Chapter 6
Banach Spaces

We begin this chapter with a quick review of the essentials of metric spaces. Then we extend our results on measurable functions and integration to complex-valued functions. After that, we rapidly review the framework of vector spaces, which allows us to consider natural collections of measurable functions that are closed under addition and scalar multiplication.

Normed vector spaces and Banach spaces, which are introduced in the third section of this chapter, play a hugely important role in modern analysis. Most interest focuses on linear maps on these vector spaces. Key results about linear maps that we develop in this chapter include the Hahn–Banach Theorem, the Open Mapping Theorem, the Closed Graph Theorem, and the Principle of Uniform Boundedness.

Market square in Lwów, a city that has been in several countries because of changing international boundaries. Before World War I, Lwów was in Austria–Hungary. During the period between World War I and World War II, Lwów was in Poland. During this time, mathematicians in Lwów, particularly Stefan Banach (1892–1945) and his colleagues, developed the basic results of modern functional analysis. After World War II, Lwów was in the USSR. Now Lwów is in Ukraine and is called Lviv.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_6

6A Metric Spaces

Open Sets, Closed Sets, and Continuity

Much of analysis takes place in the context of a metric space, which is a set with a notion of distance that satisfies certain properties. The properties we would like a distance function to have are captured in the next definition, where you should think of $d(f,g)$ as measuring the distance between f and g.

Specifically, we would like the distance between two elements of our metric space to be a nonnegative number that is 0 if and only if the two elements are the same. We would like the distance between two elements not to depend on the order in which we list them. Finally, we would like a triangle inequality (the last bullet point below), which states that the distance between two elements is less than or equal to the sum of the distances obtained when we insert an intermediate element.

Now we are ready for the formal definition.

6.1 Definition ***metric space***

A *metric* on a nonempty set V is a function $d\colon V \times V \to [0,\infty)$ such that

- $d(f,f) = 0$ for all $f \in V$;
- if $f, g \in V$ and $d(f,g) = 0$, then $f = g$;
- $d(f,g) = d(g,f)$ for all $f, g \in V$;
- $d(f,h) \leq d(f,g) + d(g,h)$ for all $f, g, h \in V$.

A *metric space* is a pair (V,d), where V is a nonempty set and d is a metric on V.

6.2 Example *metric spaces*

- Suppose V is a nonempty set. Define d on $V \times V$ by setting $d(f,g)$ to be 1 if $f \neq g$ and to be 0 if $f = g$. Then d is a metric on V.
- Define d on $\mathbf{R} \times \mathbf{R}$ by $d(x,y) = |x - y|$. Then d is a metric on $\mathbf{R}$.
- For $n \in \mathbf{Z}^+$, define d on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$d\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \max\{|x_1 - y_1|,\dots,|x_n - y_n|\}.$$

  Then d is a metric on $\mathbf{R}^n$.
- Define d on $C([0,1]) \times C([0,1])$ by $d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$; here $C([0,1])$ is the set of continuous real-valued functions on $[0,1]$. Then d is a metric on $C([0,1])$.
- Define d on $\ell^1 \times \ell^1$ by $d\big((a_1,a_2,\dots),(b_1,b_2,\dots)\big) = \sum_{k=1}^{\infty} |a_k - b_k|$; here ℓ^1 is the set of sequences $(a_1,a_2,\dots)$ of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$. Then d is a metric on ℓ^1.

The material in this section is probably review for most readers of this book. Thus more details than usual are left to the reader to verify. Verifying those details and doing the exercises is the best way to solidify your understanding of these concepts. You should be able to transfer familiar definitions and proofs from the context of $\mathbf{R}$ or $\mathbf{R}^n$ to the context of a metric space.

This book often uses symbols such as f, g, h as generic elements of a generic metric space because many of the important metric spaces in analysis are sets of functions; for example, see the fourth bullet point of Example 6.2.

We will need to use a metric space's topological features, which we introduce now.

6.3 Definition ***open ball;*** $B(f, r)$

Suppose (V, d) is a metric space, $f \in V$, and $r > 0$.

- The *open ball* centered at f with radius r is denoted $B(f, r)$ and is defined by
$$B(f, r) = \{g \in V : d(f, g) < r\}.$$
- The *closed ball* centered at f with radius r is denoted $\overline{B}(f, r)$ and is defined by
$$\overline{B}(f, r) = \{g \in V : d(f, g) \leq r\}.$$

Abusing terminology, many books (including this one) include phrases such as *suppose V is a metric space* without mentioning the metric d. When that happens, you should assume that a metric d lurks nearby, even if it is not explicitly named.

Our next definition declares a subset of a metric space to be open if every element in the subset is the center of an open ball that is contained in the set.

6.4 Definition ***open set***

A subset G of a metric space V is called *open* if for every $f \in G$, there exists $r > 0$ such that $B(f, r) \subset G$.

6.5 ***open balls are open***

Suppose V is a metric space, $f \in V$, and $r > 0$. Then $B(f, r)$ is an open subset of V.

Proof Suppose $g \in B(f, r)$. We need to show that an open ball centered at g is contained in $B(f, r)$. To do this, note that if $h \in B\bigl(g, r - d(f, g)\bigr)$, then

$$d(f, h) \leq d(f, g) + d(g, h) < d(f, g) + \bigl(r - d(f, g)\bigr) = r,$$

which implies that $h \in B(f, r)$. Thus $B\bigl(g, r - d(f, g)\bigr) \subset B(f, r)$, which implies that $B(f, r)$ is open. ∎

Closed sets are defined in terms of open sets.

6.6 Definition ***closed subset***

A subset of a metric space V is called *closed* if its complement in V is open.

For example, each closed ball $\overline{B}(f,r)$ in a metric space is closed, as you are asked to prove in Exercise 3.

Now we define the closure of a subset of a metric space.

6.7 Definition ***closure***

Suppose V is a metric space and $E \subset V$. The *closure* of E, denoted $\overline{E}$, is defined by

$$\overline{E} = \{g \in V : B(g,\varepsilon) \cap E \neq \varnothing \text{ for every } \varepsilon > 0\}.$$

Limits in a metric space are defined by reducing to the context of real numbers, where limits have already been defined.

6.8 Definition ***limit in V***

Suppose (V,d) is a metric space, $f_1, f_2, \ldots$ is a sequence in V, and $f \in V$. Then

$$\lim_{k\to\infty} f_k = f \text{ means } \lim_{k\to\infty} d(f_k, f) = 0.$$

In other words, a sequence $f_1, f_2, \ldots$ in V converges to $f \in V$ if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that

$$d(f_k, f) < \varepsilon \text{ for all integers } k \geq n.$$

The next result states that the closure of a set is the collection of all limits of elements of the set. Also, a set is closed if and only if it equals its closure. The proof of the next result is left as an exercise that provides good practice in using these concepts.

6.9 ***closure***

Suppose V is a metric space and $E \subset V$. Then

(a) $\overline{E} = \{g \in V : \text{ there exist } f_1, f_2, \ldots \text{ in } E \text{ such that } \lim_{k\to\infty} f_k = g\}$;

(b) $\overline{E}$ is the intersection of all closed subsets of V that contain E;

(c) $\overline{E}$ is a closed subset of V;

(d) E is closed if and only if $\overline{E} = E$;

(e) E is closed if and only if E contains the limit of every convergent sequence of elements of E.

The definition of continuity that follows uses the same pattern as the definition for a function from a subset of $\mathbf{R}$ to $\mathbf{R}$.

6.10 Definition ***continuity***

Suppose (V, d_V) and (W, d_W) are metric spaces and $T\colon V \to W$ is a function.

- For $f \in V$, the function T is called *continuous* at f if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d_W\big(T(f), T(g)\big) < \varepsilon$$
for all $g \in V$ with $d_V(f, g) < \delta$.
- The function T is called *continuous* if T is continuous at f for every $f \in V$.

The next result gives equivalent conditions for continuity. Recall that $T^{-1}(E)$ is called the inverse image of E and is defined to be $\{f \in V : T(f) \in E\}$. Thus the equivalence of the (a) and (c) below could be restated as saying that a function is continuous if and only if the inverse image of every open set is open. The equivalence of the (a) and (d) below could be restated as saying that a function is continuous if and only if the inverse image of every closed set is closed.

6.11 ***equivalent conditions for continuity***

Suppose V and W are metric spaces and $T\colon V \to W$ is a function. Then the following are equivalent:

(a) T is continuous.

(b) $\lim\limits_{k\to\infty} f_k = f$ in V implies $\lim\limits_{k\to\infty} T(f_k) = T(f)$ in W.

(c) $T^{-1}(G)$ is an open subset of V for every open set $G \subset W$.

(d) $T^{-1}(F)$ is a closed subset of V for every closed set $F \subset W$.

Proof We first prove that (b) implies (d). Suppose (b) holds. Suppose F is a closed subset of W. We need to prove that $T^{-1}(F)$ is closed. To do this, suppose $f_1, f_2, \dots$ is a sequence in $T^{-1}(F)$ and $\lim_{k\to\infty} f_k = f$ for some $f \in V$. Because (b) holds, we know that $\lim_{k\to\infty} T(f_k) = T(f)$. Because $f_k \in T^{-1}(F)$ for each $k \in \mathbf{Z}^+$, we know that $T(f_k) \in F$ for each $k \in \mathbf{Z}^+$. Because F is closed, this implies that $T(f) \in F$. Thus $f \in T^{-1}(F)$, which implies that $T^{-1}(F)$ is closed [by 6.9(e)], completing the proof that (b) implies (d).

The proof that (c) and (d) are equivalent follows from the equation

$$T^{-1}(W \setminus E) = V \setminus T^{-1}(E)$$

for every $E \subset W$ and the fact that a set is open if and only if its complement (in the appropriate metric space) is closed.

The proof of the remaining parts of this result are left as an exercise that should help strengthen your understanding of these concepts. ■

Cauchy Sequences and Completeness

The next definition is useful for showing (in some metric spaces) that a sequence has a limit, even when we do not have a good candidate for that limit.

6.12 Definition ***Cauchy sequence***

A sequence $f_1, f_2, \ldots$ in a metric space (V, d) is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $d(f_j, f_k) < \varepsilon$ for all integers $j \geq n$ and $k \geq n$.

6.13 ***every convergent sequence is a Cauchy sequence***

Every convergent sequence in a metric space is a Cauchy sequence.

Proof Suppose $\lim_{k\to\infty} f_k = f$ in a metric space (V, d). Suppose $\varepsilon > 0$. Then there exists $n \in \mathbf{Z}^+$ such that $d(f_k, f) < \frac{\varepsilon}{2}$ for all $k \geq n$. If $j, k \in \mathbf{Z}^+$ are such that $j \geq n$ and $k \geq n$, then

$$d(f_j, f_k) \leq d(f_j, f) + d(f, f_k) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus $f_1, f_2, \ldots$ is a Cauchy sequence, completing the proof. ■

Metric spaces that satisfy the converse of the result above have a special name.

6.14 Definition ***complete metric space***

A metric space V is called *complete* if every Cauchy sequence in V converges to some element of V.

6.15 Example

- All five of the metric spaces in Example 6.2 are complete, as you should verify.
- The metric space $\mathbf{Q}$, with metric defined by $d(x, y) = |x - y|$, is not complete. To see this, for $k \in \mathbf{Z}^+$ let

$$x_k = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{k!}}.$$

If $j < k$, then

$$|x_k - x_j| = \frac{1}{10^{(j+1)!}} + \cdots + \frac{1}{10^{k!}} < \frac{2}{10^{(j+1)!}}.$$

Thus $x_1, x_2, \ldots$ is a Cauchy sequence in $\mathbf{Q}$. However, $x_1, x_2, \ldots$ does not converge to an element of $\mathbf{Q}$ because the limit of this sequence would have a decimal expansion $0.110001000000000000000001\ldots$ that is neither a terminating decimal nor a repeating decimal. Thus $\mathbf{Q}$ is not a complete metric space.

Entrance to the École Polytechnique (Paris), where Augustin-Louis Cauchy (1789–1857) was a student and a faculty member. Cauchy wrote almost 800 mathematics papers and the highly influential textbook Cours d'Analyse *(published in 1821), which greatly influenced the development of analysis.*

Every nonempty subset of a metric space is a metric space. Specifically, suppose (V, d) is a metric space and U is a nonempty subset of V. Then restricting d to $U \times U$ gives a metric on U. Unless stated otherwise, you should assume that the metric on a subset is this restricted metric that the subset inherits from the bigger set.

Combining the two bullet points in the result below shows that a subset of a complete metric space is complete if and only if it is closed.

6.16 ***connection between complete and closed***

(a) A complete subset of a metric space is closed.

(b) A closed subset of a complete metric space is complete.

Proof We begin with a proof of (a). Suppose U is a complete subset of a metric space V. Suppose $f_1, f_2, \ldots$ is a sequence in U that converges to some $g \in V$. Then $f_1, f_2, \ldots$ is a Cauchy sequence in U (by 6.13). Hence by the completeness of U, the sequence $f_1, f_2, \ldots$ converges to some element of U, which must be g (see Exercise 7). Hence $g \in U$. Now 6.9(e) implies that U is a closed subset of V, completing the proof of (a).

To prove (b), suppose U is a closed subset of a complete metric space V. To show that U is complete, suppose $f_1, f_2, \ldots$ is a Cauchy sequence in U. Then $f_1, f_2, \ldots$ is also a Cauchy sequence in V. By the completeness of V, this sequence converges to some $f \in V$. Because U is closed, this implies that $f \in U$ (see 6.9). Thus the Cauchy sequence $f_1, f_2, \ldots$ converges to an element of U, showing that U is complete. Hence (b) has been proved. ∎

EXERCISES 6A

1 Verify that each of the claimed metrics in Example 6.2 is indeed a metric.

2 Prove that every finite subset of a metric space is closed.

3 Prove that every closed ball in a metric space is closed.

4 Suppose V is a metric space.

(a) Prove that the union of each collection of open subsets of V is an open subset of V.

(b) Prove that the intersection of each finite collection of open subsets of V is an open subset of V.

5 Suppose V is a metric space.

(a) Prove that the intersection of each collection of closed subsets of V is a closed subset of V.

(b) Prove that the union of each finite collection of closed subsets of V is a closed subset of V.

6 (a) Prove that if V is a metric space, $f \in V$, and $r > 0$, then $\overline{B(f,r)} \subset \overline{B}(f,r)$.

(b) Give an example of a metric space V, $f \in V$, and $r > 0$ such that $\overline{B(f,r)} \neq \overline{B}(f,r)$.

7 Show that a sequence in a metric space has at most one limit.

8 Prove 6.9.

9 Prove that each open subset of a metric space V is the union of some sequence of closed subsets of V.

10 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $\overline{U} \cup \overline{W} = \overline{U \cup W}$.

11 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $\overline{U} \cap \overline{W} = \overline{U \cap W}$.

12 Suppose (U, d_U), (V, d_V), and (W, d_W) are metric spaces. Suppose also that $T\colon U \to V$ and $S\colon V \to W$ are continuous functions.

(a) Using the definition of continuity, show that $S \circ T\colon U \to W$ is continuous.

(b) Using the equivalence of 6.11(a) and 6.11(b), show that $S \circ T\colon U \to W$ is continuous.

(c) Using the equivalence of 6.11(a) and 6.11(c), show that $S \circ T\colon U \to W$ is continuous.

13 Prove the parts of 6.11 that were not proved in the text.

14 Suppose a Cauchy sequence in a metric space has a convergent subsequence. Prove that the Cauchy sequence converges.

15 Verify that all five of the metric spaces in Example 6.2 are complete metric spaces.

16 Suppose (U, d) is a metric space. Let W denote the set of all Cauchy sequences of elements of U.

(a) For $(f_1, f_2, \ldots)$ and $(g_1, g_2, \ldots)$ in W, define $(f_1, f_2, \ldots) \equiv (g_1, g_2, \ldots)$ to mean that

$$\lim_{k \to \infty} d(f_k, g_k) = 0.$$

Show that $\equiv$ is an equivalence relation on W.

(b) Let V denote the set of equivalence classes of elements of W under the equivalence relation above. For $(f_1, f_2, \ldots) \in W$, let $(f_1, f_2, \ldots)\hat{}$ denote the equivalence class of $(f_1, f_2, \ldots)$. Define $d_V \colon V \times V \to [0, \infty)$ by

$$d_V\bigl((f_1, f_2, \ldots)\hat{}, (g_1, g_2, \ldots)\hat{}\bigr) = \lim_{k \to \infty} d(f_k, g_k).$$

Show that this definition of d_V makes sense and that d_V is a metric on V.

(c) Show that (V, d_V) is a complete metric space.

(d) Show that the map from U to V that takes $f \in U$ to $(f, f, f, \ldots)\hat{}$ preserves distances, meaning that

$$d(f, g) = d_V\bigl((f, f, f, \ldots)\hat{}, (g, g, g, \ldots)\hat{}\bigr)$$

for all $f, g \in U$.

(e) Explain why (d) shows that every metric space is a subset of some complete metric space.

6B Vector Spaces

Integration of Complex-Valued Functions

Complex numbers were invented so that we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted i, that obeys the usual rules of arithmetic. Here are the formal definitions:

6.17 Definition ***complex numbers;*** $\mathbf{C}$

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

- *Addition* and *multiplication* on $\mathbf{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

  here $a, b, c, d \in \mathbf{R}$.

If $a \in \mathbf{R}$, then we identify $a + 0i$ with a. Thus we think of $\mathbf{R}$ as a subset of $\mathbf{C}$. We also usually write $0 + bi$ as bi, and we usually write $0 + 1i$ as i. You should verify that $i^2 = -1$.

The symbol i was first used to denote $\sqrt{-1}$ by Leonhard Euler (1707–1783) in 1777.

With the definitions as above, $\mathbf{C}$ satisfies the usual rules of arithmetic. Specifically, with addition and multiplication defined as above, $\mathbf{C}$ is a field, as you should verify. Thus subtraction and division of complex numbers are defined as in any field.

The field $\mathbf{C}$ cannot be made into an ordered field. However, the useful concept of an absolute value can still be defined on $\mathbf{C}$.

Much of this section may be review for many readers.

6.18 Definition $\operatorname{Re} z$***;*** $\operatorname{Im} z$***; absolute value; limits***

Suppose $z = a + bi$, where a and b are real numbers.

- The *real part* of z, denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
- The *imaginary part* of z, denoted $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.
- The *absolute value* of z, denoted $|z|$, is defined by $|z| = \sqrt{a^2 + b^2}$.
- If $z_1, z_2, \ldots \in \mathbf{C}$ and $L \in \mathbf{C}$, then $\lim\limits_{k \to \infty} z_k = L$ means $\lim\limits_{k \to \infty} |z_k - L| = 0$.

For b a real number, the usual definition of $|b|$ as a real number is consistent with the new definition just given of $|b|$ with b thought of as a complex number. Note that if $z_1, z_2, \ldots$ is a sequence of complex numbers and $L \in \mathbf{C}$, then

$$\lim_{k\to\infty} z_k = L \iff \lim_{k\to\infty} \operatorname{Re} z_k = \operatorname{Re} L \text{ and } \lim_{k\to\infty} \operatorname{Im} z_k = \operatorname{Im} L.$$

We will reduce questions concerning measurability and integration of a complex-valued function to the corresponding questions about the real and imaginary parts of the function. We begin this process with the following definition.

6.19 Definition ***measurable complex-valued function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to \mathbf{C}$ is called *$\mathcal{S}$-measurable* if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both $\mathcal{S}$-measurable functions.

See Exercise 5 in this section for two natural conditions that are equivalent to measurability for complex-valued functions.

We will make frequent use of the following result. See Exercise 6 in this section for algebraic combinations of complex-valued measurable functions.

6.20 ***$|f|^p$ is measurable if f is measurable***

Suppose $(X, \mathcal{S})$ is a measurable space, $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function, and $0 < p < \infty$. Then $|f|^p$ is an $\mathcal{S}$-measurable function.

Proof The functions $(\operatorname{Re} f)^2$ and $(\operatorname{Im} f)^2$ are $\mathcal{S}$-measurable because the square of an $\mathcal{S}$-measurable function is measurable (by Example 2.45). Thus the function $(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2$ is $\mathcal{S}$-measurable (because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable by 2.46). Now $\big((\operatorname{Re} f)^2 + (\operatorname{Im} f)^2\big)^{p/2}$ is $\mathcal{S}$-measurable because it is the composition of a continuous function on $[0, \infty)$ and an $\mathcal{S}$-measurable function (see 2.44 and 2.41). In other words, $|f|^p$ is an $\mathcal{S}$-measurable function. ■

Now we define integration of a complex-valued function by separating the function into its real and imaginary parts.

6.21 Definition ***integral of a complex-valued function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function with $\int |f|\, d\mu < \infty$ [the collection of such functions is denoted $\mathcal{L}^1(\mu)$]. Then $\int f\, d\mu$ is defined by

$$\int f\, d\mu = \int (\operatorname{Re} f)\, d\mu + i \int (\operatorname{Im} f)\, d\mu.$$

The integral of a complex-valued measurable function is defined above only when the absolute value of the function has a finite integral. In contrast, the integral of every nonnegative measurable function is defined (although the value may be ∞), and if f is real valued then $\int f\, d\mu$ is defined to be $\int f^+\, d\mu - \int f^-\, d\mu$ if at least one of $\int f^+\, d\mu$ and $\int f^-\, d\mu$ is finite.

You can easily show that if $f, g \colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable functions such that $\int |f| \, d\mu < \infty$ and $\int |g| \, d\mu < \infty$, then

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Similarly, the definition of complex multiplication leads to the conclusion that

$$\int \alpha f \, d\mu = \alpha \int f \, d\mu$$

for all $\alpha \in \mathbf{C}$ (see Exercise 8).

The inequality in the result below concerning integration of complex-valued functions does not follow immediately from the corresponding result for real-valued functions. However, the small trick used in the proof below does give a reasonably simple proof.

6.22 ***bound on the absolute value of an integral***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f| \, d\mu < \infty$. Then

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.$$

Proof The result clearly holds if $\int f \, d\mu = 0$. Thus assume that $\int f \, d\mu \neq 0$.

Let

$$\alpha = \frac{\left| \int f \, d\mu \right|}{\int f \, d\mu}.$$

Then

$$\begin{aligned} \left| \int f \, d\mu \right| = \alpha \int f \, d\mu &= \int \alpha f \, d\mu \\ &= \int \operatorname{Re}(\alpha f) \, d\mu + i \int \operatorname{Im}(\alpha f) \, d\mu \\ &= \int \operatorname{Re}(\alpha f) \, d\mu \\ &\leq \int |\alpha f| \, d\mu \\ &= \int |f| \, d\mu, \end{aligned}$$

where the second equality holds by Exercise 8, the fourth equality holds because $\left| \int f \, d\mu \right| \in \mathbf{R}$, the inequality on the fourth line holds because $\operatorname{Re} z \leq |z|$ for every complex number z, and the equality in the last line holds because $|\alpha| = 1$. ■

Because of the result above, the Bounded Convergence Theorem (3.26) and the Dominated Convergence Theorem (3.31) hold if the functions $f_1, f_2, \ldots$ and f in the statements of those theorems are allowed to be complex valued.

We now define the complex conjugate of a complex number.

6.23 Definition ***complex conjugate;*** $\bar{z}$

Suppose $z \in \mathbf{C}$. The *complex conjugate* of $z \in \mathbf{C}$, denoted $\bar{z}$ (pronounced z-bar), is defined by

$$\bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$

For example, if $z = 5 + 7i$ then $\bar{z} = 5 - 7i$. Note that a complex number z is a real number if and only if $z = \bar{z}$.

The next result gives basic properties of the complex conjugate.

6.24 ***properties of complex conjugates***

Suppose $w, z \in \mathbf{C}$. Then

- **product of z and $\bar{z}$**
  $z\,\bar{z} = |z|^2$;
- **sum and difference of z and $\bar{z}$**
  $z + \bar{z} = 2\operatorname{Re} z$ and $z - \bar{z} = 2(\operatorname{Im} z)i$;
- **additivity and multiplicativity of complex conjugate**
  $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\,\bar{z}$;
- **complex conjugate of complex conjugate**
  $\bar{\bar{z}} = z$;
- **absolute value of complex conjugate**
  $|\bar{z}| = |z|$;
- **integral of complex conjugate of a function**
  $\int \bar{f}\, d\mu = \overline{\int f\, d\mu}$ for every measure μ and every $f \in \mathcal{L}^1(\mu)$.

Proof The first item holds because

$$z\bar{z} = (\operatorname{Re} z + i \operatorname{Im} z)(\operatorname{Re} z - i \operatorname{Im} z) = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = |z|^2.$$

To prove the last item, suppose μ is a measure and $f \in \mathcal{L}^1(\mu)$. Then

$$\begin{aligned} \int \bar{f}\, d\mu = \int (\operatorname{Re} f - i \operatorname{Im} f)\, d\mu &= \int \operatorname{Re} f\, d\mu - i \int \operatorname{Im} f\, d\mu \\ &= \overline{\int \operatorname{Re} f\, d\mu + i \int \operatorname{Im} f\, d\mu} \\ &= \overline{\int f\, d\mu}, \end{aligned}$$

as desired.

The straightforward proofs of the remaining items are left to the reader. ■

Vector Spaces and Subspaces

The structure and language of vector spaces will help us focus on certain features of collections of measurable functions. So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation.

6.25 Definition F

From now on, **F** stands for either **R** or **C**.

In the definitions that follow, we use f and g to denote elements of V because in the crucial examples the elements of V are functions from a set X to **F**.

6.26 Definition ***addition; scalar multiplication***

- An *addition* on a set V is a function that assigns an element $f + g \in V$ to each pair of elements $f, g \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\alpha f \in V$ to each $\alpha \in \mathbf{F}$ and each $f \in V$.

Now we are ready to give the formal definition of a vector space.

6.27 Definition ***vector space***

A *vector space* (over **F**) is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$f + g = g + f$ for all $f, g \in V$;

associativity
$(f + g) + h = f + (g + h)$ and $(\alpha\beta)f = \alpha(\beta f)$ for all $f, g, h \in V$ and $\alpha, \beta \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $f + 0 = f$ for all $f \in V$;

additive inverse
for every $f \in V$, there exists $g \in V$ such that $f + g = 0$;

multiplicative identity
$1f = f$ for all $f \in V$;

distributive properties
$\alpha(f + g) = \alpha f + \alpha g$ and $(\alpha + \beta)f = \alpha f + \beta f$ for all $\alpha, \beta \in \mathbf{F}$ and $f, g \in V$.

Most vector spaces that you will encounter are subsets of the vector space $\mathbf{F}^X$ presented in the next example.

6.28 Example *the vector space* $\mathbf{F}^X$

Suppose X is a nonempty set. Let $\mathbf{F}^X$ denote the set of functions from X to $\mathbf{F}$. Addition and scalar multiplication on $\mathbf{F}^X$ are defined as expected: for $f, g \in \mathbf{F}^X$ and $\alpha \in \mathbf{F}$, define

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha\bigl(f(x)\bigr)$$

for $x \in X$. Then, as you should verify, $\mathbf{F}^X$ is a vector space; the additive identity in this vector space is the function $0 \in \mathbf{F}^X$ defined by $0(x) = 0$ for all $x \in X$.

6.29 Example $\mathbf{F}^n$; $\mathbf{F}^{\mathbf{Z}^+}$

Special case of the previous example: if $n \in \mathbf{Z}^+$ and $X = \{1, \dots, n\}$, then $\mathbf{F}^X$ is the familiar space $\mathbf{R}^n$ or $\mathbf{C}^n$, depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

Another special case: $\mathbf{F}^{\mathbf{Z}^+}$ is the vector space of all sequences of real numbers or complex numbers, again depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

By considering subspaces, we can greatly expand our examples of vector spaces.

6.30 Definition ***subspace***

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V).

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

6.31 ***conditions for a subspace***

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- **additive identity**
  $0 \in U$;
- **closed under addition**
  $f, g \in U$ implies $f + g \in U$;
- **closed under scalar multiplication**
  $\alpha \in \mathbf{F}$ and $f \in U$ implies $\alpha f \in U$.

Proof If U is a subspace of V, then U satisfies the three conditions above by the definition of vector space.

Conversely, suppose U satisfies the three conditions above. The first condition above ensures that the additive identity of V is in U.

The second condition above ensures that addition makes sense on U. The third condition ensures that scalar multiplication makes sense on U.

If $f \in V$, then $0f = (0+0)f = 0f + 0f$. Adding the additive inverse of $0f$ to both sides of this equation shows that $0f = 0$. Now if $f \in U$, then $(-1)f$ is also in U by the third condition above. Because $f + (-1)f = \big(1 + (-1)\big)f = 0f = 0$, we see that $(-1)f$ is an additive inverse of f. Hence every element of U has an additive inverse in U.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus U is a vector space and hence is a subspace of V. ■

The three conditions in 6.31 usually enable us to determine quickly whether a given subset of V is a subspace of V, as illustrated below. All the examples below except for the first bullet point involve concepts from measure theory.

6.32 Example *subspaces of* $\mathbf{F}^X$

- The set $C([0,1])$ of continuous real-valued functions on $[0,1]$ is a vector space over $\mathbf{R}$ because the sum of two continuous functions is continuous and a constant multiple of a continuous functions is continuous. In other words, $C([0,1])$ is a subspace of $\mathbf{R}^{[0,1]}$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable and a constant multiple of an $\mathcal{S}$-measurable function is $\mathcal{S}$-measurable.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{Z}(\mu)$ of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $f = 0$ almost everywhere [meaning that $\mu\big(\{x \in X : f(x) \neq 0\}\big) = 0$] is a vector space over $\mathbf{F}$ because the union of two sets with μ-measure 0 is a set with μ-measure 0 [which implies that $\mathcal{Z}(\mu)$ is closed under addition]. Note that $\mathcal{Z}(\mu)$ is a subspace of $\mathbf{F}^X$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of bounded measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two bounded $\mathcal{S}$-measurable functions is a bounded $\mathcal{S}$-measurable function and a constant multiple of a bounded $\mathcal{S}$-measurable function is a bounded $\mathcal{S}$-measurable function.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $\int f \, d\mu = 0$ is a subspace of $\mathbf{F}^X$ because of standard properties of integration.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{L}^1(\mu)$ of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ such that $\int |f| \, d\mu < \infty$ is a subspace of $\mathbf{F}^X$ [we are now redefining $\mathcal{L}^1(\mu)$ to allow for the possibility that $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$]. The set $\mathcal{L}^1(\mu)$ is closed under addition and scalar multiplication because $\int |f + g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu$ and $\int |\alpha f| \, d\mu = |\alpha| \int |f| \, d\mu$.

- The set ℓ^1 of all sequences $(a_1, a_2, \dots)$ of elements of $\mathbf{F}$ such that $\sum_{k=1}^{\infty} |a_k| < \infty$ is a subspace of $\mathbf{F}^{\mathbf{Z}^+}$. Note that ℓ^1 is a special case of the example in the previous bullet point (take μ to be counting measure on $\mathbf{Z}^+$).

EXERCISES 6B

1 Show that if $a, b \in \mathbf{R}$ with $a + bi \neq 0$, then

$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

2 Suppose $z \in \mathbf{C}$. Prove that

$$\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} \leq |z| \leq \sqrt{2}\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}.$$

3 Suppose $z \in \mathbf{C}$. Prove that $\dfrac{|\operatorname{Re} z| + |\operatorname{Im} z|}{\sqrt{2}} \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$.

4 Suppose $w, z \in \mathbf{C}$. Prove that $|wz| = |w|\,|z|$ and $|w + z| \leq |w| + |z|$.

5 Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{C}$ is a complex-valued function. For conditions (b) and (c) below, identify $\mathbf{C}$ with $\mathbf{R}^2$. Prove that the following are equivalent:

(a) f is $\mathcal{S}$-measurable.
(b) $f^{-1}(G) \in \mathcal{S}$ for every open set G in $\mathbf{R}^2$.
(c) $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \in \mathcal{B}_2$.

6 Suppose $(X, \mathcal{S})$ is a measurable space and $f, g\colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable. Prove that

(a) $f + g$, $f - g$, and fg are $\mathcal{S}$-measurable functions;
(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

7 Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{C}$. Suppose $\lim\limits_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f\colon X \to \mathbf{C}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Prove that f is an $\mathcal{S}$-measurable function.

8 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f|\,d\mu < \infty$. Prove that if $\alpha \in \mathbf{C}$, then

$$\int \alpha f\,d\mu = \alpha \int f\,d\mu.$$

9 Suppose V is a vector space. Show that the intersection of every collection of subspaces of V is a subspace of V.

10 Suppose V and W are vector spaces. Define $V \times W$ by

$$V \times W = \{(f, g) : f \in V \text{ and } g \in W\}.$$

Define addition and scalar multiplication on $V \times W$ by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) \quad \text{and} \quad \alpha(f, g) = (\alpha f, \alpha g).$$

Prove that $V \times W$ is a vector space with these operations.

6C Normed Vector Spaces

Norms and Complete Norms

This section begins with a crucial definition.

6.33 Definition ***norm; normed vector space***

A *norm* on a vector space V (over $\mathbf{F}$) is a function $\|\cdot\| \colon V \to [0,\infty)$ such that

- $\|f\| = 0$ if and only if $f = 0$ (**positive definite**);
- $\|\alpha f\| = |\alpha|\,\|f\|$ for all $\alpha \in \mathbf{F}$ and $f \in V$ (**homogeneity**);
- $\|f + g\| \le \|f\| + \|g\|$ for all $f, g \in V$ (**triangle inequality**).

A *normed vector space* is a pair $(V, \|\cdot\|)$, where V is a vector space and $\|\cdot\|$ is a norm on V.

6.34 Example *norms*

- Suppose $n \in \mathbf{Z}^+$. Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbf{F}^n$ by
$$\|(a_1,\dots,a_n)\|_1 = |a_1| + \cdots + |a_n|$$
and
$$\|(a_1,\dots,a_n)\|_\infty = \max\{|a_1|,\dots,|a_n|\}.$$
Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbf{F}^n$, as you should verify.

- On ℓ^1 (see the last bullet point in Example 6.32 for the definition of ℓ^1), define $\|\cdot\|_1$ by
$$\|(a_1,a_2,\dots)\|_1 = \sum_{k=1}^{\infty} |a_k|.$$
Then $\|\cdot\|_1$ is a norm on ℓ^1, as you should verify.

- Suppose X is a nonempty set and $b(X)$ is the subspace of $\mathbf{F}^X$ consisting of the bounded functions from X to $\mathbf{F}$. For f a bounded function from X to $\mathbf{F}$, define $\|f\|$ by
$$\|f\| = \sup\{|f(x)| : x \in X\}.$$
Then $\|\cdot\|$ is a norm on $b(X)$, as you should verify.

- Let $C([0,1])$ denote the vector space of continuous functions from the interval $[0,1]$ to $\mathbf{F}$. Define $\|\cdot\|$ on $C([0,1])$ by
$$\|f\| = \int_0^1 |f|.$$
Then $\|\cdot\|$ is a norm on $C([0,1])$, as you should verify.

Sometimes examples that do not satisfy a definition help you gain understanding.

6.35 Example *not norms*

- Let $\mathcal{L}^1(\mathbf{R})$ denote the vector space of Borel (or Lebesgue) measurable functions $f\colon \mathbf{R} \to \mathbf{F}$ such that $\int |f|\, d\lambda < \infty$, where λ is Lebesgue measure on $\mathbf{R}$. Define $\|\cdot\|_1$ on $\mathcal{L}^1(\mathbf{R})$ by
$$\|f\|_1 = \int |f|\, d\lambda.$$
Then $\|\cdot\|_1$ satisfies the homogeneity condition and the triangle inequality on $\mathcal{L}^1(\mathbf{R})$, as you should verify. However, $\|\cdot\|_1$ is not a norm on $\mathcal{L}^1(\mathbf{R})$ because the positive definite condition is not satisfied. Specifically, if E is a nonempty Borel subset of $\mathbf{R}$ with Lebesgue measure 0 (for example, E might consist of a single element of $\mathbf{R}$), then $\|\chi_E\|_1 = 0$ but $\chi_E \neq 0$. In the next chapter, we will discuss a modification of $\mathcal{L}^1(\mathbf{R})$ that removes this problem.

- If $n \in \mathbf{Z}^+$ and $\|\cdot\|$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\| = |a_1|^{1/2} + \cdots + |a_n|^{1/2},$$
then $\|\cdot\|$ satisfies the positive definite condition and the triangle inequality (as you should verify). However, $\|\cdot\|$ as defined above is not a norm because it does not satisfy the homogeneity condition.

- If $\|\cdot\|_{1/2}$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\|_{1/2} = \bigl(|a_1|^{1/2} + \cdots + |a_n|^{1/2}\bigr)^2,$$
then $\|\cdot\|_{1/2}$ satisfies the positive definite condition and the homogeneity condition. However, if $n > 1$ then $\|\cdot\|_{1/2}$ is not a norm on $\mathbf{F}^n$ because the triangle inequality is not satisfied (as you should verify).

The next result shows that every normed vector space is also a metric space in a natural fashion.

6.36 *normed vector spaces are metric spaces*

Suppose $(V, \|\cdot\|)$ is a normed vector space. Define $d\colon V \times V \to [0, \infty)$ by
$$d(f, g) = \|f - g\|.$$
Then d is a metric on V.

Proof Suppose $f, g, h \in V$. Then
$$\begin{aligned} d(f, h) = \|f - h\| &= \|(f - g) + (g - h)\| \\ &\leq \|f - g\| + \|g - h\| \\ &= d(f, g) + d(g, h). \end{aligned}$$
Thus the triangle inequality requirement for a metric is satisfied. The verification of the other required properties for a metric are left to the reader. ∎

From now on, all metric space notions in the context of a normed vector space should be interpreted with respect to the metric introduced in the previous result. However, usually there is no need to introduce the metric d explicitly—just use the norm of the difference of two elements. For example, suppose $(V, \|\cdot\|)$ is a normed vector space, $f_1, f_2, \ldots$ is a sequence in V, and $f \in V$. Then in the context of a normed vector space, the definition of limit (6.8) becomes the following statement:

$$\lim_{k\to\infty} f_k = f \text{ means } \lim_{k\to\infty} \|f_k - f\| = 0.$$

As another example, in the context of a normed vector space, the definition of a Cauchy sequence (6.12) becomes the following statement:

> A sequence $f_1, f_2, \ldots$ in a normed vector space $(V, \|\cdot\|)$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $\|f_j - f_k\| < \varepsilon$ for all integers $j \geq n$ and $k \geq n$.

Every sequence in a normed vector space that has a limit is a Cauchy sequence (see 6.13). Normed vector spaces that satisfy the converse have a special name.

6.37 Definition ***Banach space***

A complete normed vector space is called a *Banach space*.

In other words, a normed vector space V is a Banach space if every Cauchy sequence in V converges to some element of V.

In a slight abuse of terminology, we often refer to a normed vector space V without mentioning the norm $\|\cdot\|$. When that happens, you should assume that a norm $\|\cdot\|$ lurks nearby, even if it is not explicitly displayed.

The verifications of the assertions in Examples 6.38 and 6.39 below are left to the reader as exercises.

6.38 Example *Banach spaces*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \sup\limits_{[0,1]}|f|$ is a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^{\infty}|a_k|$ is a Banach space.

6.39 Example *not a Banach space*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \int_0^1 |f|$ is not a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_\infty = \sup\limits_{k\in\mathbf{Z}^+}|a_k|$ is not a Banach space.

6.40 Definition ***infinite sum in a normed vector space***

Suppose $g_1, g_2, \ldots$ is a sequence in a normed vector space V. Then $\sum_{k=1}^{\infty} g_k$ is defined by

$$\sum_{k=1}^{\infty} g_k = \lim_{n\to\infty} \sum_{k=1}^{n} g_k$$

if this limit exists, in which case the infinite series is said to *converge*.

Recall from your calculus course that if $a_1, a_2, \ldots$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges. The next result states that the analogous property for normed vector spaces characterizes Banach spaces.

6.41 $\left(\sum_{k=1}^{\infty} \|g_k\| < \infty \implies \sum_{k=1}^{\infty} g_k\right.$ ***converges***$\left.\right) \iff$ ***Banach space***

Suppose V is a normed vector space. Then V is a Banach space if and only if $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \ldots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$.

Proof First suppose V is a Banach space. Suppose $g_1, g_2, \ldots$ is a sequence in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\sum_{m=n}^{\infty} \|g_m\| < \varepsilon$. For $j \in \mathbf{Z}^+$, let f_j denote the partial sum defined by

$$f_j = g_1 + \cdots + g_j.$$

If $k > j \geq n$, then

$$\begin{aligned} \|f_k - f_j\| &= \|g_{j+1} + \cdots + g_k\| \\ &\leq \|g_{j+1}\| + \cdots + \|g_k\| \\ &\leq \sum_{m=n}^{\infty} \|g_m\| \\ &< \varepsilon. \end{aligned}$$

Thus $f_1, f_2, \ldots$ is a Cauchy sequence in V. Because V is a Banach space, we conclude that $f_1, f_2, \ldots$ converges to some element of V, which is precisely what it means for $\sum_{k=1}^{\infty} g_k$ to converge, completing one direction of the proof.

To prove the other direction, suppose $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \ldots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $f_1, f_2, \ldots$ is a Cauchy sequence in V. We want to prove that $f_1, f_2, \ldots$ converges to some element of V. It suffices to show that some subsequence of $f_1, f_2, \ldots$ converges (by Exercise 14 in Section 6A). Dropping to a subsequence (but not relabeling) and setting $f_0 = 0$, we can assume that

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\| < \infty.$$

Hence $\sum_{k=1}^{\infty} (f_k - f_{k-1})$ converges. The partial sum of this series after n terms is f_n. Thus $\lim_{n\to\infty} f_n$ exists, completing the proof. ∎

Bounded Linear Maps

When dealing with two or more vector spaces, as in the definition below, assume that the vector spaces are over the same field (either **R** or **C**, but denoted in this book as **F** to give us the flexibility to consider both cases).

The notation Tf, in addition to the standard functional notation $T(f)$, is often used when considering linear maps, which we now define.

6.42 Definition ***linear map***

Suppose V and W are vector spaces. A function $T\colon V \to W$ is called *linear* if

- $T(f + g) = Tf + Tg$ for all $f, g \in V$;
- $T(\alpha f) = \alpha Tf$ for all $\alpha \in \mathbf{F}$ and $f \in V$.

A linear function is often called a *linear map*.

The set of linear maps from a vector space V to a vector space W is itself a vector space, using the usual operations of addition and scalar multiplication of functions. Most attention in analysis focuses on the subset of bounded linear functions, defined below, which we will see is itself a normed vector space.

In the next definition, we have two normed vector spaces, V and W, which may have different norms. However, we use the same notation $\|\cdot\|$ for both norms (and for the norm of a linear map from V to W) because the context makes the meaning clear. For example, in the definition below, f is in V and thus $\|f\|$ refers to the norm in V. Similarly, $Tf \in W$ and thus $\|Tf\|$ refers to the norm in W.

6.43 Definition ***bounded linear map;*** $\|T\|$***;*** $\mathcal{B}(V, W)$

Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map.

- The norm of T, denoted $\|T\|$, is defined by
$$\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| \leq 1\}.$$
- T is called *bounded* if $\|T\| < \infty$.
- The set of bounded linear maps from V to W is denoted $\mathcal{B}(V, W)$.

6.44 Example *bounded linear map*

Let $C([0,3])$ be the normed vector space of continuous functions from $[0,3]$ to $\mathbf{F}$, with $\|f\| = \sup\limits_{[0,3]}|f|$. Define $T\colon C([0,3]) \to C([0,3])$ by
$$(Tf)(x) = x^2 f(x).$$
Then T is a bounded linear map and $\|T\| = 9$, as you should verify.

6.45 Example *linear map that is not bounded*

Let V be the normed vector space of sequences $(a_1, a_2, \ldots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$, with $\|(a_1, a_2, \ldots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$. Define $T\colon V \to V$ by

$$T(a_1, a_2, a_3, \ldots) = (a_1, 2a_2, 3a_3, \ldots).$$

Then T is a linear map that is not bounded, as you should verify.

The next result shows that if V and W are normed vector spaces, then $\mathcal{B}(V, W)$ is a normed vector space with the norm defined above.

6.46 $\|\cdot\|$ ***is a norm on*** $\mathcal{B}(V, W)$

Suppose V and W are normed vector spaces. Then $\|S + T\| \leq \|S\| + \|T\|$ and $\|\alpha T\| = |\alpha| \, \|T\|$ for all $S, T \in \mathcal{B}(V, W)$ and all $\alpha \in \mathbf{F}$. Furthermore, the function $\|\cdot\|$ is a norm on $\mathcal{B}(V, W)$.

Proof Suppose $S, T \in \mathcal{B}(V, W)$. then

$$\begin{aligned} \|S + T\| &= \sup\{\|(S + T)f\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\leq \sup\{\|Sf\| + \|Tf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\leq \sup\{\|Sf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &\qquad\qquad + \sup\{\|Tf\| : f \in V \text{ and } \|f\| \leq 1\} \\ &= \|S\| + \|T\|. \end{aligned}$$

The inequality above shows that $\|\cdot\|$ satisfies the triangle inequality on $\mathcal{B}(V, W)$. The verification of the other properties required for a normed vector space is left to the reader. ■

Be sure that you are comfortable using all four equivalent formulas for $\|T\|$ shown in Exercise 16. For example, you should often think of $\|T\|$ as the smallest number such that $\|Tf\| \leq \|T\| \, \|f\|$ for all f in the domain of T.

Note that in the next result, the hypothesis requires W to be a Banach space but there is no requirement for V to be a Banach space.

6.47 $\mathcal{B}(V, W)$ ***is a Banach space if*** W ***is a Banach space***

Suppose V is a normed vector space and W is a Banach space. Then $\mathcal{B}(V, W)$ is a Banach space.

Proof Suppose $T_1, T_2, \ldots$ is a Cauchy sequence in $\mathcal{B}(V, W)$. If $f \in V$, then

$$\|T_j f - T_k f\| \leq \|T_j - T_k\| \, \|f\|,$$

which implies that $T_1 f, T_2 f, \ldots$ is a Cauchy sequence in W. Because W is a Banach space, this implies that $T_1 f, T_2 f, \ldots$ has a limit in W, which we call Tf.

We have now defined a function $T\colon V \to W$. The reader should verify that T is a linear map. Clearly

$$\begin{aligned}\|Tf\| &\le \sup\{\|T_k f\| : k \in \mathbf{Z}^+\}\\ &\le \bigl(\sup\{\|T_k\| : k \in \mathbf{Z}^+\}\bigr)\|f\|\end{aligned}$$

for each $f \in V$. The last supremum above is finite because every Cauchy sequence is bounded (see Exercise 4). Thus $T \in \mathcal{B}(V, W)$.

We still need to show that $\lim_{k\to\infty}\|T_k - T\| = 0$. To do this, suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\|T_j - T_k\| < \varepsilon$ for all $j \ge n$ and $k \ge n$. Suppose $j \ge n$ and suppose $f \in V$. Then

$$\begin{aligned}\|(T_j - T)f\| &= \lim_{k\to\infty}\|T_j f - T_k f\|\\ &\le \varepsilon\|f\|.\end{aligned}$$

Thus $\|T_j - T\| \le \varepsilon$, completing the proof. ■

The next result shows that the phrase *bounded linear map* means the same as the phrase *continuous linear map*.

6.48 *continuity is equivalent to boundedness for linear maps*

A linear map from one normed vector space to another normed vector space is continuous if and only if it is bounded.

Proof Suppose V and W are normed vector spaces and $T\colon V \to W$ is linear.

First suppose T is not bounded. Thus there exists a sequence $f_1, f_2, \ldots$ in V such that $\|f_k\| \le 1$ for each $k \in \mathbf{Z}^+$ and $\|Tf_k\| \to \infty$ as $k \to \infty$. Hence

$$\lim_{k\to\infty}\frac{f_k}{\|Tf_k\|} = 0 \quad\text{and}\quad T\Bigl(\frac{f_k}{\|Tf_k\|}\Bigr) = \frac{Tf_k}{\|Tf_k\|} \not\to 0,$$

where the nonconvergence to 0 holds because $Tf_k/\|Tf_k\|$ has norm 1 for every $k \in \mathbf{Z}^+$. The displayed line above implies that T is not continuous, completing the proof in one direction.

To prove the other direction, now suppose T is bounded. Suppose $f \in V$ and $f_1, f_2, \ldots$ is a sequence in V such that $\lim_{k\to\infty} f_k = f$. Then

$$\begin{aligned}\|Tf_k - Tf\| &= \|T(f_k - f)\|\\ &\le \|T\|\,\|f_k - f\|.\end{aligned}$$

Thus $\lim_{k\to\infty} Tf_k = Tf$. Hence T is continuous, completing the proof in the other direction. ■

Exercise 18 gives several additional equivalent conditions for a linear map to be continuous.

EXERCISES 6C

1 Show that the map $f \mapsto \|f\|$ from a normed vector space V to $\mathbf{F}$ is continuous (where the norm on $\mathbf{F}$ is the usual absolute value).

2 Prove that if V is a normed vector space, $f \in V$, and $r > 0$, then

$$\overline{B(f,r)} = \overline{B}(f,r).$$

3 Show that the functions defined in the last two bullet points of Example 6.35 are not norms.

4 Prove that each Cauchy sequence in a normed vector space is bounded (meaning that there is a real number that is greater than the norm of every element in the Cauchy sequence).

5 Show that if $n \in \mathbf{Z}^+$, then $\mathbf{F}^n$ is a Banach space with both the norms used in the first bullet point of Example 6.34.

6 Suppose X is a nonempty set and $b(X)$ is the vector space of bounded functions from X to $\mathbf{F}$. Prove that if $\|\cdot\|$ is defined on $b(X)$ by $\|f\| = \sup\limits_X |f|$, then $b(X)$ is a Banach space.

7 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_\infty = \sup\limits_{k \in \mathbf{Z}^+} |a_k|$ is not a Banach space.

8 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^\infty |a_k|$ is a Banach space.

9 Show that the vector space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbf{F}$ with the norm defined by $\|f\| = \int_0^1 |f|$ is not a Banach space.

10 Suppose U is a subspace of a normed vector space V such that some open ball of V is contained in U. Prove that $U = V$.

11 Prove that the only subsets of a normed vector space V that are both open and closed are $\varnothing$ and V.

12 Suppose V is a normed vector space. Prove that the closure of each subspace of V is a subspace of V.

13 Suppose U is a normed vector space. Let d be the metric on U defined by $d(f,g) = \|f - g\|$ for $f, g \in U$. Let V be the complete metric space constructed in Exercise 16 in Section 6A.

(a) Show that the set V is a vector space under natural operations of addition and scalar multiplication.

(b) Show that there is a natural way to make V into a normed vector space and that with this norm, V is a Banach space.

(c) Explain why (b) shows that every normed vector space is a subspace of some Banach space.

14 Suppose U is a subspace of a normed vector space V. Suppose also that W is a Banach space and $S\colon U \to W$ is a bounded linear map.

(a) Prove that there exists a unique continuous function $T\colon \overline{U} \to W$ such that $T|_U = S$.

(b) Prove that the function T in part (a) is a bounded linear map from $\overline{U}$ to W and $\|T\| = \|S\|$.

(c) Give an example to show that part (a) can fail if the assumption that W is a Banach space is replaced by the assumption that W is a normed vector space.

15 For readers familiar with the quotient of a vector space and a subspace: Suppose V is a normed vector space and U is a subspace of V. Define $\|\cdot\|$ on V/U by

$$\|f + U\| = \inf\{\|f + g\| : g \in U\}.$$

(a) Prove that $\|\cdot\|$ is a norm on V/U if and only if U is a closed subspace of V.

(b) Prove that if V is a Banach space and U is a closed subspace of V, then V/U (with the norm defined above) is a Banach space.

(c) Prove that if U is a Banach space (with the norm it inherits from V) and V/U is a Banach space (with the norm defined above), then V is a Banach space.

16 Suppose V and W are normed vector spaces with $V \neq \{0\}$ and $T\colon V \to W$ is a linear map.

(a) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| < 1\}$.

(b) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| = 1\}$.

(c) Show that $\|T\| = \inf\{c \in [0, \infty) : \|Tf\| \leq c\|f\| \text{ for all } f \in V\}$.

(d) Show that $\|T\| = \sup\Big\{\dfrac{\|Tf\|}{\|f\|} : f \in V \text{ and } f \neq 0\Big\}$.

17 Suppose U, V, and W are normed vector spaces and $T\colon U \to V$ and $S\colon V \to W$ are linear. Prove that $\|S \circ T\| \leq \|S\|\,\|T\|$.

18 Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map. Prove that the following are equivalent:

(a) T is bounded.

(b) There exists $f \in V$ such that T is continuous at f.

(c) T is uniformly continuous (which means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Tf - Tg\| < \varepsilon$ for all $f, g \in V$ with $\|f - g\| < \delta$).

(d) $T^{-1}\big(B(0, r)\big)$ is an open subset of V for some $r > 0$.

6D Linear Functionals

Bounded Linear Functionals

Linear maps into the scalar field $\mathbf{F}$ are so important that they get a special name.

6.49 Definition ***linear functional***

A *linear functional* on a vector space V is a linear map from V to $\mathbf{F}$.

When we think of the scalar field $\mathbf{F}$ as a normed vector space, as in the next example, the norm $\|z\|$ of a number $z \in \mathbf{F}$ is always intended to be just the usual absolute value $|z|$. This norm makes $\mathbf{F}$ a Banach space.

6.50 Example *linear functional*

Let V be the vector space of sequences $(a_1, a_2, \dots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$. Define $\varphi \colon V \to \mathbf{F}$ by

$$\varphi(a_1, a_2, \dots) = \sum_{k=1}^{\infty} a_k.$$

Then φ is a linear functional on V.

- If we make V a normed vector space with the norm $\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^{\infty} |a_k|$, then φ is a bounded linear functional on V, as you should verify.
- If we make V a normed vector space with the norm $\|(a_1, a_2, \dots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$, then φ is not a bounded linear functional on V, as you should verify.

6.51 Definition ***null space;*** null T

Suppose V and W are vector spaces and $T \colon V \to W$ is a linear map. Then the *null space* of T is denoted by null T and is defined by

$$\operatorname{null} T = \{f \in V : Tf = 0\}.$$

If T is a linear map on a vector space V, then null T is a subspace of V, as you should verify. If T is a continuous linear map from a normed vector space V to a normed vector space W, then null T is a closed subspace of V because null $T = T^{-1}(\{0\})$ and the inverse image of the closed set $\{0\}$ is closed [by 6.11(d)].

*The term **kernel** is also used in the mathematics literature with the same meaning as **null space**. This book uses **null space** instead of **kernel** because **null space** better captures the connection with* 0.

The converse of the last sentence fails, because a linear map between normed vector spaces can have a closed null space but not be continuous. For example, the linear map in 6.45 has a closed null space (equal to $\{0\}$) but it is not continuous.

However, the next result states that for linear functionals, as opposed to more general linear maps, having a closed null space is equivalent to continuity.

6.52 *bounded linear functionals*

Suppose V is a normed vector space and $\varphi \colon V \to \mathbf{F}$ is a linear functional that is not identically 0. Then the following are equivalent:

(a) φ is a bounded linear functional.

(b) φ is a continuous linear functional.

(c) $\operatorname{null}\varphi$ is a closed subspace of V.

(d) $\overline{\operatorname{null}\varphi} \neq V$.

Proof The equivalence of (a) and (b) is just a special case of 6.48.

To prove that (b) implies (c), suppose φ is a continuous linear functional. Then $\operatorname{null}\varphi$, which is the inverse image of the closed set $\{0\}$, is a closed subset of V by 6.11(d). Thus (b) implies (c).

To prove that (c) implies (a), we will show that the negation of (a) implies the negation of (c). Thus suppose φ is not bounded. Thus there is a sequence $f_1, f_2, \dots$ in V such that $\|f_k\| \leq 1$ and $|\varphi(f_k)| \geq k$ for each $k \in \mathbf{Z}^+$. Now

$$\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)} \in \operatorname{null}\varphi$$

for each $k \in \mathbf{Z}^+$ and

$$\lim_{k\to\infty}\Bigl(\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)}\Bigr) = \frac{f_1}{\varphi(f_1)}.$$

This proof makes major use of dividing by expressions of the form $\varphi(f)$, which would not make sense for a linear mapping into a vector space other than $\mathbf{F}$.

Clearly

$$\varphi\Bigl(\frac{f_1}{\varphi(f_1)}\Bigr) = 1 \text{ and thus } \frac{f_1}{\varphi(f_1)} \notin \operatorname{null}\varphi.$$

The last three displayed items imply that $\operatorname{null}\varphi$ is not closed, completing the proof that the negation of (a) implies the negation of (c). Thus (c) implies (a).

We now know that (a), (b), and (c) are equivalent to each other.

Using the hypothesis that φ is not identically 0, we see that (c) implies (d). To complete the proof, we need only show that (d) implies (c), which we will do by showing that the negation of (c) implies the negation of (d). Thus suppose $\operatorname{null}\varphi$ is not a closed subspace of V. Because $\operatorname{null}\varphi$ is a subspace of V, we know that $\overline{\operatorname{null}\varphi}$ is also a subspace of V (see Exercise 12 in Section 6C). Let $f \in \overline{\operatorname{null}\varphi} \setminus \operatorname{null}\varphi$. Suppose $g \in V$. Then

$$g = \Bigl(g - \frac{\varphi(g)}{\varphi(f)} f\Bigr) + \frac{\varphi(g)}{\varphi(f)} f.$$

The term in large parentheses above is in $\operatorname{null}\varphi$ and hence is in $\overline{\operatorname{null}\varphi}$. The term above following the plus sign is a scalar multiple of f and thus is in $\overline{\operatorname{null}\varphi}$. Because the equation above writes g as the sum of two elements of $\overline{\operatorname{null}\varphi}$, we conclude that $g \in \overline{\operatorname{null}\varphi}$. Hence we have shown that $V = \overline{\operatorname{null}\varphi}$, completing the proof that the negation of (c) implies the negation of (d). ■

Discontinuous Linear Functionals

The second bullet point in Example 6.50 shows that there exists a discontinuous linear functional on a certain normed vector space. Our next major goal is to show that every infinite-dimensional normed vector space has a discontinuous linear functional (see 6.62). Thus infinite-dimensional normed vector spaces behave in this respect much differently from $\mathbf{F}^n$, where all linear functionals are continuous (see Exercise 4).

We need to extend the notion of a basis of a finite-dimensional vector space to an infinite-dimensional context. In a finite-dimensional vector space, we might consider a basis of the form $e_1, \dots, e_n$, where $n \in \mathbf{Z}^+$ and each e_k is an element of our vector space. We can think of the list $e_1, \dots, e_n$ as a function from $\{1, \dots, n\}$ to our vector space, with the value of this function at $k \in \{1, \dots, n\}$ denoted by e_k with a subscript k instead of by the usual functional notation $e(k)$. To generalize, in the next definition we allow $\{1, \dots, n\}$ to be replaced by an arbitrary set that might not be a finite set.

6.53 Definition ***family***

A *family* $\{e_k\}_{k \in \Gamma}$ in a set V is a function e from a set Γ to V, with the value of the function e at $k \in \Gamma$ denoted by e_k.

Even though a family in V is a function mapping into V and thus is not a subset of V, the set terminology and the bracket notation $\{e_k\}_{k \in \Gamma}$ are useful, and the range of a family in V really is a subset of V.

We now restate some basic linear algebra concepts, but in the context of vector spaces that might be infinite-dimensional. Note that only finite sums appear in the definition below, even though we might be working with an infinite family.

6.54 Definition ***linearly independent; span; basis***

Suppose $\{e_k\}_{k \in \Gamma}$ is a family in a vector space V.

- $\{e_k\}_{k \in \Gamma}$ is called *linearly independent* if there does not exist a finite nonempty subset Ω of Γ and a family $\{\alpha_j\}_{j \in \Omega}$ in $\mathbf{F} \setminus \{0\}$ such that $\sum_{j \in \Omega} \alpha_j e_j = 0$.

- The span of $\{e_k\}_{k \in \Gamma}$ is denoted by $\operatorname{span}\{e_k\}_{k \in \Gamma}$ and is defined to be the set of all sums of the form
$$\sum_{j \in \Omega} \alpha_j e_j,$$
where Ω is a finite subset of Γ and $\{\alpha_j\}_{j \in \Omega}$ is a family in $\mathbf{F}$.

- A vector space V is called *finite-dimensional* if there exists a finite set Γ and a family $\{e_k\}_{k \in \Gamma}$ in V such that $\operatorname{span}\{e_k\}_{k \in \Gamma} = V$.

- A vector space is called *infinite-dimensional* if it is not finite-dimensional.

- A family in V is called a *basis* of V if it is linearly independent and its span equals V.

For example, $\{x^n\}_{n\in\{0,1,2,\ldots\}}$ is a basis of the vector space of polynomials.

The term Hamel basis *is sometimes used to denote what has been called a* basis *here. The use of the term* Hamel basis *emphasizes that only finite sums are under consideration.*

Our definition of span does not take advantage of the possibility of summing an infinite number of elements in contexts where a notion of limit exists (as is the case in normed vector spaces). When we get to Hilbert spaces in Chapter 8, we consider another kind of basis that does involve infinite sums. As we will soon see, the kind of basis as defined here is just what we need to produce discontinuous linear functionals.

Now we introduce terminology that will be needed in our proof that every vector space has a basis.

No one has ever produced a concrete example of a basis of an infinite-dimensional Banach space.

6.55 Definition ***maximal element***

Suppose $\mathcal{A}$ is a collection of subsets of a set V. A set $\Gamma \in \mathcal{A}$ is called a *maximal element* of $\mathcal{A}$ if there does not exist $\Gamma' \in \mathcal{A}$ such that $\Gamma \subsetneq \Gamma'$.

6.56 Example *maximal elements*

For $k \in \mathbf{Z}$, let $k\mathbf{Z}$ denote the set of integer multiples of k; thus $k\mathbf{Z} = \{km : m \in \mathbf{Z}\}$. Let $\mathcal{A}$ be the collection of subsets of $\mathbf{Z}$ defined by $\mathcal{A} = \{k\mathbf{Z} : k = 2, 3, 4, \ldots\}$. Suppose $k \in \mathbf{Z}^+$. Then $k\mathbf{Z}$ is a maximal element of $\mathcal{A}$ if and only if k is a prime number, as you should verify.

A subset Γ of a vector space V can be thought of as a family in V by considering $\{e_f\}_{f\in\Gamma}$, where $e_f = f$. With this convention, the next result shows that the bases of V are exactly the maximal elements among the collection of linearly independent subsets of V.

6.57 ***bases as maximal elements***

Suppose V is a vector space. Then a subset of V is a basis of V if and only if it is a maximal element of the collection of linearly independent subsets of V.

Proof Suppose Γ is a linearly independent subset of V.

First suppose also that Γ is a basis of V. If $f \in V$ but $f \notin \Gamma$, then $f \in \operatorname{span}\Gamma$, which implies that $\Gamma \cup \{f\}$ is not linearly independent. Thus Γ is a maximal element among the collection of linearly independent subsets of V, completing one direction of the proof.

To prove the other direction, suppose now that Γ is a maximal element of the collection of linearly independent subsets of V. If $f \in V$ but $f \notin \operatorname{span}\Gamma$, then $\Gamma \cup \{f\}$ is linearly independent, which would contradict the maximality of Γ among the collection of linearly independent subsets of V. Thus $\operatorname{span}\Gamma = V$, which means that Γ is a basis of V, completing the proof in the other direction. ■

The notion of a chain plays a key role in our next result.

6.58 Definition ***chain***

A collection $\mathcal{C}$ of subsets of a set V is called a *chain* if $\Omega, \Gamma \in \mathcal{C}$ implies $\Omega \subset \Gamma$ or $\Gamma \subset \Omega$.

6.59 Example *chains*

- The collection $\mathcal{C} = \{4\mathbf{Z}, 6\mathbf{Z}\}$ of subsets of $\mathbf{Z}$ is not a chain because neither of the sets $4\mathbf{Z}$ or $6\mathbf{Z}$ is a subset of the other.
- The collection $\mathcal{C} = \{2^n\mathbf{Z} : n \in \mathbf{Z}^+\}$ of subsets of $\mathbf{Z}$ is a chain because if $m, n \in \mathbf{Z}^+$, then $2^m\mathbf{Z} \subset 2^n\mathbf{Z}$ or $2^n\mathbf{Z} \subset 2^m\mathbf{Z}$.

Zorn's Lemma is named in honor of Max Zorn (1906–1993), who published a paper containing the result in 1935, when he had a postdoctoral position at Yale.

The next result follows from the Axiom of Choice, although it is not as intuitively believable as the Axiom of Choice. Because the techniques used to prove the next result are so different from techniques used elsewhere in this book, the reader is asked either to accept this result without proof or find one of the good proofs available via the internet or in other books. The version of Zorn's Lemma stated here is simpler than the standard more general version, but this version is all that we need.

6.60 ***Zorn's Lemma***

Suppose V is a set and $\mathcal{A}$ is a collection of subsets of V with the property that the union of all the sets in $\mathcal{C}$ is in $\mathcal{A}$ for every chain $\mathcal{C} \subset \mathcal{A}$. Then $\mathcal{A}$ contains a maximal element.

Zorn's Lemma now allows us to prove that every vector space has a basis. The proof does not help us find a concrete basis because Zorn's Lemma is an existence result rather than a constructive technique.

6.61 ***bases exist***

Every vector space has a basis.

Proof Suppose V is a vector space. If $\mathcal{C}$ is a chain of linearly independent subsets of V, then the union of all the sets in $\mathcal{C}$ is also a linearly independent subset of V (this holds because linear independence is a condition that is checked by considering finite subsets, and each finite subset of the union is contained in one of the elements of the chain).

Thus if $\mathcal{A}$ denotes the collection of linearly independent subsets of V, then $\mathcal{A}$ satisfies the hypothesis of Zorn's Lemma (6.60). Hence $\mathcal{A}$ contains a maximal element, which by 6.57 is a basis of V. ■

Now we can prove the promised result about the existence of discontinuous linear functionals on every infinite-dimensional normed vector space.

6.62 *discontinuous linear functionals*

Every infinite-dimensional normed vector space has a discontinuous linear functional.

Proof Suppose V is an infinite-dimensional vector space. By 6.61, V has a basis $\{e_k\}_{k\in\Gamma}$. Because V is infinite-dimensional, Γ is not a finite set. Thus we can assume $\mathbf{Z}^+ \subset \Gamma$ (by relabeling a countable subset of Γ).

Define a linear functional $\varphi\colon V \to \mathbf{F}$ by setting $\varphi(e_j)$ equal to $j\|e_j\|$ for $j \in \mathbf{Z}^+$, setting $\varphi(e_j)$ equal to 0 for $j \in \Gamma \setminus \mathbf{Z}^+$, and extending linearly. More precisely, define a linear functional $\varphi\colon V \to \mathbf{F}$ by

$$\varphi\Big(\sum_{j\in\Omega} \alpha_j e_j\Big) = \sum_{j\in\Omega\cap\mathbf{Z}^+} \alpha_j j\|e_j\|$$

for every finite subset $\Omega \subset \Gamma$ and every family $\{\alpha_j\}_{j\in\Omega}$ in $\mathbf{F}$.

Because $\varphi(e_j) = j\|e_j\|$ for each $j \in \mathbf{Z}^+$, the linear functional φ is unbounded, completing the proof. ■

Hahn–Banach Theorem

In the last subsection, we showed that there exists a discontinuous linear functional on each infinite-dimensional normed vector space. Now we turn our attention to the existence of continuous linear functionals.

The existence of a nonzero continuous linear functional on each Banach space is not obvious. For example, consider the Banach space ℓ^∞/c_0, where ℓ^∞ is the Banach space of bounded sequences in $\mathbf{F}$ with

$$\|(a_1, a_2, \dots)\|_\infty = \sup_{k\in\mathbf{Z}^+} |a_k|$$

and c_0 is the subspace of ℓ^∞ consisting of those sequences in $\mathbf{F}$ that have limit 0. The quotient space ℓ^∞/c_0 is an infinite-dimensional Banach space (see Exercise 15 in Section 6C). However, no one has ever exhibited a concrete nonzero linear functional on the Banach space ℓ^∞/c_0.

In this subsection, we show that infinite-dimensional normed vector spaces have plenty of continuous linear functionals. We do this by showing that a bounded linear functional on a subspace of a normed vector space can be extended to a bounded linear functional on the whole space without increasing its norm—this result is called the Hahn–Banach Theorem (6.69).

Completeness plays no role in this topic. Thus this subsection deals with normed vector spaces instead of Banach spaces.

We do most of the work needed to prove the Hahn–Banach Theorem in the next lemma, which shows that we can extend a linear functional to a subspace generated by one additional element, without increasing the norm. This one-element-at-a-time approach, when combined with a maximal object produced by Zorn's Lemma, gives us the desired extension to the full normed vector space.

If V is a real vector space, U is a subspace of V, and $h \in V$, then $U + \mathbf{R}h$ is the subspace of V defined by

$$U + \mathbf{R}h = \{f + \alpha h : f \in U \text{ and } \alpha \in \mathbf{R}\}.$$

6.63 *Extension Lemma*

Suppose V is a real normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{R}$ is a bounded linear functional. Suppose $h \in V \setminus U$. Then ψ can be extended to a bounded linear functional $\varphi\colon U + \mathbf{R}h \to \mathbf{R}$ such that $\|\varphi\| = \|\psi\|$.

Proof Suppose $c \in \mathbf{R}$. Define $\varphi(h)$ to be c, and then extend φ linearly to $U + \mathbf{R}h$. Specifically, define $\varphi\colon U + \mathbf{R}h \to \mathbf{R}$ by

$$\varphi(f + \alpha h) = \psi(f) + \alpha c$$

for $f \in U$ and $\alpha \in \mathbf{R}$. Then φ is a linear functional on $U + \mathbf{R}h$.

Clearly $\varphi|_U = \psi$. Thus $\|\varphi\| \geq \|\psi\|$. We need to show that for some choice of $c \in \mathbf{R}$, the linear functional φ defined above satisfies the equation $\|\varphi\| = \|\psi\|$. In other words, we want

6.64 $$|\psi(f) + \alpha c| \leq \|\psi\|\,\|f + \alpha h\| \quad \text{for all } f \in U \text{ and all } \alpha \in \mathbf{R}.$$

It would be enough to have

6.65 $$|\psi(f) + c| \leq \|\psi\|\,\|f + h\| \quad \text{for all } f \in U,$$

because replacing f by $\frac{f}{\alpha}$ in the last inequality and then multiplying both sides by $|\alpha|$ would give 6.64.

Rewriting 6.65, we want to show that there exists $c \in \mathbf{R}$ such that

$$-\|\psi\|\,\|f + h\| \leq \psi(f) + c \leq \|\psi\|\,\|f + h\| \quad \text{for all } f \in U.$$

Equivalently, we want to show that there exists $c \in \mathbf{R}$ such that

$$-\|\psi\|\,\|f + h\| - \psi(f) \leq c \leq \|\psi\|\,\|f + h\| - \psi(f) \quad \text{for all } f \in U.$$

The existence of $c \in \mathbf{R}$ satisfying the line above follows from the inequality

6.66 $$\sup_{f \in U}\bigl(-\|\psi\|\,\|f + h\| - \psi(f)\bigr) \leq \inf_{g \in U}\bigl(\|\psi\|\,\|g + h\| - \psi(g)\bigr).$$

To prove the inequality above, suppose $f, g \in U$. Then

$$\begin{aligned}
-\|\psi\|\,\|f + h\| - \psi(f) &\leq \|\psi\|(\|g + h\| - \|g - f\|) - \psi(f)\\
&= \|\psi\|(\|g + h\| - \|g - f\|) + \psi(g - f) - \psi(g)\\
&\leq \|\psi\|\,\|g + h\| - \psi(g).
\end{aligned}$$

The inequality above proves 6.66, which completes the proof. ∎

Because our simplified form of Zorn's Lemma deals with set inclusions rather than more general orderings, we need to use the notion of the graph of a function.

6.67 Definition ***graph***

Suppose $T\colon V \to W$ is a function from a set V to a set W. Then the *graph* of T is denoted $\operatorname{graph}(T)$ and is the subset of $V \times W$ defined by

$$\operatorname{graph}(T) = \{(f, T(f)) \in V \times W : f \in V\}.$$

Formally, a function from a set V to a set W equals its graph as defined above. However, because we usually think of a function more intuitively as a mapping, the separate notion of the graph of a function remains useful.

The easy proof of the next result is left to the reader. The first bullet point below uses the vector space structure of $V \times W$, which is a vector space with natural operations of addition and scalar multiplication, as given in Exercise 10 in Section 6B.

6.68 ***function properties in terms of graphs***

Suppose V and W are normed vector spaces and $T\colon V \to W$ is a function.

(a) T is a linear map if and only if $\operatorname{graph}(T)$ is a subspace of $V \times W$.

(b) Suppose $U \subset V$ and $S\colon U \to W$ is a function. Then T is an extension of S if and only if $\operatorname{graph}(S) \subset \operatorname{graph}(T)$.

(c) If $T\colon V \to W$ is a linear map and $c \in [0, \infty)$, then $\|T\| \le c$ if and only if $\|g\| \le c\|f\|$ for all $(f, g) \in \operatorname{graph}(T)$.

The proof of the Extension Lemma (6.63) used inequalities that do not make sense when $\mathbf{F} = \mathbf{C}$. Thus the proof of the Hahn–Banach Theorem below requires some extra steps when $\mathbf{F} = \mathbf{C}$.

Hans Hahn (1879–1934) was a student and later a faculty member at the University of Vienna, where one of his PhD students was Kurt Gödel (1906–1978).

6.69 ***Hahn–Banach Theorem***

Suppose V is a normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{F}$ is a bounded linear functional. Then ψ can be extended to a bounded linear functional on V whose norm equals $\|\psi\|$.

Proof First we consider the case where $\mathbf{F} = \mathbf{R}$. Let $\mathcal{A}$ be the collection of subsets E of $V \times \mathbf{R}$ that satisfy all the following conditions:

- $E = \operatorname{graph}(\varphi)$ for some linear functional φ on some subspace of V;
- $\operatorname{graph}(\psi) \subset E$;
- $|\alpha| \le \|\psi\|\,\|f\|$ for every $(f, \alpha) \in E$.

Then $\mathcal{A}$ satisfies the hypothesis of Zorn's Lemma (6.60). Thus $\mathcal{A}$ has a maximal element. The Extension Lemma (6.63) implies that this maximal element is the graph of a linear functional defined on all of V. This linear functional is an extension of ψ to V and it has norm $\|\psi\|$, completing the proof in the case where $\mathbf{F} = \mathbf{R}$.

Now consider the case where $\mathbf{F} = \mathbf{C}$. Define $\psi_1 \colon U \to \mathbf{R}$ by

$$\psi_1(f) = \operatorname{Re} \psi(f)$$

for $f \in U$. Then ψ_1 is an $\mathbf{R}$-linear map from U to $\mathbf{R}$ and $\|\psi_1\| \leq \|\psi\|$ (actually $\|\psi_1\| = \|\psi\|$, but we need only the inequality). Also,

$$\begin{aligned} \psi(f) &= \operatorname{Re} \psi(f) + i \operatorname{Im} \psi(f) \\ &= \psi_1(f) + i \operatorname{Im}\bigl(-i\psi(if)\bigr) \\ &= \psi_1(f) - i \operatorname{Re}\bigl(\psi(if)\bigr) \\ &= \psi_1(f) - i\psi_1(if) \end{aligned} \tag{6.70}$$

for all $f \in U$.

Temporarily forget that complex scalar multiplication makes sense on V and temporarily think of V as a real normed vector space. The case of the result that we have already proved then implies that there exists an extension φ_1 of ψ_1 to an $\mathbf{R}$-linear functional $\varphi_1 \colon V \to \mathbf{R}$ with $\|\varphi_1\| = \|\psi_1\| \leq \|\psi\|$.

Motivated by 6.70, we define $\varphi \colon V \to \mathbf{C}$ by

$$\varphi(f) = \varphi_1(f) - i\varphi_1(if)$$

for $f \in V$. The equation above and 6.70 imply that φ is an extension of ψ to V. The equation above also implies that $\varphi(f+g) = \varphi(f) + \varphi(g)$ and $\varphi(\alpha f) = \alpha\varphi(f)$ for all $f, g \in V$ and all $\alpha \in \mathbf{R}$. Also,

$$\varphi(if) = \varphi_1(if) - i\varphi_1(-f) = \varphi_1(if) + i\varphi_1(f) = i\bigl(\varphi_1(f) - i\varphi_1(if)\bigr) = i\varphi(f).$$

The reader should use the equation above to show that φ is a $\mathbf{C}$-linear map.

The only part of the proof that remains is to show that $\|\varphi\| \leq \|\psi\|$. To do this, note that

$$|\varphi(f)|^2 = \varphi\bigl(\overline{\varphi(f)}f\bigr) = \varphi_1\bigl(\overline{\varphi(f)}f\bigr) \leq \|\psi\|\,\|\overline{\varphi(f)}f\| = \|\psi\|\,|\varphi(f)|\,\|f\|$$

for all $f \in V$, where the second equality holds because $\varphi\bigl(\overline{\varphi(f)}f\bigr) \in \mathbf{R}$. Dividing by $|\varphi(f)|$, we see from the line above that $|\varphi(f)| \leq \|\psi\|\,\|f\|$ for all $f \in V$ (no division necessary if $\varphi(f) = 0$). This implies that $\|\varphi\| \leq \|\psi\|$, completing the proof. ■

We have given the special name *linear functionals* to linear maps into the scalar field $\mathbf{F}$. The vector space of bounded linear functionals now also gets a special name and a special notation.

6.71 Definition ***dual space***

Suppose V is a normed vector space. Then the *dual space* of V, denoted V', is the normed vector space consisting of the bounded linear functionals on V. In other words, $V' = \mathcal{B}(V, \mathbf{F})$.

By 6.47, the dual space of every normed vector space is a Banach space.

6.72 $\|f\| = \max\{|\varphi(f)| : \varphi \in V' \textit{ and } \|\varphi\| = 1\}$

Suppose V is a normed vector space and $f \in V \setminus \{0\}$. Then there exists $\varphi \in V'$ such that $\|\varphi\| = 1$ and $\|f\| = \varphi(f)$.

Proof Let U be the 1-dimensional subspace of V defined by

$$U = \{\alpha f : \alpha \in \mathbf{F}\}.$$

Define $\psi\colon U \to \mathbf{F}$ by

$$\psi(\alpha f) = \alpha\|f\|$$

for $\alpha \in \mathbf{F}$. Then ψ is a linear functional on U with $\|\psi\| = 1$ and $\psi(f) = \|f\|$. The Hahn–Banach Theorem (6.69) implies that there exists an extension of ψ to a linear functional φ on V with $\|\varphi\| = 1$, completing the proof. ■

The next result gives another beautiful application of the Hahn–Banach Theorem, with a useful necessary and sufficient condition for an element of a normed vector space to be in the closure of a subspace.

6.73 ***condition to be in the closure of a subspace***

Suppose U is a subspace of a normed vector space V and $h \in V$. Then $h \in \overline{U}$ if and only if $\varphi(h) = 0$ for every $\varphi \in V'$ such that $\varphi|_U = 0$.

Proof First suppose $h \in \overline{U}$. If $\varphi \in V'$ and $\varphi|_U = 0$, then $\varphi(h) = 0$ by the continuity of φ, completing the proof in one direction.

To prove the other direction, suppose now that $h \notin \overline{U}$. Define $\psi\colon U + \mathbf{F}h \to \mathbf{F}$ by

$$\psi(f + \alpha h) = \alpha$$

for $f \in U$ and $\alpha \in \mathbf{F}$. Then ψ is a linear functional on $U + \mathbf{F}h$ with $\operatorname{null}\psi = U$ and $\psi(h) = 1$.

Because $h \notin \overline{U}$, the closure of the null space of ψ does not equal $U + \mathbf{F}h$. Thus 6.52 implies that ψ is a bounded linear functional on $U + \mathbf{F}h$.

The Hahn–Banach Theorem (6.69) implies that ψ can be extended to a bounded linear functional φ on V. Thus we have found $\varphi \in V'$ such that $\varphi|_U = 0$ but $\varphi(h) \neq 0$, completing the proof in the other direction. ■

EXERCISES 6D

1 Suppose V is a normed vector space and φ is a linear functional on V. Suppose $\alpha \in \mathbf{F} \setminus \{0\}$. Prove that the following are equivalent:

(a) φ is a bounded linear functional.

(b) $\varphi^{-1}(\alpha)$ is a closed subset of V.

(c) $\overline{\varphi^{-1}(\alpha)} \neq V$.

2 Suppose φ is a linear functional on a vector space V. Prove that if U is a subspace of V such that $\operatorname{null}\varphi \subset U$, then $U = \operatorname{null}\varphi$ or $U = V$.

3 Suppose φ and ψ are linear functionals on the same vector space. Prove that

$$\operatorname{null}\varphi \subset \operatorname{null}\psi$$

if and only if there exists $\alpha \in \mathbf{F}$ such that $\psi = \alpha\varphi$.

For the next two exercises, $\mathbf{F}^n$ should be endowed with the norm $\|\cdot\|_\infty$ as defined in Example 6.34.

4 Suppose $n \in \mathbf{Z}^+$ and V is a normed vector space. Prove that every linear map from $\mathbf{F}^n$ to V is continuous.

5 Suppose $n \in \mathbf{Z}^+$, V is a normed vector space, and $T\colon \mathbf{F}^n \to V$ is a linear map that is one-to-one and onto V.

(a) Show that
$$\inf\{\|Tx\| : x \in \mathbf{F}^n \text{ and } \|x\|_\infty = 1\} > 0.$$

(b) Prove that $T^{-1}\colon V \to \mathbf{F}^n$ is a bounded linear map.

6 Suppose $n \in \mathbf{Z}^+$.

(a) Prove that all norms on $\mathbf{F}^n$ have the same convergent sequences, the same open sets, and the same closed sets.

(b) Prove that all norms on $\mathbf{F}^n$ make $\mathbf{F}^n$ into a Banach space.

7 Suppose V and W are normed vector spaces and V is finite-dimensional. Prove that every linear map from V to W is continuous.

8 Prove that every finite-dimensional normed vector space is a Banach space.

9 Prove that every finite-dimensional subspace of each normed vector space is closed.

10 Give a concrete example of an infinite-dimensional normed vector space and a basis of that normed vector space.

11 Show that the collection $\mathcal{A} = \{k\mathbf{Z} : k = 2, 3, 4, \ldots\}$ of subsets of $\mathbf{Z}$ satisfies the hypothesis of Zorn's Lemma (6.60).

12 Prove that every linearly independent family in a vector space can be extended to a basis of the vector space.

13 Suppose V is a normed vector space, U is a subspace of V, and $\psi\colon U \to \mathbf{R}$ is a bounded linear functional. Prove that ψ has a unique extension to a bounded linear functional φ on V with $\|\varphi\| = \|\psi\|$ if and only if

$$\sup_{f \in U}\bigl(-\|\psi\|\,\|f+h\| - \psi(f)\bigr) = \inf_{g \in U}\bigl(\|\psi\|\,\|g+h\| - \psi(g)\bigr)$$

for every $h \in V \setminus U$.

14 Show that there exists a linear functional $\varphi\colon \ell^\infty \to \mathbf{F}$ such that

$$|\varphi(a_1, a_2, \ldots)| \leq \|(a_1, a_2, \ldots)\|_\infty$$

for all $(a_1, a_2, \ldots) \in \ell^\infty$ and

$$\varphi(a_1, a_2, \ldots) = \lim_{k\to\infty} a_k$$

for all $(a_1, a_2, \ldots) \in \ell^\infty$ such that the limit above on the right exists.

15 Suppose B is an open ball in a normed vector space V such that $0 \notin B$. Prove that there exists $\varphi \in V'$ such that

$$\operatorname{Re} \varphi(f) > 0$$

for all $f \in B$.

16 Show that the dual space of each infinite-dimensional normed vector space is infinite-dimensional.

A normed vector space is called* separable *if it has a countable subset whose closure equals the whole space.

17 Suppose V is a separable normed vector space. Explain how the Hahn–Banach Theorem (6.69) for V can be proved without using any results (such as Zorn's Lemma) that depend upon the Axiom of Choice.

18 Suppose V is a normed vector space such that the dual space V' is a separable Banach space. Prove that V is separable.

19 Prove that the dual of the Banach space $C([0,1])$ is not separable; here the norm on $C([0,1])$ is defined by $\|f\| = \sup\limits_{[0,1]} |f|$.

The* double dual space *of a normed vector space is defined to be the dual space of the dual space. If V is a normed vector space, then the double dual space of V is denoted by V''; thus $V'' = (V')'$. The norm on V'' is defined to be the norm it receives as the dual space of V'.

20 Define $\Phi\colon V \to V''$ by

$$(\Phi f)(\varphi) = \varphi(f)$$

for $f \in V$ and $\varphi \in V'$. Show that $\|\Phi f\| = \|f\|$ for every $f \in V$.
[*The map Φ defined above is called the* canonical isometry *of V into V''.*]

21 Suppose V is an infinite-dimensional normed vector space. Show that there is a convex subset U of V such that $\overline{U} = V$ and such that the complement $V \setminus U$ is also a convex subset of V with $\overline{V \setminus U} = V$.
[*See 8.25 for the definition of a convex set. This exercise should stretch your geometric intuition because this behavior cannot happen in finite dimensions.*]

6E Consequences of Baire's Theorem

This section focuses on several important results about Banach spaces that depend upon Baire's Theorem. This result was first proved by René-Louis Baire (1874–1932) as part of his 1899 doctoral dissertation at École Normale Supérieure (Paris).

Even though our interest lies primarily in applications to Banach spaces, the proper setting for Baire's Theorem is the more general context of complete metric spaces.

The result here called Baire's Theorem *is often called the* Baire Category Theorem. *This book uses the shorter name of this result because we do not need the categories introduced by Baire. Furthermore, the use of the word* category *in this context can be confusing because Baire's categories have no connection with the category theory that developed decades after Baire's work.*

Baire's Theorem

We begin with some key topological notions.

6.74 Definition ***interior***

Suppose U is a subset of a metric space V. The *interior* of U, denoted $\operatorname{int} U$, is the set of $f \in U$ such that some open ball of V centered at f with positive radius is contained in U.

You should verify the following elementary facts about the interior.

- The interior of each subset of a metric space is open.
- The interior of a subset U of a metric space V is the largest open subset of V contained in U.

6.75 Definition ***dense***

A subset U of a metric space V is called *dense* in V if $\overline{U} = V$.

For example, $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$ are both dense in $\mathbf{R}$, where $\mathbf{R}$ has its standard metric $d(x, y) = |x - y|$.

You should verify the following elementary facts about dense subsets.

- A subset U of a metric space V is dense in V if and only if every nonempty open subset of V contains at least one element of U.
- A subset U of a metric space V has an empty interior if and only if $V \setminus U$ is dense in V.

The proof of the next result uses the following fact, which you should first prove: If G is an open subset of a metric space V and $f \in G$, then there exists $r > 0$ such that $\overline{B}(f, r) \subset G$.

6.76 *Baire's Theorem*

(a) A complete metric space is not the countable union of closed subsets with empty interior.

(b) The countable intersection of dense open subsets of a complete metric space is nonempty.

Proof We will prove (b) and then use (b) to prove (a).

To prove (b), suppose (V, d) is a complete metric space and $G_1, G_2, \ldots$ is a sequence of dense open subsets of V. We need to show that $\bigcap_{k=1}^{\infty} G_k \neq \varnothing$.

Let $f_1 \in G_1$ and let $r_1 \in (0, 1)$ be such that $\overline{B}(f_1, r_1) \subset G_1$. Now suppose $n \in \mathbf{Z}^+$, and $f_1, \ldots, f_n$ and $r_1, \ldots, r_n$ have been chosen such that

6.77 $$\overline{B}(f_1, r_1) \supset \overline{B}(f_2, r_2) \supset \cdots \supset \overline{B}(f_n, r_n)$$

and

6.78 $$r_j \in \left(0, \tfrac{1}{j}\right) \quad \text{and} \quad \overline{B}(f_j, r_j) \subset G_j \quad \text{for } j = 1, \ldots, n.$$

Because $B(f_n, r_n)$ is an open subset of V and G_{n+1} is dense in V, there exists $f_{n+1} \in B(f_n, r_n) \cap G_{n+1}$. Let $r_{n+1} \in \left(0, \frac{1}{n+1}\right)$ be such that

$$\overline{B}(f_{n+1}, r_{n+1}) \subset \overline{B}(f_n, r_n) \cap G_{n+1}.$$

Thus we inductively construct a sequence $f_1, f_2, \ldots$ that satisfies 6.77 and 6.78 for all $n \in \mathbf{Z}^+$.

If $j \in \mathbf{Z}^+$, then 6.77 and 6.78 imply that

6.79 $$f_k \in \overline{B}(f_j, r_j) \quad \text{and} \quad d(f_j, f_k) \leq r_j < \tfrac{1}{j} \quad \text{for all } k > j.$$

Hence $f_1, f_2, \ldots$ is a Cauchy sequence. Because (V, d) is a complete metric space, there exists $f \in V$ such that $\lim_{k\to\infty} f_k = f$.

Now 6.79 and 6.78 imply that for each $j \in \mathbf{Z}^+$, we have $f \in \overline{B}(f_j, r_j) \subset G_j$. Hence $f \in \bigcap_{k=1}^{\infty} G_k$, which means that $\bigcap_{k=1}^{\infty} G_k$ is not the empty set, completing the proof of (b).

To prove (a), suppose (V, d) is a complete metric space and $F_1, F_2, \ldots$ is a sequence of closed subsets of V with empty interior. Then $V \setminus F_1, V \setminus F_2, \ldots$ is a sequence of dense open subsets of V. Now (b) implies that

$$\varnothing \neq \bigcap_{k=1}^{\infty} (V \setminus F_k).$$

Taking complements of both sides above, we conclude that

$$V \neq \bigcup_{k=1}^{\infty} F_k,$$

completing the proof of (a). ∎

Because

$$\mathbf{R} = \bigcup_{x \in \mathbf{R}} \{x\}$$

and each set $\{x\}$ has empty interior in $\mathbf{R}$, Baire's Theorem implies $\mathbf{R}$ is uncountable. Thus we have yet another proof that $\mathbf{R}$ is uncountable, different than Cantor's original diagonal proof and different from the proof via measure theory (see 2.17).

The next result is another nice consequence of Baire's Theorem.

6.80 ***the set of irrational numbers is not a countable union of closed sets***

There does not exist a countable collection of closed subsets of $\mathbf{R}$ whose union equals $\mathbf{R} \setminus \mathbf{Q}$.

Proof This will be a proof by contradiction. Suppose $F_1, F_2, \dots$ is a countable collection of closed subsets of $\mathbf{R}$ whose union equals $\mathbf{R} \setminus \mathbf{Q}$. Thus each F_k contains no rational numbers, which implies that each F_k has empty interior. Now

$$\mathbf{R} = \Big(\bigcup_{r \in \mathbf{Q}} \{r\}\Big) \cup \Big(\bigcup_{k=1}^{\infty} F_k\Big).$$

The equation above writes the complete metric space $\mathbf{R}$ as a countable union of closed sets with empty interior, which contradicts Baire's Theorem [6.76(a)]. This contradiction completes the proof. ■

Open Mapping Theorem and Inverse Mapping Theorem

The next result shows that a surjective bounded linear map from one Banach space onto another Banach space maps open sets to open sets. As shown in Exercises 10 and 11, this result can fail if the hypothesis that both spaces are Banach spaces is weakened to allow either of the spaces to be a normed vector space.

6.81 ***Open Mapping Theorem***

Suppose V and W are Banach spaces and T is a bounded linear map of V onto W. Then $T(G)$ is an open subset of W for every open subset G of V.

Proof Let B denote the open unit ball $B(0,1) = \{f \in V : \|f\| < 1\}$ of V. For any open ball $B(f,a)$ in V, the linearity of T implies that

$$T\big(B(f,a)\big) = Tf + aT(B).$$

Suppose G is an open subset of V. If $f \in G$, then there exists $a > 0$ such that $B(f,a) \subset G$. If we can show that $0 \in \operatorname{int} T(B)$, then the equation above shows that $Tf \in \operatorname{int} T\big(B(f,a)\big)$. This would imply that $T(G)$ is an open subset of W. Thus to complete the proof we need only show that $T(B)$ contains some open ball centered at 0.

The surjectivity and linearity of T imply that

$$W = \bigcup_{k=1}^{\infty} T(kB) = \bigcup_{k=1}^{\infty} kT(B).$$

Thus $W = \bigcup_{k=1}^{\infty} \overline{kT(B)}$. Baire's Theorem [6.76(a)] now implies that $\overline{kT(B)}$ has a nonempty interior for some $k \in \mathbf{Z}^+$. The linearity of T allows us to conclude that $\overline{T(B)}$ has a nonempty interior.

Thus there exists $g \in B$ such that $Tg \in \operatorname{int} \overline{T(B)}$. Hence

$$0 \in \operatorname{int} \overline{T(B - g)} \subset \operatorname{int} \overline{T(2B)} = \operatorname{int} \overline{2T(B)}.$$

Thus there exists $r > 0$ such that $\overline{B}(0, 2r) \subset \overline{2T(B)}$ [here $\overline{B}(0, 2r)$ is the closed ball in W centered at 0 with radius $2r$]. Hence $\overline{B}(0, r) \subset \overline{T(B)}$. The definition of what it means to be in the closure of $T(B)$ [see 6.7] now shows that

$$h \in W \text{ and } \|h\| \le r \text{ and } \varepsilon > 0 \implies \exists f \in B \text{ such that } \|h - Tf\| < \varepsilon.$$

For arbitrary $h \neq 0$ in W, applying the result in the line above to $\frac{r}{\|h\|}h$ shows that

6.82 $$h \in W \text{ and } \varepsilon > 0 \implies \exists f \in \tfrac{\|h\|}{r}B \text{ such that } \|h - Tf\| < \varepsilon.$$

Now suppose $g \in W$ and $\|g\| < 1$. Applying 6.82 with $h = g$ and $\varepsilon = \frac{1}{2}$, we see that

$$\text{there exists } f_1 \in \tfrac{1}{r}B \text{ such that } \|g - Tf_1\| < \tfrac{1}{2}.$$

Now applying 6.82 with $h = g - Tf_1$ and $\varepsilon = \frac{1}{4}$, we see that

$$\text{there exists } f_2 \in \tfrac{1}{2r}B \text{ such that } \|g - Tf_1 - Tf_2\| < \tfrac{1}{4}.$$

Applying 6.82 again, this time with $h = g - Tf_1 - Tf_2$ and $\varepsilon = \frac{1}{8}$, we see that

$$\text{there exists } f_3 \in \tfrac{1}{4r}B \text{ such that } \|g - Tf_1 - Tf_2 - Tf_3\| < \tfrac{1}{8}.$$

Continue in this pattern, constructing a sequence $f_1, f_2, \dots$ in V. Let

$$f = \sum_{k=1}^{\infty} f_k,$$

where the infinite sum converges in V because

$$\sum_{k=1}^{\infty} \|f_k\| < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}r} = \frac{2}{r};$$

here we are using 6.41 (this is the place in the proof where we use the hypothesis that V is a Banach space). The inequality displayed above shows that $\|f\| < \frac{2}{r}$.

Because

$$\|g - Tf_1 - Tf_2 - \cdots - Tf_n\| < \frac{1}{2^n}$$

and because T is a continuous linear map, we have $g = Tf$.

We have now shown that $B(0, 1) \subset \frac{2}{r}T(B)$. Thus $\frac{r}{2}B(0, 1) \subset T(B)$, completing the proof. ■

The next result provides the useful information that if a bounded linear map from one Banach space to another Banach space has an algebraic inverse (meaning that the linear map is injective and surjective), then the inverse mapping is automatically bounded.

The Open Mapping Theorem was first proved by Banach and his colleague Juliusz Schauder (1899–1943) in 1929–1930.

6.83 *Bounded Inverse Theorem*

Suppose V and W are Banach spaces and T is a one-to-one bounded linear map from V onto W. Then T^{-1} is a bounded linear map from W onto V.

Proof The verification that T^{-1} is a linear map from W to V is left to the reader.

To prove that T^{-1} is bounded, suppose G is an open subset of V. Then

$$(T^{-1})^{-1}(G) = T(G).$$

By the Open Mapping Theorem (6.81), $T(G)$ is an open subset of W. Thus the equation above shows that the inverse image under the function T^{-1} of every open set is open. By the equivalence of parts (a) and (c) of 6.11, this implies that T^{-1} is continuous. Thus T^{-1} is a bounded linear map (by 6.48). ∎

The result above shows that completeness for normed vector spaces sometimes plays a role analogous to compactness for metric spaces (think of the theorem stating that a continuous one-to-one function from a compact metric space onto another compact metric space has an inverse that is also continuous).

Closed Graph Theorem

Suppose V and W are normed vector spaces. Then $V \times W$ is a vector space with the natural operations of addition and scalar multiplication as defined in Exercise 10 in Section 6B. There are several natural norms on $V \times W$ that make $V \times W$ into a normed vector space; the choice used in the next result seems to be the easiest. The proof of the next result is left to the reader as an exercise.

6.84 *product of Banach spaces*

Suppose V and W are Banach spaces. Then $V \times W$ is a Banach space if given the norm defined by

$$\|(f, g)\| = \max\{\|f\|, \|g\|\}$$

for $f \in V$ and $g \in W$. With this norm, a sequence $(f_1, g_1), (f_2, g_2), \dots$ in $V \times W$ converges to (f, g) if and only if $\lim_{k\to\infty} f_k = f$ and $\lim_{k\to\infty} g_k = g$.

The next result gives a terrific way to show that a linear map between Banach spaces is bounded. The proof is remarkably clean because the hard work has been done in the proof of the Open Mapping Theorem (which was used to prove the Bounded Inverse Theorem).

6.85 *Closed Graph Theorem*

Suppose V and W are Banach spaces and T is a function from V to W. Then T is a bounded linear map if and only if $\text{graph}(T)$ is a closed subspace of $V \times W$.

Proof First suppose T is a bounded linear map. Suppose $(f_1, Tf_1), (f_2, Tf_2), \ldots$ is a sequence in $\text{graph}(T)$ converging to $(f, g) \in V \times W$. Thus

$$\lim_{k\to\infty} f_k = f \quad \text{and} \quad \lim_{k\to\infty} Tf_k = g.$$

Because T is continuous, the first equation above implies that $\lim_{k\to\infty} Tf_k = Tf$; when combined with the second equation above this implies that $g = Tf$. Thus $(f, g) = (f, Tf) \in \text{graph}(T)$, which implies that $\text{graph}(T)$ is closed, completing the proof in one direction.

To prove the other direction, now suppose $\text{graph}(T)$ is a closed subspace of $V \times W$. Thus $\text{graph}(T)$ is a Banach space with the norm that it inherits from $V \times W$ [from 6.84 and 6.16(b)]. Consider the linear map $S\colon \text{graph}(T) \to V$ defined by

$$S(f, Tf) = f.$$

Then

$$\|S(f, Tf)\| = \|f\| \le \max\{\|f\|, \|Tf\|\} = \|(f, Tf)\|$$

for all $f \in V$. Thus S is a bounded linear map from $\text{graph}(T)$ onto V with $\|S\| \le 1$. Clearly S is injective. Thus the Bounded Inverse Theorem (6.83) implies that S^{-1} is bounded. Because $S^{-1}\colon V \to \text{graph}(T)$ satisfies the equation $S^{-1}f = (f, Tf)$, we have

$$\begin{aligned}\|Tf\| &\le \max\{\|f\|, \|Tf\|\}\\ &= \|(f, Tf)\|\\ &= \|S^{-1}f\|\\ &\le \|S^{-1}\|\,\|f\|\end{aligned}$$

for all $f \in V$. The inequality above implies that T is a bounded linear map with $\|T\| \le \|S^{-1}\|$, completing the proof. ■

Principle of Uniform Boundedness

The next result states that a family of bounded linear maps on a Banach space that is pointwise bounded is bounded in norm (which means that it is uniformly bounded as a collection of maps on the unit ball). This result is sometimes called the Banach–Steinhaus Theorem. Exercise 17 is also sometimes called the Banach–Steinhaus Theorem.

The Principle of Uniform Boundedness was proved in 1927 by Banach and Hugo Steinhaus (1887–1972). Steinhaus recruited Banach to advanced mathematics after overhearing him discuss Lebesgue integration in a park.

6.86 *Principle of Uniform Boundedness*

Suppose V is a Banach space, W is a normed vector space, and $\mathcal{A}$ is a family of bounded linear maps from V to W such that

$$\sup\{\|Tf\| : T \in \mathcal{A}\} < \infty \text{ for every } f \in V.$$

Then

$$\sup\{\|T\| : T \in \mathcal{A}\} < \infty.$$

Proof Our hypothesis implies that

$$V = \bigcup_{n=1}^{\infty} \underbrace{\{f \in V : \|Tf\| \leq n \text{ for all } T \in \mathcal{A}\}}_{V_n},$$

where V_n is defined by the expression above. Because each $T \in \mathcal{A}$ is continuous, V_n is a closed subset of V for each $n \in \mathbf{Z}^+$. Thus Baire's Theorem [6.76(a)] and the equation above imply that there exist $n \in \mathbf{Z}^+$ and $h \in V$ and $r > 0$ such that

6.87 $$B(h,r) \subset V_n.$$

Now suppose $g \in V$ and $\|g\| < 1$. Thus $rg + h \in B(h,r)$. Hence if $T \in \mathcal{A}$, then 6.87 implies $\|T(rg+h)\| \leq n$, which implies that

$$\|Tg\| = \left\| \frac{T(rg+h)}{r} - \frac{Th}{r} \right\| \leq \frac{\|T(rg+h)\|}{r} + \frac{\|Th\|}{r} \leq \frac{n + \|Th\|}{r}.$$

Thus

$$\sup\{\|T\| : T \in \mathcal{A}\} \leq \frac{n + \sup\{\|Th\| : T \in \mathcal{A}\}}{r} < \infty,$$

completing the proof. ■

EXERCISES 6E

1 Suppose U is a subset of a metric space V. Show that U is dense in V if and only if every nonempty open subset of V contains at least one element of U.

2 Suppose U is a subset of a metric space V. Show that U has an empty interior if and only if $V \setminus U$ is dense in V.

3 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $(\operatorname{int} U) \cup (\operatorname{int} W) = \operatorname{int}(U \cup W)$.

4 Prove or give a counterexample: If V is a metric space and U, W are subsets of V, then $(\operatorname{int} U) \cap (\operatorname{int} W) = \operatorname{int}(U \cap W)$.

5 Suppose

$$X = \{0\} \cup \bigcup_{k=1}^{\infty} \{\tfrac{1}{k}\}$$

and $d(x, y) = |x - y|$ for $x, y \in X$.

(a) Show that (X, d) is a complete metric space.

(b) Each set of the form $\{x\}$ for $x \in X$ is a closed subset of **R** that has an empty interior as a subset of **R**. Clearly X is a countable union of such sets. Explain why this does not violate the statement of Baire's Theorem that a complete metric space is not the countable union of closed subsets with empty interior.

6 Give an example of a metric space that is the countable union of closed subsets with empty interior.
[*This exercise shows that the completeness hypothesis in Baire's Theorem cannot be dropped.*]

7 (a) Define $f\colon \mathbf{R} \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer such that } a = \frac{m}{n} \text{ for some integer } m. \end{cases}$$

At which numbers in **R** is f continuous?

(b) Show that there does not exist a countable collection of open subsets of **R** whose intersection equals **Q**.

(c) Show that there does not exist a function $f\colon \mathbf{R} \to \mathbf{R}$ such that f is continuous at each element of **Q** and discontinuous at each element of $\mathbf{R} \setminus \mathbf{Q}$.

8 Suppose (X, d) is a complete metric space and $G_1, G_2, \ldots$ is a sequence of dense open subsets of X. Prove that $\bigcap_{k=1}^{\infty} G_k$ is a dense subset of X.

9 Prove that there does not exist an infinite-dimensional Banach space with a countable basis.
[*This exercise implies, for example, that there is not a norm that makes the vector space of polynomials with coefficients in* **F** *into a Banach space.*]

10 Give an example of a Banach space V, a normed vector space W, a bounded linear map T of V onto W, and an open subset G of V such that $T(G)$ is not an open subset of W.
[*This exercise shows that the hypothesis in the Open Mapping Theorem that W is a Banach space cannot be relaxed to the hypothesis that W is a normed vector space.*]

11 Show that there exists a normed vector space V, a Banach space W, a bounded linear map T of V onto W, and an open subset G of V such that $T(G)$ is not an open subset of W.
[*This exercise shows that the hypothesis in the Open Mapping Theorem that V is a Banach space cannot be relaxed to the hypothesis that V is a normed vector space.*]

A linear map $T: V \to W$ from a normed vector space V to a normed vector space W is called* bounded below *if there exists $c \in (0, \infty)$ such that $\|f\| \le c\|Tf\|$ for all $f \in V$.

12 Suppose $T: V \to W$ is a bounded linear map from a Banach space V to a Banach space W. Prove that T is bounded below if and only if T is injective and the range of T is a closed subspace of W.

13 Give an example of a Banach space V, a normed vector space W, and a one-to-one bounded linear map T of V onto W such that T^{-1} is not a bounded linear map of W onto V.
[*This exercise shows that the hypothesis in the Bounded Inverse Theorem* (*6.83*) *that W is a Banach space cannot be relaxed to the hypothesis that W is a normed vector space.*]

14 Show that there exists a normed space V, a Banach space W, and a one-to-one bounded linear map T of V onto W such that T^{-1} is not a bounded linear map of W onto V.
[*This exercise shows that the hypothesis in the Bounded Inverse Theorem* (*6.83*) *that V is a Banach space cannot be relaxed to the hypothesis that V is a normed vector space.*]

15 Prove 6.84.

16 Suppose V is a Banach space with norm $\|\cdot\|$ and that $\varphi: V \to \mathbf{F}$ is a linear functional. Define another norm $\|\cdot\|_\varphi$ on V by

$$\|f\|_\varphi = \|f\| + |\varphi(f)|.$$

Prove that if V is a Banach space with the norm $\|\cdot\|_\varphi$, then φ is a continuous linear functional on V (with the original norm).

17 Suppose V is a Banach space, W is a normed vector space, and $T_1, T_2, \ldots$ is a sequence of bounded linear maps from V to W such that $\lim_{k\to\infty} T_k f$ exists for each $f \in V$. Define $T: V \to W$ by

$$Tf = \lim_{k\to\infty} T_k f$$

for $f \in V$. Prove that T is a bounded linear map from V to W.
[*This result states that the pointwise limit of a sequence of bounded linear maps on a Banach space is a bounded linear map.*]

18 Suppose V is a normed vector space and B is a subset of V such that

$$\sup_{f\in B} |\varphi(f)| < \infty$$

for every $\varphi \in V'$. Prove that $\sup_{f\in B} \|f\| < \infty$.

19 Suppose $T: V \to W$ is a linear map from a Banach space V to a Banach space W such that

$$\varphi \circ T \in V' \text{ for all } \varphi \in W'.$$

Prove that T is a bounded linear map.