Optimal Morphs of Planar Orthogonal Drawings II

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Abstract. Van Goethem and Verbeek [12] recently showed how to morph between two planar orthogonal drawings Γ_I and Γ_O of a connected graph G while preserving planarity, orthogonality, and the complexity of the drawing during the morph. Necessarily drawings Γ_I and Γ_O must be equivalent, that is, there exists a homeomorphism of the plane that transforms Γ_I into Γ_O. Van Goethem and Verbeek use $O(n)$ linear morphs, where n is the maximum complexity of the input drawings. However, if the graph is disconnected their method requires $O(n^{1.5})$ linear morphs. In this paper we present a refined version of their approach that allows us to also morph between two planar orthogonal drawings of a disconnected graph with $O(n)$ linear morphs while preserving planarity, orthogonality, and linear complexity of the intermediate drawings.

Van Goethem and Verbeek measure the structural difference between the two drawings in terms of the so-called *spirality* $s = O(n)$ of Γ_I relative to Γ_O and describe a morph from Γ_I to Γ_O using $O(s)$ linear morphs. We prove that $s+1$ linear morphs are always sufficient to morph between two planar orthogonal drawings, even for disconnected graphs. The resulting morphs are quite natural and visually pleasing.

1 Introduction

Continuous morphs of planar drawings have been studied for many years, starting as early as 1944, when Cairns [7] showed that there exists a planarity-preserving continuous morph between any two (compatible) triangulations that have the same outer triangle. These results were extended by Thomassen [10] in 1983, who gave a constructive proof of the fact that two compatible straight-line drawings can be morphed into each other while maintaining planarity. The resulting algorithm to compute such a morph takes exponential time (just as Cairns' result). Thomassen also considered the orthogonal setting and showed how to morph between two rectilinear polygons with the same turn sequence. For planar straight-line drawings the question was settled by Alamdari et al. [1], following work by Angelini et al. [3]. They showed that $O(n)$ uni-directional linear morphs are sufficient to morph between any compatible pair of planar straight-line drawings of a graph with n vertices while preserving planarity. The corresponding morph can be computed in $O(n^3)$ time.

In this paper we consider the orthogonal setting, that is, we study planarity-preserving morphs between two planar orthogonal drawings Γ_I and Γ_O with maximum complexity n, of a graph G. Here the complexity of an orthogonal drawing is defined as the number of vertices and bends. All intermediate drawings must remain orthogonal, as to not disrupt the mental map of the reader. This immediately implies that the results of Alamdari et al. [1] do not apply, since they do not preserve orthogonality. Biedl et al. [5] described the first results in this setting, for so-called *parallel* drawings, where every edge has the same orientation in both drawings. They showed how to morph between two parallel drawings using $O(n)$ linear morphs while maintaining parallelity and planarity. More recently, Biedl et al. [4] showed how to morph between two planar orthogonal drawings using $O(n^2)$ linear morphs, while preserving planarity, orthogonality, and linear complexity. Van Goethem and Verbeek [12] improved this bound further to $O(n)$ linear morphs for a connected graph G. This bound is tight, based on the lower bound for straight-line graphs proven by Alamdari et al. [1].

If the graph G is disconnected, then Aloupis et al. [2] show how to connect G in a way that is compatible with both Γ_I and Γ_O while increasing the complexity of the drawings to at most $O(n^{1.5})$. They also prove a matching lower bound if G has at most $\frac{n}{4}$ connected components. This directly implies that Van Goethem and Verbeek require $O(n^{1.5})$ linear morphs for a disconnected graph G.

Paper Outline. We show how to refine the approach by Van Goethem and Verbeek [12] to also morph between two planar orthogonal drawings of a disconnected graph G using $O(n)$ linear morphs while preserving planarity, orthogonality, and linear complexity. In Sect. 2 we describe the necessary background. In particular, we discuss *wires*: equivalent sets of horizontal and vertical polylines that capture the x- and y-order of the vertices in Γ_I and Γ_O. The *spirality* of these wires guides the morph. In Sect. 3 we show how to find sets of wires with linear spirality for equivalent orthogonal planar drawings Γ_I and Γ_O of a disconnected planar graph G. Van Goethem and Verbeek are agnostic of the connectivity of the graph once they create the wires. Hence, using the wires constructed in Sect. 3, we can directly apply their approach to disconnected graphs.

In the remainder of the paper we show how to "batch" intermediate morphs. We argue solely based on sets of wires, hence the results apply to both connected and disconnected graphs. In particular, in Sect. 4 we show how to combine all intermediate morphs that act on segments of spirality s into one single linear morph. Hence we need only s linear morphs to morph from Γ_I to Γ_O. However, the rerouting and simplification operations introduced by van Goethem and Verbeek to lower the intermediate complexity are not compatible with batched linear morphs and hence intermediate drawings have complexity of $O(n^3)$. In Sect. 5 we present refined versions of both operations which allow us to maintain linear complexity through the s linear morphs. The initial setup for these operations costs one additional morph, for a total of $s + 1$ linear morphs that preserve planarity, orthogonality, and linear complexity. We implemented our algorithm and believe that the resulting morphs are natural and visually pleasing[1]. We restrict

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[1] See https://youtu.be/nOZaPtfg9TM for a short movie.

our arguments to proof sketches, all omitted details can be found in the full version [11].

2   Preliminaries

Orthogonal Drawings. A *drawing* Γ of a graph $G = (V, E)$ is a mapping from every vertex $v \in V$ to a unique point $\Gamma(v)$ in the Euclidean plane and from each edge (u, v) to a simple curve in the plane starting at $\Gamma(u)$ and ending at $\Gamma(v)$. A drawing is *planar* if no two curves intersect in an internal point, and no vertices intersect a curve in an internal point. A drawing is *orthogonal* if each edge is mapped to an orthogonal polyline consisting of horizontal and vertical segments meeting at *bends*. In a *straight-line drawing* every edge is represented by a single line-segment. Two planar drawings Γ and Γ' are *equivalent* if there exists a homeomorphism of the plane that transforms Γ into Γ'.

We consider morphs between two equivalent drawings of a graph G. To simplify the presentation, we assume that both drawings are straight-line drawings with n vertices. If this is not the case then we first *unify* Γ and Γ'. We subdivide segments, creating additional virtual bends, to ensure that every edge is represented by the same number of segments in Γ and Γ'. Next, we replace all bends with vertices. All edges of the resulting graph G^* are now represented by straight segments (horizontal or vertical) in both Γ and Γ'.

A *linear morph* of two drawings Γ and Γ' can be described by a continuous linear interpolation of all vertices and bends, which are connected by straight segments. For each $0 \leq t \leq 1$ there exists an intermediate drawing Γ_t where each vertex v is drawn at $\Gamma_t(v) = (1 - t)\Gamma_v + t\Gamma'_v$ ($\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$). A linear morph *maintains planarity* (orthogonality, linear complexity, resp.), if every intermediate drawing Γ_t is planar (orthogonal, of linear complexity, resp.).

Wires. Following van Goethem and Verbeek [12] we use orthogonal polylines called *wires* as the main tool to determine the morph. Wires consist of horizontal or vertical segments called *links*. We use two sets of wires to capture the horizontal and vertical order of the vertices in Γ_I and Γ_O. The *lr-wires* $W_\rightarrow$ traverse the drawings from left to right, and the *tb-wires* $W_\downarrow$ traverse the drawings from top to bottom. Since the horizontal and vertical order of the vertices in Γ_O are
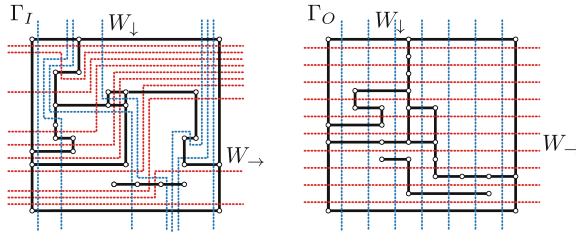

Fig. 1. Two unified drawings Γ_I and Γ_O of G (black) plus equivalent wires (red/blue). (Color figure online)

guiding our morph, the wires $W_\rightarrow$ and $W_\downarrow$ are simply horizontal and vertical lines in Γ_O separating consecutive vertices in the x- and y-order (only if their x- or y-coordinates are distinct). Γ_O and Γ_I are equivalent, hence there exist wires in Γ_I that are equivalent to the wires in Γ_O: there is a one-to-one matching between the wires of Γ_O and Γ_I such that matching wires partition the vertices identically, and cross both the segments of the drawings and the links of the other wires in the same order (see Fig. 1). Any such two wires in Γ_I do not cross if they are from the same set and cross exactly once otherwise.

Van Goethem and Verbeek use the *spirality* of wires as a measure for the distance to Γ_O (where all wires are straight lines of spirality zero). Spirality is a well-established measure in the context of orthogonal drawings and is frequently used for bend-minimization [6,8,9]. Specifically, let $w \in W_\rightarrow$ be a lr-wire, and $\ell_1, \ldots, \ell_k$ be the links ordered along w. Let b_i be the orientation of the bend from ℓ_i to ℓ_{i+1}, where $b_i = 1$ for a left turn, $b_i = -1$ for a right turn, and $b_i = 0$ otherwise. The *spirality* of a link ℓ_i is defined as $s(\ell_i) = \sum_{j=1}^{i-1} b_j$. A *maximum-spirality link* is any link with the largest absolute spirality. The spirality of a wire is the maximum absolute spirality of any link in the wire, the spirality of a set of wires is the maximum spirality of any wire in the set.

The spirality of a drawing Γ is not well defined: it is always relative to another drawing Γ' and the straight-line wires induced by Γ'. Furthermore, there are possibly multiple sets of matching wires in Γ for the straight-line wires in Γ'. Still, whenever the drawing Γ' and the matching set of wires in Γ are clear from the context, then by abuse of notation we will speak of the spirality of Γ. Unless stated otherwise, we always consider spirality relative to Γ_O.

Slides. Biedl et al. [4] introduced *slides* as a particular type of linear morph that operates on the segments of the drawing. Van Goethem and Verbeek [12] extended this concept to wires. Slides on wires may be accompanied by the insertion or deletion of bends in the drawing. In the following we exclusively consider slides on wires. A *zigzag* consists of three consecutive links of a wire and two bends β and γ that form a left turn followed by a right turn or vice versa. Consider the horizontal zigzag with bends β and γ in Fig. 2(a). Let $\mathcal{V}$ be the set of vertices and bends of both the drawing and the wires that are (1) above or at the same height as β and strictly to the left of β, (2) that are strictly above γ, and (3) β. The corresponding region is shaded in Fig. 2. A *zigzag-eliminating*
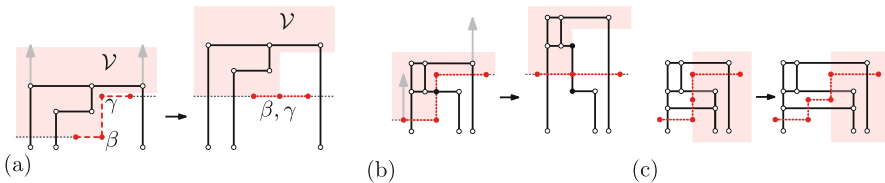

Fig. 2. A drawing (black) with vertices (open marks) and bends (closed marks). **(a)** A *zigzag-eliminating slide* with center link $\overline{\beta\gamma}$. **(b)** Introducing two additional bends in a crossing segment ensures orthogonality. **(c)** A *bend-introducing slide*.

slide is a linear morph that straightens a zigzag on a wire by moving all vertices and bends in $\mathcal{V}$ up by the initial distance between β and γ.

By definition, wires do not contain any vertices or bends of the drawing or other wires. However, the center link $\overline{\beta\gamma}$ might be crossed by a segment of the drawing or a link of a wire in the other set (see Fig. 2(b) for a crossing with a segment of the drawing). In this case we introduce two virtual bends in the segment or the link on the crossing and symbolically offset one to the right and one to the left. The left bend is thus included in $\mathcal{V}$ while the right bend is not. We can prevent that multiple segments or links cross $\overline{\beta\gamma}$ using so-called *bend-introducing slides* as discussed in [12] (see Fig. 2(c)).

3   Linear Morphs for Disconnected Graphs

Let Γ_I and Γ_O be two equivalent planar orthogonal drawings of a disconnected graph G. For a connected graph there is a unique homotopy class in Γ_I that contains all possible wires that match a given wire w from Γ_O. This statement does not hold for disconnected graphs: there might be more than one homotopy class in Γ_I that matches w (see Fig. 3(a)). If we choose homotopy classes independently for the wires in Γ_I then their union might not be equivalent to the set of wires in Γ_O, for example, wires might cross more than once (see Fig. 3(c)).

Below we show that we can choose homotopy classes for the wires in Γ_I incrementally, first for the lr-wires and then for the tb-wires, while maintaining the correct intersection pattern and hence equivalence with Γ_O. For each of the resulting equivalence classes we add the shortest wire to the set of wires. It remains to argue that the resulting set of wires has spirality $O(n)$ despite the interdependence of the homotopy classes and the fact that the arrangement of drawing and wires can have super-linear complexity (which invalidates the proofs from [12]). Below we consider only $W_{\rightarrow}$, analogous results hold for $W_{\downarrow}$.

Lemma 1. *For each right-oriented link $\ell_{\rightarrow}$ of a wire $w \in W_{\rightarrow}$ with positive (negative) spirality s there exists a vertical line L and a subsequence of $\Omega(|s|)$ links of w crossing L, such that the absolute spiralities of the links in sequence are $[0, 2, 4, \ldots, |s|-2, |s|]$, and when ordered top-to-bottom (bottom-to-top) along L form the sequence $[2, 6, 10, \ldots, |s|-2, |s|, |s|-4, \ldots, 4, 0]$.*
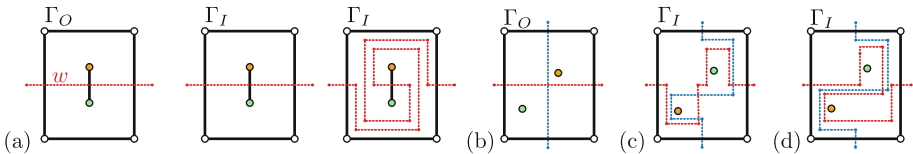

Fig. 3. (a) A (straight-line) wire w in Γ_O (red) and two possible wires in Γ_I from different homotopy classes that both match w. **(b)** A graph with three connected components. **(c)** Wires in Γ_I that cross three times. **(d)** Set of wires equivalent to Γ_O. (Color figure online)
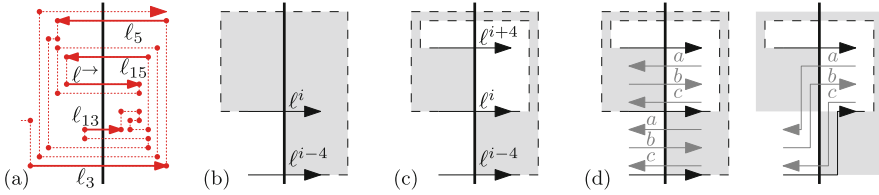
Fig. 4. **(a)** Lemma 1 for a link $\ell_\rightarrow$ and sequence $S = (\ell_3, \ell_5, \ell_{13}, \ell{15}, \ell_\rightarrow)$. **(b)** The i-core of a spiral for a link $\ell^i \in S$ (gray). **(c)** The i-layer of the spiral (gray). **(d)** A layer cannot only contain wires as then we can shorten all wires.

Figure 4(a) illustrates Lemma 1. Let $\ell_\rightarrow$ be a right-oriented link on a wire w and w.l.o.g. let $s > 0$ be the spirality of w. Further, let L be a vertical line through $\ell_\rightarrow$ and S a subsequence from w with the properties guaranteed by Lemma 1. Finally, let $\ell^i \in S$ be the unique link with spirality $0 \leq i \leq s$ in S. We define the i-*core* for S (for $4 \leq i \leq s$ and $i \pmod 4 = 0$) as the region enclosed by the wire w from the intersection between ℓ^{i-4} and L to the intersection between ℓ^i and L and the straight line segment along L connecting them (see Fig. 4(b)). We define the i-*layer* for S (for $4 \leq i \leq s - 4$ and $i \pmod 4 = 0$) as the difference of the i-core and the $(i + 4)$-core (see Fig. 4(c)).

Lemma 2. *An equivalent set of lr-wires with spirality $O(n)$ exists.*

Proof (Sketch). We prove by induction that we can add a new lr-wire with spirality $O(n)$. If a wire w has $\omega(n)$ layers, then we can argue via shortcuts (see Fig. 4(d)) that w was not shortest with respect to previously inserted wires. □

Lemma 3. *An equivalent set of wires with spirality $O(n)$ exists.*

Proof (Sketch). By Lemma 2 we can insert all lr-wires with spirality $O(n)$. By Lemma 2 from [12] the spirality of intersecting links is the same. Apply Lemma 2 for the tb-wires in the regions between the intersections with lr-wires. □

Theorem 1. *Let Γ_I and Γ_O be two unified planar orthogonal drawings of a (disconnected) graph G. We can morph Γ_I into Γ_O using $\Theta(n)$ linear morphs while maintaining planarity and orthogonality.*

Proof. By Lemma 3 an equivalent set of wires with spirality $s = O(n)$ exists. By Theorem 8 from [12] we can thus morph the drawings into each other using $O(s) = O(n)$ linear morphs. The lower bound of $\Omega(n)$ follows from [1]. □

4 Combining Intermediate Linear Morphs

The proof of Theorem 1 implies a morph between two unified planar orthogonal drawings Γ_I and Γ_O exists using $O(s)$ linear morphs, where s is the spirality of Γ_I. In this section we show how to combine consecutive linear morphs into a total number of only s linear morphs, while maintaining planarity and orthogonality.

The morphs we describe can be encoded by a sequence of drawings, starting with Γ_I and ending with Γ_O, such that every consecutive pair of drawings can be linearly interpolated while maintaining planarity and orthogonality. For notational convenience let $\Gamma_i \rightarrow \Gamma_j$ indicate that Γ_i occurs before Γ_j during the morph and $\Gamma_i \Rightarrow \Gamma_j$ that $\Gamma_i \rightarrow \Gamma_j$ or $\Gamma_i = \Gamma_j$.

Let an *iteration* of the original morph consist of all linear slides that jointly reduce spirality by one. Let the first drawing of iteration s be the first drawing in the original morph with spirality s and the last drawing be the first drawing with spirality $s-1$. Consecutive iterations overlap in exactly one drawing. These drawings in the overlap of iterations are the intermediate steps of the final morph. Within this section let $\Gamma_I \Rightarrow \Gamma_a \rightarrow \Gamma_b \Rightarrow \Gamma_O$, where Γ_a is the first drawing with spirality s and Γ_b is the first drawing with spirality $s-1$.

4.1   Staircases

Consider two distinct vertices v and w of the drawing. Define an *x-inversion* (*y-inversion*) of v and w between Γ_a and Γ_b when the sign $(+,-,0)$ of $v.x - w.x$ ($v.y - x.y$) differs in Γ_a and Γ_b. We say two vertices are *x-inverted* (*y-inverted*), or simply *inverted*. Two vertices v and w are *separated* in a drawing by a link ℓ when they are both in the vertical (horizontal) strip spanned by ℓ, and v and w are on opposite sides of ℓ.

Lemma 4. *Two vertices v and w can be inverted by a zigzag-removing slide along link ℓ, if and only if v and w are separated by ℓ.*

A *downward staircase* is a sequence of horizontal links where: (1) the left-endpoints are x-monotone increasing and y-monotone decreasing, (2) the projection on the x-axis is overlapping or touching for a pair if and only if they are consecutive in the sequence, and (3) all links have positive spirality. Two vertices v and w are *separated* by a downward staircase if v is in the vertical strip spanned by the first link of the staircase and above it and w is in the vertical strip spanned by the last link and below it. Similar concepts can be defined for upwards staircases and for vertical links.
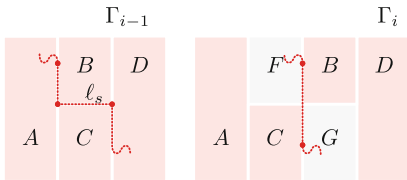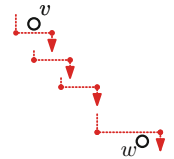

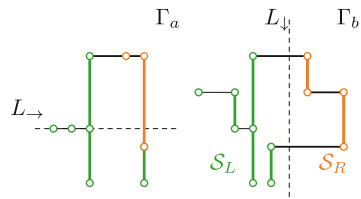**Fig. 5.** Regions surrounding ℓ_s in Γ_{i-1} and the matching regions in Γ_i.

Fig. 6. Sets $\mathcal{S}_L$ and $\mathcal{S}_R$ in Γ_a and Γ_b.

Lemma 5. *Two vertices v and w that are x-inverted (y-inverted) first during a morph from Γ_a to Γ_b, are separated by a horizontal (vertical) staircase of maximum spirality links in Γ_a.*

Proof (Sketch). Assume w.l.o.g. that only one inversion occurs and it occurs from Γ_{b-1} to Γ_b. By Lemma 4, v and w are separated by a link ℓ in Γ_{b-1}. Link ℓ must have maximum absolute spirality as it was selected for the morph. We now prove inductively that a staircase exists in all drawings from Γ_a to Γ_{b-1} by "moving backwards" through the morph. To this end we define four rectangular regions A, B, C, D surrounding ℓ_s in Γ_{i-1} (see Fig. 5). During the linear slide from Γ_{i-1} to Γ_i two new regions F and G are created, which cannot contain vertices. Using these rectangular regions and a case distinction on the type of linear slide, we can argue inductively that a staircase separating v and w must also exist in Γ_{i-1}. □

4.2   Inversions

We show that every pair of vertices is inverted along at most one axis during the morph from Γ_a to Γ_b. We then prove that Γ_a has spirality one relative to Γ_b.

Lemma 6. *Two vertices v and w can be inverted along only one axis during the morph from Γ_a to Γ_b.*

Lemma 7. *Each vertical (horizontal) line in Γ_b not crossing a vertex, can be matched to a y- (x-)monotone wire in Γ_a.*

Proof (Sketch). Consider a vertical line $L_\downarrow$ in Γ_b not intersecting any vertex. Line $L_\downarrow$ partitions the set of vertices and vertical edges in Γ_b into two subsets $\mathcal{S}_L$ and $\mathcal{S}_R$. Consider a horizontal line $L_\rightarrow$ in Γ_a and consider the maximal intervals formed along it by elements from the same set $\mathcal{S}_L$ or $\mathcal{S}_R$ (see Fig. 6). Set $\mathcal{S}_L$ and $\mathcal{S}_R$ form exactly two maximal intervals along $L_\rightarrow$. Thus a y-monotone line exists correctly splitting $\mathcal{S}_L$ and $\mathcal{S}_R$. We can show that this y-monotone line must intersect horizontal edges in the correct order as well. □

Lemma 8. *Drawing Γ_a has spirality one relative to Γ_b.*

4.3   Single Linear Morph

We now show that any two planar orthogonal drawings Γ_i and Γ_j, where Γ_i has spirality one relative to Γ_j, can be morphed into each other using a single linear morph while maintaining planarity. Two drawings are *shape-equivalent* if for each edge the sequence of left and right turns is identical and the orientation of the initial segment is identical in both drawings. We say two drawings are *degenerate shape-equivalent* if edges may contain zero-length segments but an assignment of orientations to the segments exists that is consistent with both drawings. Two (degenerate) shape-equivalent drawings are per definition also unified. We can make Γ_a degenerate shape-equivalent to Γ_b by adding zero-length edges whenever maximum absolute spirality links in Γ_a cross an edge.
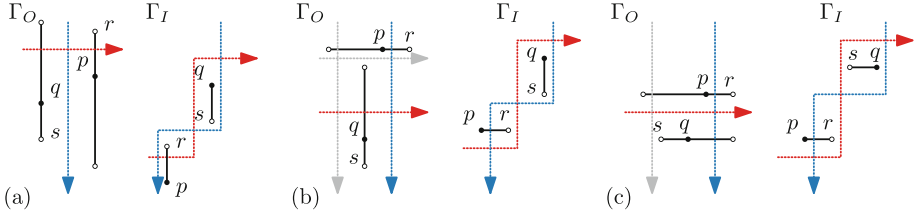
Fig. 7. (a) Two points p and q on vertical segments of the drawing that are inverted along both axes imply wires in Γ_I that are not equivalent to Γ_O. **(b)** Points p and q on a horizontal and vertical segment. **(c)** Points p and q on horizontal segments.

We say two *points* p and q on the drawing are *split* by a wire when p and q lie on different sides of the wire.

Lemma 9. *Let Γ_I and Γ_O be two degenerate shape-equivalent drawings, where Γ_I has spirality one. There exists a single linear morph from Γ_I to Γ_O that maintains planarity and orthogonality.*

Proof (Sketch). The partition of the drawing by all wires defines *cells*: regions of the plane not split by any wire. For each cell containing at least one bend or vertex, we can linearly interpolate all vertices and bends in Γ_I to the unique vertex or bend location in Γ_O. This directly defines a linear morph between Γ_I and Γ_O. To argue planarity of this morph, we assume for contradiction that there exist two points p and q on an edge or vertex of the drawing that coincide during the morph (excluding Γ_I and Γ_O). Then p and q must be x- and y-inverted in Γ_O compared to Γ_I and there must be two vertices r and s that are x- and y-inverted and split by at least a tb-wire and a lr-wire. As the lr-wire and the tb-wire are monotone they cross at least three times (see Fig. 7). Contradiction.   □

Theorem 2. *Let Γ_I and Γ_O be two unified planar orthogonal drawings of a (disconnected) graph G, where Γ_I has spirality s. We can morph Γ_I into Γ_O using exactly s linear morphs while maintaining planarity and orthogonality.*

5   Linear Complexity of Intermediate Drawings

Van Goethem and Verbeek [12] describe rerouting and a simplification operations that reduce the complexity of intermediate drawings to $O(n)$. These operations are not compatible with the batched linear morphs we described in Sect. 4. Below we show how to adapt these operations to the batched setting. These adaptations come at the cost of a single additional linear morph.
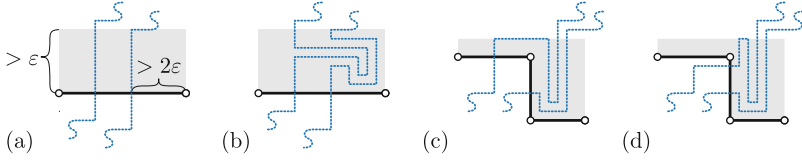
Fig. 8. (a) An ε-band adjacent to the edge. **(b)** Inserting an s-windmill. **(c–d)** Reroute wires after linear slide without introducing new crossings.

5.1   Rerouting

To avoid that the linear morphs introduce too many bends in a single iteration of the morph, we show how to route the wires such that only $O(n)$ complexity is added to the drawing in each iteration. The initial rerouting of the wires in Γ_I increases the maximum spirality by one, but it prevents any increase of spirality during the morph. Thus, using Theorem 2, $s + 1$ morphs are sufficient to morph two equivalent drawings into each other while maintaining planarity and keeping complexity of the intermediate drawings to $O(n^2)$.

We reroute the wires in $W_\downarrow$ and $W_\rightarrow$ as follows. Consider an edge e that is crossed by at least two wires in Γ_I. By Lemma 9 from [12] all crossing links have the same spirality. Assume w.l.o.g. that this spirality is positive, otherwise mirror the rotations and replace right by left. Let ε be a small distance such that the ε-band above e is empty except for the links crossing e and that there is more than a 2ε distance between the right-most crossing link and the right-endpoint of e (see Fig. 8(a)).

We insert an s-*windmill* of all crossing wires within the ε-band above e by rerouting the wires as follows. First disconnect all crossing links within the ε-band above e. Then reroute all wires in a parallel bundle to the right, beyond the right-most wire w_r crossing e. Now we spiral the bundle using right turns until the spirality of the links reaches zero. Next we unwind the bundle again within the spiral. Finally we reconnect the wires by routing back parallel to e to maintain the original crossing points (see Fig. 8(b)). This rerouting can be executed without introducing crossings between the wires. It does increase the spirality of the drawing by one.

We now change each iteration as follows. Consider a horizontal edge e crossed by $k > 1$ links of maximum absolute spirality s (assuming $s > 0$) at the start of the iteration. Instead of performing a linear slide on all crossing links, we perform a single linear slide only on the rightmost crossing link. This slide creates a new vertical segment (see Fig. 8(c)). Thanks to the introduction of the s-windmill, we can easily reroute the other crossing wires to intersect the new vertical segment instead of the horizontal segment without introducing other crossings (see Fig. 8(d)). The newly created crossing links must have spirality $s - 1$ as all links crossing the same segment have the same spirality (Lemma 9 from [12]). We can reduce all remaining spirality s links without introducing additional complexity in the drawing.

Lemma 10. *At the start of iteration i of the morph, all wires crossing an edge e with links of spirality i form an i-windmill in an empty ε-band next to e.*

Lemma 11. *Let Γ_s be the first drawing of an iteration and Γ^r_{s-1} the rerouted last drawing. The spirality of Γ_s relative to Γ^r_{s-1} is one.*

Proof (Sketch). We can argue that rerouting wires does not eliminate staircases. A link that is rerouted may have been part of a staircase, but the new links replacing it do not break any staircase properties. As rerouting links maintains staircases, Lemmata 5–8 still apply. □

Drawing Γ^r_{s-1} compared to Γ_s contains two additional bends in each edge crossed by maximum absolute spirality links in Γ_s. We can make Γ_s and Γ^r_{s-1} degenerate shape-equivalent by inserting an additional zero-length segment at the right-most (left-most for negative spirality) crossing link for each edge crossed by maximum absolute spirality links. By Lemmata 9 and 11 we can morph the resulting Γ_s into Γ^r_{s-1} in a single linear morph while maintaining planarity.

As, independently of how many wires are crossing it, each edge only introduces two new bends, complexity increases by $O(n)$ during each iteration. Thus the overall complexity is $O(s \cdot n)$. We conclude that we can morph two drawings Γ_I and Γ_O, where Γ_I has spirality s, into each other using $s + 1$ linear morphs while maintaining planarity and $O(s \cdot n)$ complexity of the drawing.

5.2   Simplification

By using rerouting we can ensure that the complexity of the drawing increases by at most $O(n)$ in every iteration, but its complexity may still grow to $O(n^2)$ over $O(n)$ iterations. In this section we show how to simplify the intermediate drawings to ensure that the complexity after each iteration is $O(n)$.

We again consider a single iteration starting with Γ_s and ending with Γ_{s-1}. Using rerouting we can find an alternative final drawing Γ^r_{s-1} that also maintains planarity. We now introduce a *redraw* step that further simplifies Γ^r_{s-1} into a straight-line drawing Γ'_{s-1} such that a linear morph from Γ_s to Γ'_{s-1} still maintains planarity. The redraw step works as follows.

For each vertex v in Γ^r_{s-1}, consider a 6ε-sized square box surrounding v that contains only v and a 3ε-part of each outgoing edge from v. If an incident edge e is crossed by a maximum absolute spirality link in Γ_s, then we reroute e inside the 6ε-box around v. Specifically, for an edge e leaving v rightwards, we reroute e within the 6ε-box using the coordinates $(v, v+(0, -\varepsilon), v+(2\varepsilon, -\varepsilon), v+(2\varepsilon, 0), v+(3\varepsilon, 0))$ (see Fig. 9(a)). Analogous rerouting can be done for edges leaving v in other directions. For an edge crossed by a negative spirality link invert the left and right turns.

Lemma 12. *We can redraw all edges in Γ_{s-1} that were crossed by a maximum-spirality link in Γ_s within 6ε-boxes while maintaining planarity of the drawing.*
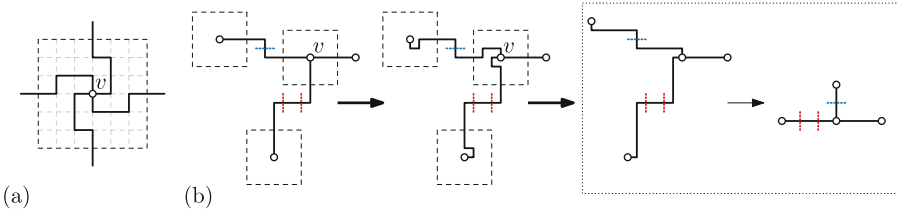
Fig. 9. (a) A 6ε-box surrounding a vertex v (dashed) with four redrawn edges. **(b)** Original drawing, rerouted drawing, and straightening the drawing.

Proof (Sketch). We can establish a relation between the spiralities of two segments incident at the same vertex. Using this relation we can argue that, after redrawing, no two edges leave a vertex in the same direction. As a result, there are no planarity violations within the 6ε-boxes around vertices. □

Lemma 13. *If Γ_s is a straight-line drawing with spirality $s > 0$ then there exists a straight-line drawing Γ'_{s-1} with spirality $s - 1$.*

Proof (Sketch). Let Γ^r_{s-1} be the drawing obtained by applying rerouting to the last drawing of iteration s. Consider an edge e crossed by maximum absolute spirality links in Γ_s. Edge e has three segments in Γ^r_{s-1} due to the two introduced bends. The first and last segment do not cross any wires. We can apply the redraw step to e, resulting in three more segments at the start and end of e. Finally we eliminate all additional segments of e by performing zigzag-eliminating slides on these segments (see Fig. 9(b)). □

Lemma 14. *The spirality of Γ'_{s-1} relative to Γ_s is one.*

Proof (Sketch). Let the *main wire set* be the set of wires used to compute the morph including rerouting from Γ_s to Γ^r_{s-1}. Consider a *reference wire grid* that is a straight-line wire grid in Γ_s. Using Lemmata 7, 8, and 11 but swapping the roles of Γ_a and Γ_b, we obtain the result that there is an equivalent monotone set of wires in Γ^r_{s-1} matching the reference grid in Γ_s. Thus the spirality of Γ_s relative to Γ^r_{s-1} is one.

When straightening Γ^r_{s-1} to Γ'_{s-1} only zigzag-removing slides are performed on segments not crossed by a wire from the main wire set. As such a segment was not crossed by a wire from the main wire set, the orientation of the segment is unchanged in Γ^r_{s-1}. Specifically, any link of a wire from the reference wire grid that crosses such a segment must have spirality zero. When straightening Γ^r_{s-1} to Γ'_{s-1} the zigzag-removing slides may insert additional bends in these reference wires, but the wires will remain monotone. □

We can make Γ_s degenerate shape-equivalent to Γ'_{s-1} as follows. For each edge e crossed by maximum absolute spirality links, we split e at the crossing with the right-most (or left-most if the links have negative spirality) crossing link and insert a zero-length segment. Furthermore, we add three zero-length segments at the endpoint of each such edge e coincident with the respective endpoint.

Theorem 3. *Let Γ_I and Γ_O be two equivalent drawings of a (disconnected) graph G, where Γ_I has spirality s. We can morph Γ_I into Γ_O using $s + 1$ linear morphs while maintaining planarity, orthogonality, and linear complexity of the drawing during the morph.*

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