



A Computational Technique for Asian Option Pricing Model

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Abstract. In the present work, the European style fixed strike Asian call option with arithmetic and continuous averaging is numerically evaluated where the volatility, the risk free interest rate and the dividend yield are functions of the time. A finite difference scheme consisting of second order HODIE scheme for spatial discretization and two-step backward differentiation formula for temporal discretization is applied. The scheme is proved to be second order accurate in space and time both. The numerical results are in accordance with analytical results.

Keywords: Asian option · Black-Scholes model · High-Order Difference approximation with Identity Expansion (HODIE) · Two-step backward differentiation formula (BDF)

1 Introduction

Asian options are path-dependent exotic options. The payoffs of Asian options depend on some sort of average of the underlying asset price over a specified time interval. The averaging can be arithmetic (mean of the asset prices) or geometric (exponential of the mean of the logarithm of the asset prices). The price of geometric average Asian option are analytically evaluated as the geometric average of the asset prices follows the lognormal distribution [9,16] whereas the distribution of arithmetic average of the asset prices is not explicitly known. Hence the analytical solution of arithmetic average Asian option is not known. The averaging can be weighted or unweighted. These options are among the most popular derivative securities as they reduce the risk of market manipulation by big and influential traders of the market near the maturity time. Also the average-value options are always economical than the standard European options [16].

Asian option in a complete and arbitrage free financial market consisting of a risky asset, with asset price S_τ , as the underlying security and a risk free asset with interest rate r , follows the following stochastic differential equation

$$dS_\tau = (r - D)S_\tau d\tau + \sigma dW_\tau, \quad (1)$$

where S_τ is the asset price at time τ , r is the risk free interest rate, D is the dividend yield, σ is the market volatility and W_τ is the standard Wiener process under the risk-neutral measure.

The payoff of European call option is

$$\max(S - K, 0). \tag{2}$$

The payoff of the “fixed strike” Asian call option is obtained by replacing S_τ in the payoff of European call option by the average of asset prices, keeping the strike price K fixed and is given as

$$\max\left(\frac{1}{T} \int_0^T S_z dz - K, 0\right). \tag{3}$$

Whereas the payoff of the “floating strike” Asian call option is obtained by replacing the strike price K in the payoff of European call option by the average of asset prices and given as

$$\max\left(S - \frac{1}{T} \int_0^T S_z dz, 0\right). \tag{4}$$

The closed form solution of Asian option pricing model is not known.

Asian options were first explained in [13]. We consider the arithmetic average fixed strike Asian option. The existence, uniqueness and regularity of solution of arithmetic and geometric average Asian options was discussed in [2]. The partial differential equation for pricing Asian options consists of Black-Scholes equation along with an advection term, which can not be reduced to heat equation and its closed form solution is not known. Different approaches were used to solve various models for pricing Asian options such as applied statistics [19,20,26], applied probability [24,25] and mathematical analysis [14,32]. Various numerical schemes are also applied to approximate the option price. The work [3] described reducing the dimension of the partial differential equation governing the option price and then solving it using finite difference scheme. An explicit and an implicit finite difference methods were described in [21] for pricing Asian options. A hybrid finite difference along with Crank-Nicolson method was used in [6] to get a second order convergent scheme for evaluating the option price. A radial basis function based finite difference approximation for spacial operator along with θ -method for temporal approximation was applied in [17] to obtain a second order convergent scheme for pricing the Asian option. In all these works where a numerical approximation was given for pricing option value, only non-dividend paying contract was considered. Hence after the dimension reducing transformation, the reaction term was not there in the partial differential equation to be solved numerically. Also the parameters σ and r were taken as constants. In the works [22,23], these parameters along with dividend yield were taken to be functions of asset price and time variables and numerical solution of such generalized Black-Scholes model in dividend paying market was computed

for different European style options. The generalizing of Black-Scholes model for pricing European call option by making the parameters r , σ and D variables was also done in [4, 8, 15, 27, 30] and then the option price was numerically approximated. The other related works were [11, 29].

In the present work we have developed a numerical scheme for a generalized Asian option pricing model, transformed from two dimensional partial differential equation to one dimensional partial differential equation. Two step backward differentiation formula for temporal discretization and the High Order Difference approximation with Identity Expansion (HODIE) scheme for spacial discretization is applied simultaneously yielding second order accuracy in both time and space.

Rest of the paper is organized as follows: Sect. 2 introduces the Asian Option pricing model and the applied modifications for solving it numerically. In Sect. 3, the discrete scheme is presented. In Sect. 4, the convergence of the approximate solution is proved. Section 5 gives the numerical experiments based on our scheme and Sect. 6 concludes the paper.

2 Asian Option Pricing Model

The European style Asian arithmetic average fixed strike call option price $V(S, A, \tau)$ is determined by [13, 31]

$$V(S, A, \tau) = \exp\left(-\int_{\tau}^T r(z)dz\right) v(S, A, \tau), \quad (5)$$

where the risk free interest rate $r(\tau)$ is taken as function of time variable, T is the maturity time of the contract, A is the continuous arithmetic running sum defined as

$$A(\tau) = \int_0^{\tau} S(z)dz \quad (6)$$

and $v(S, A, \tau)$ is governed by the following partial differential equation

$$\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2(\tau)S^2\frac{\partial^2 v}{\partial S^2} + (r(\tau) - D(\tau))S\frac{\partial v}{\partial S} + S\frac{\partial v}{\partial A} - r(\tau)v = 0, \quad (7)$$

along with the terminal condition

$$v(S, A, T) = \max\left(\frac{A}{T} - K, 0\right), \quad (8)$$

where $\sigma(\tau)$ and $D(\tau)$ are taken as functions of time, and K is the strike price. The current asset price S and the history of asset prices are independent. Hence the variables S , A and τ are independent variables. The Eq. (7) is a two dimensional ultra-parabolic partial differential equation which is numerically expensive to approximate. Inspired from Rogers and Shi [24] and Alziary, Décamps and Koehl [1], we use the following change of variables to transform the problem (7)–(8)

from final value problem to initial value problem and to reduce the dimension of the problem

$$\tilde{x} = \frac{K - A/T}{S}, \tau = T - t \text{ and } v(S, A, \tau) = S\tilde{u}(\tilde{x}, t), \tag{9}$$

where \tilde{x} and t are the new space and time variables respectively. Introducing the new variables

$$\begin{aligned} \hat{\sigma}(t) &= \sigma(\tau), \\ \hat{r}(t) &= r(\tau), \\ \hat{D}(t) &= D(\tau). \end{aligned}$$

The transformed problem is as follows

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{1}{2} \hat{\sigma}^2(t) \tilde{x}^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} - \left(\frac{1}{T} + (\hat{r}(t) - \hat{D}(t)) \tilde{x} \right) \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ &\quad - \hat{D}(t) \tilde{u}, \tilde{x} \in (0, \infty), t \in (0, T), \end{aligned} \tag{10}$$

along with the terminal condition

$$\tilde{u}(\tilde{x}, 0) = \max(-\tilde{x}, 0). \tag{11}$$

The boundary conditions for the problem (7)–(8) are well established in [12]. The left boundary condition is

$$v(0, A, \tau) = \exp\left(-\int_{\tau}^T r(z) dz\right) \max\left(\frac{A}{T} - K, 0\right). \tag{12}$$

The value of Asian call and put options are interrelated by the put-call parity. A closed form solution for the Asian option price for the case $A > KT$ is derived using the put-call parity [1, 10, 16], which is as follows

$$\begin{aligned} v(S, A, \tau) &= \left(\frac{A}{T} - K\right) \exp\left(-\int_{\tau}^T r(z) dz\right) \\ &\quad + \frac{S}{T} \int_{\tau}^T \exp\left(-\left(\int_{\tau}^y D(z) dz + \int_y^T r(z) dz\right)\right) dy, \tau \in [0, T]. \end{aligned} \tag{13}$$

Hence we will solve the problem (7)–(8) only for the case $A \leq KT$. Applying the transformations (9) to the formula (13), we get

$$\begin{aligned} \tilde{u}(\tilde{x}, t) &= -\tilde{x} \exp\left(-\int_0^t \hat{r}(z) dz\right) \\ &\quad + \frac{1}{T} \int_0^t \exp\left(-\left(\int_0^y \hat{r}(z) dz + \int_y^t \hat{D}(z) dz\right)\right) dy, \\ &\quad \text{for } A \leq KT, \tilde{x} \in [0, \infty), t \in [0, T]. \end{aligned} \tag{14}$$

The left boundary condition of the problem (10)–(11) is obtained from (14)

$$\tilde{u}(0, t) = \frac{1}{T} \int_0^t \exp \left(- \left(\int_0^y \hat{r}(z) dz + \int_y^t \hat{D}(z) dz \right) \right) dy, \tag{15}$$

for $A \leq KT, t \in [0, T]$.

The right boundary condition of the problem (10)–(11) for large \tilde{x} is obtained from (12)

$$\tilde{u}(\infty, t) = 0, \text{ for } A \leq KT, t \in [0, T]. \tag{16}$$

Let us introduce another change of variable

$$x = \exp(-\tilde{x}), \tag{17}$$

which converts the semi-infinite space domain $(0, \infty)$ into a finite space domain $(0, 1)$ [1]. Accordingly the partial differential equation (10) is transformed as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \hat{\sigma}^2(t) (\ln(x))^2 x^2 \frac{\partial^2 u}{\partial x^2} \\ &+ \left(\frac{1}{2} \hat{\sigma}^2(t) (\ln(x))^2 + \frac{1}{T} - (\hat{r}(t) - \hat{D}(t)) \ln(x) \right) x \frac{\partial u}{\partial x} - \hat{D}(t)u, \end{aligned} \tag{18}$$

and the initial and boundary conditions (11), (15) and (16) are transformed as

$$u(x, 0) = 0, x \in [0, 1], \tag{19}$$

$$u(1, t) = \frac{1}{T} \int_0^t \exp \left(- \left(\int_0^y \hat{r}(z) dz + \int_y^t \hat{D}(z) dz \right) \right) dy, \tag{20}$$

for $A \leq KT, t \in [0, T]$

and

$$u(0, t) = 0, \text{ for } A \leq KT, t \in [0, T] \tag{21}$$

respectively. Let $\Omega_x = (0, 1)$ be the space domain, $\Omega_t = (0, T)$ be the time domain and $\Omega = \Omega_x \times \Omega_t$. The final problem to be solved numerically is

$$Lu(x, t) \equiv \frac{\partial u}{\partial t} - a_2(x, t) \frac{\partial^2 u}{\partial x^2} - a_1(x, t) \frac{\partial u}{\partial x} - a_0(t)u = f(x, t), (x, t) \in \Omega, \tag{22a}$$

where

$$\begin{aligned} a_2(x, t) &= \frac{1}{2} \hat{\sigma}^2(t) (\ln(x))^2 x^2, \\ a_1(x, t) &= \left(\frac{1}{2} \hat{\sigma}^2(t) (\ln(x))^2 + \frac{1}{T} - (\hat{r}(t) - \hat{D}(t)) \ln(x) \right) x, \\ a_0(t) &= -\hat{D}(t), \\ f(x, t) &= 0, \end{aligned}$$

with the initial condition

$$u(x, 0) = 0, \quad x \in \bar{\Omega}_x, \tag{22b}$$

and the boundary conditions

$$u(0, t) = 0, \quad t \in \bar{\Omega}_t \tag{22c}$$

and

$$u(1, t) = \frac{1}{T} \int_0^t \exp \left(- \left(\int_0^y \hat{r}(z) dz + \int_y^t \hat{D}(z) dz \right) \right) dy, \quad t \in \bar{\Omega}_t. \tag{22d}$$

In a financial market, it is reasonable to assume that $\hat{r} \geq \hat{D}$ [28]. Also assume that

$$\left| \frac{\partial^{m+n} u}{\partial x^m \partial t^n}(x, t) \right| \leq \mathcal{C} \text{ on } \bar{\Omega}; \quad 0 \leq n \leq 3 \text{ and } 0 \leq m + n \leq 4. \tag{23}$$

Note that \mathcal{C} is a generic constant.

3 Discretization

Discretize the domain $\bar{\Omega}$ uniformly in both the space and time directions. Let the discrete space direction be $\bar{\Omega}_h = \{x_m | m = 0, 1, \dots, M\}$ where M is the total number of grid intervals in space direction with uniform spacing $h = x_m - x_{m-1}$, $m = 1, 2, \dots, M$. Let the discrete time direction be $\bar{\Omega}^k = \{t_n | n = 0, 1, \dots, N\}$ where N is the total number of grid intervals in time direction with uniform spacing $k = t_n - t_{n-1}$, $n = 1, 2, \dots, N$. Now the problem (22a–22d) is discretized simultaneously in both the space and time directions using two-step backward differentiation formula for temporal discretization and the High Order Difference approximation with Identity Expansion (HODIE) scheme with the stencil points $\{x_{m-1}, x_m, x_{m+1}, m = 1, 2, \dots, M - 1\}$ and two nodal auxiliary points $\{x_m, x_{m+1}, m = 1, 2, \dots, M - 1\}$, for the spacial discretization. The discretization of the partial differential equation (22a) is as follows

$$\beta_{m,c}^n (\delta_t U_m^n) + \beta_{m,+}^n (\delta_t U_{m+1}^n) + [\alpha_{m,-}^n U_{m-1}^n + \alpha_{m,c}^n U_m^n + \alpha_{m,+}^n U_{m+1}^n] = \beta_{m,c}^n f_m^n + \beta_{m,+}^n f_{m+1}^n, \quad m = 1, 2, \dots, M - 1, \quad n = 1, 2, \dots, N, \tag{24}$$

where α 's and β 's are the HODIE coefficients to be computed, U_m^n , $m = 0, 1, \dots, M$, $n = 0, 1, \dots, N$ is the numerical approximation of the solution $u(x, t)$ of the problem (22a–22d) at the grid point (x_m, t_n) , $m = 0, 1, \dots, M$, $n = 0, 1, \dots, N$ and the operator δ_t is defined as follows

$$\begin{aligned} \delta_t U_m^n &= (U_m^n - U_m^{n-1})/k, \quad n = 1, \\ \delta_t U_m^n &= \left(\frac{3}{2} U_m^n - 2 U_m^{n-1} + \frac{1}{2} U_m^{n-2} \right) /k, \quad n = 2, 3, \dots, N, \end{aligned}$$

We make the Eq. (24) exact on P_3 , the space of polynomials with degree less than or equal to 3 and use the following normalization condition

$$\beta_{m,c}^n + \beta_{m,+}^n = 1, \quad m = 1, 2, \dots, M-1, \quad n = 1, 2, \dots, N, \quad (25)$$

to compute the HODIE coefficients uniquely, which are as follows

$$\alpha_{m,-}^n = \frac{\beta_{m,c}^n(-2a_{2,m}^n + ha_{1,m}^n) + \beta_{m,+}^n(-2a_{2,m+1}^n - ha_{1,m+1}^n)}{2h^2}, \quad (26)$$

$$\alpha_{m,+}^n = \frac{\beta_{m,c}^n(-2a_{2,m}^n - ha_{1,m}^n) + \beta_{m,+}^n(-2a_{2,m+1}^n - 3ha_{1,m+1}^n - 2h^2a_0^n)}{2h^2}, \quad (27)$$

$$\alpha_{m,c}^n = \frac{\beta_{m,c}^n(4a_{2,m}^n - 2h^2a_0^n) + \beta_{m,+}^n(4a_{2,m+1}^n + 4ha_{1,m+1}^n)}{2h^2}, \quad (28)$$

$$\beta_{m,c}^n = \frac{6ha_{2,m+1}^n + 2h^2a_{1,m+1}^n}{6ha_{2,m+1}^n + 2h^2a_{1,m+1}^n + h^2a_{1,m}^n} \quad (29)$$

and

$$\beta_{m,+}^n = \frac{h^2a_{1,m}^n}{6ha_{2,m+1}^n + 2h^2a_{1,m+1}^n + h^2a_{1,m}^n}. \quad (30)$$

Now using these coefficients, the final scheme to compute the numerical approximation to the solution of the problem (22a–22d) is as follows

$$\begin{aligned} L_h^k U_m^n &\equiv \alpha_{m,-}^n U_{m-1}^n + \left(\alpha_{m,c}^n + \frac{1}{k} \beta_{m,c}^n \right) U_m^n + \left(\alpha_{m,+}^n + \frac{1}{k} \beta_{m,+}^n \right) U_{m+1}^n \\ &= \beta_{m,c}^n \left(f_m^n + \frac{1}{k} U_m^{n-1} \right) + \beta_{m,+}^n \left(f_{m+1}^n + \frac{1}{k} U_{m+1}^{n-1} \right) \\ &= F_m^n \text{ (say)}, \quad m = 1, 2, \dots, M-1, \quad n = 1, \end{aligned} \quad (31a)$$

$$\begin{aligned} L_h^k U_m^n &\equiv \alpha_{m,-}^n U_{m-1}^n + \left(\alpha_{m,c}^n + \frac{3}{2k} \beta_{m,c}^n \right) U_m^n + \left(\alpha_{m,+}^n + \frac{3}{2k} \beta_{m,+}^n \right) U_{m+1}^n \\ &= \beta_{m,c}^n \left(f_m^n + \frac{2}{k} U_m^{n-1} - \frac{1}{2k} U_m^{n-2} \right) + \beta_{m,+}^n \left(f_{m+1}^n + \frac{2}{k} U_{m+1}^{n-1} - \frac{1}{2k} U_{m+1}^{n-2} \right) \\ &= F_m^n \text{ (say)}, \quad m = 1, 2, \dots, M-1, \quad n = 2, 3, \dots, N, \end{aligned} \quad (31b)$$

$$U_m^0 = 0, \quad m = 0, 1, \dots, M, \quad (31c)$$

$$U_0^n = 0, \quad n = 0, 1, \dots, N. \quad (31d)$$

$$U_M^n = \frac{1}{T} \int_0^{t_n} \exp \left(- \left(\int_0^y \hat{r}(z) dz + \int_y^{t_n} \hat{D}(z) dz \right) \right) dy, \quad n = 0, 1, \dots, N, \quad (31e)$$

Now putting $m = 1, 2, \dots, M$ in (31a) and (31b) and shifting the boundary components to the right side of the equation, we can write the scheme in the matrix form as

$$A^n U^n = b^n, \quad n = 1, 2, \dots, N, \tag{32}$$

where $U^n = [U_1^n, U_2^n, \dots, U_{M-1}^n]^T$.

4 Error Analysis

Lemma 1. Assume that

$$h\beta_{m,c}^n a_{1,m}^n \leq 2(\beta_{m,c}^n a_{2,m}^n + \beta_{m,+}^n a_{2,m+1}^n) + h\beta_{m,+}^n a_{1,m+1}^n, \tag{33}$$

$m = 1, 2, \dots, M - 1, \quad n = 1, 2, \dots, N,$

and

$$\beta_{m,+}^n \left(-a_0^n + \frac{3}{2k} \right) \leq \frac{2(\beta_{m,c}^n a_{2,m}^n + \beta_{m,+}^n a_{2,m+1}^n) + h(\beta_{m,c}^n a_{1,m}^n + 3\beta_{m,+}^n a_{1,m+1}^n)}{2h^2}, \tag{34}$$

$m = 1, 2, \dots, M - 1, \quad n = 2, 3, \dots, N.$

then

$$\alpha_{m,-}^n \leq 0, \quad m = 1, 2, \dots, M - 1, \quad n = 1, 2, \dots, N, \tag{35}$$

$$\alpha_{m,+}^n + \frac{3}{2k}\beta_{m,+}^n \leq 0, \quad m = 1, 2, \dots, M - 1, \quad n = 2, 3, \dots, N. \tag{36}$$

Proof. The inequalities (35) and (36) can be easily obtained from (26), (27), (33) and (34).

Remark 1. Similarly assume that

$$\beta_{m,+}^n \left(-a_0^n + \frac{1}{k} \right) \leq \frac{2(\beta_{m,c}^n a_{2,m}^n + \beta_{m,+}^n a_{2,m+1}^n) + h(\beta_{m,c}^n a_{1,m}^n + 3\beta_{m,+}^n a_{1,m+1}^n)}{2h^2}, \tag{37}$$

$m = 1, 2, \dots, M - 1, \quad n = 1,$

then

$$\alpha_{m,+}^n + \frac{1}{k}\beta_{m,+}^n \leq 0, \quad m = 1, 2, \dots, M - 1, \quad n = 1. \tag{38}$$

Lemma 2. (Discrete Maximum Principle). Under the assumptions of the Lemma 1, the operator L_h^k defined by (31a)–(31b) satisfies discrete maximum principle, that is, if v_m^n and w_m^n are mesh functions that satisfy $v_0^n \leq w_0^n, v_M^n \leq w_M^n$ ($n = 0, 1, \dots, N$), $v_m^0 \leq w_m^0, (m = 0, 1, \dots, M)$ and $L_h^k v_m^n \leq L_h^k w_m^n$ ($m = 1, 2, \dots, M - 1, n = 1, 2, \dots, N$), then $v_m^n \leq w_m^n$ for all m, n .

Proof. The row sum of the matrices $A^n, n = 1, 2, \dots, N$, are

$$\alpha_{m,-}^n + \left(\alpha_{m,c}^n + \frac{3}{2k}\beta_{m,c}^n \right) + \left(\alpha_{m,+}^n + \frac{3}{2k}\beta_{m,+}^n \right) = -a_0^n + \frac{3}{2k} \tag{39}$$

$$> 0, \tag{40}$$

$m = 1, 2, \dots, M - 1, n = 1, 2, \dots, N$. $\alpha_{m,-}^n, m = 1, 2, \dots, M - 1, n = 1, 2, \dots, N$ are the sub-diagonal elements and $\alpha_{m,+}^n + \frac{3}{2k}\beta_{m,+}^n, m = 1, 2, \dots, M - 1, n = 1, 2, \dots, N$ are the super-diagonal elements of the tridiagonal matrices $A^n, n = 1, 2, \dots, N$. Under the hypothesis of the Lemma 1, the inequalities (35)–(36) and (39) show that the coefficient matrices corresponding to the discrete operator L_h^k are irreducible M-matrices which preserve the positivity. Hence the solution to the linear system of equations (32) exists and if v_m^n and w_m^n are mesh functions as defined in Lemma, then $v_m^n \leq w_m^n$ for all m, n .

The operator L_h^k , satisfying the discrete maximum principle, establishes the stability of the scheme.

Theorem 1. *Let u be the solution of the continuous problem (22a–22d) and U_m^n be the solution of the discrete problem (31a–31e). Then under the assumptions (23),*

$$|u_m^n - U_m^n| \leq C(k^2 + h^2), \quad m = 0, 1, \dots, M, \quad n = 0, 1, \dots, N, \tag{41}$$

where C is a positive constant independent of k and h .

Proof. From (24) we have

$$\begin{aligned} L_h^k u_m^n - (Lu)_m^n &= \frac{\beta_{m,c}^n}{k} \left(\frac{3}{2}u_m^n - 2u_{m-1}^{n-1} + \frac{1}{2}u_{m-2}^{n-2} \right) \\ &+ \frac{\beta_{m,+}^n}{k} \left(\frac{3}{2}u_{m+1}^n - 2u_{m+1}^{n-1} + \frac{1}{2}u_{m+1}^{n-2} \right) + [\alpha_{m,-}^n u_{m-1}^n + \alpha_{m,c}^n u_m^n + \alpha_{m,+}^n u_{m+1}^n] \\ &- \beta_{m,c}^n \left(\frac{\partial u_m^n}{\partial t} - a_{2,m}^n \frac{\partial^2 u_m^n}{\partial x^2} - a_{1,m}^n \frac{\partial u_m^n}{\partial x} - a_0^n u_m^n \right) \\ &- \beta_{m,+}^n \left(\frac{\partial u_{m+1}^n}{\partial t} - a_{2,m+1}^n \frac{\partial^2 u_{m+1}^n}{\partial x^2} - a_{1,m+1}^n \frac{\partial u_{m+1}^n}{\partial x} - a_0^n u_{m+1}^n \right). \end{aligned} \tag{42}$$

Applying the Taylor’s expansion in two variables and taking modulus, we have

$$\begin{aligned} |L_h^k(u_m^n - U_m^n)| &= |L_h^k u_m^n - (Lu)_m^n| \\ &\leq C_1 k^2 \int_0^1 \left[\left| \frac{\partial^3 u}{\partial t^3}(x_m, t_n - ky) \right| + \left| \frac{\partial^3 u}{\partial t^3}(x_m, t_n - 2ky) \right| \right. \\ &\quad \left. + \left| \frac{\partial^3 u}{\partial t^3}(x_m + hy, t_n - ky) \right| + \left| \frac{\partial^3 u}{\partial t^3}(x_m + hy, t_n - 2ky) \right| \right] dy \\ &+ C_2 h^2 \int_0^1 \left[\left| \frac{\partial^3 u}{\partial x^2 \partial t}(x_m + hy, t_n - ky) \right| + \left| \frac{\partial^3 u}{\partial x^2 \partial t}(x_m + hy, t_n - 2ky) \right| \right. \\ &\quad \left. + \left| \frac{\partial^3 u}{\partial x^2 \partial t}(x_m + hy, t_n) \right| + x_{m+1}^2 \left| \frac{\partial^4 u}{\partial x^4}(x_m + hy, t_n) \right| \right] dy \\ &\leq C_3(k^2 + h^2), \quad m = 1, 2, \dots, M - 1, \quad n = 2, 3, \dots, N. \end{aligned} \tag{43}$$

For $n = 1$, backward Euler method is used for the computation of the discrete solution, and the one-step error of the backward Euler method is $O(k^2)$ [18]. Hence we have

$$|L_h^k(u_m^n - U_m^n)| \leq C_4(k^2 + h^2), \quad m = 1, 2, \dots, M - 1, \quad n = 0, 1, \dots, N. \quad (44)$$

We construct the following barrier function

$$\varphi_m^n = C(k^2 + h^2)(1 + t_n) \pm (u_m^n - U_m^n),$$

and apply the Lemma 2 on φ_m^n , to show

$$|u_m^n - U_m^n| \leq C(k^2 + h^2) \text{ for every } m = 1, 2, \dots, M - 1, \quad n = 1, 2, \dots, N. \quad (45)$$

5 Numerical Experiments

Since the closed form solution of the considered Asian option pricing model is not known, we use double mesh principle to find the maximum error (E_{max}), the root mean square error (E_{rms}) and the corresponding order of convergence p_{max} and p_{rms} as follows

$$E_{\max}^{M,N} = \max_{0 \leq m \leq M} |U^{M,N}(x_m, t_N) - U^{2M,2N}(x_{2m}, t_{2N})|,$$

$$E_{\text{rms}}^{M,N} = \sqrt{\frac{\sum_{m=0}^M [(U^{M,N}(x_m, t_N) - U^{2M,2N}(x_{2m}, t_{2N}))^2]}{M + 1}},$$

$$p_{\max}^{M,N} = \log_2 \left(\frac{E_{\max}^{M,N}}{E_{\max}^{2M,2N}} \right) \text{ and } p_{\text{rms}}^{M,N} = \log_2 \left(\frac{E_{\text{rms}}^{M,N}}{E_{\text{rms}}^{2M,2N}} \right).$$

Example 1. Consider the initial boundary value problem (22a–22d) with $\hat{\sigma} = 0.5$, $\hat{r} = 0.09$, $\hat{D} = 0$. Take $T = 3$ and $K = 40$. The computed solution is shown in Fig. 1 and the convergence results are given in Table 1.

Example 2. Consider the initial boundary value problem (22a–22d) with $\hat{\sigma}(t) = 0.4(2 + (T - t))$, $\hat{r}(t) = 0.06(1 + t)$, $\hat{D}(t) = 0.02 \exp(-t)$. Take $T = 1$ and $K = 40$. The computed solution is shown in Fig. 2 and the convergence results are given in Table 2.

Example 3. Consider the initial boundary value problem (22a–22d) with $\hat{\sigma}(t) = 0.4(2 + \sin(T - t))$, $\hat{r}(t) = 0.06 \exp(t)$, $\hat{D}(t) = 0.02 \sin(t)$. Take $T = 1$ and $K = 40$. The computed solution is shown in Fig. 3 and the convergence results are given in Table 3.

Table 1. Maximum absolute error (E_{\max}), root mean square error (E_{rms}) and corresponding orders of convergence p_{\max} and p_{rms} for Example 1

M	10	10×2	10×2^2	10×2^3	10×2^4	10×2^5
N	6	6×2	6×2^2	6×2^3	6×2^4	6×2^5
E_{\max}		1.4876e-02	4.3894e-03	1.1038e-03	2.6960e-04	6.6220e-05
p_{\max}			1.7610	1.9915	2.0337	2.0255
E_{rms}		7.8685e-03	2.1324e-03	4.7253e-04	1.1057e-04	2.6846e-05
p_{rms}			1.8835	2.1740	2.0953	2.0422

Table 2. Maximum absolute error (E_{\max}), root mean square error (E_{rms}) and corresponding orders of convergence p_{\max} and p_{rms} for Example 2

M	8	8×2	8×2^2	8×2^3	8×2^4	8×2^5
N	2	2×2	2×2^2	2×2^3	2×2^4	2×2^5
E_{\max}		6.2322e-02	2.6239e-02	8.4130e-03	2.1647e-03	5.1478e-04
p_{\max}			1.2480	1.6411	1.9584	2.0722
E_{rms}		3.7783e-02	1.4681e-02	4.4301e-03	9.9501e-04	2.2237e-04
p_{rms}			1.3638	1.7285	2.1546	2.1617

Table 3. Maximum absolute error (E_{\max}), root mean square error (E_{rms}) and corresponding orders of convergence p_{\max} and p_{rms} for Example 3

M	8	8×2	8×2^2	8×2^3	8×2^4	8×2^5
N	2	2×2	2×2^2	2×2^3	2×2^4	2×2^5
E_{\max}		6.2810e-02	2.6458e-02	8.5204e-03	2.1947e-03	5.2182e-04
p_{\max}			1.2473	1.6347	1.9569	2.0724
E_{rms}		3.7829e-02	1.4825e-02	4.4948e-03	1.0102e-03	2.2545e-04
p_{rms}			1.3515	1.7217	2.1535	2.1639

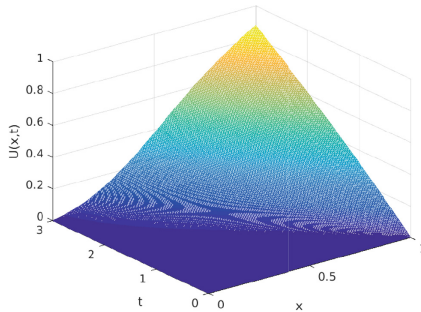


Fig. 1. Computed solution of Asian option price for Example 1

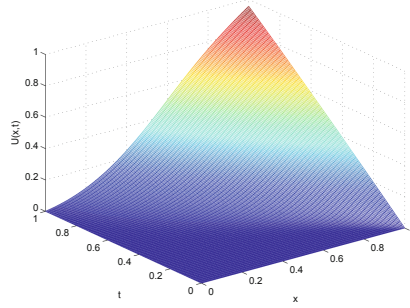


Fig. 2. Computed solution of Asian option price for Example 2

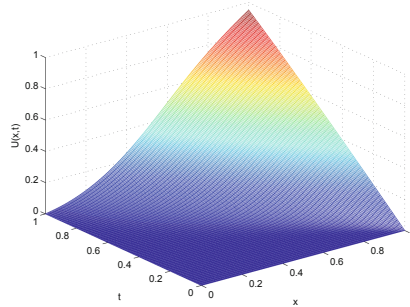


Fig. 3. Computed solution of Asian option price for Example 3

6 Conclusions

In the present work, HODIE scheme with two auxiliary points is used to achieve second order accuracy in space and two-step backward differentiation formula is applied for temporal approximation yielding second order accuracy in time also. The main features of our scheme is that it easily deals with variable coefficients occurring in the partial differential equation as we have taken volatility (σ), risk-free interest rate (r) and dividend yield (D) not as constants but as function of the time variable, which is more likely to happen in real financial market. Also this scheme can be applied simultaneously in space and time directions which leads to simpler analysis of the convergence of the solution [7]. It deals with the degeneracy issue of this problem easily without any extra efforts unlike in other works done in this field [5,6,33]. The numerical results are in accordance with the theoretical results.

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