Chapter 8
Related Topics

In this chapter we briefly review some aspects of the literature on circle packing that unfortunately we do not have space to get into in depth in this course. We hope this will be useful as a guide to further reading.

1. **Double circle packing.** If one wishes to study planar graphs that are *not* triangulations, it is often convenient to work with *double circle packings*, which enjoy similar rigidity properties to usual circle packings, but for the larger class of **polyhedral** planar graphs. Here, a planar graph is polyhedral if it is both simple and 3**-connected**, meaning that the removal of any two vertices cannot disconnect the graph. Double circle packings also satisfy a version of the ring lemma [45, Theorem 4.1], which means that they can be used to produce good straight-line embeddings of polyhedral planar graphs that have bounded face degrees but which are not necessarily triangulations.

   Let G be a planar graph with vertex set V and face set F. A double circle packing of G is a pair of circle packings $P = \{P_v : v \in V\}$ and $P^\dagger = \{P_f : f \in F\}$ satisfying the following conditions:

   (a) (**G is the tangency graph of P.**) For each pair of vertices u and v of G, the discs P_u and P_v are tangent if and only if u and v are adjacent in G.
   (b) (**$G^\dagger$ is the tangency graph of $P^\dagger$.**) For each pair of faces f and g of G, the discs P_f and P_g are tangent if and only if f and g are adjacent in $G^\dagger$.
   (c) (**Primal and dual circles are perpendicular**.) For each vertex v and face f of G, the discs P_f and P_v have non-empty intersection if and only if f is incident to v, and in this case the boundary circles of P_f and P_v intersect at right angles.

   See Fig. 8.1 for an illustration.

   Thurston's proof of the circle packing theorem also implies that every finite polyhedral planar graph admits a double circle packing. This was also shown by Brightwell and Scheinerman [13]. As with circle packings of triangulations, the double circle packing of any finite polyhedral planar map is unique up to

A. Nachmias, *Planar Maps, Random Walks and Circle Packing*, Lecture Notes in Mathematics 2243, https://doi.org/10.1007/978-3-030-27968-4_8

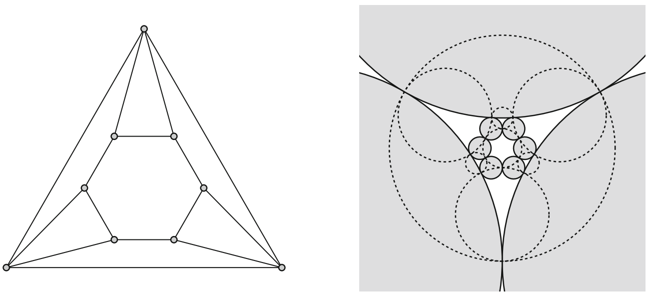

Fig. 8.1 A finite polyhedral plane graph (left) and its double circle packing (right). Primal circles are filled and have solid boundaries, dual circles have dashed boundaries

Möbius transformations or reflections. The theory of double circle packings in the infinite setting follows from the work of He [37], and is exactly analogous to the corresponding theory for triangulations. Indeed, essentially everything we have to say in these notes about circle packings of simple triangulations can be generalized to double circle packings of polyhedral planar maps (sometimes under the additional assumption that the faces are of bounded degree).

2. **Packing with other shapes**. A very powerful generalization of the circle packing theorem known as the *monster packing theorem* was proven by Schramm in his PhD thesis [76]. One consequence of this theorem is as follows: Let $T = (V, E)$ be a finite planar triangulation with a distinguished boundary vertex ∂. Specify a bounded, simply connected domain $D \subset \mathbb{C}$ with smooth boundary, and for each $v \in V \setminus \{\partial\}$ specify a strictly convex, bounded domain D_v with smooth boundary. Then there exists a collection of homotheties (compositions of translations and dilations) $\{h_v : v \in V\}$ such that

   - If $u, v \in V \setminus \{\partial\}$ are distinct, then the closure of $h_v D_v$ and $h_u D_u$ have disjoint interiors, and intersect if and only if v and u are adjacent in T.
   - If $v \in V \setminus \{\partial\}$, then the closure of $h_v D_v$ and $\mathbb{C} \setminus D$ have disjoint interiors, and intersect if and only if v is adjacent to ∂ in T.

   In other words, we can represent the triangulation of T by a packing with arbitrary smooth convex shapes that are specified up to homothety (it is quite surprising at first that rotations are not needed). The full monster packing theorem also allows one to relax the smoothness and convexity assumptions above in various ways. The proof of the monster packing theorem is based upon Brouwer's fixed point theorem, and does not give an algorithm for computing the packing.

3. **Square tiling.** Another popular method of embedding planar graphs is the *square tiling*, in which vertices are represented by horizontal line segments and edges by squares; such square tilings can take place either in a rectangle, the plane, or a cylinder. Square tiling was introduced by Brooks et al. [14], and generalized to infinite planar graphs by Benjamini and Schramm [10]. Like circle packing,

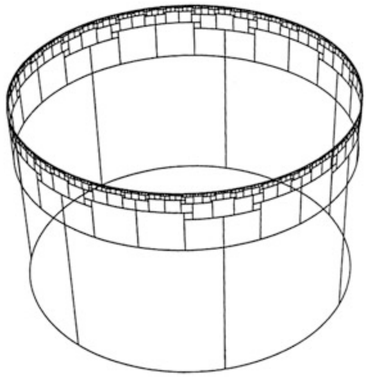

Fig. 8.2 The square tiling of the 7-regular triangulation

square tiling can be thought of as a discrete version of conformal mapping, and in particular can be used to approximate the uniformizing map from a simply connected domain with four marked boundary points to a rectangle. For studying the random walk, a very nice feature of the square tiling that is not enjoyed by the circle packing is that the height of a vertex in the cylinder is a harmonic function, so that the height of a random walk is a martingale. Furthermore, Georgakopoulos [28] observed that if one stops the random walk at the first time it hits some height, then its horizontal coordinate at this time is uniform on the circle (this takes some interpretation to make precise). Further works on square tiling include [1, 28, 46] (Fig. 8.2).

Unlike circle packing, however, square tilings do not enjoy an analogue of the ring lemma, and can be geometrically very degenerate. Indeed, it is possible for edges to be represented by squares of zero area, and is also possible for two distinct planar graphs to have the same square tiling. Furthermore, square tilings are typically defined with reference to a specified root vertex, and it is difficult to compare the two different square tilings of the same graph that are computed with respect to different root vertices. These differences tend to mean that square tilings are best suited to quite different problems than circle packing.

We also remark that a different sort of square tiling in which *vertices* are represented by squares was introduced independently by Cannon et al. [15] and Schramm [74].

4. **Multiply-connected triangulations.** Several works have studied generalizations of the circle packing theorem to triangulations that are either not simply connected or not planar. Most notably, He and Schramm [39] proved that every triangulation of a domain with countably many boundary components can be circle packed in a circle domain, that is, a domain all of whose boundary components are either circles or points: see Fig. 8.3 for examples. The corresponding statement for a triangulation of an *arbitrary* domain is a major open problem, and is closely related to the Koebe conjecture.

   Gurel-Gurevich, the current author, and Suoto [32] generalized the part of the He-Schramm Theorem concerning recurrence of the random walk as follows: A

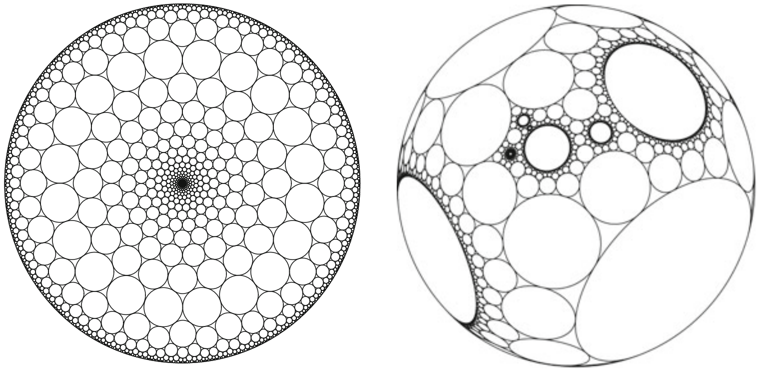

Fig. 8.3 Left: A circle packing in the multiply-connected circle domain $\mathbb{U} \setminus \{0\}$. Right: A circle packing in a circle domain with several boundary components

bounded degree triangulation circle packed in a domain D is transient if and only if Brownian motion on D is transient, i.e. leaves D in finite time almost surely.

5. **Isoperimetry of planar graphs.** In [66], Miller, Teng, Thurston, and Vavasis used circle packing to give a new proof of the *Lipton-Tarjan planar separator theorem* [60], which concerns sparse cuts in planar graphs. Precisely, the theorem states that for any n-vertex planar graph, one can find a set of vertices of size at most $O(\sqrt{n})$ such that if this vertex set is deleted from the graph then every connected component that remains has size at most $3n/4$. More precisely, the authors of [66] showed that if one circle packs a planar graph in the unit sphere of $\mathbb{R}^3$, normalizes by applying an appropriate Möbius transformation, and takes a random plane passing through the origin in $\mathbb{R}^3$, then the set of vertices whose corresponding discs intersect the plane will have the desired properties with high probability.

   A related result of Jonasson and Schramm [47] concerns the *cover time* of planar graphs, i.e., the expected number of steps for a random walk on the graph to visit every vertex of the graph. They used circle packing to prove that the cover time of an n-vertex planar graph with maximum degree M is always at least $c_M n \log^2 n$ for some positive constant c_M depending only on M. This bound is attained (up to the constant) for large boxes $[-n, n]^2$ in $\mathbb{Z}^2$. In general, it is possible for n-vertex graphs to have cover time as small as $(1 + o(1))n \log n$.

6. **Boundary theory.** Benjamini and Schramm [9] proved that if P is a circle packing of a bounded degree triangulation in the unit disc $\mathbb{U}$, then the simple random walk on the circle packed triangulation converges to a point in the boundary of $\mathbb{U}$, and that the law of the limit point is non-atomic and has full support. (That is, the walk has probability zero of converging to any specific boundary point, and has positive probability of converging to any positive-length interval.) They used this result to deduce that a bounded degree planar graph admits non-constant bounded harmonic functions if and only if it is transient (equivalently, the invariant sigma-algebra of the random walk on the triangulation

is non-trivial if and only if the walk is transient), and in this case it also admits non-constant bounded harmonic functions of finite Dirichlet energy. They also gave an alternative proof of the same result using square tiling instead of circle packing in [10].

Indeed, given the result of Benjamini and Schramm, one may construct a non-constant bounded harmonic function h on T by taking any bounded, measurable function $f : \partial\mathbb{U} \to \mathbb{R}$ and defining h to be the *harmonic extension* of f, that is,

$$h(v) = \mathbf{E}_v \left[f\left(\lim_{n\to\infty} z(X_n) \right) \right],$$

where $\mathbf{E}_v$ denotes expectation taken with respect to the random walk X started at v, and $z(u)$ denotes the center of the circle in P corresponding to u. Angel et al. [6] proved that, in fact, *every* bounded harmonic function on a bounded degree triangulation can be represented in this way. In other words, the boundary $\partial\mathbb{U}$ can be identified with the **Poisson boundary** of the triangulation. Probabilistically, this means that the entire invariant σ-algebra of the random walk coincides with the σ-algebra generated by the limit point. They also proved the stronger result that $\partial\mathbb{U}$ can be identified with the **Martin boundary** of the triangulation. Roughly speaking, this means that every *positive* harmonic function on the triangulation admits a representation as the harmonic extension of some *measure* on $\partial\mathbb{U}$. A related representation theorem for harmonic functions of *finite Dirichlet energy* on bounded degree triangulations was established by Hutchcroft [43].

The results of [6] regarding the Poisson boundary followed earlier work by Georgakopoulos [28], which established a corresponding result for square tilings. Both results were revisited in the work of Hutchcroft and Peres [46], which gave a simplified and unified proof that works for both embeddings.

A parallel boundary theory for circle packings of **unimodular random triangulations** of *unbounded* degree was developed by Angel, Hutchcroft, the current author, and Ray in [7].

7. **Harnack inequalities, Poincaré inequalities, and comparison to Brownian motion.** The work of Angel et al. [6] also established various quite strong estimates for random walk on circle packings of bounded degree triangulations. Roughly speaking, these estimates show that the random walk behaves similarly to the image of a Brownian motion under a *quasi-conformal map*, that is, a bijective map that distorts angles by at most a bounded amount (i.e., maps infinitesimal circles to infinitesimal ellipses of bounded eccentricity). These estimates were central to their result concerning the Martin boundary of the triangulation, and are also interesting in their own right. Further related estimates have also been established by Chelkak [17].

   Recent work of Murugan [67] has built further upon these methods to establish very precise control of the random walk on (graphical approximations of) various deterministic self-similar fractal surfaces.
8. **Liouville quantum gravity and the KPZ correspondence.** Statistical physics in two dimensions has been one of the hottest areas of probability theory in

recent years. The introduction of Schramm's SLE [75] and further breakthrough developments by Lawler, Schramm and Werner (see [53, 54] and the references within) on the one hand, and the application of discrete complex analysis, pioneered by Smirnov [79], on the other, have led to several breakthroughs and to the resolution of a number of long-standing conjectures. These include the conformally invariant scaling limits of critical percolation [77] and Ising models [78], and the determination of critical exponents and dimensions of sets associated with planar Brownian motion [53] (such as the frontier and the set of cut points). It is manifest that much progress will follow, possibly including the treatment of self-avoiding walk (the connective constant of the hexagonal lattice was calculated in the breakthrough work [22]), the $O(n)$ loop model and the Potts model. While the bulk of this body of work applies to specific lattices, there are many fascinating problems in extending results to arbitrary planar graphs.

The next natural step is to study the classical models of statistical physics in the context of random planar maps (see Le Gall's 2014 ICM proceedings [57]). There are deep conjectured connections between the behaviour of the models in the random setting versus the Euclidean setting, most significantly the KPZ formula of Knizhnik et al. [50] from conformal field theory. This formula relates the dimensions of certain sets in Euclidean geometry to the dimensions of corresponding sets in the random geometry. It may provide a systematic way to analyze models on the two dimensional Euclidean lattice: first study the model in the random geometry setting, where the Markovian properties of the underlying space make the model tractable; then use the KPZ formula to translate the critical exponents from the random setting to the Euclidean one.

Much of this picture is conjectural but a definite step towards this goal was taken in the influential paper of Duplantier and Sheffield [23]. Let us describe their formulation. Let G_n be a random triangulation on n vertices and consider its circle packing (or any other "natural" embedding) in the unit sphere. The embedding induces a random measure μ_n on the sphere by putting $\mu_n(A)$ to be the proportion of circle centers that are in A. The Duplantier-Sheffield conjecture asserts that the measures μ_n converge in distribution to a random measure μ on the sphere that has density given by an exponential of the Gaussian free field—the latter is carefully defined and constructed in [23]. This measure is what is known as *Liouville quantum gravity* (LQG).

Next, given a deterministic or random set K on the sphere, one can calculate its expected dimension using the random measure given by LQG, and using the usual Lebesgue measure—one gets two different numbers. Duplantier and Sheffield [23] obtain a quadratic formula allowing to compute one number from the other in the spirit of [50]; this is the first rigorous instance of the KPZ correspondence. It allows one to compute the dimension of random sets in the $\mathbb{Z}^2$ lattice (corresponding to Lebesgue measure) by first calculating the corresponding dimension in the random geometry setting and then appealing to the KPZ formula.

Many difficult models of statistical physics are tractable on a random planar map due to the inherent randomness of the space. For instance, it can be shown that the self avoiding walk on the UIPT behaves diffusively, that is, the endpoint of a self avoiding walk of length n is typically of distance $n^{1/2+o(1)}$ from the origin [19, 34]. A straightforward calculation with the KPZ formula allows one to predict that the typical displacement of the self-avoiding walk of length n on the lattice $\mathbb{Z}^2$ is $n^{3/4+o(1)}$—a notoriously hard open problem with endless simulations supporting it.

LQG and the KPZ correspondence thus pose a path to solving many difficult problems in classical two-dimensional statistical physics. We refer the interested reader to Garban's excellent survey [27] of the topic.