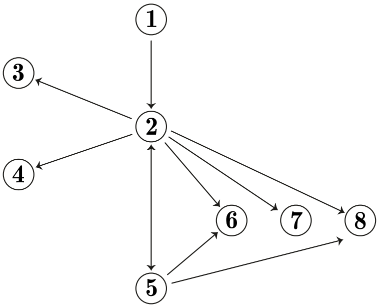
From the early 1940s on E.C. Titchmarsh turned his attention to the singular Sturm–Liouville equation. He put aside Weyl's method of handling the Sturm–Liouville problem on the basis of integral equations and also bypassed the use of the general theory of linear operators in Hilbert spaces as in Stone's book [739]. Instead, Titchmarsh used contour integration and the Cauchy calculus of residues, influenced by the work of E. Hilb [417, 418, 419], a contemporary of Weyl. In this way he found a simple formula to determine the spectral measure; this last formula was also discovered by K. Kodaira around the same time [469, 470]. A complete survey of the work of Titchmarsh, both for ordinary and partial differential operators, is given in his two books on eigenfunction expansions [740, 741]. A different approach, followed by B.M. Levitan [541, 542], N. Levinson [539, 540], and K. Yosida [780, 781], is based on the fact that the resolvent operator of the self-adjoint realization of a singular differential operator can be approximated by compact resolvents corresponding to Sturm–Liouville problems for proper closed subintervals. Closely connected with this is an abstract approach to eigenfunction expansions generated by differential operators that was introduced by Kreĭn [495] in the form of directing functionals.

Influenced by questions from mathematical physics, von Neumann posed the following problem in the middle of the 1930s: can one extend a densely defined semibounded symmetric operator to a self-adjoint operator with the same lower bound? There were contributions by M.H. Stone [724] and K.O. Friedrichs [310] (whose work was simplified by H. Freudenthal [309]). The Friedrichs extension was the solution to von Neumann's problem. For Sturm–Liouville operators the Friedrichs extension was determined in various cases by K.O. Friedrichs [311] in 1935 and by F. Rellich [654] in 1950. Another semibounded extension, the so-called Kreĭn–von Neumann extension (going back to Stone) has particularly interesting properties. It was Kreĭn [493, 494] who established a complete theory of semibounded extensions. In the middle of the 1950s this circle of ideas was carried forward, and it inspired contributions by M.S. Birman [139], and also M.I. Vishik [747], who was particularly interested in the case of elliptic partial differential operators. Building on the work of J.L. Lions and E. Magenes [544] on Sobolev spaces and trace mappings G. Grubb [352, 353] gave a characterization of all closed extensions of a minimal elliptic operator by nonlocal boundary conditions in her 1966 Stanford doctoral dissertation, written under the direction of R.S. Phillips.

The context of symmetric operators which are densely defined was soon felt to be too restrictive. Already in 1949 M.A. Krasnoselskiĭ [490] described all self-adjoint operator extensions of a not necessarily densely defined symmetric operator. The appearance of the work on linear relations by R. Arens [42] in 1961 made all the difference. B.C. Orcutt [619] in a 1969 dissertation written under the direction of J. Rovnyak treated the spectral theory of canonical systems of differential equations in terms of linear relations. Subsequently, E.A. Coddington [202] in 1973 gave a description of all self-adjoint relation extensions of a symmetric relation. In fact, it turned out that many of the earlier results concerning extensions of symmetric operators could be put in the framework of relations. The new context made it also possible to consider nonstandard boundary conditions (involving integrals, for instance). Furthermore, in terms of relations the Kreĭn–von Neumann extension of a semibounded relation could be simply expressed in terms of the Friedrichs extension. There has been an abundance of papers devoted to linear relations in Hilbert spaces, and later also to linear relations in indefinite inner product spaces.

In the middle of the 1970s boundary triplets were introduced independently by V.M. Bruk [176] and A.N. Kochubei [466] as a convenient tool for the description of boundary values of abstract Hilbert space operators; they applied them to, e.g., Sturm–Liouville operators with an operator-valued potential. The main feature is that under a given boundary triplet there is a natural correspondence between self-adjoint extensions of a symmetric operator and self-adjoint relations in the parameter space. An overview of the theory with applications to differential operators is contained in the 1984 book by M.L. Gorbachuk and V.I. Gorbachuk [346]. Around the same time V.A. Derkach and M.M. Malamud [244, 246] continued the work on boundary triplets by associating the notion of Weyl function to a boundary triplet; their later work was written in the context of symmetric operators that are not necessarily densely defined. The Weyl function is a very useful tool in spectral analysis; it turns out to be a special choice of a Q-function (which is uniquely determined by the boundary triplet) and hence the analytic properties and the limit behavior of the Weyl function towards the real line reflect the spectral properties of the self-adjoint extensions. Broadly speaking, boundary triplets and Weyl functions placed the work of Titchmarsh, and others, in a more abstract setting while retaining the flavor of concrete boundary value problems. The link to form methods and the Birman–Kreĭn–Vishik approach to semibounded self-adjoint extensions is made with the help of so-called boundary pairs. The origin of the concept of boundary pair lies in the work of Kreĭn and Vishik; it was formalized and studied by V.E. Lyantse and O.G. Storozh [552] in the early 1980s. Its connection with boundary triplets was later established by Yu.M. Arlinskiĭ [44].

It is the main objective of this monograph to present the theory of boundary triplets and Weyl functions in an easily accessible and self-contained manner. The exposition is detailed and kept as simple as possible; the reader is only assumed to be familiar with the basic principles of functional analysis and some fundamentals of the spectral theory of self-adjoint operators in Hilbert spaces. The monograph is divided into the abstract part Chapters 1–5, the applied part Chapters 6–8, and Appendices A–D. The heart of the monograph is Chapter 2 and it is complemented by Chapter 5; for a rough idea on the general techniques the reader may first look through these chapters and examine one of the applications (which may also be read independently) afterwards: Sturm–Liouville operators, canonical systems, or Schrödinger operators – up to personal taste and preferences.



The monograph opens in Chapter 1 with a detailed introduction to the theory of linear operators and relations in Hilbert spaces. A large part of this material is preparatory and may be used for reference purposes in the rest of the text.

The heart of the matter in this book is contained in Chapter 2, where boundary value problems are presented as extension problems of symmetric operators or relations. Here the notions of boundary triplets and their Weyl functions are introduced, and the fundamental properties of these objects are provided. Particular attention is paid to the question of existence and uniqueness of boundary triplets. Closely connected with a boundary triplet is Kreĭn's resolvent formula for canonical extensions and self-adjoint extensions in larger Hilbert spaces.

Chapter 3 is a continuation and further refinement of the techniques in the previous chapter. Here the main objective is to give a detailed description of the complete spectrum of the self-adjoint extensions of a symmetric relation in terms of the Weyl function. The connection between the limit properties of the Weyl function and the spectrum of the self-adjoint extension is explained via the Borel transform of the spectral measure.

Most of the topics in Chapter 4 are supplementary to the main text as they are concerned with a certain type of inverse problem. More precisely, it will be shown that any (uniformly strict) operator-valued Nevanlinna function can be realized as the Weyl function corresponding to a boundary triplet for a symmetric relation in a reproducing kernel Hilbert model space. Of independent interest is the discussion around the orthogonal coupling of boundary triplets with a view to exit space extensions.

Another central theme in this monograph is presented in Chapter 5, where the important case of semibounded symmetric relations is treated in more detail; here the general methods from Chapter 2 are further developed. The chapter starts with an introduction to closed semibounded forms and the corresponding representation theorems, and continues with the Friedrichs extension, the so-called Kreĭn type extensions, and the Kreĭn–von Neumann extension. The ultimate result is a description of the semibounded self-adjoint extensions of a semibounded relation via the notions of a boundary triplet and a boundary pair; this establishes the connection with the Kreĭn–Birman–Vishik theory.

The general theory is applied to boundary value problems for differential operators in Chapters 6–8 in three different situations. In each case the presentation follows a similar scheme: After the necessary preparations to keep these chapters mostly self-contained, explicit boundary triplets and Weyl functions for the particular operators or relations under consideration, are provided. A further spectral analysis, depending on the nature of problem is presented. The class of Sturm–Liouville operators that is discussed in Chapter 6 covers also the example given earlier in this introduction. A good deal of preparation is needed to construct closed semibounded forms and corresponding boundary pairs in the singular situation. Chapter 7 deals with 2×2 canonical systems of differential equations and also illustrates the role of linear relations in the analysis of such systems. Finally, in Chapter 8 Schrödinger operators on bounded domains $\Omega \subset \mathbb{R}^n$ are treated, where one of the main challenges is to construct Dirichlet and Neumann traces on the maximal domain.

For the reader's convenience a number of appendices have been added: they contain material concerning Nevanlinna functions and some useful elementary observations on operators and subspaces in Hilbert spaces. At the end of the text a few notes and some (historical) comments, as well as a list of recent and earlier references, can be found. Here the reader is also referred to some recent literature for topics that go beyond this monograph.