IPIES for Uncertainly Defined Shape of Boundary, Boundary Conditions and Other Parameters in Elasticity Problems

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Abstract. The main purpose of this paper is modelling and solving boundary value problems simultaneously considering uncertainty of all of input data such as: shape of boundary, boundary conditions and other parameters. The strategy is presented on the basis of problems described by Navier-Lamé equations. Therefore, the uncertainty of parameters here, means the uncertainty of the Poisson's ratio and Young's modulus. For solving uncertainly defined problems we use implementation of interval parametric integral equations system method (IPIES). In this method we propose modification of directed interval arithmetic for modeling and solving uncertainly defined problems. We consider an examples of uncertainly defined, 2D elasticity problems. We present boundary value problems with linear as well as curvelinear (modelled using NURBS curves) shape of boundary. We verify obtained interval solutions by comparison with precisely defined (without uncertainty) analytical solutions. Additionally, to obtain errors of such solutions, we decided to use the total differential method. We also analyze influence of input data uncertainty on interval solutions.

Keywords: Boundary problems · Uncertainty · Interval arithmetic · Parametric integral equations system

1 Introduction

Modeling of uncertainty is a very important problem and it generates considerable interest among researchers. However direct application of existing mathematical apparatuses is often useless in practice. In this paper we present interval parametric integral equations system (IPIES) [8,10] for solving uncertainly defined boundary value problems on examples of 2D elasticity problems. The parametric integral equations system method (PIES) was previously developed and widely tested for precisely (exactly) defined problems [7,11]. Many studies

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have confirmed PIES advantages over other methods, such as well known FEM and BEM methods.

Mentioned methods have also corresponding interval methods, such as IFEM [1] and IBEM [4]. However, in these methods accuracy of solutions depends on (finite or boundary) elements number. So, discretization increases the number of interval data and results in solutions overestimation. Therefore, the IFEM and IBEM researchers have mainly focused on modeling uncertainty of boundary conditions and other parameters only.

The article presents the impact of all uncertainly defined input data (necessary to define the problem) on IPIES solutions, obtained based on implemented program of the method. We consider 2D elasticity problems modeled by Navier-Lamé equations and we define uncertainty of the shape of boundary, boundary conditions and other parameters (Poisson's ratio and Young's modulus). We model uncertainty using modified directed interval arithmetic (applied in IPIES). To verify obtained interval solutions we use analytical solutions with errors obtained using total differential method [3]. Additionally we test an impact of change in data uncertainty on interval solutions.

2 Mathematical Foundations of IPIES

Till now, PIES was applied for precisely defined boundary problems [7,11]. It is an analytical modification of boundary integral equations. Now, to include uncertainly defined input data, we can present PIES (on example of Navier-Lamé equation) using intervals:

$$0.5\boldsymbol{u}_l(s_1) = \sum_{j=1}^{n} \int_{\widehat{s}_{j-1}}^{\widehat{s}_j} \left\{ \boldsymbol{U}_{lj}^*(s_1, s)\boldsymbol{p}_j(s) - \boldsymbol{P}_{lj}^*(s_1, s)\boldsymbol{u}_j(s) \right\} \boldsymbol{J}_j(s) ds, \qquad (1)$$

where $l = 1, 2, ..., n$, $\widehat{s}_{l-1} \leq s_1 \leq \widehat{s}_l$, $\widehat{s}_{j-1} \leq s \leq \widehat{s}_j$, where $\widehat{s}_{l-1}, \widehat{s}_{j-1}$ are the beginnings and $\widehat{s}_l, \widehat{s}_j$ are the ends of segments with index l or j. Function $\boldsymbol{J}_j(s) = [\underline{J}_j(s), \overline{J}_j(s)]$ is the Jacobian for interval curve segment $\boldsymbol{S}_j = [\boldsymbol{S}_j{}^{(1)}(s), \boldsymbol{S}_j{}^{(2)}(s)]^T$ with index j, where $\boldsymbol{S}_j{}^{(1)}(s) = [\underline{S}_j^{(1)}(s), \overline{S}_j^{(1)}(s)], \boldsymbol{S}_j{}^{(2)}(s) = [\underline{S}_j^{(2)}(s), \overline{S}_j^{(2)}(s)]$. Integral functions $\boldsymbol{p}_j(s) = \left\{ [\underline{p}_j^{(1)}(s), \overline{p}_j^{(1)}(s)], [\underline{p}_j^{(2)}(s), \overline{p}_j^{(2)}(s)] \right\}, \boldsymbol{u}_j(s) = \left\{ [\underline{u}_j^{(1)}(s), \overline{u}_j^{(1)}(s)], [\underline{u}_j^{(2)}(s), \overline{u}_j^{(2)}(s)] \right\}$ are the interval parametric boundary functions on corresponding boundary segments $\boldsymbol{S}_j$ (on which the boundary was theoretically divided). One of these functions will always be defined as uncertain (interval) boundary conditions on segment $\mathbf{S}_j$, then the other will be obtained as a result of numerical solution of IPIES (1).

Including uncertainty in the first integral $\boldsymbol{U}_{lj}^*(s_1, s)$ for plane state of strain we obtained following interval matrix:

$$\boldsymbol{U}_{lj}^*(s_1, s) = -\frac{1}{8\pi(1-\boldsymbol{\nu})\boldsymbol{\mu}} \begin{bmatrix} (3-4\boldsymbol{\nu})\ln(\boldsymbol{\eta}) - \frac{\boldsymbol{\eta}_1^2}{\boldsymbol{\eta}^2} & -\frac{\boldsymbol{\eta}_1\boldsymbol{\eta}_2}{\boldsymbol{\eta}^2} \\ -\frac{\boldsymbol{\eta}_1\boldsymbol{\eta}_2}{\boldsymbol{\eta}^2} & (3-4\boldsymbol{\nu})\ln(\boldsymbol{\eta}) - \frac{\boldsymbol{\eta}_2^2}{\boldsymbol{\eta}^2} \end{bmatrix}, \qquad (2)$$

where $l, j = 1, 2, ..., n$, $\boldsymbol{\mu} = 0.5 \cdot \boldsymbol{E}/(1+\boldsymbol{\nu})$ is an interval Lamé parameter, $\boldsymbol{\nu} = [\underline{\nu}, \overline{\nu}]$ is an interval Poisson's ratio, $\boldsymbol{E} = [\underline{E}, \overline{E}]$ is an interval Young's modulus and the formulas to obtain $\boldsymbol{\eta} = [\underline{\eta}, \overline{\eta}]$, $\boldsymbol{\eta}_1 = [\underline{\eta}_1, \overline{\eta}_1]$ and $\boldsymbol{\eta}_2 = [\underline{\eta}_2, \overline{\eta}_2]$ are:

$$\boldsymbol{\eta} = [\boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_2^2]^{0.5}, \qquad \boldsymbol{\eta}_1 = \boldsymbol{S}_l^{(1)}(s_1) - \boldsymbol{S}_j^{(1)}(s), \quad \boldsymbol{\eta}_2 = \boldsymbol{S}_l^{(2)}(s_1) - \boldsymbol{S}_j^{(2)}(s). \quad (3)$$

The second integral $\mathbf{P}_{lj}^*(s_1, s)$ we also defined using intervals:

$$\boldsymbol{P}_{lj}^*(s_1, s) = -\frac{1}{4\pi(1-\boldsymbol{\nu})\boldsymbol{\eta}} \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}, \quad (4)$$

where $l, j = 1, 2, ..., n$ and $\boldsymbol{P}_{ik} = [\underline{P}_{ik}, \overline{P}_{ik}]$ (where i, k = 1, 2) are defined as:

$$\boldsymbol{P}_{ii} = \left\{ (1-2\boldsymbol{\nu}) + 2\frac{\boldsymbol{\eta}_i^2}{\boldsymbol{\eta}^2} \right\} \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{n}}, \quad (5)$$

$$\boldsymbol{P}_{ik} = \left\{ 2\frac{\boldsymbol{\eta}_i \boldsymbol{\eta}_k}{\boldsymbol{\eta}^2} \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{n}} - (1-2\boldsymbol{\nu}) \left[\frac{\boldsymbol{\eta}_i}{\boldsymbol{\eta}} \boldsymbol{n}_k(s) + \frac{\boldsymbol{\eta}_k}{\boldsymbol{\eta}} \boldsymbol{n}_i(s) \right] \right\}, \quad (6)$$

$$\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{n}} = \frac{\boldsymbol{\eta}_1}{\boldsymbol{\eta}} \mathbf{n}_1(s) + \frac{\boldsymbol{\eta}_2}{\boldsymbol{\eta}} \boldsymbol{n}_2(s). \quad (7)$$

We use directed intervals for modeling uncertainty and modified directed interval arithmetic for calculations:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \begin{cases} \boldsymbol{x}_s \cdot \boldsymbol{y}_s - \boldsymbol{x}_s \cdot y_m - x_m \cdot \boldsymbol{y}_s + x_m \cdot y_m & \text{for } \boldsymbol{x} \le 0, \boldsymbol{y} \le 0 \\ \boldsymbol{x}_s \cdot \boldsymbol{y} - x_m \cdot \boldsymbol{y} & \text{for } \boldsymbol{x} > 0, \boldsymbol{y} \le 0 \\ \boldsymbol{x} \cdot \boldsymbol{y}_s - \boldsymbol{x} \cdot y_m & \text{for } \boldsymbol{x} \le 0, \boldsymbol{y} > 0 \\ \boldsymbol{x} \cdot \boldsymbol{y} & \text{for } \boldsymbol{x} > 0, \boldsymbol{y} > 0 \end{cases}, \quad (8)$$

where for any interval number $\boldsymbol{a} = [\underline{a}, \overline{a}]$ we define $\boldsymbol{a}_s = \boldsymbol{a} + a_m$ and $a_m = \begin{cases} |\overline{a}| & \text{for } \overline{a} > \underline{a} \\ |\underline{a}| & \text{for } \overline{a} < \underline{a} \end{cases}$, where $\boldsymbol{a} > 0$ means $\underline{a} > 0$ and $\overline{a} > 0$ i $\boldsymbol{a} \le 0$ means $\underline{a} < 0$ or $\overline{a} < 0$ then multiplication $(\cdot)$ is an interval multiplication. Research on such modification was widely discussed in [8].

3 Verification of IPIES on Examples

To obtain interval solutions, presented mathematical apparatus was implemented as computer program of IPIES method. We decided to verify obtained interval solutions using analytical solutions with errors obtained by total differential method (used directly to define errors in arithmetic operations). This method allows us to obtain error of the function, when the errors of all function arguments are known. If the function $u = f(x_1, x_2, ..., x_n)$ is differentiable and we define $|\Delta x_i| (i = 1, 2, ..., n)$ as absolute errors of function arguments, then the general formula to obtain an absolute error of the function is [3]:

$$\Delta u = \sum_{i=0}^{n} \left| \frac{\partial f}{\partial x_i} \right| |\Delta x_i|, \tag{9}$$

where in boundary value problem the function f correspond to analytical solution.

Example 1. The first of considered problems is known as the Lamé problem [6]. We define simultaneously uncertainty of the boundary shape and boundary conditions (Fig. 1). We analyze thick-walled pipe, which length of the internal radius ($a = 10\,\text{cm}$) and external radius ($b = 25\,\text{cm}$) is defined with the width of uncertainty band (the width of interval) $\varepsilon_a = \varepsilon_b = 1$. So defined pipe is subjected to a uniform internal pressure p_a. Therefore, we define the uncertainty of the boundary condition by interval value $\boldsymbol{p}_a = [99, 101]$. The problem is defined in plane state of strain and the material parameters are defined as degenerate intervals with values: $E = 2 \cdot 10^5$ MPa i $\nu = 0.25$. We use NURBS curves [5,9] of second degree (defined using interval points) to model uncertainty of the shape of boundary.

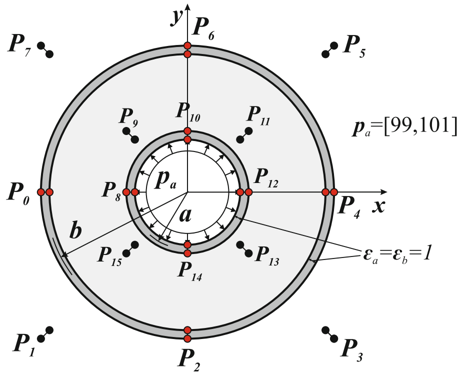

Fig. 1. Modelling uncertainty of the shape of boundary and boundary condition.

As already mentioned, we decided to compare obtained interval solutions with analytical ones [6]:

$$\sigma_x = \frac{p_a a^2 - p_b b^2}{b^2 - a^2} - \frac{(p_a - p_b) a^2 b^2}{r^2 (b^2 - a^2)}, \qquad \sigma_y = \frac{p_a a^2 - p_b b^2}{b^2 - a^2} + \frac{(p_a - p_b) a^2 b^2}{r^2 (b^2 - a^2)}, \tag{10}$$

where r is the one of polar coordinates $r^2 = x^2 + y^2$, $a < r < b$. We obtain solution error using total differential method, where as the error of function arguments ($|\Delta x_i|$ from the (9)) we assume the half of the interval width ($\Delta a = |\overline{a} - \underline{a}|/2$) for uncertainly defined shape of boundary $\boldsymbol{a} = [\underline{a}, \overline{a}]$, $\boldsymbol{b} = [\underline{b}, \overline{b}]$ and boundary condition $\boldsymbol{p}_a = [\underline{p_a}, \overline{p_a}]$, $\boldsymbol{p}_b = [\underline{p_b}, \overline{p_b}]$. Additionally we assume $p_b = [0, 0]$ and omit the uncertainty of material constant in this example, because the Young's modulus and Poisson's ratio are not defined in analytical solution (10).

We present interval solutions and analytical solutions with errors obtained using total differential method in Table 1 in cross section: $x = 12, 14, ..., 24$ and $y = 0$. To allow direct comparison, we present interval solutions using the middle of interval ($\mathrm{mid}(a) = (\overline{a} + \underline{a})/2$) and the half of interval width ($\Delta a = |\overline{a} - \underline{a}|/2$). Both obtained middle values of interval solutions and analytical solutions, as well as obtained halves of the width of interval solutions and analytical solutions errors are almost equal. Average relative error is ca one percent, so the example confirms correctness of proposed strategy.

Table 1. Interval solutions in domain with compare to analytical ones.

x	Analytical solution				Interval solution (IPIES)			
	σ_x	σ_y	$\Delta\sigma_x$	$\Delta\sigma_y$	$mid(\boldsymbol{\sigma}_x)$	$mid(\boldsymbol{\sigma}_y)$	$\Delta\sigma_x$	$\Delta\sigma_y$
12	−63.624	101.720	8.488	14.664	−64.074	102.686	8.535	14.714
14	−41.691	79.786	5.824	11.666	−41.980	80.573	5.833	11.708
16	−27.455	65.551	4.096	9.721	−27.670	66.223	4.101	9.758
18	−17.695	55.791	2.911	8.387	−17.861	56.385	2.914	8.421
20	−10.714	48.810	2.063	7.433	−10.846	49.347	2.066	7.464
22	−5.549	43.644	1.436	6.727	−5.649	44.135	1.441	6.754
24	−1.620	39.716	0.959	6.190	−1.629	40.180	1.003	6.230
Average relative error [%]					0.95	1.06	0.87	0.43

Example 2. In the next example we decided to examine the influence of input data uncertainty on interval solutions of IPIES method. We consider 2×2 square plate presented in [2] and uncertainly defined in Fig. 2. The shape of boundary is defined with width of uncertainty band $\varepsilon = 0.1$. Material constants are defined as follow: Young's modulus $\boldsymbol{E} = [0.9, 1.1]$ and Poisson's ratio $\boldsymbol{\nu} = [0.29, 0.31]$. In Fig. 2, we also present interval force $\boldsymbol{p}$ acting to one side of the plate. We obtain solutions of the problem in plane state of strain.

Analytical solutions, of above mentioned example, are defined exactly (without uncertainty) as follow [2]:

$$u_x = -0.195x^2 - 0.455(y-1)^2, \qquad u_y = 0.91x(y+1), \tag{11}$$

and corresponding stress [2]:

$$\sigma_x = 0, \qquad \sigma_y = x. \tag{12}$$

We obtain interval solutions in cross-section presented on Fig. 2, where $y = 0$ and x changing from -1 to 1. Results are presented in Fig. 3. We denote the width

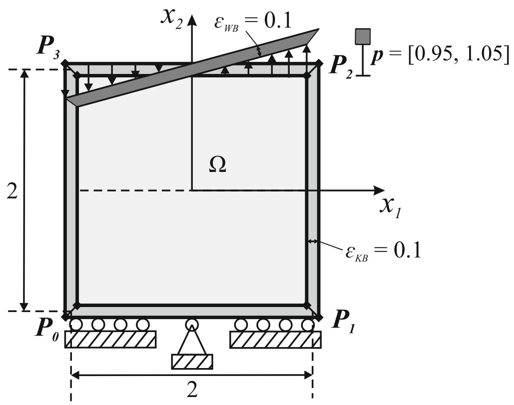

Fig. 2. Uncertainly defined boundary value problem.

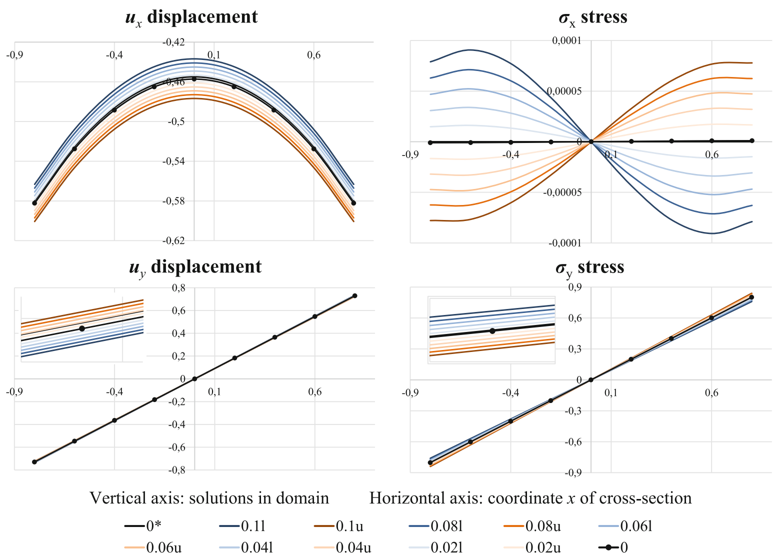

Fig. 3. Interval solutions in domain.

of uncertainty band as values 0.1 – 0. For example, value 0.08 means the width of uncertainty band equal to $\varepsilon_{KB} = 0.08$ for the shape of boundary, $\varepsilon_{WB} = 0.08$ for boundary conditions, $\varepsilon_E = 0.16$ for Young's modulus and $\varepsilon_\nu = 0.016$ for Poisson's ratio. Interval solutions are presented as lower (l) and upper (u) bound.

So, we can note, that the strategy works well for problems where all of input data are defined uncertainly. As we expect, by increasing the width of interval input data, the width of interval solutions is also increased. Additionally presented IPIES solutions with the width of uncertainty band $\varepsilon = 0$ and analytical ones 0* (for precisely defined problem) are almost equal and are located inside of each interval solution.

4 Conclusions

We presented mathematical foundation of IPIES with uncertainly defined all of input data simultaneously. Based on such mathematical model the program of IPIES method has been implemented. We test the program on examples of the problems described by Navier-Lamé equations. Therefore we modeled uncertainly defined shape of boundary, boundary conditions and parameters: Young's modulus and Poisson's ratio. To verify interval solutions we decided to use analytical solutions and its errors obtained by total differential method. Obtained solutions are almost equal. We also tested an impact of the width of interval input data on width of interval solutions. We noted, that the width of interval solution, as expected, increases with the width of interval input data. Additionally exact solutions of PIES (almost equal to the analytical ones) are located inside all of considered interval solutions. Therefore, in conclusion, it is very difficult and time-consuming to define exactly what kind of problems can be solved by IPIES, but we present a high potential of the method, in solving problems with all of input data defined uncertainly, for further investigations.

References

1. Muhanna, R.L., Mullen, R.L., Rao, M.V.R.: Nonlinear finite element analysis of frames under interval material and load uncertainty. ASCE-ASME J. Risk Uncertain. Eng. Syst. Part B Mech. Eng. **1**(4), 041003 (2015)
2. Panzeca, T., Fujita Yashima, H., Salerno, M.: Direct stiffness matrices of BEs in the Galerkin BEM formulation. Eur. J. Mech. Solids **20**, 277–298 (2001)
3. Peng, F.Y., Ma, J.Y., Wang, W., Duan, X.Y., Sun, P.P., Yan, R.: Int. J. Mach. Tools Manuf. **70**, 53–62 (2013)
4. Piasecka-Belkhayat, A.: Interval boundary element method for 2D transient diffusion problem using the directed interval arithmetic. Eng. Anal. Bound. Elem. **35**(3), 259–263 (2011)
5. Piegl, L., Tiller, W.: The NURBS Book, 2nd edn. Springer, Heidelberg (1997). https://doi.org/10.1007/978-3-642-59223-2
6. Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity. McGraw-Hill, Tokyo (1970)
7. Zieniuk, E.: A new integral identity for potential polygonal domain problems described by parametric linear functions. Eng. Anal. Bound. Elem. **26**(10), 897–904 (2002)
8. Zieniuk, E., Kapturczak, M., Kużelewski, A.: Concept of modeling uncertainly defined shape of the boundary in two-dimensional boundary value problems and verification of its reliability. Appl. Math. Model. **40**(23-24), 10274–10285 (2016)
9. Zieniuk, E., Kapturczak, M.: Modeling the shape of boundary using NURBS curves directly in modified boundary integral equations for Laplace's equation. Comput. Appl. Math. **37**(4), 4835–4855 (2018)

10. Zieniuk, E., Kapturczak, M., Kużelewski, A.: Modification of interval arithmetic for modelling and solving uncertainly defined problems by interval parametric integral equations system. In: Shi, Y., et al. (eds.) ICCS 2018, Part III. LNCS, vol. 10862, pp. 231–240. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-93713-7_19
11. Zieniuk, E., Kużelewski, A.: GPU-based acceleration of computations in elasticity problems solving by parametric integral equations system. Adv. Eng. Softw. **79**, 27–35 (2015)