Chapter 15
The Emergence of Meaningful Geometry

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Abstract This chapter is about a change in geometry education that took place in the last century. We discuss the emergence of meaningful geometry in the Netherlands. Of course, this was not an isolated reform. Worldwide, mathematicians and mathematics educators came up with new ideas as an alternative for the traditional axiomatic approach to teaching geometry. Already at the end of the 19th century, Klein had made a start with this by advocating a transformation geometry, but in this approach the axiomatic structure still played a main role for ordering activities. This was not the case in the work of Fröbel and Montessori who by building on students' intuitions and their attention for students' development of spatial insight were important driving forces towards a meaningful approach to geometry education. In the Netherlands, the pioneers of such a geometry were Tatiana Ehrenfest and Dieke van Hiele–Geldof. Freudenthal was a great promoter of their ideas. For him, geometry is 'grasping space', meaning that geometrical experiences should start with the observation of phenomena in reality. Supported by Freudenthal, from the 1970s on, experiments were carried out in the Netherlands to develop a new intuitive and meaningful approach to geometry education, in which the focus was on spatial orientation. How big the change in geometry education that resulted from these experiments was, is illustrated in this chapter by comparing geometry problems from two Dutch mathematics textbooks: one from 1976 and one from 2002.

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15.1   A World-Wide Change in Geometry Education

The New Math movement which commenced in the United States around the 1960s, and which controlled mathematics education for about two decades in Europe as well, brought about a big change in the mathematics curriculum by giving a central role to mathematical structures based on set theory. This approach corresponded largely with the work of the Bourbaki group that started in the 1930s with reformulating the foundations of mathematics. Although the New Math movement is often referred to as a result of the Sputnik effect requiring the Western world to modernise its education to meet the demands of the upcoming technological innovations, the urge to adapt the content of mathematics education to new developments in science and society had already begun in the 19th century. For example, in 1872, Felix Klein pointed out the importance of attention for structures in mathematics in his Erlanger Programm. He advocated the unification of group theory and transformation geometry (Klein, 1872). According to Botsch (Barbin & Menghini, 2014), Klein inspired most secondary schools in Germany to replace Euclidean geometry with so-called 'motion geometry'. This geometry was a simplified version of transformation geometry.

Klein seemed far ahead of the age of New Math, in which the structural character of mathematics was the central element. Also, by his rejection of Euclidian geometry he was anticipating the ideas of the Bourbaki group. It was almost one century later when during the epoch making Royaumont seminar in 1959, Jean Dieudonné, one of the leading figures of Bourbaki, launched his famous slogan 'A bas Euclide!'. With his slogan Dieudonné drew attention to the outdated content of geometry in secondary schools, still too much based on Euclid, taking geometry as the ideal context for teaching the axiomatic construction of mathematics. This teaching approach did not meet the needs of the new technical society nor the modern language of mathematicians and scientists. In geometry, modern topics were needed, which, to Dieudonné, included vector spaces in finite dimensions. Linear algebra was supposed to provide a 'royal road' to geometry (Choquet, 1964).

However, these 'modern' views on geometry teaching still had a characteristic of the traditional approach: the structure of a mathematical system was still more or less the main guideline for the learning process. Structuralism dominated mathematics education and resulted in a view on geometry as a means to rush on to analytical geometry, to the world of algebra describing space, and an axiomatic approach of linear algebra. Almost no opportunities were created for students to first develop spatial insight and to become familiar with 'space'.

Yet new developments towards a meaningful approach to geometry education with attention for students' development of spatial insight, had already been proposed early in the 19th century, in particular by German philosophers, pedagogues and psychologists who had been engaged in the content of mathematics education during that era. One of them was the German pedagogue Friedrich Fröbel (1782–1852) who came up with a new programme for geometry education for children aged 4–14, with his blocks, mosaics and other educational toys (Fröbel, 1826). He advocated practical activities for enabling children to get acquainted with characteristics of geometrical

shapes in an early phase. The work of Fröbel and also that of Maria Montessori (1870–1952) inspired attention for spatial orientation beginning at the kindergarten level. Choices of activities were based on educational psychological research on spatial insight with young children, and on experiences with playing as a context for geometrical explorations.

15.2   First Steps Towards a New Geometry Education in the Netherlands

Following the international trend, the first steps to reform the then prevailing geometry education were made in the Netherlands as well. Inspired by Klein's Erlanger Programm and influenced by mathematicians like Dieudonne, from the 1960s on, textbooks containing transformation geometry (e.g., Troelstra, Habermann, De Groot, & Bulens, 1962) were also published in the Netherlands. In this approach students were involved in constructing and transforming shapes instead of an emphasis on analysing given angles and triangles and reasoning about congruency. Although the importance of building on students' intuitions was emphasised in the introductions of these new textbooks, the formal axiomatic structure—in this case for reasoning with symmetry—again played an important role for ordering the activities. Figure 15.1 shows an example of a problem that illustrates the logic-deductive reasoning underlying this approach.

The students were expected to draw the two lines PE and QF perpendicular on a and b, resulting in the rectangle $PFQE$ with diagonals PQ and EF. This rectangle has two lines of symmetry, p and q. Reflection in p shows that angle P_1 equals F_1 and reflection in q shows that angle F_1 equals Q_3. From this it can be concluded that angle P_1 equals angle Q_3.

Given two parallel lines a and b and line l.

Prove that the angles P_1 and Q_3 are equal.
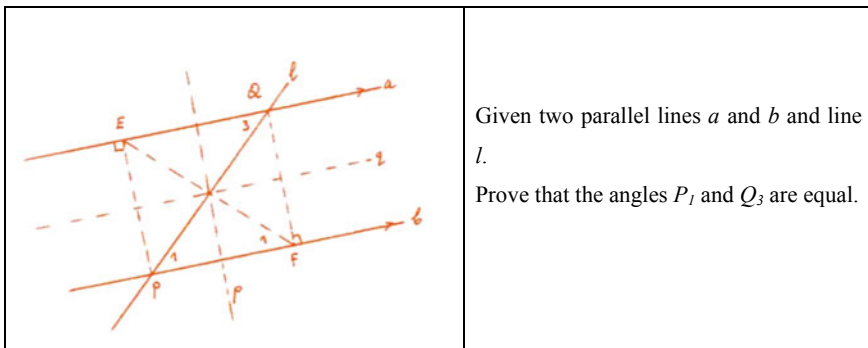
Fig. 15.1 Proving with transformations (from Troelstra et al., 1962, included in Groen, 2004, p. 299)

One of the most systematic investigations into the possibilities of transformation geometry in the early years of secondary school was carried out by the Dutch psychologist A. D. de Groot (De Groot et al., 1968). In this study, the new approach of transformation geometry was compared to the traditional approach through a large-scale teaching experiment involving 12- to 13-years old students who were in their first year of general secondary education. The results showed that between the two approaches in general no difference in performance was found and the students also did not differ in being motivated for geometry.

15.3 Precursors of Meaningful Geometry Education in the Netherlands

Someone who contributed significantly to introducing an approach to geometry education with attention for students' development of spatial insight in the Netherlands, was Tatiana Ehrenfest (1876–1964). She was originally from Russia and lived in the Netherlands for a long time from 1912 on. Ehrenfest had a great interest in teaching and education and gave this interest a practical expression by organising monthly mathematical-didactical colloquia for teachers at her house. Here, spirited discussions were held about the, in her view, fossilised mathematics education in the Netherlands (La Bastide-Van Gemert, 2006, 2015). Among other things, she developed an introductory geometry course with exercises in spatial geometry, titled *Übungensammlung zu einer Geometrische Propädeuse* (Ehrenfest-Afanassjewa, 1931), in which she took geometrical phenomena as a starting point for developing geometrical concepts. With this course, she enriched the domain of geometry with how we experience space. Ehrenfest-Afanassjewa considered activities of looking along two objects, identifying parallel lines in a classroom and lines as light beams and determining angles, basic for an intuitive understanding of the straight line as a mathematical object. In her introduction of the course she motivates the importance of such a phenomenological introduction by contrasting it with the geometrical method that emphasises a logical-deductive approach:

> Den *Weg vom Chaos zum System* und den *Segen*, welchen die *systematische Behandlung* des Stoffes mit sich bringt, zeigen die Logiker nicht, und so erscheint bei ihnen "die Geometrie", als ein von allem Materiellen losgelostes Denkspiel, und anstatt mit *Begriffen* zu operieren – welche ja nur durch eigenen Abstraktionsakt aus eigener lebendigen Erfahrung gewonnen werden können - haben die Schüler mit *Namen* und *Zeichnungen* zu tun, die sie oft an nichts Bekanntes erinnern (Ehrenfest-Afanassjewa, 1931, p. 5, italics in original).

> The *road from chaos to system* and the *blessing* resulting from the *systematic dealing* with the learning content, is not shown by the logicians. Therefore, for them "Geometry" becomes a game with thought objects that are isolated of all concreteness, and instead of operating with *concepts* – which can be acquired through the act of abstraction of one's own living experiences – students have to work with *names* and *drawings* which do often not refer to anything they know (Ehrenfest-Afanassjewa, 1931, p. 5; translated from German by the authors).

Halfway through the 20th century, a further impetus to change geometry education in the Netherlands came from the couple Van Hiele (1957) and Van Hiele-Geldof (1957) who proposed introductory activities with concrete materials like folding, cutting, gluing, and paving. As an example, Dieke van Hiele-Geldof started one of her geometry courses with physical cubes. She did not define a cube, but gave the students different kinds of solid cubes and cubes as wire figures of different materials. She discussed with her 12- to 13-years-old students, who were in the first year of the lowest level of secondary vocational education, similarities and differences between these objects, which led to an activity of constructing cardboard cubes. In this process, the students became acquainted with the geometrical objects and with fundamental notions of concepts such as right angle (defined by folding). During subsequent analyses of the objects, other characteristics, patterns and symmetries were identified and relationships were constructed (Van Hiele-Geldof, 1957). This example illustrates a learning process which completely differed from starting with a deductive structure of mathematics. The process that the Van Hieles advocated, passed different levels of understanding, labelled as visualisation (Ground Level 0), analysis (Level 1), informal deduction (Level 2), generalisation and the construction of a formal system of relationships and deduction (Level 3), and rigor (Level 4) (Van Hiele, 1957).

A next step towards a meaningful geometry education was made by Freudenthal who was involved with the work of the Van Hieles. Freudenthal highly appreciated and admired the analysis of classroom observations by Dieke van Hiele and the intuitive approach she promoted in introductory geometry education (La Bastide-Van Gemert, 2006, 2015). Freudenthal was also fond of Ehrenfest's *Übungensammlung*, although he did not agree with the deductive system of teaching geometry that he initially recognised in it. Later however, he understood better what a masterpiece Ehrenfest's publication was (Freudenthal, 1987). For him the relevance of her work was her plea for a resource-based approach to teaching geometry and for the need for an explorative and student-oriented approach to geometry which can be described as 'watching, acting, thinking and seeing'. Geometrical experiences start with the observation of a phenomenon in the surrounding environment. After that you make a model or a drawing to describe the phenomenon with geometrical means. Reasoning about these means will help you to develop mathematics and to understand the modelled phenomena. Freudenthal labelled the research underlying these activities as didactical phenomenology, and he summarised the resulting geometrical experiences and activities more concisely with the term 'grasping space'.

> [G]eometry is grasping space (…) that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it (Freudenthal, 1973, p. 403).

By saying this, Freudenthal criticised geometry education as the teaching of structures, an approach that was inspired by New Math and that isolated geometry from reality. He emphasised that apart from some applications of the Pythagorean theorem and some measurement problems about area and volume, the criterion of use

entirely failed in geometry (Freudenthal, 1973). As an alternative, he and his colleagues started working on developing a geometry education that later became known as 'Realistic Geometry Education'.

15.4   The Early Experiments: The Focus on Spatial Insight

From the 1970s on, experiments were carried out to develop a new intuitive and meaningful approach to geometry education (De Moor & Groen, 2012; Groen & De Moor, 2013). These experiments were carried out in educational practice through working with teachers and students in real classrooms. Initially, the plan was not to build a learning line for geometry, but to look for themes and problems that result in meaningful mathematical activities.

The designers of this new approach to geometry were focussed on developing the students' understanding of and skills in working with traditionally familiar subjects such as angles, area, symmetry, and the Pythagorean theorem. The intention was to find empirical support for a phenomenological approach to these subjects. To highlight the new character of the geometrical activities, the term 'vision geometry' was used. The experiments in class were aimed at the development of reasoning with vision lines, vision angles, sighting, rays of light, projecting, shadowing and perspective. In particular, this latter subject, perspective, was considered to play a central role in learning basic geometrical concepts and reasoning and the development of a deeper spatial insight. The set-up of the designs was not axiomatic, but based on phenomena and experiences in daily life.

15.4.1   Five Examples of Vision Geometry

The following examples are from tasks designed and tried out during the years 1970–1980. They all reflect the importance of starting with three-dimensional problem situations to evoke and further develop meaningful geometrical reasoning.

15.4.1.1   The Task 'The Singer'

The first example is the task 'The singer' (Fig. 15.2), which was developed, tried out and finally published in a geometry unit for lower secondary education (Schoemaker, 1980). The task is about a singer whose performance is filmed by four cameras. In this task, students can explore the way an object is seen from a certain viewpoint, which is one of the core ideas of the geometry of vision. Because the task fits well within the range of daily available student experiences and intuitions, there was not much need for further explanation. The experiment confirmed that the task is indeed easily accessible for students. Not even a question was needed to put the students to work.
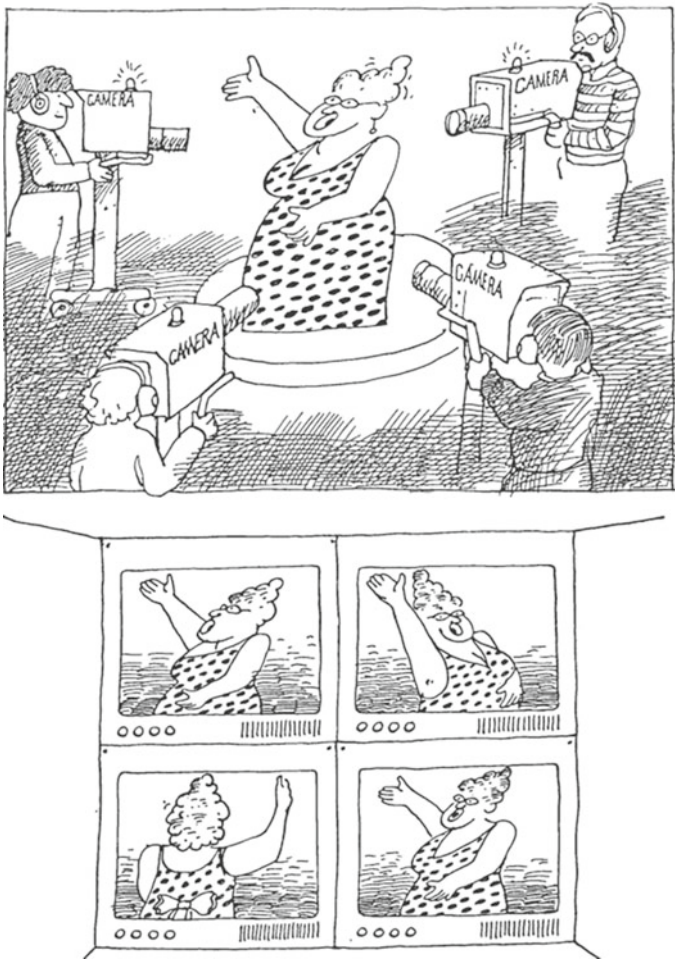
Fig. 15.2 Which image comes from which camera? (Schoemaker, 1980, p. 24)

The students immediately started connecting cameras with the images displayed on the four screens in the control room and they easily determined which camera saw the back of the singer and which camera had the slightly less decent look at the armpit. Deciding which of the four cameras were responsible for the two other images required more advanced reasoning, but the two easy images gave the students a good basis to find which cameras went with the remaining images.

15.4.1.2   The Task 'Rabbits Behind a Lighthouse'

The second example is the task 'Rabbits behind a lighthouse', which is about a boy walking in the dunes (Fig. 15.3). The students were asked whether the boy can see (some of) the rabbits behind the lighthouse, and whether that number changes when he is walking towards the lighthouse. The task evokes the need for drawing lines from the boy to the lighthouse. Students are expected to experience that drawing these lines is difficult in the presented figure and that a top view of the situation would be helpful to be able to decide which rabbits can be seen.

The tasks 'The singer' and 'Rabbits behind a lighthouse' probe to what extent exploring reality—investigating what we see and how we see things—can be used as a context for geometry. The geometry that emerges is related to becoming aware that space can be projected on a plane, that vision lines and different views of a situation can be used for explanations, and that drawings like top views and side views with vision lines are important tools for reasoning (Goddijn, 1980b).

15.4.1.3   The Task 'Tower and Bridge'

The third example is the task 'Tower and bridge' (Fig. 15.4). This task further elaborates the need for constructing vision lines and reasoning with these lines when a particular situation is shown from another view. This task was used in an experiment for introducing scale and geometrical reasoning in a 3D context (Goddijn, 1979; Schoemaker, 1980). The task was meant to create opportunities for students
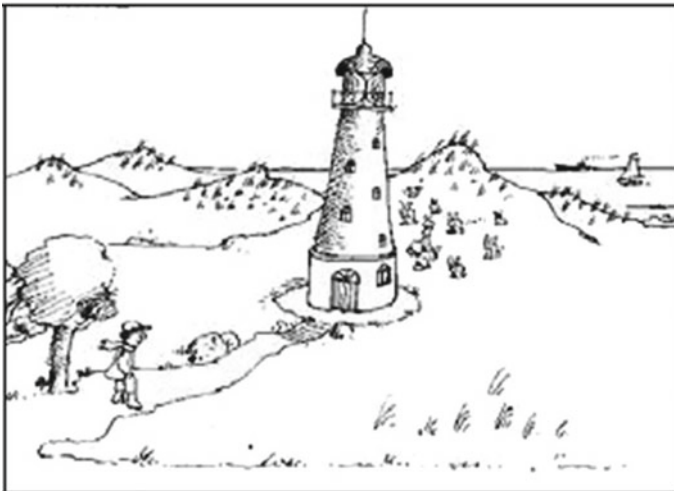

Fig. 15.3 Can he see any rabbits? (Schoemaker, 1980, p. 16)

Fig. 15.4   What is higher, the tower or the bridge? (Goddijn, 1979, p. 2)

to recognise the connection with situations in reality, how they can question them, and how they can use geometry to explain phenomena.

In the left picture, the bridge seems higher than the church, while in the picture on the right the church is higher than the bridge. By constructing a side view of this situation and drawing triangles based on vision lines students can explain this phenomenon and argue that the church must be higher than the bridge. This example shows again how the teaching and learning of geometry can be a constructive and creative activity and that the geometry that focusses on grasping space starts with looking, analysing and creating drawings like top views or side views and the vision lines as tools for explaining phenomena of vision.

15.4.1.4   The Task 'Shadows of a Cube'

In the fourth example the geometry also comes with just looking. This 'Shadows of a cube' task (Fig. 15.5) is about the polygons that can be created from projections of a cube. The question asked students was, what kind of shadows a cube can have (Goddijn, 1980c). It is obvious that a square must be possible, and a rectangle is also not too difficult. But what other polygons can be created? Can you have a pentagon, hexagon or heptagon as a shadow? Explore and explain.

15.4.1.5   The Task 'Shadows from the Sun and a Lamp'

The last example is the task 'Shadows from the sun and a lamp'. This task is also about shadows and addresses different projection methods caused by two different light sources. On the left side of Fig. 15.6 it is night and the street lamp is on. On the right side, it is daytime and the sun is shining. In both cases shadows of posts around the lamp need to be created. Students are asked to describe and explain differences and similarities between the shadows created by the sun and by the lamp.

Fig. 15.5 What kind of shadows can a cube have? (Goddijn, 1980c, p. 20)

A CIRCLE OF POSTS

>31. Here's a top-view of a street-lamp and some posts.
The lamp is on. One shadow has already been drawn in.
Draw the other shadows as well.
Do it as exactly as possible!

>32. The same situation, only now the street-lamp
is out and the sun is shining.
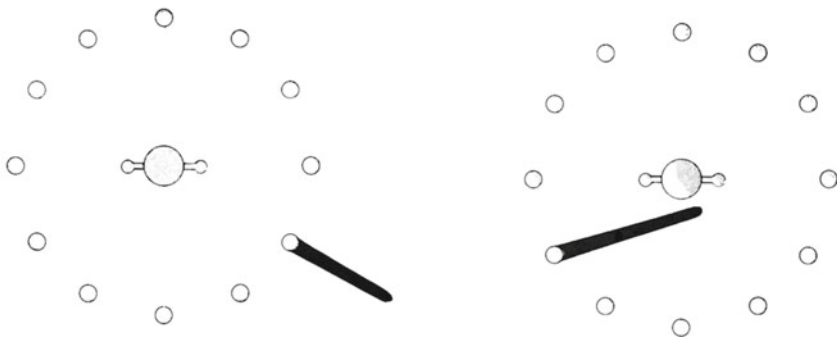Draw the rest of the shadows.

Fig. 15.6 The different shadows (Goddijn, 1980c, pp. 12–13)

This task illustrates the potential of explorative activities with vision lines, rays of light, projecting, shadows and perspective (see also Goddijn, 1980a). One can also speak of an 'intuitive geometry of the straight line'. The principles of 'parallel perspective' and 'central perspective' and reasoning about what you see and how or why you see it, are the core of this vision geometry. By tasks like these, students are given the opportunity to experience that straight lines of light are the central elements for understanding shadow. Reasoning with these lines in different views, properties of bundles of lines like 'being parallel' or 'all intersecting in one point' come to

the fore as natural tools for reasoning in this context. Students are expected to truly experience the characteristics of these situations by experimenting with parallel light beams (sunlight) and a central light source (a lamp). This could intuitively lead to a base for an understanding of invariant characteristics of the two perspective methods.

15.4.2   What These Tasks Have in Common

All foregoing tasks show an approach to geometry education in which fundamental geometrical insights are strongly connected to phenomena that students can experience in everyday life. The tasks that are used for developing these insights are characteristic for Realistic Geometry Education. More specifically this approach to geometry education implies:

– Starting with 'realistic' problems
– Considering students as active and creative explainers of problems
– Giving students opportunities for explorative activities through which they can further develop their geometrical intuitions and by which preliminary constructions can emerge
– Eliciting mathematisation in students by focussing on the development of 'situation models' like vision lines which bring the students from the informal to the more formal geometry.

We can conclude that these characteristics are in line with the ideas of Ehrenfest-Afanassjewa (1931). In her introductory geometry course with exercises in spatial geometry she also tried to have students develop geometrical concepts from their own living experiences and to prevent that students would work with names and drawings that do not refer to something they know.

15.5   A Change in Geometry Education: Geometry Problems in 1976 and in 2002

The experiments that have been carried out since the beginning of the 1970s differed hugely from the then prevailing approach to teaching geometry. These experiments brought about a big change in geometry education. Therefore, there is a sharp contrast between geometry as it was offered in textbooks in the 1970s and geometry in current textbooks. Geometry became a discipline that was no longer isolated within the world of mathematics, but connections were made to the daily life situations of students. For example, an important attainment target for students in the lower grades of secondary school was: Students can interpret, describe, spatially imagine and create two-dimensional representations of spatial situations, such as photos, sewing patterns, maps, plans, and blueprints (OC&W, 1997).

What this means for the daily practice of teaching and learning geometry and how this differs from the previous approach to teaching geometry comes clearly to the fore when a textbook series from, for example, 1976 is compared with a more recent one that is published in 2002. The first textbook series is *Moderne Wiskunde voor Voortgezet Onderwijs* written by Jacobs et al. (1976). The second textbook is the series *Moderne Wiskunde* written by Van der Eijk et al. (2002). For the comparison, we took the books for Grade 7, which are meant for the first year of secondary school, and we chose the topics: (a) introduction to 3D shapes, (b) location, in particular the introduction of coordinate systems, and (c) reasoning with lines and angles. Due to space limitations, we can only give a few examples which never will do full justice to the two carefully designed textbook series. Nevertheless, the three examples we provide give a clear impression of the changes that have taken place at the end of the twentieth century in the Netherlands.

The first example is about 3D shapes. As a start for this topic, in the 1976 textbook, the students are shown drawings of two kinds of boxes (Fig. 15.7). The drawings are used to introduce the mathematical terms that describe the elements of 3D shapes (faces, vertices and edges) and characteristics of them. One of the following assignments for the students is to list the edges that are parallel to each other and to learn to draw the mathematical shapes on grid paper. In contrast, the 2002 textbook focusses on providing opportunities to students to explore and analyse shapes that they can see in daily life. Students are stimulated to figure out all kinds of characteristics of the shapes. For example, which objects can roll and what are the similarities and differences between the sides of each of the shapes?

The second example is about the topic of location. Figure 15.8 shows how differently coordinate systems are introduced to students in 1976 and in 2002. In 1976,
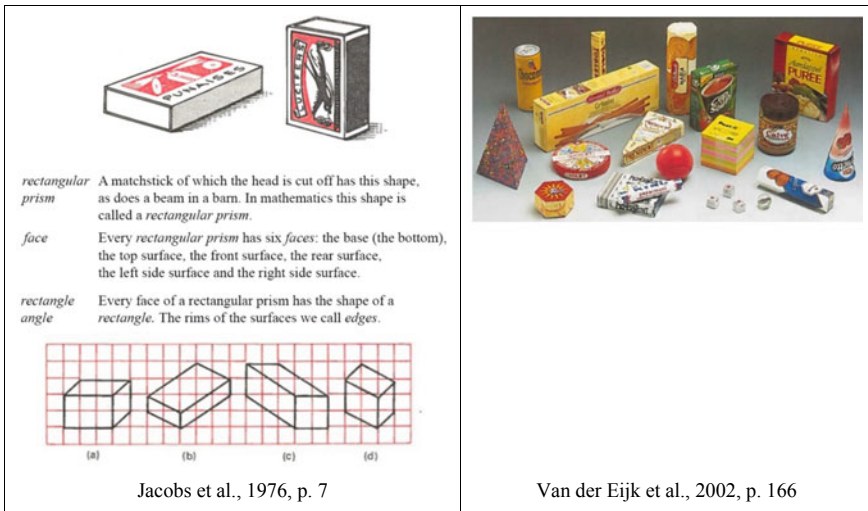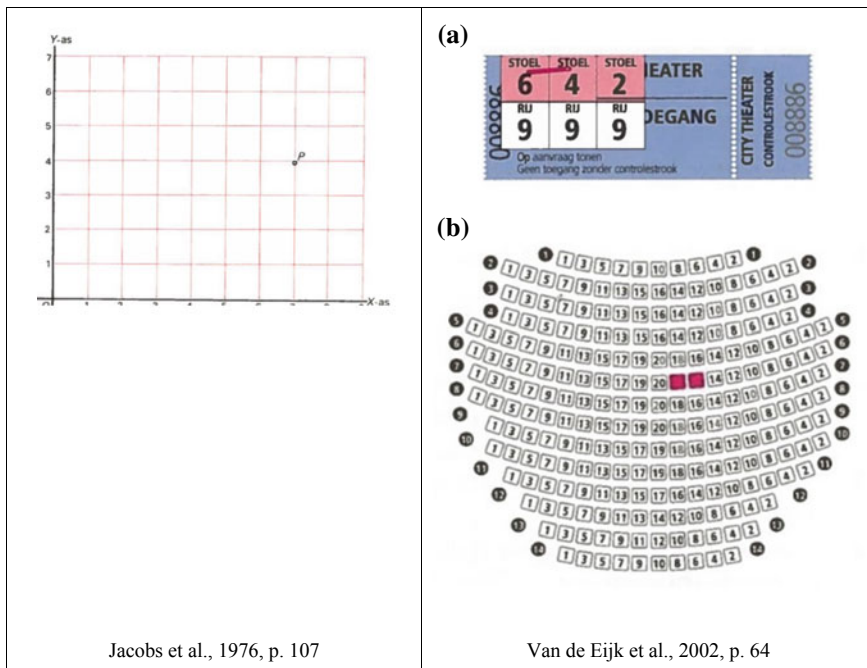

| Jacobs et al., 1976, p. 7 | Van der Eijk et al., 2002, p. 166 |

Fig. 15.7 Introduction to 3D shapes

Fig. 15.8 Introduction to coordinate systems

the idea of a coordinate system is posed as a way to organise a plane presented as a grid. The accompanying text in the textbook introduces the students to the language of a coordinate system:

> Start counting from the origin: first seven lines to the right, then four up. We arrive at point *P*. […] The pair of numbers (7, 4) are called the 'coordinates' of *P*.

Next, they have to locate other points following a similar recipe of counting lines to the right and up starting at the origin. In the 2002 textbook, the introduction to coordinate systems is preceded with activities that are connected to the need for such systems. Students are provided with problems in which they can use a coordinate system for reasoning about locations in daily life situations. In the problem from the 2002 textbook, the context of seating people in a theatre is used. The students are asked (a) to figure out where the seats are when you have bought tickets that tell you the chair number and the row number, and (b) to determine what information will be on the tickets when you are seated on the two coloured locations on the floor map of the theatre.

The third example illustrates the differences between the introduction in both textbooks of reasoning with lines and angles. In the 1976 textbook (Fig. 15.9 on the left), the students have to explain that triangles *ABC* and *CDA* are congruent.

In the 2002 textbook (Fig. 15.9 on the right), the topic of reasoning with lines and angles has changed into reasoning about vision lines and angles starting in 3D

| Jacobs et al., 1976, p. 126 | Van der Eijk et al., 2002, p. 14 |

Fig. 15.9 Reasoning with lines and angles

contexts. Doing geometry is not limited to reasoning with lines and angles in the plane, but can also start with spatial situations that refer to reality. The students are provided with a picture showing the top view of a room in which a boy and a girl are sitting and showing a garden where there is a cat and two birds are flying around. In the room, there are two windows. The girl who is sitting on a sofa warns that the birds are in danger, but the boy does not understand her. The students are asked to explain this. The purpose of the problem is to introduce students to a situation which they can 'organise' with geometrical means. The students are asked to construct top and side views and to draw vision lines and angles in them that can be used to explain what is seen and how it is seen in reality.

Another remarkable difference between the 1976 and the 2002 textbook is how the topics are ordered. The 1976 textbook starts with teaching the names of 3D shapes on page 7 (see Fig. 15.7). Many pages later, on page 107 (see Fig. 15.8), this is followed with the introduction to coordinate systems and finally, from page 126 (see Fig. 15.9) on, reasoning with lines and angles in the plane is addressed. In contrast, the sequence in the 2002 textbook is the other way around. Here, the introduction to reasoning with vision lines and angles is situated in the beginning of the textbook, on page 14 (see Fig. 15.9). Later, on page 64 (see Fig. 15.8), coordinate systems are introduced with reference to coordinate systems in various real situations. Only in the end, on page 166 (see Fig. 15.7), spatial shapes are explored and geometrical terms for describing these shapes are introduced.

Although in the 2002 examples many of the original ideas for a more meaningful approach to geometry education that were developed in the years 1970–1980 can be recognised, the ideal of geometry as a real constructive activity appeared to be difficult to implement in textbooks. The design of rather closed tasks is more feasible in textbooks than having open tasks that ask for classroom experiments and discussion. Take, for example, a task that deals with the concept of vision angle. Getting a good understanding of this concept requires that it is really experienced through a whole class activity and interactive discussion in which so-called 'why-questions'

are asked. However, such questions are often missing in textbooks. Also, in class, attention is seldom paid to reasoning with vision lines and demonstrating their use.

The task that mostly reflects the ideas behind the experiments that started in the 1970s is the task on the right in Fig. 15.9, where the students are provided with a top view of a room and an adjoining garden where birds seemed to be in danger. The power of this task is that the students are offered the opportunity to geometrically organise the situation to understand and know for sure what is going on. According to Freudenthal (1971), this so-called 'local organisation' is the way to develop the concepts and reasoning schemes and has the potential to create the need for axioms, definitions and a logic-deductive system. A further example of this idea is presented in the next section.

15.6 An Example of Local Organisation: The Nearest Neighbour Principle

One of the reasons for teaching geometry at secondary school is that the deductive system of definitions, axioms and theorems offers an excellent context for students to experience the mathematics of proof, of being sure and of being creative. However, it requires some maturity of the students to really value and use the very precise definitions of geometrical objects and to understand which constructions are allowed. Therefore, we think it is appropriate to deal with this formal geometry with students who are in the higher grades of pre-university secondary education and who have chosen a science or technology track. Nevertheless, for these students as well, geometry should not start on a formal, abstract level, but with problems that are experienced by the students as real problems and that create the need for further formalisation. The local organisation at the problem level that is necessary for this can be considered as a geometrical activity to re-invent principles of Euclidean geometry. How this works is illustrated by the following example which originates from a unit designed by Goddijn et al. (2014) meant for students in the higher grades of pre-university secondary education (see also Goddijn, 2017).

The task in Fig. 15.10 is part of a series that deals with the topic of the nearest neighbour principle. The so-called 'Voronoi diagrams' that can be used to express this principle have many applications that are relevant in reality; for example, in the case of resolving territory conflicts.

To introduce the notion of the nearest neighbour principle the students are shown a map of a desert with five water wells (see Fig. 15.10). The students are asked to colour areas in the desert in such a way that for each possible point (e.g., point J) in a coloured area the corresponding well should be the one that is the closest to that point.

This situation is expected to evoke strategies, like drawing circles and lines. Students are challenged to find the borders between the areas and discover then that these borders seem to be straight lines, which meet each other in one point. After solving
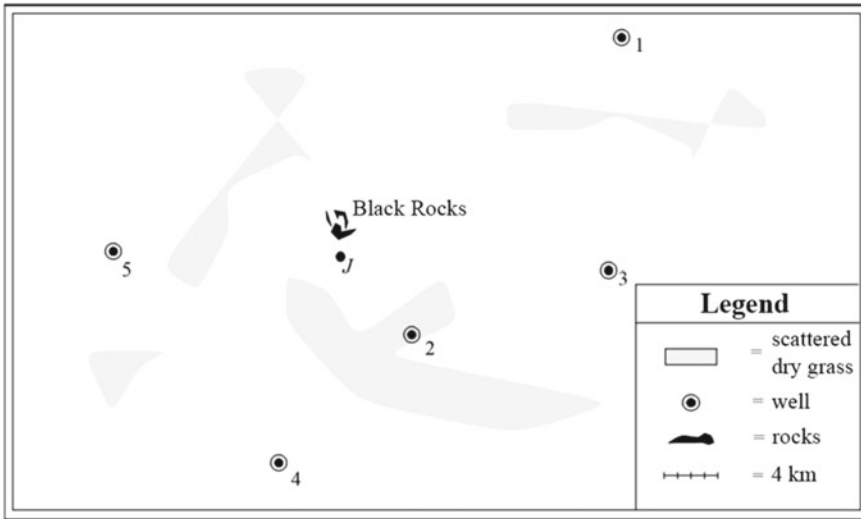
Fig. 15.10 The task 'Desert map' (Goddijn et al., 2014, p. 45)

a series of such contextual problems, the focus is changed towards the mathematical characteristics of the diagrams. One of the questions for the students is why these lines, which are called 'Voronoi edges', always meet in one point (Fig. 15.11).

Initially this sounds like a useless question, because it is quite obvious for the students that this is the case. Nevertheless, they are invited to look for an answer to this *why* question. To tackle this question, they have to realise that the Voronoi edge between, for example, the centres *A* and *B* (the wells in the desert problem) is a set of points *P* for which the distance to *A* equals the distance to *B*. This can be expressed with the distance notation $d(..,..)$. Then, this verbal description is written down as: $d(P, A) = d(P, B)$. This description defines the property of the Voronoi edge and makes the proof that these edges always meet in one point rather straightforward. Assume there is a point *M* that is the meeting point of the Voronoi edge between *A*
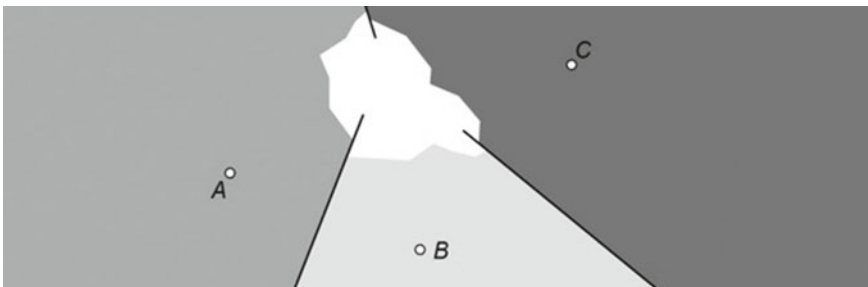

Fig. 15.11 How do three Voronoi edges meet?

and B and the Voronoi edge between B and C, then we have $d(M, A) = d(M, B)$ and $d(M, B) = d(M, C)$. This means that $d(M, A) = d(M, C)$, so M is also on the Voronoi edge between A and C. Consequently, it can be concluded that the three Voronoi edges meet each other in one point M.

However, this is not a full proof. Actually, in this proof it is assumed that there is a meeting point of the Voronoi edges which we started with (the edge between A and B and between B and C). Yet it might also be possible that this is not the case. So, there is a gap in the argument. Students can detect this, because they already have experienced that when A, B and C are in line, that the Voronoi edges between A and B and between B and C are parallel and do not meet. Of course, this can be considered as an exception. Nevertheless, again we can ask whether the proof is complete. Are we sure that the Voronoi edges meet in all other cases?

For example, if there are Voronoi edges that are curved, then it is possible that they do not meet. So, that means that next it should be proved that the Voronoi edge of A and B is *always* a straight line. This proof is difficult, because it seems so obvious that the Voronoi edge of A and B, being the collection of all points with equal distance to A and B, is similar to the perpendicular bisector of A and B, which is a straight line. But does this mean that if a point is *not* on the perpendicular bisector of A *and B*, that is, not on $pbs(A, B)$, that then this point is also *not* on the Voronoi edge of A and B.

Suppose, point Q is not on the perpendicular bisector of A and B (see Fig. 15.12). When Q is on the left side of the bisector, then the line from Q to B meets the bisector in R. Because R is on the bisector it can be concluded that $d(A, R) = d(B, R)$. As soon as it can be established that $d(A, Q) < d(A, R) + d(R, Q)$, it can be inferred that $d(A, Q) < d(B, R) + d(R, Q)$ and consequently that $d(A, Q) < d(B, Q)$. This proves that Q does not belong to the Voronoi edge of A and B. The only thing to be determined is

**Fig. 15.12**   Point Q is not on the perpendicular bisector of A and B
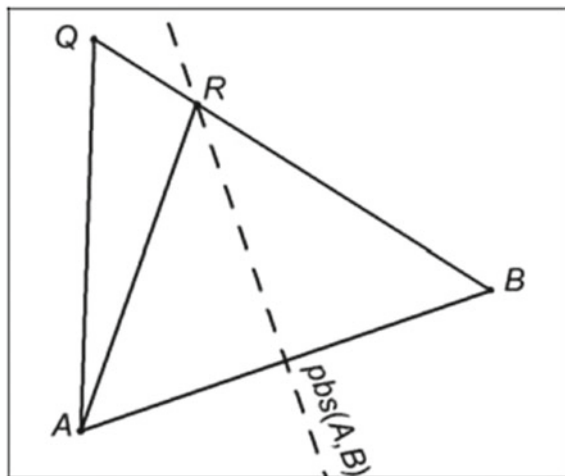
whether $d(A, Q) < d(A, R) + d(R, Q)$. The famous triangle inequality says that this is true if A, Q and R are not on a line.

Students can experience now that there is a bottom in this process of asking *why*-questions towards more fundamental elements, and that this bottom is chosen consciously. In a course which takes distances and 'the nearest neighbour principle' as a topic of departure it is natural to take the triangle inequality as one of the basic truths. However, some protest can be raised against this choice by students who defend the Pythagorean theorem as being a more sure thing. In that case, students can be kindly requested to derive the triangle inequality from the Pythagorean theorem.

The aforementioned example of local organisation around the nearest neighbour principle and the Voronoi diagrams, illustrates the path from exploration to geometry as a logic-deductive system. First, the students have to answer the question about the three meeting Voronoi edges, then they have to explore the character of the Voronoi edge and answering the question whether it is similar to the perpendicular bisector, and finally they arrive at something that is more fundamental and belongs to the geometry as a logic-deductive system: the triangle inequality (Fig. 15.13).

This approach contrasts with a traditional approach starting with the known things at the bottom of the logic-deductive system and building step-by-step theorems with logical arguments. This is the path in which the teaching of geometry stays within the logic-deductive system (Fig. 15.14).

The local organisation described in this section that emerges from the explorative solutions of situational problems results in a reflection on the kinds of definitions that are needed to be able to prove theorems and to establish a strong foundation for (deductive) reasoning. That process guides students from situational problems into the world of geometry and supports them in the development of heuristics for searching for answers to why-questions.
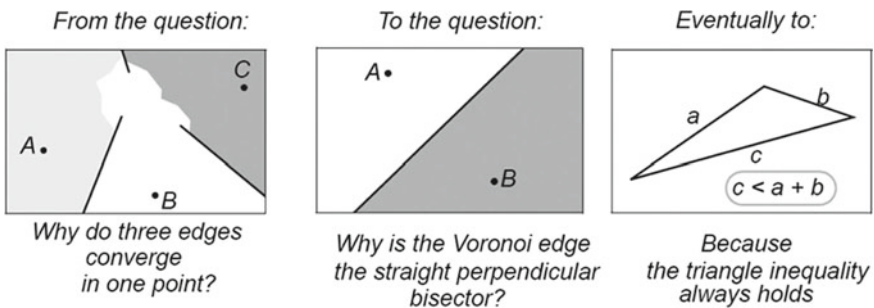
Fig. 15.13 The path from exploration to finding an underpinning argumentation (Goddijn, 2017, p. 30)
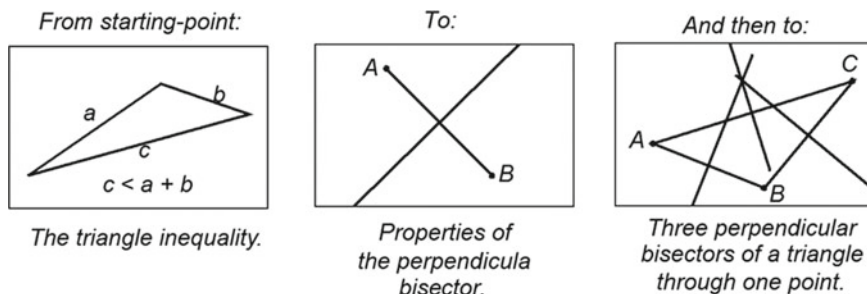
Fig. 15.14   The path from axioms to proving statements (Goddijn, 2017, p. 30)

15.7   Final Remarks

In this chapter, we have tried to shed light on the change that took place in the Netherlands in which an axiomatic approach to teaching geometry was gradually superseded by an intuitive and meaningful approach focussed on spatial orientation. Characteristic of the reformed approach, that in the Netherlands later became known under the term 'Realistic Geometry Education', is that students are introduced to the world of geometry (the language, the objects and the constructions) by providing them with tasks in 3D contexts that can elicit their intuitive geometrical reasoning. Starting geometry education by developing spatial intuition and 'grasping space' was very much supported by Freudenthal (1973) and is exactly at the heart of the ideal of Ehrenfest-Afanassjewa (1931). The result of this reform is that in the Netherlands geometry education nowadays mostly starts with an intuitive introduction (see, e.g., De Lange, 1986; De Moor, 1991; Van den Heuvel-Panhuizen & Buys, 2008), after which it continues in a context-rich course for 12 to 16-year olds (see, e.g., Goddijn, 1991), ending in reflections on definitions and axioms, that is, geometry as a deductive system, by the end of secondary school (see, e.g., Goddijn et al., 2014).

What needs to be stated here is that the reformed approach not only made geometry more meaningful for students, but that this change also widened the scope of the geometry trajectory both in terms of students involved and topics. On the one hand, due to the intuitive introduction some topics, such as vision lines, can now already be dealt with in primary education or even earlier. On the other hand, older students who have reached a certain mathematical maturity can be provided with meaningful imaginable contexts that can be organised locally which gives them access to further learning towards more formal geometry. In this way, at the end of the geometry trajectory, a topic like proofs can become interesting and intriguing for more students.

Furthermore, the change in approach also implies that the structure of the geometry trajectory has changed. Traditionally, structure in a teaching-learning trajectory for geometry was provided by a deductive system starting with formal definitions and basic axioms. This deductive system also dominated the structure of the textbooks,

whether they were based on Euclid or on transformation geometry. The traditional trajectory introduced students into a mathematical world without developing their intuitions about this world. Freudenthal and his collaborators criticised this approach to geometry education that is based on geometry as a logic-deductive system. Freudenthal (1973) called this an anti-didactical inversions of learning sequences. This means that this approach takes the final state of the work of mathematicians as a starting point for mathematics education. As an alternative for such an inversion Freudenthal advocated that mathematics education should take its starting point in mathematics as an activity (Freudenthal, 1973, 1991). For him the core mathematical activity was mathematising, that is, organising from a mathematical perspective. Finally, 45 years after Freudenthal wrote his famous paper 'Geometry between the devil and the deep sea' (1971), the experiences in the past decades have shown what Realistic Geometry Education can offer students at all educational levels.

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