Chapter 2
Random Walks and Electric Networks

An extremely useful tool and viewpoint for the study of random walks is Kirchhoff's theory of electric networks. Our treatment here roughly follows [69, Chapter 8], we also refer the reader to [61] for an in-depth comprehensive study.

Definition 2.1 A **network** is a connected graph $G = (V, E)$ endowed with positive edge weights, $\{c_e\}_{e \in E}$ (called **conductances**). The reciprocals $r_e = 1/c_e$ are called **resistances**.

In Sects. 2.1–2.4 below we discuss finite networks. We extend our treatment to infinite networks in Sect. 2.5.

2.1 Harmonic Functions and Voltages

Let $G = (V, E)$ be a finite network. In physics classes it is taught that when we impose specific voltages at fixed vertices a and z, then current flows through the network according to certain laws (such as the series and parallel laws). An immediate consequence of these laws is that the function from V to $\mathbb{R}$ giving the voltage at each vertex is harmonic at each $x \in V \setminus \{a, z\}$.

Definition 2.2 A function $h : V \to \mathbb{R}$ is **harmonic** at a vertex x if

$$h(x) = \frac{1}{\pi_x} \sum_{y:y \sim x} c_{xy} h(y) \qquad \text{where} \qquad \pi_x := \sum_{y:y\sim x} c_{xy}. \tag{2.1}$$

Instead of starting with the physical laws and proving that voltage is harmonic, we now take the axiomatically equivalent approach of defining voltage to be a harmonic function and deriving the laws as corollaries.

A. Nachmias, *Planar Maps, Random Walks and Circle Packing*, Lecture Notes in Mathematics 2243, https://doi.org/10.1007/978-3-030-27968-4_2

Definition 2.3 Given a network $G = (V, E)$ and two distinct vertices $a, z \in V$, a **voltage** is a function $h : V \to \mathbb{R}$ that is harmonic at any $x \in V \setminus \{a, z\}$.

We will show in Claim 2.8 and Corollary 2.7 that for any $\alpha, \beta \in \mathbb{R}$, there is a unique voltage h such that $h(a) = \alpha$ and $h(z) = \beta$ (this assertion is true only when the network is finite).

Claim 2.4 If h_1, h_2 are harmonic at x then so is any linear combination of h_1, h_2.

Proof Let $\bar{h} = \alpha h_1 + \beta h_2$ for some $\alpha, \beta \in \mathbb{R}$. It holds that

$$\bar{h}(x) = \alpha h_1(x) + \beta h_2(x) = \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} \alpha h_1(y) + \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} \beta h_2(y)$$
$$= \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} \bar{h}(y).$$
□

Claim 2.5 If $h : V \to \mathbb{R}$ is harmonic at all the vertices of a finite network, then it is constant.

Proof Let $M = \sup_x h(x)$ be the maximum value of h. Let $A = \{x \in V : h(x) = M\}$. Since G is finite, $A \neq \emptyset$. Given $x \in A$, we have that $h(y) \leq h(x)$ for all neighbors y of x. By harmonicity, $h(x)$ is the weighted average of the values of $h(y)$ at the neighbors; but this can only happen if all neighbors of x are also in A. Since G is connected we obtain that $A = V$ implying that h is constant. □

We now show that a voltage is determined by its boundary values, i.e., by its values at a, z.

Claim 2.6 If h is a voltage satisfying $h(a) = h(z) = 0$, then $h \equiv 0$.

Proof Put $M = \max_x h(x)$ (which is attained since G is finite) and let $A = \{x \in V : h(x) = M\}$. As before, by harmonicity, if $x \in A \setminus \{a, z\}$ then all of its neighbors are also in A. Since G is connected, there exists a simple path from x to either a or z such that only its endpoint is in $\{a, z\}$. Since $h(a) = h(z) = 0$ we learn that $M = 0$, that is, h is non-positive. Similarly, one proves that h is non-negative, thus $h \equiv 0$. □

Corollary 2.7 (Voltage Uniqueness) *For every $\alpha, \beta \in \mathbb{R}$, if h, h' are voltages satisfying $h(a) = h'(a) = \alpha$ and $h(z) = h'(z) = \beta$, then $h \equiv h'$.*

Proof By Claim 2.4, the function $h - h'$ is a voltage, taking the value 0 at a and z, hence by Claim 2.6 we get $h \equiv h'$. □

Claim 2.8 For every $\alpha, \beta \in \mathbb{R}$, there exists a voltage h satisfying $h(a) = \alpha$,$h(z) = \beta$.

Proof 1 We write $n = |V|$. Observe that a voltage h with $h(a) = \alpha$ and $h(z) = \beta$ is defined by a system of $n - 2$ linear equations of the form (2.1) in $n - 2$ variables (which are the values $h(x)$ for $x \in V \setminus \{a, z\}$). Corollary 2.7 guarantees that the matrix representing that system has empty kernel, hence it is invertible. □

We present an alternative proof of existence based on the random walk on the network. Consider the Markov chain $\{X_n\}$ on the state space V with transition probabilities

$$p_{xy} := \mathbb{P}(X_{t+1} = y \mid X_t = x) = \frac{c_{xy}}{\pi_x}. \tag{2.2}$$

This Markov chain is a **weighted random walk** (note that if c_{xy} are all 1 then the described chain is the so-called **simple random walk**). We write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability and expectation, respectively, conditioned on $X_0 = x$. For a vertex x, define the **hitting time** of x by

$$\tau_x := \min\{t \geq 0 \mid X_t = x\}.$$

Proof 2 We will find a voltage g satisfying $g(a) = 0$ and $g(z) = 1$ by setting

$$g(x) = \mathbb{P}_x(\tau_z < \tau_a).$$

Indeed, g is harmonic at $x \neq a, z$, since by the law of total probability and the Markov property we have

$$\begin{aligned} g(x) = \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} \mathbb{P}_x(\tau_z < \tau_a \mid X_1 = y) &= \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} \mathbb{P}_y(\tau_z < \tau_a) \\ &= \frac{1}{\pi_x} \sum_{y:y\sim x} c_{xy} g(y). \end{aligned}$$

For general boundary conditions α, β we define h by

$$h(x) = g(x) \cdot (\beta - \alpha) + \alpha\,.$$

By Claim 2.4, h is a voltage, and clearly $h(a) = \alpha$ and $h(z) = \beta$, concluding the proof. $\square$

This proof justifies the equality between simple random walk probabilities and voltages that was discussed at the start of this chapter: since the function $x \mapsto \mathbb{P}_x(\tau_z < \tau_a)$ is harmonic on $V \setminus \{a, z\}$ and takes values 0, 1 at a, z respectively, it must be equal to the voltage at x when voltages 0, 1 are imposed at a, z.

Claim 2.9 If h is a voltage with $h(a) \leq h(z)$, then $h(a) \leq h(x) \leq h(z)$ for all $x \in V$.

Furthermore, if $h(a) < h(z)$ and $x \in V \setminus \{a, z\}$ is a vertex such that x is in the connected component of z in the graph $G \setminus \{a\}$, and x is in the connected component of a in the graph $G \setminus \{z\}$, then $h(a) < h(x) < h(z)$.

Proof This follows directly from the construction of h in Proof 2 of Claim 2.8 and the uniqueness statement of Corollary 2.7. Alternatively, one can argue as in the proof of Claim 2.6 that if $M = \max_x h(x)$ and $m = \min_x h(x)$, then the sets $A = \{x \in V : h(x) = M\}$ and $B = \{x \in V : h(x) = m\}$ must each contain at least one element of $\{a, z\}$.

To prove the second assertion, we note that by Claim 2.8 and Corollary 2.7 it is enough to check when h is the voltage with boundary values $h(a) = 0$ and $h(z) = 1$. In this case, the condition on x guarantees that the probabilities that the random walk started at x visits a before z or visits z before a are positive. By proof 2 of Claim 2.8 we find that $h(x) \in (0, 1)$. □

2.2 Flows and Currents

For a graph $G = (V, E)$, denote by $\vec{E}$ the set of edges of G, each endowed with the two possible orientations. That is, $(x, y) \in \vec{E}$ iff $\{x, y\} \in E$ (and in that case, $(y, x) \in \vec{E}$ as well).

Definition 2.10 A **flow from** a **to** z in a network G is a function $\theta : \vec{E} \to \mathbb{R}$ satisfying

1. For any $\{x, y\} \in E$ we have $\theta(xy) = -\theta(yx)$ (**antisymmetry**), and
2. $\forall x \notin \{a, z\}$ we have $\sum_{y:y\sim x} \theta(xy) = 0$ (**Kirchhoff's node law**).

Claim 2.11 If θ_1, θ_2 are flows then, so is any linear combination of θ_1, θ_2.

Proof Let $\bar{\theta} = \alpha\theta_1 + \beta\theta_2$ for some $\alpha, \beta \in \mathbb{R}$. It holds that

$$\bar{\theta}(xy) = \alpha\theta_1(xy) + \beta\theta_2(xy) = -\alpha\theta_1(yx) - \beta\theta_2(yx) = -\bar{\theta}(yx),$$

and for $x \neq a, z$,

$$\sum_{y:y\sim x} \bar{\theta}(xy) = \alpha \sum_{y:y\sim x} \theta_1(xy) + \beta \sum_{y:y\sim x} \theta_2(xy) = 0.$$ □

Definition 2.12 Given a voltage h, the **current flow** $\theta = \theta_h$ associated with h is defined by $\theta(xy) = c_{xy}(h(y) - h(x))$.

In other words, the voltage difference across an edge is the product of the current flowing along the edge with the resistance of the edge. This is known as **Ohm's law**. According to this definition, the current flows from vertices with lower voltage to vertices with higher voltage. We will use this convention throughout, but the reader should be advised that some other sources use the opposite convention.

Claim 2.13 The current flow associated with a voltage is indeed a flow.

Proof The current flow is clearly antisymmetric by definition. To show that it satisfies the node law, observe that for $x \neq a, z$, since h is harmonic,

$$\sum_{y:y\sim x} \theta(xy) = \overbrace{\sum_{y:y\sim x} c_{xy}h(y)}^{=\pi_x h(x)} - \overbrace{\sum_{y:y\sim x} c_{xy}h(x)}^{=\pi_x h(x)} = 0. \qquad \square$$

Claim 2.14 The current flow associated with a voltage h satisfies **Kirchhoff's cycle law**, that is, for every directed cycle $\vec{e}_1, \ldots, \vec{e}_m$,

$$\sum_{i=1}^{r} r_{e_i}\theta(\vec{e}_i) = 0.$$

Proof Write $\vec{e}_i = (x_{i-1}, x_i)$, and observe that $x_0 = x_m$. We have that

$$\sum_{i=1}^{m} r_{e_i}\theta(\vec{e}_i) = \sum_{i=1}^{m} r_{x_{i-1}x_i}c_{x_{i-1}x_i}(h(x_i) - h(x_{i-1})) = \sum_{i=1}^{m}(h(x_i) - h(x_{i-1})) = 0. \qquad \square$$

For examples of a flow which does not satisfy the cycle law and a current flow, see Fig. 2.1.

Claim 2.15 Given a flow θ which satisfies the cycle law, there exists a voltage $h = h_\theta$ such that θ is the current flow associated with h. Furthermore, this voltage is unique up to an additive constant.

Proof For every vertex x, let $\vec{e}_1, \ldots, \vec{e}_k$ be a path from a to x, and define

$$h(x) = \sum_{i=1}^{k} r_{e_i}\theta(\vec{e}_i). \tag{2.3}$$

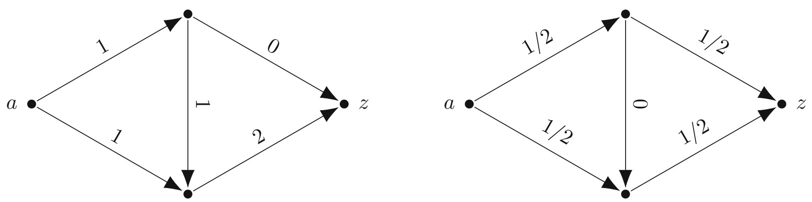

Fig. 2.1 On the left, a flow of strength 2 in which the cycle law is violated. On the right, the unit (i.e., strength 1) current flow

Note that since θ satisfies the cycle law, the right hand side of (2.3) does not depend on the choice of the path, hence $h(x)$ is well defined. Let $x \in V$, and consider a given path $\vec{e}_1, \ldots, \vec{e}_k$ from a to x (if $x = a$ we take the empty path). To evaluate $h(y)$ for $y \sim x$, consider the path $\vec{e}_1, \ldots, \vec{e}_k, xy$ from a to y, so $h(y) = h(x) + r_{xy}\theta(xy)$. It follows that $h(y) - h(x) = r_{xy}\theta(xy)$, hence $\theta(xy) = c_{xy}(h(y) - h(x))$, meaning that θ is indeed the current flow associated with h.

Since $\theta(xy) = c_{xy}(h(y) - h(x))$ for any $x \sim y$, the node law of immediately implies that h is a voltage. To show that h is unique up to an additive constant, suppose that $g : V \to \mathbb{R}$ is another voltage such that $r_{xy}\theta(xy) = g(y) - g(x)$. It follows that $g(y) - h(y) = g(x) - h(x)$ for any $x \sim y$. Since G is connected it follows that $g - h$ is the constant function on V. □

Definition 2.16 The **strength** of a flow θ is

$$\|\theta\| = \sum_{x:x\sim a} \theta(ax).$$

Claim 2.17 For every flow θ,

$$\sum_{x:x\sim z} \theta(xz) = \|\theta\|.$$

Proof We have that

$$\begin{aligned} 0 &= \sum_{x\in V} \sum_{y:y\sim x} \theta(xy) \\ &= \sum_{x\in V\setminus\{a,z\}} \sum_{y:y\sim x} \theta(xy) + \sum_{y:y\sim a} \theta(ay) + \sum_{y:y\sim z} \theta(zy) \\ &= \sum_{y:y\sim a} \theta(ay) + \sum_{y:y\sim z} \theta(zy) \end{aligned}$$

where the first equality is due to antisymmetry, and the third equality is due to the node law. The claim follows again by antisymmetry. □

Claim 2.18 If θ_1, θ_2 are flows satisfying the cycle law and $\|\theta_1\| = \|\theta_2\|$, then $\theta_1 = \theta_2$.

Proof Let $\bar{\theta} = \theta_1 - \theta_2$. According to Claim 2.11, $\bar{\theta}$ is a flow. It also satisfies the cycle law, as for every cycle $\vec{e}_1, \ldots, \vec{e}_m$,

$$\sum_{i=1}^{m} r_{e_i}\bar{\theta}(\vec{e}_i) = \sum_{i=1}^{m} r_{e_i}\theta_1(\vec{e}_i) - \sum_{i=1}^{m} r_{e_i}\theta_2(\vec{e}_i) = 0.$$

Observe in addition that $\|\bar{\theta}\| = \|\theta_1\| - \|\theta_2\| = 0$. Now, let $h = h_{\bar{\theta}}$ be the voltage defined in Claim 2.15, chosen so that $h(a) = 0$. Note that it is harmonic at a, since

$$\frac{1}{\pi_a}\sum_{x:x\sim a} c_{ax}h(x) = \frac{1}{\pi_a}\sum_{x:x\sim a} c_{ax}(h(a) + r_{ax}\bar{\theta}(ax))$$
$$= \frac{1}{\pi_a}\sum_{x:x\sim a} c_{ax}h(a) + \frac{1}{\pi_a}\sum_{x:x\sim a}\bar{\theta}(ax) = h(a) + \frac{\|\bar{\theta}\|}{\pi_a} = h(a).$$

Similarly, using Claim 2.17 it is also harmonic at z. Since h is harmonic everywhere, it is constant by Claim 2.5, and thus $h \equiv 0$, hence $\bar{\theta} \equiv 0$ and so $\theta_1 = \theta_2$. □

This last claim prompts the following useful definition.

Definition 2.19 The **unit current flow** from a to z is the unique current flow from a to z of strength 1.

2.3 The Effective Resistance of a Network

Suppose we are given a voltage h on a network G with fixed vertices a and z. Scaling h by a constant multiple causes the associated current flow to scale by the same multiple, while adding a constant to h does not change the current flow at all. Therefore, the strength of the current flow is proportional to the difference $h(z) - h(a)$.

Claim 2.20 For every non-constant voltage h and a current flow θ corresponding to h, the ratio

$$\frac{h(z) - h(a)}{\|\theta\|} \tag{2.4}$$

is a positive constant which does not depend on h.

Proof Let h_1, h_2 be two non-constant voltages, and let θ_1, θ_2 be their associated current flows. For $i = 1, 2$, let $\bar{h}_i = h_i/\|\theta_i\|$ and let $\bar{\theta}_i$ be the current flow associated with $\bar{h}_i$ (note that since h_i is non-constant $\|\theta_i\| \neq 0$). Thus, $\|\bar{\theta}_i\| = 1$. By Claim 2.18 we get $\bar{\theta}_1 = \bar{\theta}_2$ and therefore $\bar{h}_1 = \bar{h}_2 + c$ for some constant c by Claim 2.15. It follows that $\bar{h}_1(z) - \bar{h}_1(a) = \bar{h}_2(z) - \bar{h}_2(a)$.

To see that this constant is positive, it is enough to check one particular choice of a voltage. By Claim 2.8, let h be the voltage with $h(a) = 0$ and $h(z) = 1$. By Claim 2.9 and since G is connected, we have that $h(x) > 0$ for at least one neighbor x of a. Thus, the corresponding current flow θ has $\|\theta\| > 0$ making (2.4) positive. □

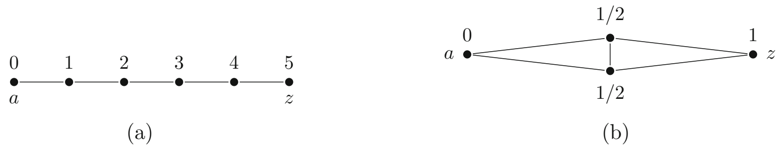

Fig. 2.2 Examples for effective resistances of two networks with unit edge conductances. (**a**) For the voltage depicted, the voltage difference between a and z is 5, and the current flow's strength is 1, hence the effective resistance is $5/1 = 5$. (**b**) For the voltage depicted, the voltage difference between a and z is 1, and the current flow's strength is 1, hence the effective resistance is $1/1 = 1$

Claim 2.20 is the mathematical manifestation of Ohm's law which states that the voltage difference across an electric circuit is proportional to the current through it. The constant of proportionality is usually called the *effective resistance* of the circuit.

Definition 2.21 The number defined in (2.4) is called the **effective resistance** between a and z in the network, and is denoted $\mathcal{R}_{\text{eff}}(a \leftrightarrow z)$. We call its reciprocal the **effective conductance** between a and z and is denoted $\mathcal{C}_{\text{eff}}(a \leftrightarrow z) := \mathcal{R}_{\text{eff}}(a \leftrightarrow z)^{-1}$.

For examples of computing the effective resistances of networks, see Fig. 2.2.

Notation In most cases we write $\mathcal{R}_{\text{eff}}(a \leftrightarrow z)$ and suppress the notation of which network we are working on. However, when it is important to us what the network is, we will write $\mathcal{R}_{\text{eff}}(a \leftrightarrow z; G)$ for the effective resistance in the network G with unit edge conductances and $\mathcal{R}_{\text{eff}}(a \leftrightarrow z; (G, \{r_e\}))$ for the effective resistance in the network G with edge resistances $\{r_e\}_{e \in E}$. Furthermore, given disjoint subsets A and Z of vertices in a graph G, we write $\mathcal{R}_{\text{eff}}(A \leftrightarrow Z)$ for the effective resistance between a and z in the network obtained from the original network by identifying all the vertices of A into a single vertex a, and all the vertices of Z into a single vertex z.

Probabilistic Interpretation For a vertex x we write τ_x^+ for the stopping time

$$\tau_x^+ = \min\{t \geq 1 \mid X_t = x\}, \tag{2.5}$$

where X_t is the weighted random walk on the network, as defined in (2.2). Note that if $X_0 \neq x$ then $\tau_x = \tau_x^+$ with probability 1.

Claim 2.22

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z) = \frac{1}{\pi_a \mathbb{P}_a(\tau_z < \tau_a^+)}.$$

Proof Consider the voltage h satisfying $h(a) = 0$ and $h(z) = 1$, and let θ be the current flow associated with h. Due to uniqueness of h (Corollary 2.7) we have that for $x \neq a, z$,

$$h(x) = \mathbb{P}_x(\tau_z < \tau_a),$$

hence

$$\begin{aligned}\mathbb{P}_a(\tau_z < \tau_a^+) &= \frac{1}{\pi_a} \sum_{x \sim a} c_{ax} \mathbb{P}_x(\tau_z < \tau_a) \\ &= \frac{1}{\pi_a} \sum_{x \sim a} c_{ax} h(x) \\ &= \frac{1}{\pi_a} \sum_{x \sim a} \theta(ax) = \frac{\|\theta\|}{\pi_a} = \frac{1}{\pi_a \mathcal{R}_{\text{eff}}(a \leftrightarrow z)}.\end{aligned}$$ □

Network Simplifications Sometimes a network can be replaced by a simpler network, without changing the effective resistance between a pair of vertices.

Claim 2.23 (Parallel Law) Conductances add in parallel. Suppose e_1, e_2 are parallel edges between a pair of vertices, with conductances c_1 and c_2, respectively. If we replace them with a single edge e' with conductance $c_1 + c_2$, then the effective resistance between a and z is unchanged.

A demonstration of the parallel law appears in Fig. 2.3.

Proof Let G' be the graph where e_1 and e_2 are replaced with e' with conductance $c_1 + c_2$. Then it is immediate that if h is any voltage function on G, then it remains a voltage function on the network G'. The claim follows. □

Claim 2.24 (Series Law) Resistances add in series. Suppose that $u \notin \{a, z\}$ is a vertex of degree 2 and that $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ are the two edges touching u with edge resistances r_1 and r_2, respectively. If we erase u and replace e_1 and e_2 by a single edge $e' = (v_1, v_2)$ of resistance $r_1 + r_2$, then the effective resistance between a and z is unchanged.

The series law is depicted in Fig. 2.4.

Proof Denote by G' the graph in which u is erased and e_1 and e_2 are replaced by a single edge (v_1, v_2) of resistance $r_1 + r_2$. Let θ be a current flow from a to z in G, and define a flow θ' from a to z in G' by putting $\theta'(e) = \theta(e)$ for any $e \neq e_1, e_2$ and $\theta'(v_1, v_2) = \theta(v_1, u)$. Since u had degree 2, it must be that $\theta(v_1, u) = \theta(u, v_2)$.

Fig. 2.3 Demonstrating the parallel law. Two parallel edges are replaced by a single edge

Fig. 2.4 An example of a network G where edges in series are replaced by a single edge

Thus θ' satisfies the node law at any $x \notin \{a, z\}$ and $\|\theta\| = \|\theta'\|$. Furthermore, since θ satisfies the cycle law, so does θ'. We conclude θ' is a current flow of the same strength as θ and the voltage difference they induce is the same. □

The operation of **gluing** a subset of vertices $S \subset V$ consists of identifying the vertices of S into a single vertex and keeping all the edges and their conductances. In this process we may generate parallel edges or loops.

Claim 2.25 (Gluing) Gluing vertices of the same voltage does not change the effective resistance between a and z.

Proof This is immediate since the voltage on the glued graph is still harmonic. □

Example: Spherically Symmetric Tree Let Γ be a **spherically symmetric tree**, that is, a rooted tree where all vertices at the same distance from the root have the same number of children. Denote by ρ the root of the tree, and let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers. Every vertex at distance n from the root ρ has d_n children. Denote by Γ_n the set of all vertices of height n. We would like to calculate $\mathcal{R}_{\text{eff}}(\rho \leftrightarrow \Gamma_n)$. Due to the tree's symmetry, all vertices at the same level have the same voltage and therefore by Claim 2.25 we can identify them. Our simplified network has now one vertex for each level, denoted by $\{v_i\}_{i\in\mathbb{N}}$ (where $\rho = v_0$), with $|\Gamma_{n+1}|$ edges between v_n and v_{n+1}. Using the parallel law (Claim 2.23), we can reduce each set of $|\Gamma_n|$ edges to a single edge with resistance $\frac{1}{|\Gamma_n|}$, then, using the series law (Claim 2.24) we get

$$\mathcal{R}_{\text{eff}}(\rho \leftrightarrow \Gamma_n) = \sum_{i=1}^{n} \frac{1}{|\Gamma_i|} = \sum_{i=1}^{n} \frac{1}{d_0 \cdots d_{i-1}} ,$$

see Fig. 2.5.

By Claim 2.22 we learn that

$$\mathbb{P}_\rho(\tau_n < \tau_\rho^+) = \frac{1}{d_0 \sum_{i=1}^{n} \frac{1}{d_0 \cdots d_{i-1}}} , \tag{2.6}$$

where τ_n is the hitting time of Γ_n for the random walk on Γ. Observe that

$$\mathbb{P}_\rho\left(\tau_n < \tau_\rho^+ \text{ for all } n\right) = \mathbb{P}_\rho\left(X_t \text{ never returns to } \rho\right) ,$$

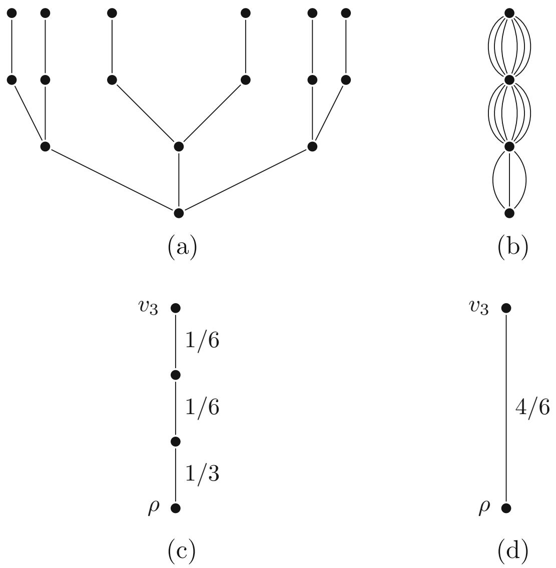

Fig. 2.5 Using network simplifications. (**a**) The first four levels of a spherically symmetric tree with $\{d_n\} = \{3, 2, 1, \ldots\}$. (**b**) Gluing nodes on the same level. (**c**) Applying the parallel law. (**d**) Applying the series law

so by (2.6) we reach an interesting dichotomy. If $\sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} = \infty$, then the random walker returns to ρ with probability 1, and hence returns to ρ infinitely often almost surely. If $\sum_{i=1}^{\infty} \frac{1}{d_1 \cdots d_i} < \infty$, then with positive probability the walker never returns to ρ, and hence visits ρ only finitely many times almost surely.

The former graph is called a **recurrent** graph and the latter is called **transient**. We will get back to this dichotomy in Sect. 2.5.

The Commute Time Identity

The following lemma shows that the effective resistance between a and z is proportional to the expected time it takes the random walk starting at a to visit z and then return to a, in other words, the expected *commute time* between a and z. We will use this lemma only in Chap. 6 so the impatient reader may skip this section and return to it later.

Lemma 2.26 (Commute Time Identity) *Let $G = (V, E)$ be a finite network and $a \neq z$ two vertices. Then*

$$\mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a] = 2\mathcal{R}_{\text{eff}}(a \leftrightarrow z) \sum_{e \in E} c_e$$

Proof We denote by $G_z : V \times V \to \mathbb{R}$ the so-called Green function

$$G_z(a, x) = \mathbb{E}_a[\text{number of visits to } x \text{ before } z]$$

and note that

$$\mathbb{E}_a[\tau_z] = \sum_{x \in V} G_z(a, x).$$

It is straightforward to show that the function $v(x) = G_z(a, x)/\pi_x$ is harmonic in $V \setminus \{a, z\}$. Also, we have that $G_z(a, z) = 0$ and $G_z(a, a) = \frac{1}{\mathbb{P}_a(\tau_z < \tau_a)} = \pi_a \mathcal{R}_{\text{eff}}(a \leftrightarrow z)$. Thus, v is a voltage function with boundary conditions $v(z) = 0$ and $v(a) = \mathcal{R}_{\text{eff}}(a \leftrightarrow z)$ which satisfies

$$\mathbb{E}_a[\tau_z] = \sum_{x \in V} v(x)\pi_x \, .$$

Similarly, the same analysis for $\mathbb{E}_z[\tau_a]$ yields the same result, with the voltage function η which has boundary conditions $\eta(z) = \mathcal{R}_{\text{eff}}(a \leftrightarrow z)$ and $\eta(a) = 0$. Therefore, $\eta(x) = v(a) - v(x)$ for all $x \in V$ since both sides are harmonic functions in $V \setminus \{a, z\}$ that receive the same boundary values. This implies that

$$\mathbb{E}_z[\tau_a] = \sum_{x \in V} \pi_x \left(v(a) - v(x)\right).$$

Summing these up gives

$$\mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a] = \sum_{x \in V} \pi_x v(a) = 2 \sum_{e \in E} c_e \mathcal{R}_{\text{eff}}(a \leftrightarrow z). \qquad \square$$

2.4 Energy

So far we have seen how to compute the effective resistance of a network via harmonic functions and current flows. However, in typical situations it is hard to find a flow satisfying the circle law. Luckily, an extremely useful property of the effective resistance is that it can be represented by a variational problem. Our intuition from highschool physics suggests that the *energy* of the unit current flow is minimal among all unit flows from a to z. The notion of energy can be made precise and will allow us to obtain valuable monotonicity properties. For instance, removing any edge from an electric network can only increase its effective resistance. Hence, any recurrent graph remains recurrent after removing any subset of edges from it. Two variational problems govern the effective resistance, Thomson's principle, which is used to bound the effective resistance from above, and Dirichlet's principle, used to bound it from below.

Definition 2.27 The **energy** of a flow θ from a to z, denoted by $\mathcal{E}(\theta)$, is defined to be

$$\mathcal{E}(\theta) := \frac{1}{2}\sum_{\vec{e}\in\vec{E}} r_{\vec{e}}\,\theta(\vec{e})^2 = \sum_{e\in E}\theta(e)^2 r_e.$$

Note that in the second sum we sum over undirected edges, but since $\theta(xy)^2 = \theta(yx)^2$, this is well defined.

Theorem 2.28 (Thomson's Principle)

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z) = \inf\{\mathcal{E}(\theta) : \|\theta\| = 1, \theta \textit{ is a flow from } a \textit{ to } z\}$$

and the unique minimizer is the unit current flow.

Proof First, we will show that the energy of the unit current flow is the effective resistance. Let I be the unit current flow, and h the corresponding (Claim 2.15) voltage function.

$$\begin{aligned}
\mathcal{E}(I) &= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy} I(xy)^2 = \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}\left(\frac{h(y)-h(x)}{r_{xy}}\right) I(xy) \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} (h(y)-h(x))\, I(xy) \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} h(y)I(xy) - \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} h(x)I(xy).
\end{aligned}$$

Observe that in the second term of the right hand side, for every $x \neq a, z$ the sum over all $y \sim x$ is 0 due to the node law, hence the entire term equals $\frac{1}{2}(h(a)-h(z))$. From antisymmetry of I, the first term on the right hand side equals $-\frac{1}{2}(h(a) - h(z))$, hence the right hand side equals altogether $h(z) - h(a) = \mathcal{R}_{\text{eff}}(a \leftrightarrow z)$.

We will now show that every other flow J with $\|J\| = 1$ has $\mathcal{E}(J) \geq \mathcal{E}(I)$. Let J be such flow and write $J = I + (J - I)$. Set $\theta = J - I$ and note that $\|\theta\| = 0$. We have

$$\begin{aligned}
\mathcal{E}(J) &= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}(I(xy)+\theta(xy))^2 \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}I(xy)^2 + \frac{1}{2}\sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta(xy)^2 \\
&\quad + \sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta(xy)I(xy) \\
&= \mathcal{E}(I) + \mathcal{E}(\theta) + \sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta(xy)I(xy).
\end{aligned}$$

Now,

$$\sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta(xy)I(xy) = \sum_{x\in V}\sum_{y:y\sim x} r_{xy}\theta(xy)\frac{(h(y)-h(x))}{r_{xy}}$$
$$= \sum_{x\in V}\sum_{y:y\sim x} \theta(xy)\,(h(y)-h(x))$$
$$= 2\cdot\|\theta\|\cdot(h(z)-h(a)) = 0,$$

where the last inequality follows from the same reasoning as before. We conclude that $\mathcal{E}(J)\geq\mathcal{E}(I)$ as required and that equality holds if and only if $\mathcal{E}(\theta)=0$, that is, if and only if $J=I$. □

Corollary 2.29 (Rayleigh's Monotonicity Law) *If $\{r_e\}_{e\in E}$ and $\{r'_e\}_{e\in E}$ are edge resistances on the same graph G so that $r_e\leq r'_e$ for all edges $e\in E$, then*

$$\mathcal{R}_{\text{eff}}(a\leftrightarrow z;\,(G,\{r_e\}))\leq\mathcal{R}_{\text{eff}}(a\leftrightarrow z;\,(G,\{r'_e\})).$$

Proof Let θ be a flow on G, then

$$\sum_{e\in E} r_e\theta(e)^2\leq\sum_{e\in E} r'_e\theta(e)^2.$$

This inequality is preserved while taking infimum over all flows with strength 1. Applying Theorem 2.28 finishes the proof. □

Corollary 2.30 *Gluing vertices cannot increase the effective resistance between a and z.*

Proof Denote by G the original network and by G' the network obtained from gluing a subset of vertices. Then every flow θ on G (viewed as a function on the edges) is a flow on G'. Hence the infimum in Theorem 2.28 taken over flows in G' is taken over a larger subset of flows. □

Definition 2.31 The **energy** of a *function* $h:V\to\mathbb{R}$, denoted by $\mathcal{E}(h)$, is defined to be

$$\mathcal{E}(h):=\sum_{\{x,y\}\in E} c_{xy}(h(x)-h(y))^2.$$

Compare the following lemma with Thomson's principle (Theorem 2.28).

Lemma 2.32 (Dirichlet's Principle) *Let G be a finite network with source a and sink z. Then*

$$\frac{1}{\mathcal{R}_{\text{eff}}(a\leftrightarrow z)}=\inf\big\{\mathcal{E}(h):h:V\to\mathbb{R},\ h(a)=0,\ h(z)=1\big\}.$$

Proof The infimum is obtained when h is *the* harmonic function taking 0 and 1 at a, z respectively. The reason is that if there exists $v \neq a, z$ with

$$h(v) \neq \sum_{u \sim v} \frac{c_{vu}}{\pi_v} h(u), \tag{2.7}$$

then we can change the value of h at v to be the right hand side of (2.7) and the energy will only decrease. One way to see this is that if X is a random variable with a second moment, then the value $\mathbb{E}(X)$ minimizes the function $f(x) = \mathbb{E}\left((X - x)^2\right)$.

Let h be that harmonic function and let I be its current flow, so $I(xy) = c_{xy}(h(y) - h(x))$. Write $\hat{I} = \mathcal{R}_{\text{eff}}(a \leftrightarrow z) \cdot I$, so $\|\hat{I}\| = 1$. By Thomson's principle,

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z) = \mathcal{E}(\hat{I}) = \sum_{e \in E} r_e \hat{I}(e)^2 = \sum_{\{x,y\} \in E} r_{xy} \mathcal{R}_{\text{eff}}(a \leftrightarrow z)^2 c_{xy}^2 (h(y) - h(x))^2,$$

hence

$$\frac{1}{\mathcal{R}_{\text{eff}}(a \leftrightarrow z)} = \mathcal{E}(h).$$

□

2.5 Infinite Graphs

Let $G = (V, E)$ be an infinite connected graph with edge resistances $\{r_e\}_{e \in E}$. We assume henceforth that this network is **locally finite**, that is, for any vertex $x \in V$ we have $\sum_{y: y \sim x} c_{xy} < \infty$. Let $\{G_n\}$ be a sequence of finite subgraphs of G such that $\bigcup_{n \in \mathbb{N}} G_n = G$ and $G_n \subset G_{n+1}$; we call such a sequence an **exhaustive sequence** of G. Identify all vertices of $G \setminus G_n$ with a single vertex z_n.

Claim 2.33 Given an exhaustive sequence $\{G_n\}$ of G, the limit

$$\lim_{n \to \infty} \mathcal{R}_{\text{eff}}(a \leftrightarrow z_n; G_n \cup \{z_n\}) \tag{2.8}$$

exists.

Proof The graph $G_n \cup \{z_n\}$ can be obtained from $G_{n+1} \cup \{z_{n+1}\}$ by gluing the vertices in $G_{n+1} \setminus G_n$ with z_{n+1} and labeling the new vertex z_n. By Corollary 2.30, the effective resistance $\mathcal{R}_{\text{eff}}(a \leftrightarrow z_n; G_n \cup \{z_n\})$ is increasing in n. □

Claim 2.34 The limit in (2.8) does not depend on the choice of exhaustive sequence $\{G_n\}$.

Proof Indeed, let $\{G_n\}$ and $\{G'_n\}$ be two exhaustive sequences of G. We can find subsequences $\{i_k\}_{k\geq 1}$ and $\{j_k\}_{k\geq 1}$ such that

$$G_{i_1} \subseteq G'_{j_1} \subseteq G_{i_2} \subseteq \dots$$

Since $\{G_{i_1}, G'_{j_1}, G_{i_2}, \dots\}$ is itself an exhaustive sequence of G, the limit of effective resistances for this sequence exists and equals the limits of effective resistances for the subsequences $\{G_{i_k}\}$ and $\{G'_{j_k}\}$. In turn, these are equal to the limits of effective resistances for the original sequences $\{G_n\}$ and $\{G'_n\}$, respectively. □

Definition 2.35 In an infinite network, the effective resistance from a vertex a and ∞ is

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow \infty) := \lim_{n\to\infty} \mathcal{R}_{\text{eff}}(a \leftrightarrow z_n; G_n \cup \{z_n\}) .$$

We are now able to address the question of recurrence versus transience of a graph systematically. Recall the definition of τ_x^+ in (2.5). In an infinite network we define $\tau_a^+ = \infty$ when there is no time $t \geq 1$ such that $X_t = a$.

Definition 2.36 A network $(G, \{r_e\}_{e\in E})$ is called **recurrent** if $\mathbb{P}_a(\tau_a^+ = \infty) = 0$, that is, if the probability of the random walker started at a never returning to a is 0. Otherwise, it is called **transient** .

Observe that since G is connected, if $\mathbb{P}_a(\tau_a^+ = \infty) = 0$ for one vertex a, then it holds for all vertices in the network. As we have seen, if n is large enough so that $a \in G_n$, then

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z_n; G_n \cup \{z_n\}) = \frac{1}{\pi_a \cdot \mathbb{P}_a\left(\tau_{G\setminus G_n} < \tau_a^+\right)} .$$

Since $\bigcap_n \{\tau_{G\setminus G_n} < \tau_a^+\} = \{\tau_a^+ = \infty\}$ we have

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow \infty) = \frac{1}{\pi_a \cdot \mathbb{P}_a(\tau_a^+ = \infty)} ,$$

with the convention that $1/0 = \infty$.

Definition 2.37 Let G be an infinite network. A function $\theta : E(G) \to \mathbb{R}$ is a **flow from** a **to** ∞ if it is anti-symmetric and satisfies the node law on each vertex $v \neq a$.

The following follows easily from Theorem 2.28, we omit the proof.

Theorem 2.38 (Thomson's Principle for Infinite Networks) *Let G be an infinite network, then*

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow \infty) = \inf\{\mathcal{E}(\theta) : \theta \textit{ is a flow from } a \textit{ to } \infty \textit{ of strength } 1\}.$$

Corollary 2.39 *Let G be an infinite graph. The following are equivalent:*

1. *G is transient.*
2. *There exists a vertex $a \in V$ such that $\mathcal{R}_{\text{eff}}(a \leftrightarrow \infty) < \infty$. Hence all vertices satisfy this.*
3. *There exists a vertex $a \in V$ and a unit flow θ from a to ∞ with $\mathcal{E}(\theta) < \infty$. Hence all vertices satisfy this.*

We will now develop a useful method for bounding effective resistances from below. This will lead us to a popular sufficient criterion for recurrence in Corollary 2.43.

Definition 2.40 A **cutset** $\Gamma \subseteq E(G)$ separating a from z is a set of edges such that every path from a to z must use an edge from Γ.

Claim 2.41 Let θ be a flow from a to z in a finite network, and let Γ a cutset separating a from z. Then

$$\sum_{e\in\Gamma} |\theta(e)| \geq \|\theta\|.$$

Proof Denote by Z the set of vertices separated from a by Γ. Denote by G' the network where Z is identified to a single vertex x and all edges having both endpoints in Z are removed. Now, the restriction of θ to the edges of the new network is a flow from a to x. By Claim 2.17, we have $\sum_{y:y\sim x} \theta(yx) = \|\theta\|$. Also, all edges incident to x must be in Γ, since otherwise x is not separated from a by Γ. Therefore

$$\sum_{e\in\Gamma} |\theta(e)| \geq \sum_{y:y\sim x} \theta(yx) = \|\theta\|. \qquad \square$$

Theorem 2.42 (Nash-Williams Inequality) *Let $\{\Gamma_n\}$ be disjoint cutsets separating a from z in a finite network. Then*

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z) \geq \sum_n \left(\sum_{e\in\Gamma_n} c_e \right)^{-1}.$$

Proof Let θ be a flow from a to z with $\|\theta\| = 1$. From Cauchy-Schwarz, for each n we have

$$\left(\sum_{e\in\Gamma_n} \sqrt{r_e}\sqrt{c_e}|\theta(e)| \right)^2 \leq \sum_{e\in\Gamma_n} c_e \sum_{e\in\Gamma_n} r_e\theta(e)^2 .$$

Also, since Γ_n is a cutset, the flow passing through Γ_n is at least $\|\theta\|$, by Claim 2.41. So

$$\left(\sum_{e\in\Gamma_n} \sqrt{r_e}\sqrt{c_e}|\theta(e)|\right)^2 \geq \|\theta\|^2 = 1.$$

Combining them, we get that

$$\sum_{e\in\Gamma_n} r_e\theta(e)^2 \geq \frac{1}{\sum_{e\in\Gamma_n} c_e}.$$

Summing over all n gives

$$\mathcal{E}(\theta) \geq \sum_n \sum_{e\in\Gamma_n} r_e\theta(e)^2 \geq \sum_n \left(\sum_{e\in\Gamma_n} c_e\right)^{-1}.$$

Applying Thomson's principle (Theorem 2.28) yields the result. □

Consider now an infinite network $G = (V, E)$. We say that $\Gamma \subset E$ is a cutset separating a from ∞ if any infinite simple path from a must intersect Γ.

Corollary 2.43 *In any infinite network, if there exists a collection $\{\Gamma_n\}$ of disjoint cutsets separating a from ∞ such that*

$$\sum_n \left(\sum_{e\in\Gamma_n} c_e\right)^{-1} = \infty,$$

then the network is recurrent.

Example 2.44 ($\mathbb{Z}^2$ is Recurrent) Define Γ_n as the set of vertical edges $\{(x, y), (x, y+1)\}$ with $|x| \leq n$ and $\min\{|y|, |y+1|\} = n$ along with the horizontal edges $\{(x, y), (x+1, y)\}$ with $|y| \leq n$ and $\min\{|x|, |x+1|\} = n$, see Fig. 2.6. Then

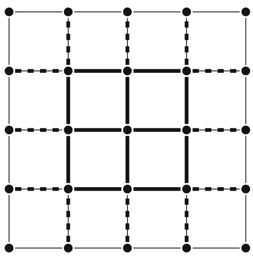

Fig. 2.6 A part of $\mathbb{Z}^2$: the edges in $\{-1, 0, 1\}^2$ are drawn in bold. Γ_1 is dashed

$\{\Gamma_n\}$ is a collection of disjoint cutsets separating 0 from ∞. Also, $|\Gamma_n| = 4(2n+1)$ and therefore $\sum_n \left(\sum_{e\in\Gamma_n} c_e\right)^{-1} = \infty$. We deduce by Corollary 2.43 that $\mathbb{Z}^2$ is recurrent.

Remark 2.45 There are recurrent graphs for which there exists $M < \infty$ such that for every collection $\{\Gamma_n\}$ of disjoint cutsets, $\sum_n \left(\sum_{e\in\Gamma_n} c_e\right)^{-1} \le M$. Therefore, the Nash-Williams inequality is not sharp. See Exercise 4 of this chapter.

2.6 Random Paths

We now present the method of random paths, which is one of the most useful methods for generating unit flows on a network and bounding their energy. In fact, it is possible to show that the electric flow can be represented by such a random path. Suppose G is a network with fixed vertices a, z and μ is a probability measure on the set of paths from a to z.

Claim 2.46 For a path γ sampled from μ, let

$$\theta_\gamma(\vec{e}) = (\#\text{ of times } \vec{e} \text{ was traversed by } \gamma) - (\#\text{ of times } \overleftarrow{e} \text{ was traversed by } \gamma),$$

where by $\vec{e}$ and $\overleftarrow{e}$ we mean the two orientations of an edge e of G. Set

$$\theta(\vec{e}) = \mathbb{E}\theta_\gamma(\vec{e}).$$

Then θ is a flow from a to z with $\|\theta\| = 1$.

Proof θ is antisymmetric since θ_γ is antisymmetric for every γ, and it satisfies the node law since θ_γ satisfies the node law. Similarly, the "strength" of θ_γ (i.e., $\sum_{x\sim a} \theta_\gamma(ax)$) is 1, hence $\|\theta\| = 1$. □

An example of the use of this method is the following classical result.

Theorem 2.47 *$\mathbb{Z}^3$ is transient.*

Proof For $R > 0$ denote by $B_R = \{(x, y, z) : x^2 + y^2 + z^2 \le R^2\}$ the ball of radius R in $\mathbb{R}^3$. Put $V_R = B_R \cap \mathbb{Z}^3$ and let ∂V_R be the external vertex boundary of V_R, that is, the set of vertices not in V_R which belong to an edge with an endpoint in V_R.

We construct a random path μ from the origin $\mathbf{0}$ to ∂V_R by choosing a uniform random point $\mathbf{p}$ in $\partial B_R = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$, drawing a straight line between $\mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^3$, considering the set of distance at most 10 in $\mathbb{R}^3$ from the line, and then choosing (in some arbitrary fashion) a path in $\mathbb{Z}^3$ which is contained inside this set. The non-optimal constant 10 was chosen in order to guarantee that such a discrete path exists for any point $\mathbf{p} \in \partial B_R$.

By Claim 2.46, the measure μ corresponds to a flow from $\mathbf{0}$ to ∂V_R. To estimate the energy of this flow, we note that if $\vec{e}$ is an edge at distance $r \le R$ from the origin, then the probability that it is traversed by a path drawn by μ is $O(r^{-2})$. Furthermore, there are $O(r^2)$ such edges. Hence the energy of the flow is at most

$$\mathcal{E} = O\Big(\sum_{r=1}^{R} r^2 \cdot (r^{-2})^2\Big) \le C\,,$$

for some constant $C < \infty$ which does not depend on R. By Claim 2.46 and Theorem 2.28 we learn that $\mathcal{R}_{\text{eff}}(\mathbf{0} \leftrightarrow \partial V_R) \le C$ for all R, and so by Corollary 2.39 we deduce that $\mathbb{Z}^3$ is transient. $\square$

2.7 Exercises

1. Let $G_z(a, x)$ be the Green's function, that is,

$$G_z(a, x) = \mathbb{E}_a\big[\#\text{visits to } x \text{ before visiting } z\big]\,.$$

   Show that the function $h(x) = G_z(a, x)/\pi(x)$ is a voltage.
2. Show that the effective resistance satisfies the triangle inequality. That is, for any three vertices x, y, z we have

$$\mathcal{R}_{\text{eff}}(x \leftrightarrow z) \le \mathcal{R}_{\text{eff}}(x \leftrightarrow y) + \mathcal{R}_{\text{eff}}(y \leftrightarrow z)\,. \tag{2.9}$$

3. Let a, z be two vertices of a finite network and let τ_a, τ_z be the first visit time to a and z, respectively, of the weighted random walk. Show that for any vertex x

$$\mathbf{P}_x(\tau_a < \tau_z) \le \frac{\mathcal{R}_{\text{eff}}(x \leftrightarrow \{a, z\})}{\mathcal{R}_{\text{eff}}(x \leftrightarrow a)}\,.$$

4. Consider the following tree T. At height n it has 2^n vertices (the root is at height $n = 0$) and if $(v_1, \ldots, v_{2^n})$ are the vertices at level n we make it so that v_k has 1 child at level $n + 1$ and if $1 \le k \le 2^{n-1}$ and v_k has 3 children at level $n + 1$ for all other k.

   (a) Show that T is recurrent.
   (b) Show that for any disjoint edge cutsets Π_n we have that $\sum_n |\Pi_n|^{-1} < \infty$. (So, the Nash-Williams criterion for recurrence is not sharp)

5. (a) Let G be a finite planar graph with two distinct vertices $a \ne z$ such that a, z are on the outer face. Consider an embedding of G so that a is the left most point on the real axis and z is the right most point on the real axis. Split the outer face of G into two by adding the ray from a to $-\infty$ and the ray from z to $+\infty$. Consider the dual graph G^* of G and write a^* and z^* for the two vertices corresponding to the split outer face of G. Assume that all edge

resistances are 1. Show that

$$\mathcal{R}_{\text{eff}}(a \leftrightarrow z; G) = \frac{1}{\mathcal{R}_{\text{eff}}(a^* \leftrightarrow z^*; G^*)} .$$

(b) Show that the probability that a simple random walk on $\mathbb{Z}^2$ started at $(0, 0)$ has probability $1/2$ to visit $(0, 1)$ before returning to $(0, 0)$.

6. Let $G = (V, E)$ be a graph so that $V = \mathbb{Z}$ and the edge set $E = \cup_{k \geq 0} E_k$ where $E_0 = \{(i, i + 1) : i \in \mathbb{Z}\}$ and for $k > 0$

$$E_k = \Big\{\big(2^k(n - 1/2), 2^k(n + 1/2)\big) : n \in \mathbb{Z}\Big\} .$$

Is G recurrent or transient?