Chapter 11
Fourier Analysis

This chapter uses Hilbert space theory to motivate the introduction of Fourier coefficients and Fourier series. The classical setting applies these concepts to functions defined on bounded intervals of the real line. However, the theory becomes easier and cleaner when we instead use a modern approach by considering functions defined on the unit circle of the complex plane.

The first section of this chapter shows how consideration of Fourier series leads us to harmonic functions and a solution to the Dirichlet problem. In the second section of this chapter, convolution becomes a major tool for the L^p theory.

The third section of this chapter changes the context to functions defined on the real line. Many of the techniques introduced in the first two sections of the chapter transfer easily to provide results about the Fourier transform on the real line. The highlights of our treatment of the Fourier transform are the Fourier Inversion Formula and the extension of the Fourier transform to a unitary operator on $L^2(\mathbf{R})$.

The vast field of Fourier analysis cannot be completely covered in a single chapter. Thus this chapter gives readers just a taste of the subject. Readers who go on from this chapter to one of the many book-length treatments of Fourier analysis will then already be familiar with the terminology and techniques of the subject.

The Giza pyramids, near where the Battle of Pyramids took place in 1798 during Napoleon's invasion of Egypt. Joseph Fourier (1768–1830) was one of the scientific advisors to Napoleon in Egypt. While in Egypt as part of Napoleon's invading force, Fourier began thinking about the mathematical theory of heat propagation, which eventually led to what we now call Fourier series and the Fourier transform.

S. Axler, *Measure, Integration & Real Analysis*, Graduate Texts in Mathematics 282, https://doi.org/10.1007/978-3-030-33143-6_11

11A Fourier Series and Poisson Integral

Fourier Coefficients and Riemann–Lebesgue Lemma

For $k \in \mathbf{Z}$, suppose $e_k \colon (-\pi, \pi] \to \mathbf{R}$ is defined by

11.1
$$e_k(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(kt) & \text{if } k > 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \\ \frac{1}{\sqrt{\pi}} \cos(kt) & \text{if } k < 0. \end{cases}$$

The classical theory of Fourier series features $\{e_k\}_{k \in \mathbf{Z}}$ as an orthonormal basis of $L^2\big((-\pi, \pi]\big)$. The trigonometric formulas displayed in Exercise 1 in Section 8C can be used to show that $\{e_k\}_{k \in \mathbf{Z}}$ is indeed an orthonormal family in $L^2\big((-\pi, \pi]\big)$.

To show that $\{e_k\}_{k \in \mathbf{Z}}$ is an orthonormal basis of $L^2\big((-\pi, \pi]\big)$ requires more work. One slick possibility is to note that the Spectral Theorem for compact operators produces orthonormal bases; an appropriate choice of a compact normal operator can then be used to show that $\{e_k\}_{k \in \mathbf{Z}}$ is an orthonormal basis of $L^2\big((-\pi, \pi]\big)$ [see Exercise 11(c) in Section 10D].

In this chapter we take a cleaner approach to Fourier series by working on the unit circle in the complex plane instead of on the interval $(-\pi, \pi]$. The map

11.2
$$t \mapsto e^{it} = \cos t + i \sin t$$

can be used to identify the interval $(-\pi, \pi]$ with the unit circle; thus the two approaches are equivalent. However, the calculations are easier in the unit circle context. In addition, we will see that the unit circle context provides the huge benefit of making a connection with harmonic functions.

We begin by introducing notation for the open unit disk and the unit circle in the complex plane.

11.3 Definition $\mathbf{D}$; $\partial\mathbf{D}$

- $\mathbf{D}$ denotes the open unit disk in the complex plane:
$$\mathbf{D} = \{w \in \mathbf{C} : |w| < 1\}.$$
- $\partial\mathbf{D}$ is the unit circle in the complex plane:
$$\partial\mathbf{D} = \{z \in \mathbf{C} : |z| = 1\}.$$

The function given in 11.2 is a one-to-one map of $(-\pi, \pi]$ onto $\partial\mathbf{D}$. We use this map to define a σ-algebra on $\partial\mathbf{D}$ by transferring the Borel subsets of $(-\pi, \pi]$ to subsets of $\partial\mathbf{D}$ that we will call the *measurable subsets* of $\partial\mathbf{D}$. We also transfer Lebesgue measure on the Borel subsets of $(\pi, \pi]$ to a measure called σ on the measurable subsets of $\partial\mathbf{D}$, except that for convenience we normalize by dividing by 2π so that the measure of $\partial\mathbf{D}$ is 1 rather than 2π. We are now ready to give the formal definitions.

11.4 Definition ***measurable subsets of $\partial\mathbf{D}$; σ***

- A subset E of $\partial\mathbf{D}$ is *measurable* if $\{t \in (-\pi, \pi] : e^{it} \in E\}$ is a Borel subset of $\mathbf{R}$.
- σ is the measure on the measurable subsets of $\partial\mathbf{D}$ obtained by transferring Lebesgue measure from $(-\pi, \pi]$ to $\partial\mathbf{D}$, normalized so that $\sigma(\partial\mathbf{D}) = 1$. In other words, if $E \subset \partial\mathbf{D}$ is measurable, then

$$\sigma(E) = \frac{|\{t \in (-\pi, \pi] : e^{it} \in E\}|}{2\pi}.$$

Our definition of the measure σ on $\partial\mathbf{D}$ allows us to transfer integration on $\partial\mathbf{D}$ to the familiar context of integration on $(-\pi, \pi]$. Specifically,

$$\int_{\partial\mathbf{D}} f\, d\sigma = \int_{\partial\mathbf{D}} f(z)\, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it})\, \frac{dt}{2\pi}$$

for all measurable functions $f : \partial\mathbf{D} \to \mathbf{C}$ such that any of these integrals is defined.

Throughout this chapter, we assume that the scalar field $\mathbf{F}$ is the complex field $\mathbf{C}$. Furthermore, $L^p(\partial\mathbf{D})$ is defined as follows.

11.5 Definition $L^p(\partial\mathbf{D})$

For $1 \leq p \leq \infty$, define $L^p(\partial\mathbf{D})$ to mean the complex version ($\mathbf{F} = \mathbf{C}$) of $L^p(\sigma)$.

Note that if $z = e^{it}$ for some $t \in \mathbf{R}$, then $\overline{z} = e^{-it} = \frac{1}{z}$ and $z^n = e^{int}$ and $\overline{z^n} = e^{-int}$ for all $n \in \mathbf{Z}$. These observations make the proof of the next result much simpler than the proof of the corresponding result for the trigonometric family defined by 11.1.

In the statement of the next result, z^n means the function on $\partial\mathbf{D}$ defined by $z \mapsto z^n$.

11.6 ***orthonormal family in*** $L^2(\partial\mathbf{D})$

$\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal family in $L^2(\partial\mathbf{D})$.

Proof If $n \in \mathbf{Z}$, then

$$\langle z^n, z^n \rangle = \int_{\partial\mathbf{D}} |z^n|^2\, d\sigma(z) = \int_{\partial\mathbf{D}} 1\, d\sigma = 1.$$

If $m, n \in \mathbf{Z}$ with $m \neq n$, then

$$\langle z^m, z^n \rangle = \int_{-\pi}^{\pi} e^{imt} e^{-int}\, \frac{dt}{2\pi} = \int_{-\pi}^{\pi} e^{i(m-n)t}\, \frac{dt}{2\pi} = \frac{e^{i(m-n)t}}{i(m-n)2\pi}\Big]_{t=-\pi}^{t=\pi} = 0,$$

as desired. ■

In the next section, we improve the result above by showing that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$ (see 11.30).

Hilbert space theory tells us that if f is in the closure in $L^2(\partial\mathbf{D})$ of $\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}$, then

$$f = \sum_{n\in\mathbf{Z}} \langle f, z^n\rangle z^n,$$

where the infinite sum above converges as an unordered sum in the norm of $L^2(\partial\mathbf{D})$ (see 8.58). The inner product $\langle f, z^n\rangle$ above equals

$$\int_{\partial\mathbf{D}} f(z)\overline{z^n}\, d\sigma(z).$$

Because $|z^n| = 1$ for every $z \in \partial\mathbf{D}$, the integral above makes sense not only for $f \in L^2(\partial\mathbf{D})$ but also for f in the larger space $L^1(\partial\mathbf{D})$. Thus we make the following definition.

11.7 Definition ***Fourier coefficient***

Suppose $f \in L^1(\partial\mathbf{D})$.

- For $n \in \mathbf{Z}$, the n^{th} *Fourier coefficient* of f is denoted $\hat{f}(n)$ and is defined by

$$\hat{f}(n) = \int_{\partial\mathbf{D}} f(z)\overline{z^n}\, d\sigma(z) = \int_{-\pi}^{\pi} f(e^{it})e^{-int}\,\frac{dt}{2\pi}.$$

- The *Fourier series* of f is the formal sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n.$$

As we will see, Fourier analysis helps describe the sense in which the Fourier series of f represents f.

11.8 Example *Fourier coefficients*

- Suppose h is an analytic function on an open set that contains $\overline{D}$. Then h has a power series representation

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the sum on the right converges uniformly on $\overline{D}$ to h. Because uniform convergence on $\partial\mathbf{D}$ implies convergence in $L^2(\partial\mathbf{D})$, 8.58(b) and 11.6 now imply that

$$(h|_{\partial\mathbf{D}})\hat{}\,(n) = \begin{cases} a_n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0 \end{cases}$$

for all $n \in \mathbf{Z}$. In other words, for functions analytic on an open set containing $\overline{D}$, the Fourier series is the same as the Taylor series.

- Suppose $f \colon \partial\mathbf{D} \to \mathbf{R}$ is defined by

$$f(z) = \frac{1}{|3 - z|^2}.$$

Then for $z \in \partial\mathbf{D}$ we have

$$\begin{aligned} f(z) &= \frac{1}{(3 - z)(3 - \overline{z})} \\ &= \frac{1}{8}\Big(\frac{z}{3 - z} + \frac{3}{3 - \overline{z}}\Big) \\ &= \frac{1}{8}\Big(\frac{\frac{z}{3}}{1 - \frac{z}{3}} + \frac{1}{1 - \frac{\overline{z}}{3}}\Big) \\ &= \frac{1}{8}\Big(\frac{z}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \sum_{n=0}^{\infty}\frac{(\overline{z})^n}{3^n}\Big) \\ &= \frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{z^n}{3^{|n|}}, \end{aligned}$$

where the infinite sums above converge uniformly on $\partial\mathbf{D}$. Thus we see that

$$\hat{f}(n) = \frac{1}{8} \cdot \frac{1}{3^{|n|}}$$

for all $n \in \mathbf{Z}$.

We begin with some simple algebraic properties of Fourier coefficients, whose proof is left to the reader.

11.9 ***algebraic properties of Fourier coefficients***

Suppose $f, g \in L^1(\partial\mathbf{D})$ and $n \in \mathbf{Z}$. Then

(a) $(f + g)\hat{}\,(n) = \hat{f}(n) + \hat{g}(n)$;

(b) $(\alpha f)\hat{}\,(n) = \alpha\hat{f}(n)$ for all $\alpha \in \mathbf{C}$;

(c) $|\hat{f}(n)| \leq \|f\|_1$.

Parts (a) and (b) above could be restated by saying that for each $n \in \mathbf{Z}$, the function $f \mapsto \hat{f}(n)$ is a linear functional from $L^1(\partial\mathbf{D})$ to $\mathbf{C}$. Part (c) could be restated by saying that this linear functional has norm at most 1.

Part (c) above implies that the set of Fourier coefficients $\{\hat{f}(n)\}_{n\in\mathbf{Z}}$ is bounded for each $f \in L^1(\partial\mathbf{D})$. The Fourier coefficients of the functions in Example 11.8 have the stronger property that $\lim_{n\to\pm\infty} \hat{f}(n) = 0$. The next result shows that this stronger conclusion holds for all functions in $L^1(\partial\mathbf{D})$.

11.10 ***Riemann–Lebesgue Lemma***

Suppose $f \in L^1(\partial\mathbf{D})$. Then $\lim_{n\to\pm\infty} \hat{f}(n) = 0$.

Proof Suppose $\varepsilon > 0$. There exists $g \in L^2(\partial\mathbf{D})$ such that $\|f - g\|_1 < \varepsilon$ (by 3.44). By 11.6 and Bessel's inequality (8.57), we have

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 \le \|g\|_2^2 < \infty.$$

Thus there exists $M \in \mathbf{Z}^+$ such that $|\hat{g}(n)| < \varepsilon$ for all $n \in \mathbf{Z}$ with $|n| \ge M$. Now if $n \in \mathbf{Z}$ and $|n| \ge M$, then

$$\begin{aligned} |\hat{f}(n)| &\le |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)| \\ &< |(f-g)\hat{\ }(n)| + \varepsilon \\ &\le \|f-g\|_1 + \varepsilon \\ &< 2\varepsilon. \end{aligned}$$

Thus $\lim_{n\to\pm\infty} \hat{f}(n) = 0$. ■

Poisson Kernel

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is continuous and $z \in \partial\mathbf{D}$. For this fixed $z \in \partial\mathbf{D}$, the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$$

is a series of complex numbers. It would be nice if $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$, but this is not necessarily true because the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ might not converge, as you can see in Exercise 11.

Various techniques exist for trying to assign some meaning to a series of complex numbers that does not converge. In one such technique, called *Abel summation*, the n^{th}-term of the series is multiplied by r^n and then the limit is taken as $r \uparrow 1$. For example, if the n^{th}-term of the divergent series

$$1 - 1 + 1 - 1 + \cdots$$

is multiplied by r^n for $r \in [0,1)$, we get a convergent series whose sum equals $\frac{r}{1+r}$. Taking the limit of this sum as $r \uparrow 1$ then gives $\frac{1}{2}$ as the value of the Abel sum of the series above.

The next definition can be motivated by applying a similar technique to the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$. Here we have a series of complex numbers whose terms are indexed by $\mathbf{Z}$ rather than by $\mathbf{Z}^+$. Thus we use $r^{|n|}$ rather than r^n because we want these multipliers to have limit 0 as $n \to \pm\infty$ for each $r \in [0,1)$ (and to have limit 1 as $r \uparrow 1$ for each $n \in \mathbf{Z}$).

11.11 Definition $\mathcal{P}_r f$

For $f \in L^1(\partial\mathbf{D})$ and $0 \le r < 1$, define $\mathcal{P}_r f : \partial\mathbf{D} \to \mathbf{C}$ by

$$(\mathcal{P}_r f)(z) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) z^n.$$

No convergence problems arise in the series above because

$$\left| r^{|n|} \hat{f}(n) z^n \right| \le \|f\|_1 \, r^{|n|}$$

for each $z \in \partial\mathbf{D}$, which implies that

$$\sum_{n=-\infty}^{\infty} |r^{|n|} \hat{f}(n) z^n| \le \|f\|_1 \frac{1+r}{1-r} < \infty.$$

Thus for each $r \in [0, 1)$, the partial sums of the series above converge uniformly on $\partial\mathbf{D}$, which implies that $\mathcal{P}_r f$ is a continuous function from $\partial\mathbf{D}$ to $\mathbf{C}$ (for $r = 0$ and $n = 0$, interpret the expression 0^0 to be 1).

Let's unravel the formula in 11.11. If $f \in L^1(\partial\mathbf{D})$, $0 \le r < 1$, and $z \in \partial\mathbf{D}$, then

$$\begin{aligned}(\mathcal{P}_r f)(z) &= \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) z^n \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \int_{\partial\mathbf{D}} f(w) \overline{w^n} \, d\sigma(w) z^n \\ &= \int_{\partial\mathbf{D}} f(w) \Big(\sum_{n=-\infty}^{\infty} r^{|n|} (z\overline{w})^n \Big) \, d\sigma(w), \qquad (11.12)\end{aligned}$$

where interchanging the sum and integral above is justified by the uniform convergence of the series on $\partial\mathbf{D}$. To evaluate the sum in parentheses in the last line above, let $\zeta \in \partial\mathbf{D}$ (think of $\zeta = z\overline{w}$ in the formula above). Thus $(\zeta)^{-n} = (\overline{\zeta})^n$ and

$$\begin{aligned}\sum_{n=-\infty}^{\infty} r^{|n|} \zeta^n &= \sum_{n=0}^{\infty} (r\zeta)^n + \sum_{n=1}^{\infty} (r\overline{\zeta})^n \\ &= \frac{1}{1 - r\zeta} + \frac{r\overline{\zeta}}{1 - r\overline{\zeta}} \\ &= \frac{(1 - r\overline{\zeta}) + (1 - r\zeta) r\overline{\zeta}}{|1 - r\zeta|^2} \\ &= \frac{1 - r^2}{|1 - r\zeta|^2}. \qquad (11.13)\end{aligned}$$

Motivated by the formula above, we now make the following definition. Notice that 11.11 uses calligraphic $\mathcal{P}$, while the next definition uses italic P.

11.14 Definition *$P_r(\zeta)$; Poisson kernel*

- For $0 \le r < 1$, define $P_r \colon \partial\mathbf{D} \to (0, \infty)$ by

$$P_r(\zeta) = \frac{1 - r^2}{|1 - r\zeta|^2}.$$

- The family of functions $\{P_r\}_{r \in [0,1)}$ is called the *Poisson kernel* on $\mathbf{D}$.

Combining 11.12 and 11.13 now gives the following result.

11.15 *integral formula for $\mathcal{P}_r f$*

If $f \in L^1(\partial\mathbf{D})$, $0 \le r < 1$, and $z \in \partial\mathbf{D}$, then

$$(\mathcal{P}_r f)(z) = \int_{\partial\mathbf{D}} f(w) P_r(z\overline{w})\, d\sigma(w) = \int_{\partial\mathbf{D}} f(w) \frac{1 - r^2}{|1 - rz\overline{w}|^2}\, d\sigma(w).$$

The terminology *approximate identity* is sometimes used to describe the three properties for the Poisson kernel given in the next result.

11.16 *properties of P_r*

(a) $P_r(\zeta) > 0$ for all $r \in [0,1)$ and all $\zeta \in \partial\mathbf{D}$.

(b) $\displaystyle\int_{\partial\mathbf{D}} P_r(\zeta)\, d\sigma(\zeta) = 1$ for each $r \in [0,1)$.

(c) $\displaystyle\lim_{r \uparrow 1} \int_{\{\zeta \in \partial\mathbf{D} : |1-\zeta| \ge \delta\}} P_r(\zeta)\, d\sigma(\zeta) = 0$ for each $\delta > 0$.

Proof Part (a) follows immediately from the definition of $P_r(\zeta)$ given in 11.14.

Part (b) follows from integrating the series representation for P_r given by 11.13 termwise and noting that

$$\int_{\partial\mathbf{D}} \zeta^n\, d\sigma(\zeta) = \int_{-\pi}^{\pi} e^{int} \frac{dt}{2\pi} = \frac{e^{int}}{in2\pi}\bigg]_{t=-\pi}^{t=\pi} = 0 \text{ for all } n \in \mathbf{Z} \setminus \{0\};$$

for $n = 0$, we have $\int_{\partial\mathbf{D}} \zeta^n\, d\sigma(\zeta) = 1$.

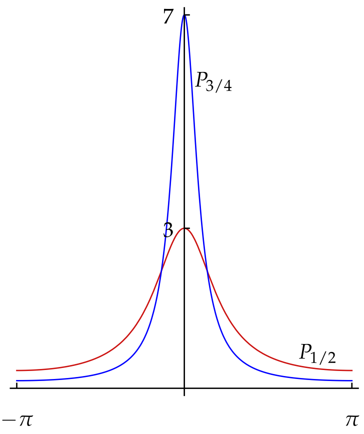
To prove part (c), suppose $\delta > 0$. If $\zeta \in \partial\mathbf{D}$, $|1 - \zeta| \ge \delta$, and $1 - r < \frac{\delta}{2}$, then

$$\begin{aligned} |1 - r\zeta| &= |1 - \zeta - (r-1)\zeta| \\ &\ge |1 - \zeta| - (1 - r) \\ &> \tfrac{\delta}{2}. \end{aligned}$$

Thus as $r \uparrow 1$, the denominator in the definition of $P_r(\zeta)$ is uniformly bounded away from 0 on $\{\zeta \in \partial\mathbf{D} : |1 - \zeta| \ge \delta\}$ and the numerator goes to 0. Thus the integral of P_r over $\{\zeta \in \partial\mathbf{D} : |1 - \zeta| \ge \delta\}$ goes to 0 as $r \uparrow 1$. ∎

Here is the intuition behind the proof of the next result: Parts (a) and (b) of the previous result and 11.15 mean that $(\mathcal{P}_r f)(z)$ is a weighted average of f. Part (c) of the previous result says that for r close to 1, most of the weight in this weighted average is concentrated near z. Thus $(\mathcal{P}_r f)(z) \to f(z)$ as $r \uparrow 1$.

The figure here transfers the context from $\partial\mathbf{D}$ to $(-\pi, \pi]$. The area under both curves is 2π [corresponding to 11.16(b)] and $P_r(e^{it})$ becomes more concentrated near $t = 0$ as $r \uparrow 1$ [corresponding to 11.16(c)]. See Exercise 3 for the formula for $P_r(e^{it})$.



The graphs of $P_{\frac{1}{2}}(e^{it})$ [red] and $P_{\frac{3}{4}}(e^{it})$ [blue] on $(-\pi, \pi]$.

One more ingredient is needed for the next proof: If $h \in L^1(\partial\mathbf{D})$ and $z \in \partial\mathbf{D}$, then

11.17 $$\int_{\partial\mathbf{D}} h(z\overline{w})\,d\sigma(w) = \int_{\partial\mathbf{D}} h(\zeta)\,d\sigma(\zeta).$$

The equation above holds because the measure σ is rotation and reflection invariant. In other words, $\sigma(\{w \in \partial\mathbf{D} : h(z\overline{w}) \in E\}) = \sigma(\{\zeta \in \partial\mathbf{D} : h(\zeta) \in E\})$ for all measurable $E \subset \partial\mathbf{D}$.

11.18 ***if f is continuous, then*** $\lim\limits_{r\uparrow 1}\|f - \mathcal{P}_r f\|_\infty = 0$

Suppose $f : \partial\mathbf{D} \to \mathbf{C}$ is continuous. Then $\mathcal{P}_r f$ converges uniformly to f on $\partial\mathbf{D}$ as $r \uparrow 1$.

Proof Suppose $\varepsilon > 0$. Because f is uniformly continuous on $\partial\mathbf{D}$, there exists $\delta > 0$ such that

$$|f(z) - f(w)| < \varepsilon \text{ for all } z, w \in \partial\mathbf{D} \text{ with } |z - w| < \delta.$$

If $z \in \partial\mathbf{D}$, then

$$\begin{aligned}
|f(z) - (\mathcal{P}_r f)(z)| &= \left|f(z) - \int_{\partial\mathbf{D}} f(w) P_r(z\overline{w})\,d\sigma(w)\right| \\
&= \left|\int_{\partial\mathbf{D}} \bigl(f(z) - f(w)\bigr) P_r(z\overline{w})\,d\sigma(w)\right| \\
&\le \varepsilon \int_{\{w\in\partial\mathbf{D} : |z-w|<\delta\}} P_r(z\overline{w})\,d\sigma(w) \\
&\qquad + 2\|f\|_\infty \int_{\{w\in\partial\mathbf{D} : |z-w|\ge\delta\}} P_r(z\overline{w})\,d\sigma(w) \\
&\le \varepsilon + 2\|f\|_\infty \int_{\{\zeta\in\partial\mathbf{D} : |1-\zeta|\ge\delta\}} P_r(\zeta)\,d\sigma(\zeta),
\end{aligned}$$

where we have used 11.17, 11.16(a), and 11.16(b); the last line uses the equality $|z - w| = |1 - \zeta|$, which holds when $\zeta = z\overline{w}$. Now 11.16(c) shows that the value of the last integral above has uniform (with respect to $z \in \partial\mathbf{D}$) limit 0 as $r \uparrow 1$. ■

Solution to Dirichlet Problem on Disk

As a bonus to our investigation into Fourier series, the previous result provides the solution to the Dirichlet problem on the unit disk. To state the Dirichlet problem, we first need a few definitions. As usual, we identify $\mathbf{C}$ with $\mathbf{R}^2$. Thus for $x, y \in \mathbf{R}$, we can think of $w = x + yi \in \mathbf{C}$ or $w = (x, y) \in \mathbf{R}^2$. Hence

$$\mathbf{D} = \{w \in \mathbf{C} : |w| < 1\} = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}.$$

For a function $f \colon G \to \mathbf{C}$ on an open subset G of $\mathbf{C}$ (or an open subset G of $\mathbf{R}^2$), the partial derivatives $D_1 f$ and $D_2 f$ are defined as in 5.46 except that now we allow f to be a complex-valued function. Clearly $D_j f = D_j(\operatorname{Re} f) + i D_j(\operatorname{Im} f)$ for $j = 1, 2$.

11.19 Definition ***harmonic function***

A function $u \colon G \to \mathbf{C}$ on an open subset G of $\mathbf{R}^2$ is called *harmonic* if

$$\big(D_1(D_1 f)\big)(w) + \big(D_2(D_2 f)\big)(w) = 0$$

for all $w \in G$. The left side of the equation above is called the *Laplacian* of f at w and is often denoted by $(\Delta f)(w)$.

11.20 Example *harmonic functions*

- If $f \colon G \to \mathbf{C}$ is an analytic function on an open set $G \subset \mathbf{C}$, then the functions $\operatorname{Re} f$, $\operatorname{Im} f$, f, and $\overline{f}$ are all harmonic functions on G, as is usually discussed near the beginning of a course on complex analysis.
- If $\zeta \in \partial\mathbf{D}$, then the function

$$w \mapsto \frac{1 - |w|^2}{|1 - \overline{\zeta} w|^2}$$

  is harmonic on $\mathbf{C} \setminus \{\zeta\}$ (see Exercise 7).
- The function $u \colon \mathbf{C} \setminus \{0\} \to \mathbf{R}$ defined by $u(w) = \log|w|$ is harmonic on $\mathbf{C} \setminus \{0\}$, as you should verify. However, there does not exist a function f analytic on $\mathbf{C} \setminus \{0\}$ such that $u = \operatorname{Re} f$.

The Dirichlet problem asks to extend a continuous function on the boundary of an open subset of $\mathbf{R}^2$ to a function that is harmonic on the open set and continuous on the closure of the open set. Here is a more formal statement:

11.21 **Dirichlet problem on G:** Suppose $G \subset \mathbf{R}^2$ is an open set and $f \colon \partial G \to \mathbf{C}$ is a continuous function. Find a continuous function $u \colon \overline{G} \to \mathbf{C}$ such that $u|_G$ is harmonic and $u|_{\partial G} = f$.

For some open sets $G \subset \mathbf{R}^2$, there exist continuous functions f on ∂G whose Dirichlet problem has no solution. However, the situation on the open unit disk $\mathbf{D}$ is much nicer, as we will soon see.

The function u defined in the result below is called the *Poisson integral* of f on $\mathbf{D}$.

11.22 *Poisson integral is harmonic*

Suppose $f \in L^1(\partial\mathbf{D})$. Define $u\colon \mathbf{D} \to \mathbf{C}$ by

$$u(rz) = (\mathcal{P}_r f)(z)$$

for $r \in [0,1)$ and $z \in \partial\mathbf{D}$. Then u is harmonic on $\mathbf{D}$.

Proof If $w \in \mathbf{D}$, then $w = rz$ for some $r \in [0,1)$ and some $z \in \partial\mathbf{D}$. Thus

$$\begin{aligned} u(w) &= (\mathcal{P}_r f)(z) \\ &= \sum_{n=0}^{\infty} \hat{f}(n)(rz)^n + \sum_{n=1}^{\infty} \hat{f}(-n)(r\bar{z})^n \\ &= \sum_{n=0}^{\infty} \hat{f}(n)w^n + \overline{\sum_{n=1}^{\infty} \overline{\hat{f}(-n)}w^n}. \end{aligned}$$

Every function that has a power series representation on $\mathbf{D}$ is analytic on $\mathbf{D}$. Thus the equation above shows that u is the sum of an analytic function and the complex conjugate of an analytic function. Hence u is harmonic. ■

11.23 *Poisson integral solves Dirichlet problem on unit disk*

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is continuous. Define $u\colon \overline{\mathbf{D}} \to \mathbf{C}$ by

$$u(rz) = \begin{cases} (\mathcal{P}_r f)(z) & \text{if } 0 \le r < 1 \text{ and } z \in \partial\mathbf{D}, \\ f(z) & \text{if } r = 1 \text{ and } z \in \partial\mathbf{D}. \end{cases}$$

Then u is continuous on $\overline{\mathbf{D}}$, $u|_{\mathbf{D}}$ is harmonic, and $u|_{\partial\mathbf{D}} = f$.

Proof Suppose $\zeta \in \partial\mathbf{D}$. To prove that u is continuous at ζ, we need to show that if $w \in \overline{\mathbf{D}}$ is close to ζ, then $u(w)$ is close to $u(\zeta)$. Because $u|_{\partial\mathbf{D}} = f$ and f is continuous on $\partial\mathbf{D}$, we do not need to worry about the case where $w \in \partial\mathbf{D}$. Thus assume $w \in \mathbf{D}$. We can write $w = rz$, where $r \in [0,1)$ and $z \in \partial\mathbf{D}$. Now

$$\begin{aligned} |u(\zeta) - u(w)| &= |f(\zeta) - (\mathcal{P}_r f)(z)| \\ &\le |f(\zeta) - f(z)| + |f(z) - (\mathcal{P}_r f)(z)|. \end{aligned}$$

If w is close to ζ, then z is also close to ζ, and hence by the continuity of f the first term in the last line above is small. Also, if w is close to ζ, then r is close to 1, and hence by 11.18 the second term in the last line above is small. Thus if w is close to ζ, then $u(w)$ is close to $u(\zeta)$, as desired.

The function $u|_{\mathbf{D}}$ is harmonic on $\mathbf{D}$ (and hence continuous on $\mathbf{D}$) by 11.22.

The definition of u immediately implies that $u|_{\partial\mathbf{D}} = f$. ■

Fourier Series of Smooth Functions

The Fourier series of a continuous function on $\partial\mathbf{D}$ need not converge pointwise (see Exercise 11). However, in this subsection we will see that Fourier series behave well for functions that are twice continuously differentiable.

First we need to define what we mean for a function on $\partial\mathbf{D}$ to be differentiable. The formal definition is given below, along with the introduction of the notation $\tilde{f}$ for the transfer of f to $(-\pi, \pi]$ and $f^{[k]}$ for the transfer back to $\partial\mathbf{D}$ of the k^{th}-derivative of $\tilde{f}$.

The idea here is that we transfer a function defined on $\partial\mathbf{D}$ to $(-\pi, \pi]$, take the usual derivative there, then transfer back to $\partial\mathbf{D}$.

11.24 Definition *$\tilde{f}$; k times continuously differentiable; $f^{[k]}$*

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is a complex-valued function on $\partial\mathbf{D}$ and $k \in \mathbf{Z}^+ \cup \{0\}$.

- Define $\tilde{f}\colon \mathbf{R} \to \mathbf{C}$ by $\tilde{f}(t) = f(e^{it})$.
- f is called k times *continuously differentiable* if $\tilde{f}$ is k times differentiable everywhere on $\mathbf{R}$ and its k^{th}-derivative $\tilde{f}^{(k)}\colon \mathbf{R} \to \mathbf{C}$ is continuous.
- If f is k times continuously differentiable, then $f^{[k]}\colon \partial\mathbf{D} \to \mathbf{C}$ is defined by
$$f^{[k]}(e^{it}) = \tilde{f}^{(k)}(t)$$
for $t \in \mathbf{R}$. Here $\tilde{f}^{(0)}$ is defined to be $\tilde{f}$, which means that $f^{[0]} = f$.

Note that the function $\tilde{f}$ defined above is periodic on $\mathbf{R}$ because $\tilde{f}(t + 2\pi) = \tilde{f}(t)$ for all $t \in \mathbf{R}$. Thus all derivatives of $\tilde{f}$ are also periodic on $\mathbf{R}$.

11.25 Example Suppose $n \in \mathbf{Z}$ and $f\colon \partial\mathbf{D} \to \mathbf{C}$ is defined by $f(z) = z^n$. Then $\tilde{f}\colon \mathbf{R} \to \mathbf{C}$ is defined by $\tilde{f}(t) = e^{int}$.

If $k \in \mathbf{Z}^+$, then $\tilde{f}^{(k)}(t) = i^k n^k e^{int}$. Thus $f^{[k]}(z) = i^k n^k z^n$ for $z \in \partial\mathbf{D}$.

Our next result gives a formula for the Fourier coefficients of a derivative.

11.26 ***Fourier coefficients of differentiable functions***

Suppose $k \in \mathbf{Z}^+$ and $f\colon \partial\mathbf{D} \to \mathbf{C}$ is k times continuously differentiable. Then
$$\bigl(f^{[k]}\bigr)\hat{}\,(n) = i^k n^k \hat{f}(n)$$
for every $n \in \mathbf{Z}$.

Proof First suppose $n = 0$. By the Fundamental Theorem of Calculus, we have
$$\bigl(f^{[k]}\bigr)\hat{}\,(0) = \int_{-\pi}^{\pi} f^{[k]}(e^{it})\,\frac{dt}{2\pi} = \int_{-\pi}^{\pi} \tilde{f}^{(k)}(t)\,\frac{dt}{2\pi} = \tilde{f}^{(k-1)}(t)\Big]_{t=-\pi}^{t=\pi} = 0,$$
which is the desired result for $n = 0$.

Now suppose $n \in \mathbf{Z} \setminus \{0\}$. Then

$$\begin{aligned}(f^{[k]})\hat{\ }(n) &= \int_{-\pi}^{\pi} \tilde{f}^{(k)}(t)e^{-int}\,\frac{dt}{2\pi}\\ &= \frac{1}{2\pi}\tilde{f}^{(k-1)}(t)e^{-int}\Big]_{t=-\pi}^{t=\pi} + in\int_{-\pi}^{\pi} \tilde{f}^{(k-1)}(t)e^{-int}\,\frac{dt}{2\pi}\\ &= in\big(f^{[k-1]}\big)\hat{\ }(n),\end{aligned}$$

where the second equality above follows from integration by parts.

Iterating the equation above now produces the desired result. ■

Now we can prove the beautiful result that a twice continuously differentiable function on $\partial\mathbf{D}$ equals its Fourier series, with uniform convergence of the Fourier series. This conclusion holds with the weaker hypothesis that the function is continuously differentiable, but the proof is easier with the hypothesis used here.

11.27 ***Fourier series of twice continuously differentiable functions converge***

Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is twice continuously differentiable. Then

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$$

for all $z \in \partial\mathbf{D}$. Furthermore, the partial sums $\sum_{n=-K}^{M} \hat{f}(n)z^n$ converge uniformly on $\partial\mathbf{D}$ to f as $K, M \to \infty$.

Proof If $n \in \mathbf{Z} \setminus \{0\}$, then

11.28 $$|\hat{f}(n)| = \frac{|(f^{[2]})\hat{\ }(n)|}{n^2} \le \frac{\|f^{[2]}\|_1}{n^2},$$

where the equality above follows from 11.26 and the inequality above follows from 11.9(c). Now 11.28 implies that

11.29 $$\sum_{n=-\infty}^{\infty} |\hat{f}(n)z^n| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

for all $z \in \partial\mathbf{D}$. The inequality above implies that $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ converges and that the partial sums converge uniformly on $\partial\mathbf{D}$.

Furthermore, for each $\in \partial\mathbf{D}$ we have

$$f(z) = \lim_{r\uparrow 1} \sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}(n)z^n = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n,$$

where the first equality holds by 11.18 and 11.11, and the second equality holds by the Dominated Convergence Theorem (use counting measure on $\mathbf{Z}$) and 11.29. ■

In 1923 Andrey Kolmogorov (1903–1987) published a proof that there exists a function in $L^1(\partial\mathbf{D})$ whose Fourier series diverges almost everywhere on $\partial\mathbf{D}$. Kolmogorov's result and the result in Exercise 11 probably led most mathematicians to suspect that there exists a continuous function on $\partial\mathbf{D}$ whose Fourier series diverges almost everywhere. However, in 1966 Lennart Carleson (1928–) showed that if $f \in L^2(\partial\mathbf{D})$ (and in particular if f is continuous on $\partial\mathbf{D}$), then the Fourier series of f converges to f almost everywhere.

EXERCISES 11A

1 Prove that $(\overline{f})\hat{\ }(n) = \overline{\hat{f}(-n)}$ for all $f \in L^1(\partial\mathbf{D})$ and all $n \in \mathbf{Z}$.

2 Suppose $1 \le p \le \infty$ and $n \in \mathbf{Z}$.

(a) Show that the function $f \mapsto \hat{f}(n)$ is a bounded linear functional on $L^p(\partial\mathbf{D})$ with norm 1.

(b) Find all $f \in L^p(\partial\mathbf{D})$ such that $\|f\|_p = 1$ and $|\hat{f}(n)| = 1$.

3 Show that if $0 \le r < 1$ and $t \in \mathbf{R}$, then

$$P_r(e^{it}) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

4 Suppose $f \in \mathcal{L}^1(\partial\mathbf{D})$, $z \in \partial\mathbf{D}$, and f is continuous at z. Prove that

$$\lim_{r\uparrow 1}(\mathcal{P}_r f)(z) = f(z).$$

[*Here $\mathcal{L}^1(\partial\mathbf{D})$ means the complex version of $\mathcal{L}^1(\sigma)$. The result in this exercise differs from 11.18 because here we are assuming continuity only at a single point and we are not even assuming that f is bounded, as compared to 11.18, which assumed continuity at all points of* $\partial\mathbf{D}$.]

5 Suppose $f \in \mathcal{L}^1(\partial\mathbf{D})$, $z \in \partial\mathbf{D}$, $\lim_{t\downarrow 0} f(e^{it}z) = a$, and $\lim_{t\uparrow 0} f(e^{it}z) = b$. Prove that

$$\lim_{r\uparrow 1}(\mathcal{P}_r f)(z) = \frac{a+b}{2}.$$

[*If $a \ne b$, then f is said to have a* jump discontinuity *at z.*]

6 Prove that for each $p \in [1, \infty)$, there exists $f \in L^1(\partial\mathbf{D})$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p = \infty.$$

7 Suppose $\zeta \in \partial\mathbf{D}$. Show that the function

$$w \mapsto \frac{1-|w|^2}{|1-\overline{\zeta}w|^2}$$

is harmonic on $\mathbf{C} \setminus \{\zeta\}$ by finding an analytic function on $\mathbf{C} \setminus \{\zeta\}$ whose real part is the function above.

8 Suppose $f\colon \partial\mathbf{D} \to \mathbf{R}$ is the function defined by

$$f(x,y) = x^4 y$$

for $(x,y) \in \mathbf{R}^2$ with $x^2 + y^2 = 1$. Find a polynomial u of two variables x, y such that u is harmonic on $\mathbf{R}^2$ and $u|_{\partial\mathbf{D}} = f$.
[*Of course, $u|_{\mathbf{D}}$ is the Poisson integral of f. However, here you are asked to find an explicit formula for u in closed form, without involving or computing an integral. It may help to think of f as defined by $f(z) = (\operatorname{Re} z)^4(\operatorname{Im} z)$ for $z \in \partial\mathbf{D}$.*]

9 Find a formula (in closed form, not as an infinite sum) for $\mathcal{P}_r f$, where f is the function in the second bullet point of Example 11.8.

10 Suppose $f\colon \partial\mathbf{D} \to \mathbf{C}$ is three times continuously differentiable. Prove that

$$f^{[1]}(z) = i \sum_{n=-\infty}^{\infty} n\hat{f}(n)z^n$$

for all $z \in \partial\mathbf{D}$.

11 Let $C(\partial\mathbf{D})$ denote the Banach space of continuous function from $\partial\mathbf{D}$ to $\mathbf{C}$, with the supremum norm. For $M \in \mathbf{Z}^+$, define a linear functional $\varphi_M\colon C(\partial\mathbf{D}) \to \mathbf{C}$ by

$$\varphi_M(f) = \sum_{n=-M}^{M} \hat{f}(n).$$

Thus $\varphi_M(f)$ is a partial sum of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$, evaluated at $z = 1$.

(a) Show that

$$\varphi_M(f) = \int_{-\pi}^{\pi} f(e^{it}) \frac{\sin(M+\frac{1}{2})t}{\sin\frac{t}{2}} \frac{dt}{2\pi}$$

for every $f \in C(\partial\mathbf{D})$ and every $M \in \mathbf{Z}^+$.

(b) Show that

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left| \frac{\sin(M+\frac{1}{2})t}{\sin\frac{t}{2}} \right| \frac{dt}{2\pi} = \infty.$$

(c) Show that $\lim_{M\to\infty} \|\varphi_M\| = \infty$.

(d) Show that there exists $f \in C(\partial\mathbf{D})$ such that $\lim_{M\to\infty} \sum_{n=-M}^{M} \hat{f}(n)$ does not exist (as an element of $\mathbf{C}$).

[*Because the sum in part (d) is a partial sum of the Fourier series evaluated at $z = 1$, part (d) shows that the Fourier series of a continuous function on $\partial\mathbf{D}$ need not converge pointwise on $\partial\mathbf{D}$.*
The family of functions (one for each $M \in \mathbf{Z}^+$) on $\partial\mathbf{D}$ defined by

$$e^{it} \mapsto \frac{\sin(M+\frac{1}{2})t}{\sin\frac{t}{2}}$$

is called the Dirichlet kernel.]

12 Define $f\colon \partial\mathbf{D} \to \mathbf{R}$ by

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Im} z > 0, \\ -1 & \text{if } \operatorname{Im} z < 0, \\ 0 & \text{if } \operatorname{Im} z = 0. \end{cases}$$

(a) Show that if $n \in \mathbf{Z}$, then

$$\hat{f}(n) = \begin{cases} -\frac{2i}{n\pi} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

(b) Show that

$$(\mathcal{P}_r f)(z) = \frac{2}{\pi} \arctan \frac{2r \operatorname{Im} z}{1 - r^2}$$

for every $r \in [0, 1)$ and every $z \in \partial\mathbf{D}$.

(c) Verify that $\lim_{r\uparrow 1} (\mathcal{P}_r f)(z) = f(z)$ for every $z \in \partial\mathbf{D}$.

(d) Prove that $\mathcal{P}_r f$ does not converge uniformly to f on $\partial\mathbf{D}$.

11B Fourier Series and L^p of Unit Circle

The last paragraph of the previous section mentioned the result that the Fourier series of a function in $L^2(\partial\mathbf{D})$ converges pointwise to the function almost everywhere. This terrific result had been an open question until 1966. Its proof is not included in this book, partly because the proof is difficult and partly because pointwise convergence has turned out to be less useful than norm convergence.

Thus we begin this section with the easy proof that the Fourier series converges in the norm of $L^2(\partial\mathbf{D})$. The remainder of this section then concentrates on issues connected with norm convergence.

Orthonormal Basis for L^2 of Unit Circle

We already showed that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal family in $L^2(\partial\mathbf{D})$ (see 11.6). Now we show that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$.

11.30 *orthonormal basis of* $L^2(\partial\mathbf{D})$

The family $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$.

Proof Suppose $f \in \big(\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}\big)^\perp$. Thus $\langle f, z^n\rangle = 0$ for all $n \in \mathbf{Z}$. In other words, $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$.

Suppose $\varepsilon > 0$. Let $g\colon \partial\mathbf{D} \to \mathbf{C}$ be a twice continuously differentiable function such that $\|f - g\|_2 < \varepsilon$. [To prove the existence of $g \in L^2(\partial\mathbf{D})$ with this property, first approximate f by step functions as in 3.47, but use the L^2-norm instead of the L^1-norm. Then approximate the characteristic function of an interval as in 3.48, but again use the L^2-norm and round the corners of the graph in the proof of 3.48 to get a twice continuously differentiable function.]

Now

$$\begin{aligned}
\|f\|_2 &\le \|f-g\|_2 + \|g\|_2\\
&= \|f-g\|_2 + \Big(\sum_{n\in\mathbf{Z}} |\hat{g}(n)|^2\Big)^{1/2}\\
&= \|f-g\|_2 + \Big(\sum_{n\in\mathbf{Z}} |(g-f)\hat{\ }(n)|^2\Big)^{1/2}\\
&\le \|f-g\|_2 + \|g-f\|_2\\
&< 2\varepsilon,
\end{aligned}$$

where the second line above follows from 11.27, the third line above holds because $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$, and the fourth line above follows from Bessel's inequality (8.57).

Because the inequality above holds for all $\varepsilon > 0$, we conclude that $f = 0$. We have now shown that $\big(\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}\big)^\perp = \{0\}$. Hence $\overline{\operatorname{span}\{z^n\}_{n\in\mathbf{Z}}} = L^2(\partial\mathbf{D})$ by 8.42, which implies that $\{z^n\}_{n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\partial\mathbf{D})$. ∎

Now the convergence of the Fourier series of $f \in L^2(\partial \mathbf{D})$ to f follows immediately from standard Hilbert space theory [see 8.63(a)] and the previous result. Thus with no further proof needed, we have the following important result.

11.31 *convergence of Fourier series in the norm of $L^2(\partial \mathbf{D})$*

Suppose $f \in L^2(\partial \mathbf{D})$. Then

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

where the infinite sum converges to f in the norm of $L^2(\partial \mathbf{D})$.

The next example is a spectacular application of Hilbert space theory and the orthonormal basis $\{z^n\}_{n \in \mathbf{Z}}$ of $L^2(\partial \mathbf{D})$. The evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ had been an open question until Euler discovered in 1734 that this infinite sum equals $\frac{\pi^2}{6}$.

Euler's proof, which would not be considered sufficiently rigorous by today's standards, was quite different from the technique used in the example below.

11.32 Example $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$

Define $f \in L^2(\partial \mathbf{D})$ by $f(e^{it}) = t$ for $t \in (-\pi, \pi]$. Then $\hat{f}(0) = \int_{-\pi}^{\pi} t \, \frac{dt}{2\pi} = 0$. For $n \in \mathbf{Z} \setminus \{0\}$, we have

$$\begin{aligned} \hat{f}(n) &= \int_{-\pi}^{\pi} t e^{-int} \, \frac{dt}{2\pi} \\ &= \frac{t e^{int}}{-i2\pi n} \Big]_{t=-\pi}^{t=\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-int} \, \frac{dt}{2\pi} \\ &= \frac{(-1)^n i}{n}, \end{aligned}$$

where the second line above follows from integration by parts. The equation above implies that

11.33 $$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Also,

11.34 $$\|f\|_2^2 = \int_{-\pi}^{\pi} t^2 \, \frac{dt}{2\pi} = \frac{\pi^2}{3}.$$

Parseval's identity [8.63(c)] implies that the left side of 11.33 equals the left side of 11.34. Setting the right side of 11.33 equal to the right side of 11.34 shows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Convolution on Unit Circle

Recall that

11.35
$$(\mathcal{P}_r f)(z) = \int_{\partial\mathbf{D}} f(w) P_r(z\overline{w})\, d\sigma(w)$$

for $f \in L^1(\partial\mathbf{D})$, $0 \le r < 1$, and $z \in \partial\mathbf{D}$ (see 11.15). The kind of integral formula that appears in the result above is so useful that it gets a special name and notation.

11.36 Definition ***convolution; $f * g$***

Suppose $f, g \in L^1(\partial\mathbf{D})$. The *convolution* of f and g is denoted $f * g$ and is the function defined by

$$(f * g)(z) = \int_{\partial\mathbf{D}} f(w) g(z\overline{w})\, d\sigma(w)$$

for those $z \in \partial\mathbf{D}$ for which the integral above makes sense.

Thus 11.35 states that $\mathcal{P}_r f = f * P_r$. Here $f \in L^1(\partial\mathbf{D})$ and $P_r \in L^\infty(\partial\mathbf{D})$; hence there is no problem with the integral in the definition of $f * P_r$ being defined for all $z \in \partial\mathbf{D}$. See Exercise 11 for an interpretation of convolution when the functions are transferred to the real line.

The definition above of the convolution of two functions allows both functions to be in $L^1(\partial\mathbf{D})$. The product of two functions in $L^1(\partial\mathbf{D})$ is not, in general, in $L^1(\partial\mathbf{D})$. Thus it is not obvious that the convolution of two functions in $L^1(\partial\mathbf{D})$ is defined anywhere. However, the next result shows that all is well.

11.37 ***convolution of two functions in $L^1(\partial\mathbf{D})$ is in $L^1(\partial\mathbf{D})$***

If $f, g \in L^1(\partial\mathbf{D})$, then $(f * g)(z)$ is defined for almost every $z \in \partial\mathbf{D}$. Furthermore, $f * g \in L^1(\partial\mathbf{D})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

Proof Suppose $f, g \in \mathcal{L}^1(\partial\mathbf{D})$. The function $(w, z) \mapsto f(w) g(z\overline{w})$ is a measurable function on $\partial\mathbf{D} \times \partial\mathbf{D}$, as you are asked to show in Exercise 4. Now Tonelli's Theorem (5.28) and 11.17 imply that

$$\begin{aligned}\int_{\partial\mathbf{D}} \int_{\partial\mathbf{D}} |f(w) g(z\overline{w})|\, d\sigma(w)\, d\sigma(z) &= \int_{\partial\mathbf{D}} |f(w)| \int_{\partial\mathbf{D}} |g(z\overline{w})|\, d\sigma(z)\, d\sigma(w) \\ &= \int_{\partial\mathbf{D}} |f(w)| \|g\|_1\, d\sigma(w) \\ &= \|f\|_1 \|g\|_1.\end{aligned}$$

The equation above implies that $\int_{\partial\mathbf{D}} |f(w) g(z\overline{w})|\, d\sigma(w) < \infty$ for almost every $z \in \partial\mathbf{D}$. Thus $(f * g)(z)$ is defined for almost every $z \in \partial\mathbf{D}$.

The equation above also implies that $\|f * g\|_1 \le \|f\|_1 \|g\|_1$. ■

Soon we will apply convolution results to Poisson integrals. However, first we need to extend the previous result by bounding $\|f * g\|_p$ when $g \in L^p(\partial\mathbf{D})$.

11.38 ***L^p-norm of a convolution***

Suppose $1 \le p \le \infty$, $f \in L^1(\partial\mathbf{D})$, and $g \in L^p(\partial\mathbf{D})$. Then

$$\|f * g\|_p \le \|f\|_1 \, \|g\|_p.$$

Proof We use the following result to estimate the norm in $L^p(\partial\mathbf{D})$:

If $F\colon \partial\mathbf{D} \to \mathbf{C}$ is measurable and $1 \le p \le \infty$, then

11.39 $$\|F\|_p = \sup\{\int_{\partial\mathbf{D}} |Fh| \, d\sigma : h \in L^{p'}(\partial\mathbf{D}) \text{ and } \|h\|_{p'} = 1\}.$$

Hölder's inequality (7.9) shows that the left side of the equation above is greater than or equal to the right side. The inequality in the other direction almost follows from 7.12, but 7.12 would require the hypothesis that $f \in L^p(\partial\mathbf{D})$ (and we want the equation above to hold even if $\|f\|_p = \infty$). To get around this problem, apply 7.12 to truncations of F and use the Monotone Convergence Theorem (3.11); the details of verifying 11.39 are left to the reader.

Suppose $h \in L^{p'}(\partial\mathbf{D})$ and $\|h\|_{p'} = 1$. Then

$$\begin{aligned}
\int_{\partial\mathbf{D}} |(f*g)(z)h(z)| \, d\sigma(z) &\le \int_{\partial\mathbf{D}} \Big(\int_{\partial\mathbf{D}} |f(w)g(z\overline{w})| \, d\sigma(w) |h(z)| \Big) \, d\sigma(z) \\
&= \int_{\partial\mathbf{D}} |f(w)| \int_{\partial\mathbf{D}} |g(z\overline{w})h(z)| \, d\sigma(z) \, d\sigma(w) \\
&\le \int_{\partial\mathbf{D}} |f(w)| \|g\|_p \|h\|_{p'} \, d\sigma(w)
\end{aligned}$$

11.40 $$= \|f\|_1 \, \|g\|_p,$$

where the second line above follows from Tonelli's Theorem (5.28) and the third line follows from Hölder's inequality (7.9) and 11.17. Now 11.39 (with $F = f * g$) and 11.40 imply that $\|f * g\|_p \le \|f\|_1 \, \|g\|_p$. ■

Order does not matter in convolutions, as we now prove.

11.41 ***convolution is commutative***

Suppose $f, g \in L^1(\partial\mathbf{D})$. Then $f * g = g * f$.

Proof Suppose $z \in \partial\mathbf{D}$ is such that $(f * g)(z)$ is defined. Then

$$(f*g)(z) = \int_{\partial\mathbf{D}} f(w)g(z\overline{w}) \, d\sigma(w) = \int_{\partial\mathbf{D}} f(z\overline{\zeta})g(\zeta) \, d\sigma(\zeta) = (g*f)(z),$$

where the second equality follows from making the substitution $\zeta = z\overline{w}$ (which implies that $w = z\overline{\zeta}$); the invariance of the integral under this substitution is explained in connection with 11.17. ■

Now we come to a major result, stating that for $p \in [1, \infty)$, the Poisson integrals of functions in $L^p(\partial \mathbf{D})$ converge in the norm of $L^p(\partial \mathbf{D})$. This result fails for $p = \infty$ [see, for example, Exercise 12(d) in Section 11A].

11.42 ***if $f \in L^p(\partial \mathbf{D})$, then $\mathcal{P}_r f$ converges to f in $L^p(\partial \mathbf{D})$***

Suppose $1 \leq p < \infty$ and $f \in L^p(\partial \mathbf{D})$. Then $\lim_{r \uparrow 1} \|f - \mathcal{P}_r f\|_p = 0$.

Proof Suppose $\varepsilon > 0$. Let $g \colon \partial \mathbf{D} \to \mathbf{C}$ be a continuous function on $\partial \mathbf{D}$ such that

$$\|f - g\|_p < \varepsilon.$$

By 11.18, there exists $R \in [0, 1)$ such that

$$\|g - \mathcal{P}_r g\|_\infty < \varepsilon$$

for all $r \in (R, 1)$. If $r \in (R, 1)$, then

$$\begin{aligned}
\|f - \mathcal{P}_r f\|_p &\leq \|f - g\|_p + \|g - \mathcal{P}_r g\|_p + \|\mathcal{P}_r g - \mathcal{P}_r f\|_p \\
&< \varepsilon + \|g - \mathcal{P}_r g\|_\infty + \|\mathcal{P}_r (g - f)\|_p \\
&< 2\varepsilon + \|P_r * (g - f)\|_p \\
&\leq 2\varepsilon + \|P_r\|_1 \, \|g - f\|_p \\
&< 3\varepsilon,
\end{aligned}$$

where the third line above is justified by 11.41, the fourth line above is justified by 11.38, and the last line above is justified by the equation $\|P_r\|_1 = 1$, which follows from 11.16(a) and 11.16(b). The last inequality implies that $\lim_{r \uparrow 1} \|f - \mathcal{P}_r f\|_p = 0$. ■

As a consequence of the result above, we can now prove that functions in $L^1(\partial \mathbf{D})$, and thus functions in $L^p(\partial \mathbf{D})$ for every $p \in [1, \infty]$, are uniquely determined by their Fourier coefficients. Specifically, if $g, h \in L^1(\partial \mathbf{D})$ and $\hat{g}(n) = \hat{h}(n)$ for every $n \in \mathbf{Z}$, then applying the result below to $g - h$ shows that $g = h$.

11.43 ***functions are determined by their Fourier coefficients***

Suppose $f \in L^1(\partial \mathbf{D})$ and $\hat{f}(n) = 0$ for every $n \in \mathbf{Z}$. Then $f = 0$.

Proof Because $\mathcal{P}_r f$ is defined in terms of Fourier coefficients (see 11.11), we know that $\mathcal{P}_r f = 0$ for all $r \in [0, 1)$. Because $\mathcal{P}_r f \to f$ in $L^1(\partial \mathbf{D})$ as $r \uparrow 1$ [by 11.42]), this implies that $f = 0$. ■

Our next result shows that multiplication of Fourier coefficients corresponds to convolution of the corresponding functions.

11.44 *Fourier coefficients of a convolution*

Suppose $f, g \in L^1(\partial\mathbf{D})$. Then

$$(f * g)\hat{}(n) = \hat{f}(n)\,\hat{g}(n)$$

for every $n \in \mathbf{Z}$.

Proof First note that if $w \in \partial\mathbf{D}$ and $n \in \mathbf{Z}$, then

11.45 $$\int_{\partial\mathbf{D}} g(z\overline{w})\overline{z^n}\,d\sigma(z) = \int_{\partial\mathbf{D}} g(\zeta)\overline{\zeta^n w^n}\,d\sigma(\zeta) = \overline{w^n}\hat{g}(n),$$

where the first equality comes from the substitution $\zeta = z\overline{w}$ (equivalent to $z = \zeta w$), which is justified by the rotation invariance of σ.

Now

$$\begin{aligned}(f * g)\hat{}(n) &= \int_{\partial\mathbf{D}} (f * g)(z)\overline{z^n}\,d\sigma(z)\\ &= \int_{\partial\mathbf{D}} \overline{z^n} \int_{\partial\mathbf{D}} f(w)g(z\overline{w})\,d\sigma(w)\,d\sigma(z)\\ &= \int_{\partial\mathbf{D}} f(w) \int_{\partial\mathbf{D}} g(z\overline{w})\overline{z^n}\,d\sigma(z)\,d\sigma(w)\\ &= \int_{\partial\mathbf{D}} f(w)\overline{w^n}\hat{g}(n)\,d\sigma(w)\\ &= \hat{f}(n)\,\hat{g}(n),\end{aligned}$$

where the interchange of integration order in the third equality is justified by the same steps used in the proof of 11.37 and the fourth equality above is justified by 11.45. ■

The next result could be proved by appropriate uses of Tonelli's Theorem and Fubini's Theorem. However, the slick proof technique used in the proof below should be useful in dealing with some of the exercises.

11.46 *convolution is associative*

Suppose $f, g, h \in L^1(\partial\mathbf{D})$. Then $(f * g) * h = f * (g * h)$.

Proof Suppose $n \in \mathbf{Z}$. Using 11.44 twice, we have

$$((f * g) * h)\hat{}(n) = (f * g)\hat{}(n)\hat{h}(n) = \hat{f}(n)\hat{g}(n)\hat{h}(n).$$

Similarly,

$$(f * (g * h))\hat{}(n) = \hat{f}(n)(g * h)\hat{}(n) = \hat{f}(n)\hat{g}(n)\hat{h}(n).$$

Hence $(f * g) * h$ and $f * (g * h)$ have the same Fourier coefficients. Because functions in $L^1(\partial\mathbf{D})$ are determined by their Fourier coefficients (see 11.43), this implies that $(f * g) * h = f * (g * h)$. ■

EXERCISES 11B

1 Show that the family $\{e_k\}_{k\in\mathbf{Z}}$ of trigonometric functions defined by 11.1 is an orthonormal basis of $L^2\big((-\pi,\pi]\big)$.

2 Use the result of Exercise 12(a) in Section 11A to show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$$

3 Use techniques similar to Example 11.32 to evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$.

[*If you feel industrious, you may also want to evaluate $\sum_{n=1}^{\infty}1/n^6$. Similar techniques work to evaluate $\sum_{n=1}^{\infty}1/n^k$ for each positive even integer k. You can become famous if you figure out how to evaluate $\sum_{n=1}^{\infty}1/n^3$, which currently is an open question.*]

4 Suppose $f,g\colon\partial\mathbf{D}\to\mathbf{C}$ are measurable functions. Prove that the function $(w,z)\mapsto f(w)g(z\overline{w})$ is a measurable function from $\partial\mathbf{D}\times\partial\mathbf{D}$ to $\mathbf{C}$.
[*Here the σ-algebra on $\partial\mathbf{D}\times\partial\mathbf{D}$ is the usual product σ-algebra as defined in 5.2.*]

5 Where does the proof of 11.42 fail when $p=\infty$?

6 Suppose $f\in L^1(\partial\mathbf{D})$. Prove that f is real valued (almost everywhere) if and only if $\hat{f}(-n)=\overline{\hat{f}(n)}$ for every $n\in\mathbf{Z}$.

7 Suppose $f\in L^1(\partial\mathbf{D})$. Show that $f\in L^2(\partial\mathbf{D})$ if and only if $\displaystyle\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2<\infty$.

8 Suppose $f\in L^2(\partial\mathbf{D})$. Prove that $|f(z)|=1$ for almost every $z\in\partial\mathbf{D}$ if and only if

$$\sum_{k=-\infty}^{\infty}\hat{f}(k)\overline{\hat{f}(k-n)}=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{if } n\neq 0\end{cases}$$

for all $n\in\mathbf{Z}$.

9 For this exercise, for each $r\in[0,1)$ think of $\mathcal{P}_r$ as an operator on $L^2(\partial\mathbf{D})$.

(a) Show that $\mathcal{P}_r$ is a self-adjoint compact operator for each $r\in[0,1)$.

(b) For each $r\in[0,1)$, find all eigenvalues and eigenvectors of $\mathcal{P}_r$.

(c) Prove or disprove: $\lim_{r\uparrow 1}\|I-\mathcal{P}_r\|=0$.

10 Suppose $f\in L^1(\partial\mathbf{D})$. Define $T\colon L^2(\partial\mathbf{D})\to L^2(\partial\mathbf{D})$ by $Tg=f*g$.

(a) Show that T is a compact operator on $L^2(\partial\mathbf{D})$.

(b) Prove that T is injective if and only if $\hat{f}(n)\neq 0$ for every $n\in\mathbf{Z}$.

(c) Find a formula for T^*.

(d) Prove: T is self-adjoint if and only if all Fourier coefficients of f are real.

(e) Show that T is a normal operator.

11 Show that if $f, g \in L^1(\partial\mathbf{D})$ then

$$(f * g)\tilde{}(t) = \int_{-\pi}^{\pi} \tilde{f}(x)\tilde{g}(t - x)\, dx,$$

for those $t \in \mathbf{R}$ such that $(f * g)(e^{it})$ makes sense; here $(f * g)\tilde{}$, $\tilde{f}$, and $\tilde{g}$ denote the transfers to the real line as defined in 11.24.

12 Suppose $1 \le p \le \infty$. Prove that if $f \in L^p(\partial\mathbf{D})$ and $g \in L^{p'}(\partial\mathbf{D})$, then $f * g$ is a continuous function on $\partial\mathbf{D}$.

13 Suppose $g \in L^1(\partial\mathbf{D})$ is such that $\hat{g}(n) \neq 0$ for infinitely many $n \in \mathbf{Z}$. Prove that if $f \in L^1(\partial\mathbf{D})$ and $f * g = g$, then $f = 0$.

14 Show that there exists a two-sided sequence $\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots$ such that $\lim\limits_{n\to\pm\infty} b_n = 0$ but there does not exist $f \in L^1(\partial\mathbf{D})$ with $\hat{f}(n) = b_n$ for all $n \in \mathbf{Z}$.

15 Prove that if $f, g \in L^2(\partial\mathbf{D})$, then

$$(fg)\hat{}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n - k)$$

for every $n \in \mathbf{Z}$.

16 Suppose $f \in L^1(\partial\mathbf{D})$. Prove that $\mathcal{P}_r(\mathcal{P}_s f) = \mathcal{P}_{rs} f$ for all $r, s \in [0, 1)$.

17 Suppose $p \in [1, \infty]$ and $f \in L^p(\partial\mathbf{D})$. Prove that if $0 \le r < s < 1$, then

$$\|\mathcal{P}_r f\|_p \le \|\mathcal{P}_s f\|_p.$$

18 Prove Wirtinger's inequality: If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuously differentiable 2π-periodic function and $\int_{-\pi}^{\pi} f(t)\, dt = 0$, then

$$\int_{-\pi}^{\pi} \big(f(t)\big)^2\, dt \le \int_{-\pi}^{\pi} \big(f'(t)\big)^2\, dt,$$

with equality if and only if $f(t) = a\sin(t) + b\cos(t)$ for some constants a, b.

11C Fourier Transform

Fourier Transform on $L^1(\mathbf{R})$

We now switch from consideration of functions defined on the unit circle $\partial\mathbf{D}$ to consideration of functions defined on the real line $\mathbf{R}$. Instead of dealing with Fourier coefficients and Fourier series, we now deal with Fourier transforms.

Recall that $\int_{-\infty}^{\infty} f(x)\,dx$ means $\int_{\mathbf{R}} f\,d\lambda$, where λ denotes Lebesgue measure on $\mathbf{R}$, and similarly if a dummy variable other than x is used (see 3.39). Similarly, $L^p(\mathbf{R})$ means $L^p(\lambda)$ (the version that allows the functions to be complex valued). Thus in this section, $\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}$ for $1 \le p < \infty$.

11.47 Definition ***Fourier transform***

For $f \in L^1(\mathbf{R})$, the *Fourier transform* of f is the function $\hat{f}\colon \mathbf{R} \to \mathbf{C}$ defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx.$$

We use the same notation $\hat{f}$ for the Fourier transform as we did for Fourier coefficients. The analogies that we will see between the two concepts makes using the same notation reasonable. The context should make it clear whether this notation refers to Fourier transforms (when we are working with functions defined on $\mathbf{R}$) or whether the notation refers to Fourier coefficients (when we are working with functions defined on $\partial\mathbf{D}$).

The factor 2π that appears in the exponent in the definition above of the Fourier transform is a normalization factor. Without this normalization, we would lose the beautiful result that $\|\hat{f}\|_2 = \|f\|_2$ (see 11.82). Another possible normalization, which is used by some books, is to define the Fourier transform of f at t to be

$$\int_{-\infty}^{\infty} f(x)e^{-itx}\,\frac{dx}{\sqrt{2\pi}}.$$

There is no right or wrong way to do the normalization—pesky π's will pop up somewhere regardless of the normalization or lack of normalization. However, the choice made in 11.47 seems to cause fewer problems than other choices.

11.48 Example *Fourier transforms*

(a) Suppose $b \le c$. If $t \in \mathbf{R}$, then

$$\begin{aligned}(\chi_{[b,c]})\hat{}\,(t) &= \int_b^c e^{-2\pi i t x}\,dx \\ &= \begin{cases} \dfrac{i\left(e^{-2\pi i c t} - e^{-2\pi i b t}\right)}{2\pi t} & \text{if } t \ne 0, \\ c - b & \text{if } t = 0. \end{cases}\end{aligned}$$

(b) Suppose $f(x) = e^{-2\pi|x|}$ for $x \in \mathbf{R}$. If $t \in \mathbf{R}$, then

$$\begin{aligned}\hat{f}(t) &= \int_{-\infty}^{\infty} e^{-2\pi|x|} e^{-2\pi i t x}\,dx \\ &= \int_{-\infty}^{0} e^{2\pi x} e^{-2\pi i t x}\,dx + \int_{0}^{\infty} e^{-2\pi x} e^{-2\pi i t x}\,dx \\ &= \frac{1}{2\pi(1-it)} + \frac{1}{2\pi(1+it)} \\ &= \frac{1}{\pi(t^2+1)}.\end{aligned}$$

Recall that the Riemann–Lebesgue Lemma on the unit circle $\partial\mathbf{D}$ states that if $f \in L^1(\partial\mathbf{D})$, then $\lim_{n\to\pm\infty} \hat{f}(n) = 0$ (see 11.10). Now we come to the analogous result in the context of the real line.

11.49 *Riemann–Lebesgue Lemma*

Suppose $f \in L^1(\mathbf{R})$. Then $\hat{f}$ is uniformly continuous on $\mathbf{R}$. Furthermore,

$$\|\hat{f}\|_\infty \le \|f\|_1 \quad \text{and} \quad \lim_{t\to\pm\infty} \hat{f}(t) = 0.$$

Proof Because $|e^{-2\pi i t x}| = 1$ for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}$, the definition of the Fourier transform implies that if $t \in \mathbf{R}$ then

$$|\hat{f}(t)| \le \int_{-\infty}^{\infty} |f(x)|\,dx = \|f\|_1.$$

Thus $\|\hat{f}\|_\infty \le \|f\|_1$.

If f is the characteristic function of a bounded interval, then the formula in Example 11.48(a) shows that $\hat{f}$ is uniformly continuous on $\mathbf{R}$ and $\lim_{t\to\pm\infty} \hat{f}(t) = 0$. Thus the same result holds for finite linear combinations of such functions. Such finite linear combinations are called step functions (see 3.46).

Now consider arbitrary $f \in L^1(\mathbf{R})$. There exists a sequence $f_1, f_2, \ldots$ of step functions in $L^1(\mathbf{R})$ such that $\lim_{k\to\infty} \|f - f_k\|_1 = 0$ (by 3.47). Thus

$$\lim_{k\to\infty} \|\hat{f} - \hat{f}_k\|_\infty = 0.$$

In other words, the sequence $\hat{f}_1, \hat{f}_2, \ldots$ converges uniformly on $\mathbf{R}$ to $\hat{f}$. Because the uniform limit of uniformly continuous functions is uniformly continuous, we can conclude that $\hat{f}$ is uniformly continuous on $\mathbf{R}$. Furthermore, the uniform limit of functions on $\mathbf{R}$ each of which has limit 0 at $\pm\infty$ also has limit 0 at $\pm\infty$, completing the proof. ■

The next result gives a condition that forces the Fourier transform of a function to be continuously differentiable. This result also gives a formula for the derivative of the Fourier transform. See Exercise 8 for a formula for the n^{th} derivative.

11.50 *derivative of a Fourier transform*

Suppose $f \in L^1(\mathbf{R})$. Define $g\colon \mathbf{R} \to \mathbf{C}$ by $g(x) = xf(x)$. If $g \in L^1(\mathbf{R})$, then $\hat{f}$ is a continuously differentiable function on $\mathbf{R}$ and

$$(\hat{f})'(t) = -2\pi i \hat{g}(t)$$

for all $t \in \mathbf{R}$.

Proof Fix $t \in \mathbf{R}$. Then

$$\begin{aligned}
\lim_{s\to 0} \frac{\hat{f}(t+s) - \hat{f}(t)}{s} &= \lim_{s\to 0} \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \Big(\frac{e^{-2\pi i s x} - 1}{s} \Big)\, dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \Big(\lim_{s\to 0} \frac{e^{-2\pi i s x} - 1}{s} \Big)\, dx \\
&= -2\pi i \int_{-\infty}^{\infty} x f(x) e^{-2\pi i t x}\, dx \\
&= -2\pi i \hat{g}(t),
\end{aligned}$$

where the second equality is justified by using the inequality $|e^{i\theta} - 1| \le \theta$ (valid for all $\theta \in \mathbf{R}$, as the reader should verify) to show that $|(e^{-2\pi i s x} - 1)/s| \le 2\pi |x|$ for all $s \in \mathbf{R} \setminus \{0\}$ and all $x \in \mathbf{R}$; the hypothesis that $xf(x) \in L^1(\mathbf{R})$ and the Dominated Convergence Theorem (3.31) then allow for the interchange of the limit and the integral that is used in the second equality above.

The equation above shows that $\hat{f}$ is differentiable and that $(\hat{f})'(t) = -2\pi i \hat{g}(t)$ for all $t \in \mathbf{R}$. Because $\hat{g}$ is continuous on $\mathbf{R}$ (by 11.49), we can also conclude that $\hat{f}$ is continuously differentiable. ■

11.51 Example $e^{-\pi x^2}$ *equals its Fourier transform*

Suppose $f \in L^1(\mathbf{R})$ is defined by $f(x) = e^{-\pi x^2}$. Then the function $g\colon \mathbf{R} \to \mathbf{C}$ defined by $g(x) = xf(x) = xe^{-\pi x^2}$ is in $L^1(\mathbf{R})$. Hence 11.50 implies that if $t \in \mathbf{R}$ then

$$\begin{aligned}
(\hat{f})'(t) &= -2\pi i \int_{-\infty}^{\infty} x e^{-\pi x^2} e^{-2\pi i t x}\, dx \\
&= \big(i e^{-\pi x^2} e^{-2\pi i t x} \big) \Big]_{x=-\infty}^{x=\infty} - 2\pi t \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i t x}\, dx
\end{aligned}$$

11.52 $$= -2\pi t \hat{f}(t),$$

where the second equality follows from integration by parts (if you are nervous about doing an integration by parts from $-\infty$ to ∞, change each integral to be the limit as $M \to \infty$ of the integral from $-M$ to M).

Note that $f'(t) = -2\pi t e^{-\pi t^2} = -2\pi t f(t)$. Combining this equation with 11.52 shows that

$$\left(\frac{\hat{f}}{f}\right)'(t) = \frac{f(t)(\hat{f})'(t) - f'(t)\hat{f}(t)}{\big(f(t)\big)^2} = -2\pi t \frac{f(t)\hat{f}(t) - f(t)\hat{f}(t)}{\big(f(t)\big)^2} = 0$$

for all $t \in \mathbf{R}$. Thus $\hat{f}/f$ is a constant function. In other words, there exists $c \in \mathbf{C}$ such that $\hat{f} = cf$. To evaluate c, note that

11.53 $$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1 = f(0),$$

where the integral above is evaluated by writing its square as the integral times the same integral but using y instead of x for the dummy variable and then converting to polar coordinates ($dx\,dy = r\,dr\,d\theta$).

Clearly 11.53 implies that $c = 1$. Thus $\hat{f} = f$.

The next result gives a formula for the Fourier transform of a derivative. See Exercise 9 for a formula for the Fourier transform of the n^{th} derivative.

11.54 *Fourier transform of a derivative*

Suppose $f \in L^1(\mathbf{R})$ is a continuously differentiable function and $f' \in L^1(\mathbf{R})$. If $t \in \mathbf{R}$, then

$$(f')\hat{}\,(t) = 2\pi i t \hat{f}(t).$$

Proof Suppose $\varepsilon > 0$. Because f and f' are in $L^1(\mathbf{R})$, there exists $a \in \mathbf{R}$ such that

$$\int_a^{\infty} |f'(x)|\,dx < \varepsilon \quad \text{and} \quad |f(a)| < \varepsilon.$$

Now if $b > a$ then

$$|f(b)| = \left|\int_a^b f'(x)\,dx + f(a)\right| \le \int_a^{\infty} |f'(x)|\,dx + |f(a)| < 2\varepsilon.$$

Hence $\lim_{x\to\infty} f(x) = 0$. Similarly, $\lim_{x\to-\infty} f(x) = 0$.

If $t \in \mathbf{R}$, then

$$\begin{aligned}(f')\hat{}\,(t) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi i t x}\,dx \\ &= f(x)e^{-2\pi i t x}\Big]_{x=-\infty}^{x=\infty} + 2\pi i t \int_{-\infty}^{\infty} f(x)e^{-2\pi i t x}\,dx \\ &= 2\pi i t \hat{f}(t),\end{aligned}$$

where the second equality comes from integration by parts and the third equality holds because we showed in the paragraph above that $\lim_{x\to\pm\infty} f(x) = 0$. ■

The next result gives formulas for the Fourier transforms of some algebraic transformations of a function. Proofs of these formulas are left to the reader.

11.55 *Fourier transforms of translations, rotations, and dilations*

Suppose $f \in L^1(\mathbf{R})$, $b \in \mathbf{R}$, and $t \in \mathbf{R}$.

(a) If $g(x) = f(x-b)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = e^{-2\pi ibt}\hat{f}(t)$.

(b) If $g(x) = e^{2\pi ibx} f(x)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = \hat{f}(t-b)$.

(c) If $b \neq 0$ and $g(x) = f(bx)$ for all $x \in \mathbf{R}$, then $\hat{g}(t) = \frac{1}{|b|}\hat{f}\big(\frac{t}{b}\big)$.

11.56 **Example** *Fourier transform of a rotation of an exponential function*

Suppose $y > 0$, $x \in \mathbf{R}$, and $h(t) = e^{-2\pi y|t|}e^{2\pi ixt}$. To find the Fourier transform of h, first consider the function g defined by $g(t) = e^{-2\pi y|t|}$. By 11.48(b) and 11.55(c), we have

11.57
$$\hat{g}(t) = \frac{1}{y}\,\frac{1}{\pi\Big(\big(\frac{t}{y}\big)^2+1\Big)} = \frac{1}{\pi}\,\frac{y}{t^2+y^2}.$$

Now 11.55(b) implies that

11.58
$$\hat{h}(t) = \frac{1}{\pi}\,\frac{y}{(t-x)^2+y^2};$$

note that x is a constant in the definition of h, which has t as the variable, but x is the variable in 11.55(b)—this slightly awkward permutation of variables is done in this example to make a later reference to 11.58 come out cleaner.

The next result will be immensely useful later in this section.

11.59 *integral of a function times a Fourier transform*

Suppose $f, g \in L^1(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} \hat{f}(t)g(t)\,dt = \int_{-\infty}^{\infty} f(t)\hat{g}(t)\,dt.$$

Proof Both integrals in the equation above make sense because $f, g \in L^1(\mathbf{R})$ and $\hat{f}, \hat{g} \in L^\infty(\mathbf{R})$ (by 11.49). Using the definition of the Fourier transform, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(t)g(t)\,dt &= \int_{-\infty}^{\infty} g(t)\int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx\,dt \\
&= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(t)e^{-2\pi itx}\,dt\,dx \\
&= \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx,
\end{aligned}$$

where Tonelli's Theorem and Fubini's Theorem justify the second equality. Changing the dummy variable x to t in the last expression gives the desired result. ∎

Convolution on $\mathbf{R}$

Our next big goal is to prove the Fourier Inversion Formula. This remarkable formula, discovered by Fourier, states that if $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$, then

11.60
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi ixt}\, dt$$

for almost every $x \in \mathbf{R}$. We will eventually prove this result (see 11.76), but first we need to develop some tools that will be used in the proof. To motivate these tools, we look at the right side of the equation above for fixed $x \in \mathbf{R}$ and see what we would need to prove that it equals $f(x)$.

To get from the right side of 11.60 to an expression involving f rather than $\hat{f}$, we should be tempted to use 11.59. However, we cannot use 11.59 because the function $t \mapsto e^{2\pi ixt}$ is not in $L^1(\mathbf{R})$, which is a hypothesis needed for 11.59. Thus we throw in a convenient convergence factor, fixing $y > 0$ and considering the integral

11.61
$$\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi ixt}\, dt.$$

The convergence factor above is a good choice because for fixed $y > 0$ the function $t \mapsto e^{-2\pi y|t|}$ is in $L^1(\mathbf{R})$, and $\lim_{y \downarrow 0} e^{-2\pi y|t|} = 1$ for every $t \in \mathbf{R}$ (which means that 11.61 may be a good approximation to 11.60 for y close to 0).

Now let's be rigorous. Suppose $f \in L^1(\mathbf{R})$. Fix $y > 0$ and $x \in \mathbf{R}$. Define $h\colon \mathbf{R} \to \mathbf{C}$ by $h(t) = e^{-2\pi y|t|} e^{2\pi ixt}$. Then $h \in L^1(\mathbf{R})$ and

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi ixt}\, dt &= \int_{-\infty}^{\infty} \hat{f}(t) h(t)\, dt \\ &= \int_{-\infty}^{\infty} f(t) \hat{h}(t)\, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2}\, dt, \qquad (11.62)\end{aligned}$$

where the second equality comes from 11.59 and the third equality comes from 11.58. We will come back to the specific formula in 11.62 later, but for now we use 11.62 as motivation for study of expressions of the form $\int_{-\infty}^{\infty} f(t) g(x-t)\, dt$. Thus we have been led to the following definition.

11.63 Definition ***convolution; $f * g$***

Suppose $f, g\colon \mathbf{R} \to \mathbf{C}$ are measurable functions. The *convolution* of f and g is denoted $f * g$ and is the function defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

for those $x \in \mathbf{R}$ for which the integral above makes sense.

Here we are using the same terminology and notation as was used for the convolution of functions on the unit circle. Recall that if $F, G \in L^1(\partial \mathbf{D})$, then

$$(F * G)(e^{i\theta}) = \int_{-\pi}^{\pi} F(e^{is}) G(e^{i(\theta - s)}) \, \frac{ds}{2\pi}$$

for $\theta \in \mathbf{R}$ (see 11.36). The context should always indicate whether $f * g$ denotes convolution on the unit circle or convolution on the real line. The formal similarities between the two notions of convolution make many of the proofs transfer in either direction from one context to the other.

If $f, g \in L^1(\mathbf{R})$, then $f * g$ is defined for almost every $x \in \mathbf{R}$, and furthermore $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ (as you should verify by translating the proof of 11.37 to the context of $\mathbf{R}$).

If $p \in (1, \infty]$, then neither $L^1(\mathbf{R})$ nor $L^p(\mathbf{R})$ is a subset of the other [unlike the inclusion $L^p(\partial \mathbf{D}) \subset L^1(\partial \mathbf{D})$]. Thus we do not yet know that $f * g$ makes sense for $f \in L^1(\mathbf{R})$ and $g \in L^p(\mathbf{R})$. However, the next result shows that all is well.

If $1 \leq p \leq \infty$, $f \in L^p(\mathbf{R})$, and $g \in L^{p'}(\mathbf{R})$, then Hölder's inequality (7.9) and the translation invariance of Lebesgue measure imply $(f * g)(x)$ is defined for all $x \in \mathbf{R}$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$ (more is true; with these hypothesis, $f * g$ is a uniformly continuous function on $\mathbf{R}$, as you are asked to show in Exercise 10).

11.64 *L^p-norm of a convolution*

Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbf{R})$, and $g \in L^p(\mathbf{R})$. Then $(f * g)(x)$ is defined for almost every $x \in \mathbf{R}$. Furthermore,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Proof First consider the case where $f(x) \geq 0$ and $g(x) \geq 0$ for almost every $x \in \mathbf{R}$. Thus $(f * g)(x)$ is defined for each $x \in \mathbf{R}$, although its value might equal ∞. Apply the proof of 11.38 to the context of $\mathbf{R}$, concluding that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ [which implies that $(f * g)(x) < \infty$ for almost every $x \in \mathbf{R}$].

Now consider arbitrary $f \in L^1(\mathbf{R})$, and $g \in L^p(\mathbf{R})$. Apply the case of the previous paragraph to $|f|$ and $|g|$ to get the desired conclusions. ■

The next proof, as is the case for several other proofs in this section, asks the reader to transfer the proof of the analogous result from the context of the unit circle to the context of the real line. This should require only minor adjustments of a proof from one of the two previous sections. The best way to learn this material is to write out for yourself the required proof in the context of the real line.

11.65 *convolution is commutative*

Suppose $f, g \colon \mathbf{R} \to \mathbf{C}$ are measurable functions and $x \in \mathbf{R}$ is such that $(f * g)(x)$ is defined. Then $(f * g)(x) = (g * f)(x)$.

Proof Adjust the proof of 11.41 to the context of $\mathbf{R}$. ■

Our next result shows that multiplication of Fourier transforms corresponds to convolution of the corresponding functions.

11.66 ***Fourier transform of a convolution***

Suppose $f, g \in L^1(\mathbf{R})$. Then

$$(f * g)\hat{}(t) = \hat{f}(t)\,\hat{g}(t)$$

for every $t \in \mathbf{R}$.

Proof Adjust the proof of 11.44 to the context of $\mathbf{R}$. ■

Poisson Kernel on Upper Half-Plane

As usual, we identify $\mathbf{R}^2$ with $\mathbf{C}$, as illustrated in the following definition. We will see that the upper half-plane plays a role in the context of $\mathbf{R}$ similar to the role that the open unit disk plays in the context of $\partial\mathbf{D}$.

11.67 Definition $\mathbf{H}$***; upper half-plane***

- $\mathbf{H}$ denotes the open upper half-plane in $\mathbf{R}^2$:

$$\mathbf{H} = \{(x, y) \in \mathbf{R}^2 : y > 0\} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}.$$

- $\partial\mathbf{H}$ is identified with the real line:

$$\partial\mathbf{H} = \{(x, y) \in \mathbf{R}^2 : y = 0\} = \{z \in \mathbf{C} : \operatorname{Im} z = 0\} = \mathbf{R}.$$

Recall that we defined a family of functions on $\partial\mathbf{D}$ called the Poisson kernel on $\mathbf{D}$ (see 11.14, where the family is called the Poisson kernel on $\mathbf{D}$ because $0 \le r < 1$ and $\zeta \in \partial\mathbf{D}$ implies $r\zeta \in \mathbf{D}$). Now we are ready to define a family of functions on $\mathbf{R}$ that is called the Poisson kernel on $\mathbf{H}$ [because $x \in \mathbf{R}$ and $y > 0$ implies $(x, y) \in \mathbf{H}$].

The following definition is motivated by 11.62. The notation P_r for the Poisson kernel on $\mathbf{D}$ and P_y for the Poisson kernel on $\mathbf{H}$ is potentially ambiguous (what is $P_{1/2}$?), but the intended meaning should always be clear from the context.

11.68 Definition P_y***; Poisson kernel***

- For $y > 0$, define $P_y \colon \mathbf{R} \to (0, \infty)$ by

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$

- The family of functions $\{P_y\}_{y>0}$ is called the *Poisson kernel* on $\mathbf{H}$.

The properties of the Poisson kernel on $\mathbf{H}$ listed in the result below should be compared to the corresponding properties (see 11.16) of the Poisson kernel on $\mathbf{D}$.

11.69 ***properties of*** P_y

(a) $P_y(x) > 0$ for all $y > 0$ and all $x \in \mathbf{R}$.

(b) $\displaystyle\int_{-\infty}^{\infty} P_y(x)\,dx = 1$ for each $y > 0$.

(c) $\displaystyle\lim_{y \downarrow 0} \int_{\{x \in \mathbf{R} : |x| \geq \delta\}} P_y(x)\,dx = 0$ for each $\delta > 0$.

Proof Part (a) follows immediately from the definition of $P_y(x)$ given in 11.68.

Parts (b) and (c) follow from explicitly evaluating the integrals, using the result that for each $y > 0$, an anti-derivative of $P_y(x)$ (as a function of x) is $\frac{1}{\pi} \arctan \frac{x}{y}$. ■

If $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$ and $y > 0$, then $f * P_y$ makes sense because $P_y \in L^{p'}(\mathbf{R})$. Thus the following definition makes sense.

11.70 Definition $\mathcal{P}_y f$

For $f \in L^p(\mathbf{R})$ for some $p \in [1, \infty]$ and for $y > 0$, define $\mathcal{P}_y f \colon \mathbf{R} \to \mathbf{C}$ by

$$(\mathcal{P}_y f)(x) = \int_{-\infty}^{\infty} f(t) P_y(x - t)\,dt = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2}\,dt$$

for $x \in \mathbf{R}$. In other words, $\mathcal{P}_y f = f * P_y$.

The next result is analogous to 11.18, except that now we need to include in the hypothesis that our function is uniformly continuous and bounded (those conditions follow automatically from continuity in the context of the unit circle).

For the proof of the result below, you should use the properties in 11.69 instead of the corresponding properties in 11.16.

When Napoleon appointed Fourier to an administrative position in 1806, Siméon Poisson (1781–1840) was appointed to the professor position at École Polytechnique vacated by Fourier. Poisson published over 300 mathematical papers during his lifetime.

11.71 ***if*** f ***is uniformly continuous and bounded, then*** $\displaystyle\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_\infty = 0$

Suppose $f \colon \mathbf{R} \to \mathbf{C}$ is uniformly continuous and bounded. Then $\mathcal{P}_y f$ converges uniformly to f on $\mathbf{R}$ as $y \downarrow 0$.

Proof Adjust the proof of 11.18 to the context of $\mathbf{R}$. ■

The function u defined in the result below is called the *Poisson integral* of f on $\mathbf{H}$.

11.72 *Poisson integral is harmonic*

Suppose $f \in L^p(\mathbf{R})$ for some $p \in [1,\infty]$. Define $u\colon \mathbf{H} \to \mathbf{C}$ by

$$u(x,y) = (\mathcal{P}_y f)(x)$$

for $x \in \mathbf{R}$ and $y > 0$. Then u is harmonic on $\mathbf{H}$.

Proof First we consider the case where f is real valued. For $x \in \mathbf{R}$ and $y > 0$, let $z = x + iy$. Then

$$\frac{y}{(x-t)^2 + y^2} = -\operatorname{Im} \frac{1}{z-t}$$

for $t \in \mathbf{R}$. Thus

$$u(x,y) = -\operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt.$$

The function $z \mapsto -\int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt$ is analytic on $\mathbf{H}$; its derivative is the function $z \mapsto \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt$ (justification for this statement is in the next paragraph). In other words, we can differentiate (with respect to z) under the integral sign in the expression above. Because u is the imaginary part of an analytic function, u is harmonic on $\mathbf{H}$, as desired.

To justify the differentiation under the integral sign, fix $z \in \mathbf{H}$ and define a function $g\colon \mathbf{H} \to \mathbf{C}$ by $g(z) = -\int_{-\infty}^{\infty} f(t) \frac{1}{z-t}\, dt$. Then

$$\frac{g(z) - g(w)}{z - w} - \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt = \int_{-\infty}^{\infty} f(t) \frac{z-w}{(z-t)^2(w-t)}\, dt.$$

As $w \to z$, the function $t \mapsto \frac{z-w}{(z-t)^2(w-t)}$ goes to 0 in the norm of $L^{p'}(\mathbf{R})$. Thus Hölder's inequality (7.9) and the equation above imply that $g'(z)$ exists and that $g'(z) = \int_{-\infty}^{\infty} f(t) \frac{1}{(z-t)^2}\, dt$, as desired. ■

We have now solved the Dirichlet problem on the half-space for uniformly continuous, bounded functions on $\mathbf{R}$ (see 11.21 for the statement of the Dirichlet problem).

11.73 *Poisson integral solves Dirichlet problem on half-plane*

Suppose $f\colon \mathbf{R} \to \mathbf{C}$ is uniformly continuous and bounded. Define $u\colon \overline{\mathbf{H}} \to \mathbf{C}$ by

$$u(x,y) = \begin{cases} (\mathcal{P}_y f)(x) & \text{if } x \in \mathbf{R} \text{ and } y > 0, \\ f(x) & \text{if } x \in \mathbf{R} \text{ and } y = 0. \end{cases}$$

Then u is continuous on $\overline{\mathbf{H}}$, $u|_{\mathbf{H}}$ is harmonic, and $u|_{\mathbf{R}} = f$.

Proof Adjust the proof of 11.23 to the context of $\mathbf{R}$; now you will need to use 11.71 and 11.72 instead of the corresponding results for the unit circle. ■

The next result, which states that the Poisson integrals of functions in $L^p(\mathbf{R})$ converge in the norm of $L^p(\mathbf{R})$, will be a major tool in proving the Fourier Inversion Formula and other results later in this section.

Poisson and Fourier are two of the 72 mathematicians/scientists whose names are prominently inscribed on the Eiffel Tower in Paris.

For the result below, the proof of the corresponding result on the unit circle (11.42) does not transfer to the context of $\mathbf{R}$ (because the inequality $\|\cdot\|_p \le \|\cdot\|_\infty$ fails in the context of $\mathbf{R}$).

11.74 *if $f \in L^p(\mathbf{R})$, then $\mathcal{P}_y f$ converges to f in $L^p(\mathbf{R})$*

Suppose $1 \le p < \infty$ and $f \in L^p(\mathbf{R})$. Then $\lim\limits_{y \downarrow 0} \|f - \mathcal{P}_y f\|_p = 0$.

Proof If $y > 0$ and $x \in \mathbf{R}$, then

$$|f(x) - (\mathcal{P}_y f)(x)| = \Big| f(x) - \int_{-\infty}^{\infty} f(x-t) P_y(t)\, dt \Big|$$

$$= \Big| \int_{-\infty}^{\infty} \big(f(x) - f(x-t)\big) P_y(t)\, dt \Big|$$

11.75
$$\le \Big(\int_{-\infty}^{\infty} |f(x) - f(x-t)|^p P_y(t)\, dt \Big)^{1/p},$$

where the inequality comes from applying 7.10 to the measure $P_y\, dt$ (note that the measure of $\mathbf{R}$ with respect to this measure is 1).

Define $h\colon \mathbf{R} \to [0,\infty)$ by

$$h(t) = \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p\, dx.$$

Then h is a bounded function that is uniformly continuous on $\mathbf{R}$ [by Exercise 23(a) in Section 7A]. Furthermore, $h(0) = 0$.

Raising both sides of 11.75 to the p^{th} power and then integrating over $\mathbf{R}$ with respect to x, we have

$$\|f - \mathcal{P}_y f\|_p^p \le \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p P_y(t)\, dt\, dx$$

$$= \int_{-\infty}^{\infty} P_y(t) \int_{-\infty}^{\infty} |f(x) - f(x-t)|^p\, dx\, dt$$

$$= \int_{-\infty}^{\infty} P_y(-t) h(t)\, dt$$

$$= (\mathcal{P}_y h)(0).$$

Now 11.71 implies that $\lim\limits_{y \downarrow 0} (\mathcal{P}_y h)(0) = h(0) = 0$. Hence the last inequality above implies that $\lim\limits_{y \downarrow 0} \|f - \mathcal{P}_y f\|_p = 0$. ■

Fourier Inversion Formula

Now we can prove the remarkable Fourier Inversion Formula.

11.76 *Fourier Inversion Formula*

Suppose $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$. Then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i x t}\, dt$$

for almost every $x \in \mathbf{R}$. In other words,

$$f(x) = (\hat{f})\hat{\ }(-x)$$

for almost every $x \in \mathbf{R}$.

Proof Equation 11.62 states that

11.77
$$\int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi y|t|} e^{2\pi i x t}\, dt = (\mathcal{P}_y f)(x)$$

for every $x \in \mathbf{R}$ and every $y > 0$.

Because $\hat{f} \in L^1(\mathbf{R})$, the Dominated Convergence Theorem (3.31) implies that for every $x \in \mathbf{R}$, the left side of 11.77 has limit $(\hat{f})\hat{\ }(-x)$ as $y \downarrow 0$.

Because $f \in L^1(\mathbf{R})$, 11.74 implies that $\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_1 = 0$. Now 7.23 implies that there is a sequence of positive numbers $y_1, y_2, \ldots$ such that $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} (\mathcal{P}_{y_n} f)(x) = f(x)$ for almost every $x \in \mathbf{R}$.

Combining the results in the two previous paragraphs and equation 11.77 shows that $f(x) = (\hat{f})\hat{\ }(-x)$ for almost every $x \in \mathbf{R}$. ■

The Fourier transform of a function in $L^1(\mathbf{R})$ is a uniformly continuous function on $\mathbf{R}$ (by 11.49). Thus the Fourier Inversion Formula (11.76) implies that if $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$, then f can be modified on a set of measure zero to become a uniformly continuous function on $\mathbf{R}$.

The Fourier Inversion Formula now allows us to calculate the Fourier transform of P_y for each $y > 0$.

11.78 Example *Fourier transform of P_y*

Suppose $y > 0$. Define $f \colon \mathbf{R} \to (0, 1]$ by

$$f(t) = e^{-2\pi y|t|}.$$

Then $\hat{f} = P_y$ by 11.57. Hence both f and $\hat{f}$ are in $L^1(\mathbf{R})$. Thus we can apply the Fourier Inversion Formula (11.76), concluding that

11.79
$$(P_y)\hat{\ }(x) = (\hat{f})\hat{\ }(x) = f(-x) = e^{-2\pi y|x|}$$

for all $x \in \mathbf{R}$.

Now we can prove that the map on $L^1(\mathbf{R})$ defined by $f \mapsto \hat{f}$ is one-to-one.

11.80 *functions are determined by their Fourier transforms*

Suppose $f \in \mathcal{L}^1(\mathbf{R})$ and $\hat{f}(t) = 0$ for every $t \in \mathbf{R}$. Then $f = 0$.

Proof Because $\hat{f} = 0$, we also have $(\hat{f})\hat{} = 0$. The Fourier Inversion Formula (11.76) now implies that $f = 0$. ∎

The next result could be proved directly using the definition of convolution and Tonelli's/Fubini's Theorems. However, the following cute proof deserves to be seen.

11.81 *convolution is associative*

Suppose $f, g, h \in L^1(\mathbf{R})$. Then $(f * g) * h = f * (g * h)$.

Proof The Fourier transform of $(f * g) * h$ and the Fourier transform of $f * (g * h)$ both equal $\hat{f}\hat{g}\hat{h}$ (by 11.66). Because the Fourier transform is a one-to-one mapping on $L^1(\mathbf{R})$ [see 11.80], this implies that $(f * g) * h = f * (g * h)$. ∎

Extending Fourier Transform to $L^2(\mathbf{R})$

We now prove that the map $f \mapsto \hat{f}$ preserves $L^2(\mathbf{R})$ norms on $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

11.82 *Plancherel's Theorem: Fourier transform preserves $L^2(\mathbf{R})$ norms*

Suppose $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then $\|\hat{f}\|_2 = \|f\|_2$.

Proof First consider the case where $\hat{f} \in L^1(\mathbf{R})$ in addition to the hypothesis that $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Define $g \colon \mathbf{R} \to \mathbf{C}$ by $g(x) = \overline{f(-x)}$. Then $\hat{g}(t) = \overline{\hat{f}(t)}$ for all $t \in \mathbf{R}$, as is easy to verify. Now

$$\|f\|_2^2 = \int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx$$

$$= \int_{-\infty}^{\infty} f(-x)\overline{f(-x)}\,dx$$

11.83
$$= \int_{-\infty}^{\infty} (\hat{f})\hat{}(x) g(x)\,dx$$

11.84
$$= \int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(x)\,dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{f}(x)}\,dx$$

$$= \|\hat{f}\|_2^2,$$

where 11.83 holds by the Fourier Inversion Formula (11.76) and 11.84 follows from 11.59. The equation above shows that our desired result holds in the case when $\hat{f} \in L^1(\mathbf{R})$.

Now consider arbitrary $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. If $y > 0$, then $f * P_y \in L^1(\mathbf{R})$ by 11.64. If $x \in \mathbf{R}$, then

$$(f * P_y)\hat{\ }(x) = \hat{f}(x)(P_y)\hat{\ }(x)$$

11.85
$$= \hat{f}(x)e^{-2\pi y|x|},$$

where the first equality above comes from 11.66 and the second equality comes from 11.79. The equation above shows that $(f * P_y)\hat{\ } \in L^1(\mathbf{R})$. Thus we can apply the first case to $f * P_y$, concluding that

$$\|f * P_y\|_2 = \|(f * P_y)\hat{\ }\|_2.$$

As $y \downarrow 0$, the left side of the equation above converges to $\|f\|_2$ [by 11.74]. As $y \downarrow 0$, the right side of the equation above converges to $\|\hat{f}\|_2$ [by the explicit formula for $f * P_y$ given in 11.85 and the Monotone Convergence Theorem (3.11)]. Thus the equation above implies that $\|\hat{f}\|_2 = \|f\|_2$. ∎

Because $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, Plancherel's Theorem (11.82) allows us to extend the map $f \mapsto \hat{f}$ uniquely to a bounded linear map from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ (see Exercise 14 in Section 6C). This extension is called the *Fourier transform* on $L^2(\mathbf{R})$; it gets its own notation, as shown below.

11.86 Definition ***Fourier transform on*** $L^2(\mathbf{R})$

The *Fourier transform* $\mathcal{F}$ on $L^2(\mathbf{R})$ is the bounded operator on $L^2(\mathbf{R})$ such that $\mathcal{F}f = \hat{f}$ for all $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$.

For $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use either $\hat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of f. But if $f \in L^1(\mathbf{R}) \setminus L^2(\mathbf{R})$, we will use only the notation $\hat{f}$, and if $f \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$, we will use only the notation $\mathcal{F}f$.

Suppose $f \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$ and $t \in \mathbf{R}$. Do not make the mistake of thinking that $(\mathcal{F}f)(t)$ equals

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi itx}\,dx.$$

Indeed, the integral above makes no sense because $\left|f(x)e^{-2\pi itx}\right| = |f(x)|$ and $f \notin L^1(\mathbf{R})$. Instead of defining $\mathcal{F}f$ via the equation above, $\mathcal{F}f$ must be defined as the limit in $L^2(\mathbf{R})$ of $(f_1)\hat{\ }, (f_2)\hat{\ }, \ldots$, where $f_1, f_2, \ldots$ is a sequence in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$\|f - f_n\|_2 \to 0 \text{ as } n \to \infty.$$

For example, one could take $f_n = f\chi_{[-n,n]}$ because $\|f - f\chi_{[-n,n]}\|_2 \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem (3.31).

Because $\mathcal{F}$ is obtained by continuously extending [in the norm of $L^2(\mathbf{R})$] the Fourier transform from $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ to $L^2(\mathbf{R})$, we know that $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L^2(\mathbf{R})$. In other words, $\mathcal{F}$ is an isometry on $L^2(\mathbf{R})$. The next result shows that even more is true.

11.87 *properties of the Fourier transform on $L^2(\mathbf{R})$*

(a) $\mathcal{F}$ is a unitary operator on $L^2(\mathbf{R})$.

(b) $\mathcal{F}^4 = I$.

(c) $\operatorname{sp}(\mathcal{F}) = \{1, i, -1, -i\}$.

Proof First we prove (b). Suppose $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. If $y > 0$, then $P_y \in L^1(\mathbf{R})$ and hence 11.64 implies that

11.88 $$f * P_y \in L^1(\mathbf{R}) \cap L^2(\mathbf{R}).$$

Also,

11.89 $$(f * P_y)\hat{\ } \in L^1(\mathbf{R}) \cap L^2(\mathbf{R}),$$

as follows from the equation $(f * P_y)\hat{\ } = \hat{f} \cdot (P_y)\hat{\ }$ [see 11.66] and the observation that $\hat{f} \in L^\infty(\mathbf{R})$, $(P_y)\hat{\ } \in L^1(\mathbf{R})$ [see 11.49 and 11.79] and the observation that $\hat{f} \in L^2(\mathbf{R})$, $(P_y)\hat{\ } \in L^\infty(\mathbf{R})$ [see 11.82 and 11.49].

Now the Fourier Inversion Formula (11.76) as applied to $f * P_y$ (which is valid by 11.88 and 11.89) implies that

$$\mathcal{F}^4(f * P_y) = f * P_y.$$

Taking the limit in $L^2(\mathbf{R})$ of both sides of the equation above as $y \downarrow 0$, we have $\mathcal{F}^4 f = f$ (by 11.74), completing the proof of (b).

Plancherel's Theorem (11.82) tells us that $\mathcal{F}$ is an isometry on $\mathcal{L}^2(\mathbf{R})$. Part (a) implies that $\mathcal{F}$ is surjective. Because a surjective isometry is unitary (see 10.61), we conclude that $\mathcal{F}$ is unitary, completing the proof of (a).

The Spectral Mapping Theorem [see 10.40—take $p(z) = z^4$] and (b) imply that $\alpha^4 = 1$ for each $\alpha \in \operatorname{sp}(T)$. In other words, $\operatorname{sp}(T) \subset \{1, i, -1, -i\}$. However, 1, i, -1, $-i$ are all eigenvalues of $\mathcal{F}$ (see Example 11.51 and Exercises 2, 3, and 4) and thus are all in $\operatorname{sp}(T)$. Hence $\operatorname{sp}(T) = \{1, i, -1, -i\}$, completing the proof of (c). ■

EXERCISES 11C

1 Suppose $f \in L^1(\mathbf{R})$. Prove that $\|\hat{f}\|_\infty = \|f\|_1$ if and only if there exists $\zeta \in \partial\mathbf{D}$ and $t \in \mathbf{R}$ such that $\zeta f(x) e^{-itx} \geq 0$ for almost every $x \in \mathbf{R}$.

2 Suppose $f(x) = xe^{-\pi x^2}$ for all $x \in \mathbf{R}$. Show that $\hat{f} = -if$.

3 Suppose $f(x) = 4\pi x^2 e^{-\pi x^2} - e^{-\pi x^2}$ for all $x \in \mathbf{R}$. Show that $\hat{f} = -f$.

4 Find $f \in L^1(\mathbf{R})$ such that $f \neq 0$ and $\hat{f} = if$.

5 Prove that if p is a polynomial on $\mathbf{R}$ with complex coefficients and $f: \mathbf{R} \to \mathbf{C}$ is defined by $f(x) = p(x)e^{-\pi x^2}$, then there exists a polynomial q on $\mathbf{R}$ with complex coefficients such that $\deg q = \deg p$ and $\hat{f}(t) = q(t)e^{-\pi t^2}$ for all $t \in \mathbf{R}$.

6 Suppose

$$f(x) = \begin{cases} xe^{-2\pi x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that $\hat{f}(t) = \dfrac{1}{4\pi^2(1+it)^2}$ for all $t \in \mathbf{R}$.

7 Prove the formulas in 11.55 for the Fourier transforms of translations, rotations, and dilations.

8 Suppose $f \in L^1(\mathbf{R})$ and $n \in \mathbf{Z}^+$. Define $g: \mathbf{R} \to \mathbf{C}$ by $g(x) = x^n f(x)$. Prove that if $g \in L^1(\mathbf{R})$, then $\hat{f}$ is n times continuously differentiable on $\mathbf{R}$ and

$$(\hat{f})^{(n)}(t) = (-2\pi i)^n \hat{g}(t)$$

for all $t \in \mathbf{R}$.

9 Suppose $n \in \mathbf{Z}^+$ and $f \in L^1(\mathbf{R})$ is n times continuously differentiable and $f^{(n)} \in L^1(\mathbf{R})$. Prove that if $t \in \mathbf{R}$, then

$$(f^{(n)})\hat{\ }(t) = (2\pi i t)^n \hat{f}(t).$$

10 Suppose $1 \leq p \leq \infty$, $f \in L^p(\mathbf{R})$, and $g \in L^{p'}(\mathbf{R})$. Prove that $f * g$ is a uniformly continuous function on $\mathbf{R}$.

11 Suppose $f \in \mathcal{L}^\infty(\mathbf{R})$, $x \in \mathbf{R}$, and f is continuous at x. Prove that

$$\lim_{y \downarrow 0} (\mathcal{P}_y f)(x) = f(x).$$

12 Suppose $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$. Prove that $\mathcal{P}_y(\mathcal{P}_{y'} f) = \mathcal{P}_{y+y'} f$ for all $y, y' > 0$.

13 Suppose $p \in [1, \infty]$ and $f \in L^p(\mathbf{R})$. Prove that if $0 < y < y'$, then

$$\|\mathcal{P}_y f\|_p \geq \|\mathcal{P}_{y'} f\|_p.$$

14 Suppose $f \in L^1(\mathbf{R})$.

(a) Prove that $(\overline{f})\hat{\ }(t) = \overline{\hat{f}(-t)}$ for all $t \in \mathbf{R}$.

(b) Prove that $f(x) \in \mathbf{R}$ for almost every $x \in \mathbf{R}$ if and only if $\hat{f}(t) = \overline{\hat{f}(-t)}$ for all $t \in \mathbf{R}$.

15 Define $f \in L^1(\mathbf{R})$ by $f(x) = e^{-x^4}\chi_{[0,\infty)}(x)$. Show that $\hat{f} \notin L^1(\mathbf{R})$.

16 Suppose $f \in L^1(\mathbf{R})$ and $\hat{f} \in L^1(\mathbf{R})$. Prove that $f \in L^2(\mathbf{R})$ and $\hat{f} \in L^2(\mathbf{R})$.

17 Prove there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $\lim_{t\to\pm\infty} g(t) = 0$ and $g \notin \{\hat{f} : f \in L^1(\mathbf{R})\}$.

18 Prove that if $f \in L^1(\mathbf{R})$, then $\|\hat{f}\|_2 = \|f\|_2$.
[*This exercise slightly improves Plancherel's Theorem (11.82) because here we have the weaker hypothesis that $f \in L^1(\mathbf{R})$ instead of $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Because of Plancherel's Theorem, here you need only prove that if $f \in L^1(\mathbf{R})$ and $\|f\|_2 = \infty$, then $\|\hat{f}\|_2 = \infty$.*]

19 Suppose $y > 0$. Define on operator T on $L^2(\mathbf{R})$ by $Tf = f * P_y$.

(a) Show that T is a self-adjoint operator on $L^2(\mathbf{R})$.

(b) Show that $\operatorname{sp}(T) = [0, 1]$.

[*Because the spectrum of each compact operator is a countable set (by 10.93), part (b) above implies that T is not a compact operator. This conclusion differs from the situation on the unit circle—see Exercise 9 in Section 11B.*]

20 Prove that if $f \in L^1(\mathbf{R})$ and $g \in L^2(\mathbf{R})$, then $\mathcal{F}(f * g) = \hat{f} \cdot \mathcal{F}g$.

21 Prove that $f, g \in L^2(\mathbf{R})$, then $(fg)\hat{} = (\mathcal{F}f) * (\mathcal{F}g)$.