Distant Neighbors and Interscalar Contiguities

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Abstract. This paper studies the "integration" problem of nineteenth-century harmony—the question whether the novel chromatic chord transitions in this time are a radical break from or a natural extension of the conventional diatonic system. We examine the connections between the local behavior of voice leading among diatonic triads and their generalizations on one hand, and the global properties of voice-leading spaces on the other. In particular, we aim to identify those neo-Riemannian chord connections which can be integrated into the diatonic system and those which cannot. Starting from Jack Douthett's approach of filtered point symmetries, we generalize diatonic triads as second-order Clough-Myerson scales and compare the resulting Douthett graph to the respective Betweenness graph. This paper generally strengthens the integrationist position, for example by presenting a construction of the hexatonic and octatonic cycles that uses the principle of minimal voice leading in the diatonic system. At the same time it provides a method to detect chromatic wormholes, i.e. parsimonious connections between diatonic chords, which are not contiguous in the system of second order Clough-Myerson scales.

Keywords: Diatonic theory · Hexatonic cycles · Neo-Riemannian transformations · Maximally even scales · Voice-leading parsimony

1 Introduction

Music exhibits complex structures across its various dimensions such as harmony, voice-leading, and rhythm. The structural relations are subject to cultural evolution observable in the historic development of music. A famous transition in

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harmony took place in the Western classical music of the 19th century when composers started to use triads and seventh-chords in novel chromatic connections that exceed the previous diatonic usage [6,10,16]. An ongoing debate concerns the nature of this transition, called the "integration" problem [18]: Was the novel usage of the chords a radical break from or a natural extension of conventional tonal harmony? This paper aims at supporting the latter position by reconstructing the advanced harmonic structures of late 19th century music—also found in Jazz—in a generalized diatonic framework. We draw from definitions and results of neo-Riemannian theory, mathematical scale theory, voice-leading spaces, and Fourier analysis and reveal connections between these approaches.

As a means of precise formalization, mathematical music theory comprises descriptions and propositions of musical objects and their relations. The strands of this discipline originated and evolved independently, concentrating on the characterization of particular musical phenomena. Many proposals have for example been made to explain the musical relevance of the triad from various different angles. These include its status as a building block of a three-triad-system in Hauptmann's concept of the major and minor keys [12], its voice-leading parsimony in connection to other triads [5,8], its status as a second-order maximally even set [4], its transformational stability properties as a pitch class set [13], and its position as a neighbor of the singularity on the orbifold of 3-chords [17]. The logical dependencies between these characterizations triggered an exciting process of integration between previously separated lines of research and constitute ongoing debates to date. The existence of structural connections between the different approaches would moreover have an interesting interpretation concerning the development of tonality in Western music of the 19th century: It would suggest that Romantic composers gradually widened the diatonic usage of triads and seventh-chords to explore a complex space of extended tonality. The contrary position argues for several independent utilizations of the triad as the basic building block of tonality. For instance, Richard Cohn interprets the "over-determined triad" [5,6] as the seed for the destruction of the tonal system from within and emphasizes triads as inhabitants of the chromatic system, released from their diatonic affiliations. The voice-leading connections within a hexatonic cycle, Cohn's paradigmatic example for the disengagement of the triad from diatonic control, exemplifies two distinct types of diatonically motivated chord connections. In the hexatonic cycle

$$\mathrm{C} \xmapsto{P} \mathrm{Cm} \xmapsto{L} \mathrm{A}\flat \xmapsto{P} \mathrm{A}\flat\mathrm{m} \xmapsto{L} \mathrm{E} \xmapsto{P} \mathrm{Em} \xmapsto{L} \mathrm{C},$$

the neo-Riemannian operations P and L are alternately applied to the respective triads in 12-tone equal temperament. The *leading-tone exchange* L is a parsimonious diatonic connection (e.g. $\mathrm{C} \mapsto \mathrm{Em}$). The *parallel* transformation P can be interpreted as the consequence of an alteration of the underlying scale ($\flat$ or $\sharp$, e.g. $\mathrm{Em} \mapsto \mathrm{E}$).

The present article examines the connections between the local voice-leading behavior of diatonic triads and their generalizations on one hand, and the global properties of voice-leading graphs on the other. We thereby identify those neo-Riemannian chord connections which can be integrated into our generalization of the diatonic system and those which cannot. John Clough, Jack Douthett and their co-authors made significant contributions to both local and global aspects, which we take as our points of departure. Main results with respect to characterizing properties of diatonic sets were achieved in their joint paper on *Maximally Even Sets* [4]. Pathbreaking for the understanding of voice-leading graphs was the joint paper of Douthett with Peter Steinbach on *parsimonious graphs* [8] using neo-Riemannian approaches. Programmatic steps towards a systematic combination of both aspects can be found in Douthett's approach of filtered point symmetry and dynamical voice leading [7].

2 Triads as Second-Order Clough-Myerson Scales

The goal of this section is to generalize diatonic triads and their PLR transformations in order to investigate the generalized triadic connections in the consequent sections.

Definition 1 (Clough-Myerson Scale). *A* (first-order) Clough-Myerson scale *is a maximally even scale of which its chromatic and diatonic cardinalities are co-prime.*

Notably, all Clough-Myerson scales are non-degenerate well-formed [2,3].[1] Each such scale is thus uniquely determined (up to chromatic transposition) by its cardinality d and the cardinality c of its ambient chromatic scale ($d < c$). Within the chromatic scale $\mathbb{Z}_c$, we have $d \cdot c$ different modes of the c scale transpositions. Each mode is given by one instance of Clough and Douthett's J-function $J^m_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$, where the mode index m varies from 0 to $d \cdot c - 1$. Note that in this terminology, modes have more structure than scales, as they are scales with a distinguished root tone just like C Ionian, F Lydian, and E Phrygian are modes of the diatonic scale $\{C, D, E, F, G, A, B\}$. The specific pitch class of scale degree k of the mode with index m is given by

$$J^m_{c,d}(k) := \left\lfloor \frac{ck + m}{d} \right\rfloor \bmod c.$$

Clough-Myerson scales are well-formedly generated by the specific interval $\bar{d}$, which is the multiplicative inverse of the scale cardinality d modulo the chromatic cardinality c, $\bar{d}d = 1 \bmod c$. The following proposition shows how each Clough-Myerson mode can be expressed in generation order.

Proposition 1. *Each mode first-order Clough-Myerson mode $J^m_{c,d}$ can be expressed in generation order by virtue of the map $G^m_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$ where*

$$G^m_{c,d}(k) := \bar{d}(m - k) \ mod \ c.$$

[1] Clough and Meyerson [3] show that Clough-Myerson scales have Myhill's property, i.e. every non-zero generic interval appears in two species. Carey and Clampitt [2] show that Myhill's property is equivalent to non-degenerate well-formedness.

Proof. For each mode index m, the well-formedness property can be expressed by a commutative triangle as follows.

$$\begin{tikzcd}
\mathbb{Z}_d \arrow[dr, "J^m_{c,d}"] \arrow[dd, shift right, "f^{-1}"'] & \\
& \mathbb{Z}_c \\
\mathbb{Z}_d \arrow[ur, "G^m_{c,d}"'] \arrow[uu, shift right, "f"'] &
\end{tikzcd}$$

The affine automorphism $f : \mathbb{Z}_d \to \mathbb{Z}_d$ and its inverse f^{-1} are given by $f(k) = g(m-k) \bmod d$ and $f^{-1}(k) = m - wk \bmod d$. The linear factor $g \in \mathbb{Z}_d$ in the automorphism f is the residue class mod d of the step span (= generic size) of the generator and can be calculated as $\left\lfloor \frac{d\bar{d}}{c} \right\rfloor$. The inverse linear factor $w = g^{-1} \in \mathbb{Z}_d$ (in the inverse automorphism f^{-1}) is the winding number of the scale, i.e. the number of covered octave ambits in the scale generation. Obviously, the formulae represent mutually inverse morphisms: $f^{-1}(f((k)) = m - wg(m-k) = m - m + k = k$. We check the commutativity of the diagram by verifying that $J^m_{c,d} \circ f = G^m_{c,d}$ for $m \in \{0, ..., d \cdot c - 1\}$ and $k \in \{0, ..., d-1\}$. This follows from the following equalities in $\mathbb{N}$:

$$\left\lfloor \frac{c \left\lfloor \frac{d\bar{d}}{c} \right\rfloor (m-k) + m}{d} \right\rfloor = \left\lfloor \frac{(d\bar{d}-1)(m-k)+m}{d} \right\rfloor = \left\lfloor \bar{d}(m-k) + \frac{k}{d} \right\rfloor = \bar{d}(m-k).$$

Definition 2. *Consider three natural numbers* $0 < e < d < c$ *such that the greatest common divisors* $(e,d) = (d,c) = 1$. *A* second-order Clough-Myerson mode $J^{m,n}_{c,d,e} := J^m_{c,d} \circ J^n_{d,e} : \mathbb{Z}_e \to \mathbb{Z}_c$ *is defined as the concatenation of the two Clough-Myerson modes* $J^n_{d,e} : \mathbb{Z}_e \to \mathbb{Z}_d$ *and* $J^m_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$.

3 Diatonic Contiguity and Its Violation

Observation. Whenever there is a most parsimonious connection between two diatonic triads (one voice moves by one semitone), they either belong to a common diatonic collection (leading tone exchange) or they belong to a pair of neighboring diatonic collections along the circle of fifths (parallel transformation, e.g. C in F major to Cm in B♭ major). This shall be called the property of *diatonic contiguity.*

The observation is relevant, because its general formulation would strengthen the integrationist position of Yust [18] as described in Sect. 1. In the controversy on the autonomy or the diatonic dependency of triads, it is therefore interesting to know whether this property holds for a broader family of second-order Clough-Meyerson scales. The contiguity notably does not hold for all configurations of second-order Clough-Myerson scales. The following is a counter-example.

We consider a generic seven-note scale $\mathbb{Z}_7$ in the role of the generalized chromatic space and use the note names C, D, E, F, G, A, and B. Therein, we have the parsimonious cycle of the fourth-generated pentatonic scales and within each scale we have the complete parsimonious cycle of all five chords with four tones. In other words, we consider the second-order Clough-Myerson scales for $c = 7$, $d = 5$, and $e = 4$. In each of the seven 4-chord cycles, there is exactly one diatonic-seventh chord being maximally even in $\mathbb{Z}_7$. These diatonic seventh chords violate the—in this case—“pentatonic” contiguity. They have most parsimonious voice leading (one voice moves by one diatonic step), but the pentatonic distance (generalized key distance) between them is 3, as shown in the array below using the Manhatten (or taxicab) distance. Consider for instance the seventh-chords $\mathrm{F}^{\mathrm{maj7}}$ and Dm^7. They have a generalized chromatic distance of 1 implying that they are most parsimonious, but the pentatonic scales in which they occur ($\{C, E, F, A, B\}$ and $\{C, D, F, G, A\}$) are not neighboring scales.

$$
\begin{array}{ll}
\{C,D,F,G,A\}: & \{C,D,F,G\}, \underline{\{C,D,F,A\}}, \{C,D,G,A\}, \{C,F,G,A\}, \{D,F,G,A\} \\
\{C,D,E,G,A\}: & \{C,D,E,G\}, \{C,D,E,A\}, \{C,D,G,A\}, \underline{\{C,E,G,A\}}, \{D,E,G,A\} \\
\{D,E,G,A,B\}: & \underline{\{D,E,G,B\}}, \{D,E,A,B\}, \{D,G,A,B\}, \{E,G,A,B\}, \{E,G,A,B\} \\
\{D,E,F,A,B\}: & \{D,E,F,B\}, \{D,E,A,B\}, \underline{\{D,F,A,B\}}, \{E,F,A,B\}, \{D,E,F,A\} \\
\{C,E,F,A,B\}: & \{C,E,F,B\}, \{C,E,A,B\}, \{C,F,A,B\}, \{E,F,A,B\}, \underline{\{C,E,F,A\}} \\
\{C,E,F,G,B\}: & \{C,E,F,B\}, \underline{\{C,E,G,B\}}, \{C,F,G,B\}, \{E,F,G,B\}, \{C,E,F,G\} \\
\{C,D,F,G,B\}: & \{C,D,F,B\}, \{C,D,G,B\}, \{C,F,G,B\}, \underline{\{D,F,G,B\}}, \{C,D,F,G\}.
\end{array}
$$

The construction used to derive the counter-example can be generalized to arbitrary second-order Clough-Myerson configurations using the concept of Douthett graphs, similar to coordinate spaces [14,15] or configuration spaces [19].

Definition 3 (Douthett Graph). *For given cardinalities* $0 < e < d < c$*, the* Douthett graph $\mathcal{D}_{c,d,e}$ *has second-order Clough-Myerson scales as nodes, and edges between any two* intrascalar neighbors $J^m_{c,d}(J^{n+1}_{d,e}(\mathbb{Z}_e))$ *and* $J^m_{c,d}(J^n_{d,e}(\mathbb{Z}_e))$*, as well as between any* neighbors under chromatic alteration*. This is whenever* $J^{m+1}_{c,d}(J^n_{d,e}(\mathbb{Z}_e)) \neq J^m_{c,d}(J^n_{d,e}(\mathbb{Z}_e))$.

Intrascalar neighbors generalize chords that are related by the neo-Riemannian transformations L and R while neighbors under chromatic alteration generalize chords that are related by the transformation P.[2]

4 Distant Neighbors and Interscalar Contiguities

This section compares Douthett graphs with their corresponding betweenness graphs using the concepts of distant (intrascalar) neighbors and interscalar

[2] Here we have to dispense with a transformational treatment of chord progressions. The Douthett graph for the diatonic triads contains the chicken-wire graph (being the Cayley-graph of the action of the Schritt-Wechsel group on the 24 major and minor triads with respect to the generators P, L and R). The diminished triads and the corresponding voice leading connections are not included.

contiguities as defined below. Figure 1 (top) shows the Douthett graph $\mathcal{D}_{12,7,3}$ of the major, minor, and diminished triads in 12-tone equal temperament. Edges are colored black if they are also edges of the corresponding betweenness graph (see below) or colored orange otherwise. The chords of the diatonic scales go from left to right. For example, the triads of the C major scale Em, C, Am, F, Dm, Bdim, and G form an intrascalar (in this case diatonic) voice-leading cycle of the graph. The chords C and Cm, represented by $\{0, 4, 7\}$ and $\{0, 3, 7\}$, are an example of neighbors under chromatic alteration. The following concept of voice-leading distance (also known as taxicab metric [17] or voice-leading work [6]) is used to consequently define betweenness graphs.

Definition 4 (Voice-Leading Distance). *For each $c \in \mathbb{N}$, the generalized chromatic scale $\mathbb{N}_c = \{0, 1, \ldots, c-1\}$ (here used as a set of integers) forms a metric space together with the* Lee distance

$$d(x, y) = \min(|x - y|, c - |y - x|),$$

where $x, y \in \mathbb{N}_c$. The (minimal) voice-leading distance *between to chords (or scales) $X, Y \subseteq \mathbb{N}_c$ is then defined as the summed movement of a minimal bijection,*

$$D(X, Y) = \min_{f : X \xrightarrow{\cong} Y} \sum_{x \in X} d(x, f(x)).$$

A minimal bijection is also called minimal voice leading *from X to Y.*

Note that the voice-leading distance is a metric on any set of equally sized chords. There is in particular always a minimal voice leading which fixes all notes in the intersection of the chords. For proofs and examples see [11].

Definition 5 (Betweenness Graph). *A chord $Y \subseteq \mathbb{Z}_c$ is* in between *two chords $X, Z \subseteq \mathbb{Z}_c$ if $D(X, Z) = D(X, Y) + D(Y, Z)$. The betweenness graph of a set of equally sized chords $\mathcal{X} \subseteq 2^{\mathbb{N}_c}$ (where $2^{\mathbb{N}_c}$ denotes the powerset of $\mathbb{N}_c$) has an edge between two chords X and Z iff there is no other chord in between them. For given cardinalities $0 < e < d < c$, the betweenness graph of second-order Clough-Myerson scales is denoted by $\mathcal{B}_{c,d,e}$.*

Figure 1 (bottom) shows the betweenness graph $\mathcal{B}_{12,7,3}$ that corresponds to the Douthett graph $\mathcal{D}_{12,7,3}$ shown in Fig. 1 (top). In Fig. 1 (bottom), edges are colored black if they are also edges of $\mathcal{D}_{12,7,3}$ or colored orange otherwise. Since the chord pairs Dm and Bdim as well as Bdim and G are not connected directly, but through the out-of-scale chords B♭ respectively Bm, the triads of the C major scale form a different cycle in the betweenness graph than the intrascalar voice-leading cycle in the Douthett graph $\mathcal{D}_{12,7,3}$.

Douthett graphs focus on the local aspect of voice-leading transformations. From the definition, the path distance of two given chords cannot be obtained directly, but for any given chord, its neighbors can be accessed directly. In contrast, betweenness graphs focus on the global aspect of minimal voice-leading.

The voice-leading distance of two given chords can be calculated using the definition, but it is not straightforward to decide if two chords are adjacent in the betweenness graph. In general, Douthett graphs and betweenness graphs have overlapping edges, but neither of them is a subgraph of the other. To compare them, we name edges that are in the Douthett graph $\mathcal{D}_{c,d,e}$, but not in the betweenness graph $\mathcal{B}_{c,d,e}$ and vice versa.

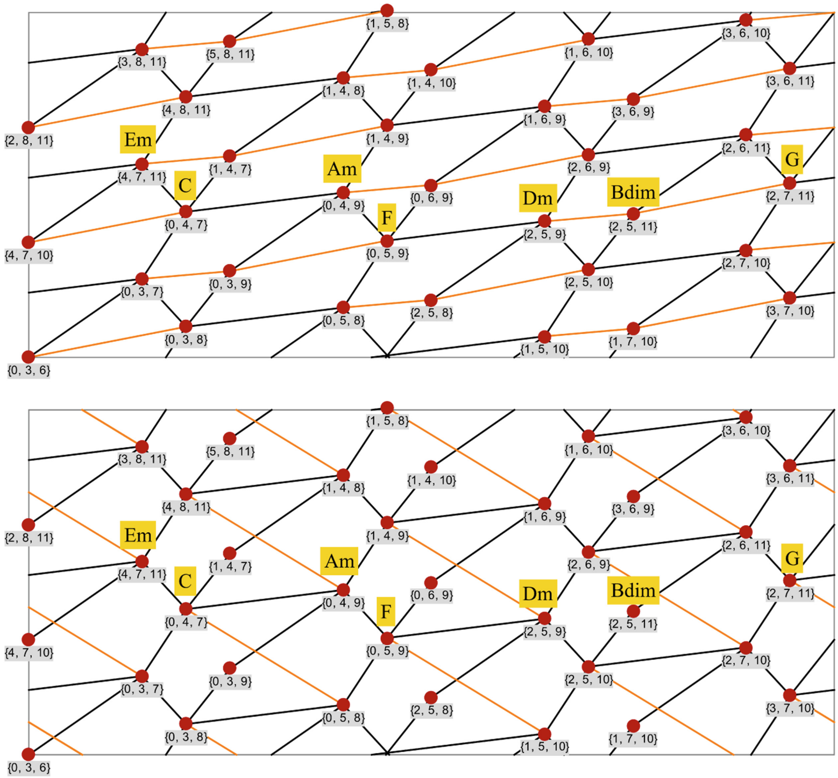

Fig. 1. Douthett and Betweenness graphs for $c = 12, d = 7$ and $e = 3$. (Color figure online)

Definition 6 (Distant Neighbors). *Edges of a Douthett graph $\mathcal{D}_{c,d,e}$ which are not edges of the corresponding betweenness graph $\mathcal{B}_{c,d,e}$ are called* distant (intrascalar) neighbors.

The orange edges in Fig. 1 (top) show the distant neighbors of the second-order Clough-Myerson scales with cardinalities $c = 12, d = 7$ and $e = 3$. Consider for example the chords Dm and Bdim. They are both chords in the C major scale, but not adjacent in the betweenness graph $\mathcal{B}_{12,7,3}$ shown in Fig. 1 (bottom), because the chord B♭ (represented by $\{2, 5, 10\}$) is located between them with respect to the voice-leading distance.

Definition 7 (Interscalar Contiguities). *Edges of the betweenness graph $\mathcal{B}_{c,d,e}$, which are not edges of the Douthett graph $\mathcal{D}_{c,d,e}$ are called* interscalar contiguities*.*

The orange edges in Fig. 1 (bottom) show the interscalar contiguities of the second-order Clough-Myerson scales with cardinalities $c = 12, d = 7$ and $e = 3$. Consider for example the chords Cm and G, represented by $\{0, 3, 7\}$ and $\{2, 7, 11\}$, respectively. Since there is no triad Y such that $D(Cm, G) = D(Cm, Y) + D(Y, G)$, Cm and G are adjacent in the betweenness graph $\mathcal{B}_{12,7,3}$, but they are neither intrascalar neighbors nor neighbors under chromatic alteration.

5 Path Characterization of Hexatonic and Octatonic Cycles in Betweenness Graphs

As described in the introduction, hexatonic cycles are commonly defined using the neo-Riemannian transformations L and P in the Douthett graph $\mathcal{D}_{12,7,3}$ of diatonic triads. In the corresponding betweenness graph $\mathcal{B}_{12,7,3}$, hexatonic cycles can be characterized as cycles of toggling voice leadings with voice-leading distance 1 (zigzag voice leadings with alternating voice-movement direction). In the cycle of C, Cm, A♭, A♭m, E, and Em for example, the tone E is moving *down* to E♭, G is moving *up* to A♭, C is moving *down* to C♭, etc. The characterization in terms of toggling voice leadings is, in particular, independent of the inner structure of the chords. It is an outer characterization that can be applied to any space of equally sized chords. For example in the case of 12-7-4 second-order Clough-Myerson scales (diatonic seventh-chords), the cycles of toggling voice leadings with voice-leading distance 1 are exactly the octatonic cycles such as C^7 Am^7 A^7 F♯m^7 F♯^{7} E♭m^7 E♭^{7} Cm^7 C^7. In general, however, toggling voice-leading cycles with voice-leading distance 1 do not always exists in second-order Clough-Myerson spaces. The betweenness graph $\mathcal{B}_{17,8,3}$ does for instance not contain any.

The observed characterization of hexatonic and octatonic cycles moreover allows for their construction using solely the principle of minimal voice leading and the set of diatonic triads and seventh-chords, respectively. If taken as a definition of a *generalized hexatonic cycle*, this characterization allows for the investigation of sufficient and necessary conditions for Clough-Myerson configurations *c*-*d*-*e* to have a generalized hexatonic.

6 Chromatic Saturation

Distant intrascalar neighbors occur when a scale-external chord sneaks in between neighbor chords of a scale. These intermediate chromatic passing chords can easily be detected in terms of their voice-leading behavior. They are reached and left with motions in one and the same voice. It is therefore a consistent further step in the modelling of voice-leading connections to include these passing chords, which we do not require to be second-order Clough-Myerson scales.

Recalling the fact that second-order Clough-Myerson scales are in particular *second order maximally even*, we find that such single-voice-motion passing chords turn out to be at least as even as the less even one among the two connected chords. To make this explicit we subscribe to Emmanuel Amiot's proposal to measure the evenness of chords or scales $X \subset \mathbb{Z}_c$ in terms of the magnitude of the 1st Fourier coefficient of its representation on the unit circle [1].

For every chromatic pitch class space $\mathbb{Z}_c$, consider the embedding $\iota : \mathbb{Z}_c \to \mathbb{T} \subset \mathbb{C}$ via c-th roots of unity into the unit circle $\mathbb{T}$ within the complex numbers $\mathbb{C}$, $\iota(k) := \exp(2\pi i k/c)$ for any integral representative $k \in \mathbb{Z}$ of the residue classes in $\mathbb{Z}_c$. For every e-tone mode $\sigma : \mathbb{Z}_e \to \mathbb{Z}_c$, we have the concatenation $\iota \circ \sigma : \mathbb{Z}_e \to \mathbb{C}$. Recall that for any map $\phi : \mathbb{Z}_e \to \mathbb{C}$ we may consider its finite Fourier transform $\widehat{\phi} : \mathbb{Z}_e \to \mathbb{C}$ by virtue of the formula

$$\widehat{\phi}(t) = \frac{1}{e} \sum_{k=0}^{e-1} \phi(k) \exp\left(-\frac{2\pi i k t}{e}\right).$$

Definition 8. *For a given sequence $X : \mathbb{Z}_e \to \mathbb{T}$ of points $X_k = \exp(2\pi i t_k), t_k \in [0, 1)$ on the unit circle we define its* evenness *in terms of the absolute value evenness$(X) = |\widehat{X}(1)|$. For an e-tone mode $\sigma : \mathbb{Z}_e \to \mathbb{Z}_c$, we define its evenness as the evenness of the concatenation: $\iota \circ \sigma : \mathbb{Z}_e \to \mathbb{C}$.*

Our finding on the passing chords is in fact of more general nature and can be best understood with a continuous moving voice.

Definition 9. *For a given sequence $X = (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_{e-1})) \in \mathbb{T}^{e-1}$ of $e-1$ points on the unit circle, consider the continuous family $X_{\_} : [0, 1) \to \mathbb{T}^e$ with $X_t := (\exp(2\pi i t), \exp(2\pi i t_1), \ldots, \exp(2\pi i t_{e-1}))$, together with the Single-Zero-Padding of X, namely the vector*

$$X^0 := (0, \exp(2\pi i t_1), \ldots, \exp(2\pi i t_{e-1})) \in \mathbb{C}^e.$$

The family X_t is called a chord with a sliding voice *and the vector X^0 shall be called the "muted slider."*

Proposition 2. *Consider a chord with a sliding voice $X_t : [0, 1) \to \mathbb{T}^e$ and the associated "muted slider" X^0. Then the associated one-parameter family $\{\widehat{X_t}(1) \in \mathbb{C} \,|\, t \in [0, 1)\}$ of the first Fourier-Coefficients of the vectors X_t forms a circle of radius $1/e$ around the first Fourier-Coefficient $\widehat{X^0}(1)$ of X^0. The most even and uneven chords X_v and X_u correspond to the parameters $v = \arg(\widehat{X^0}(1))/2\pi$ and $u = v + \frac{1}{2}$ mod 1, respectively.*

Proof. For $t \in [0, 1)$ one finds:

$$\widehat{X_t}(1) = \frac{1}{e}\left(\exp(2\pi i t) + \sum_{k=1}^{e-1} \exp(2\pi i (t_k - k/e))\right) = \frac{\exp(2\pi i t)}{e} + \widehat{X^0}(1). \quad (1)$$

Let $l = |\widehat{X^0}(1)|$ and $\psi = \arg(\widehat{X^0}(1))$ denote the magnitude and the phase of the first Fourier-coefficient of X^0, respectively. Then we obtain the first Fourier-Coefficient of the most even chord X_v by adding the radius $1/e$ to the magnitude l, while keeping the same phase ψ, $\widehat{X_v}(1) = (l+1/e)\exp(i\psi)$. Likewise, we obtain $\widehat{X_u}(1) = (l-1/e)\exp(i\psi)$, for the most uneven chord. In order to see that the parameter of X_v is actually $v = \psi/2\pi$ we insert this value for t in Eq. (1).

$$\widehat{X_{\psi/2\pi}}(1) = \frac{\exp(\frac{2\pi i\psi}{2\pi})}{e} + \widehat{X^0}(1) = \frac{\exp(i\psi)}{e} + l\exp(i\psi) = (l+\frac{1}{e})\exp(i\psi) = \widehat{X_v}. \tag{1}$$

Moving the sliding voice from $\exp(2\pi i\psi)$ about half the unit circle to the opposite point $\exp(2\pi i(\psi + 1/2))$ corresponds to a half circle movement of the Fourier-coefficient $(l+1/e)\exp(i\psi)$ in the small circle around $\widehat{X^0}(1)$ to the opposite point $(l-1/e)\exp(i\psi)$, and hence $u = v + \frac{1}{2}$.

This proposition has the following consequence for passing chords.

Corollary 1. *Consider two instances X_r and X_s of a chord with a sliding voice $X_t : [0,1) \to \mathbb{T}^e$, such that all other points (the fixed tones) are located in the circular segment between s and r and no points between r and s, i.e. $0 \le s < t_1 < \ldots t_{e-1} < r < 1$. Consider the corresponding 1st Fourier coefficients $\widehat{X_r}(1)$ and $\widehat{X_s}(1)$. If the location of the 1st Fourier coefficient $\widehat{X_u}(1)$ of the most uneven chord X_u in the family X_t is not between $\widehat{X_r}(1)$ and $\widehat{X_s}(1)$ in counter-clockwise direction, then for any t between r and s ($r \le t < 1$ or $0 \le t \le s$), we have $|\widehat{X_t}(1)| \ge \min(|\widehat{X_r}(1)|, |\widehat{X_s}(1)|)$.*

Proof. The assertion is an immediate consequence of the fact that the curve $\{\widehat{X_t}(1) \,|\, t \in [0,1)\}$ is a circle. Within any segment of the circle that does neither contain $\widehat{X_v}(1)$ nor $\widehat{X_u}(1)$, the magnitude $|\widehat{X_t}(1)|$ is monotonously increasing (or decreasing). If only $\widehat{X_v}(1)$ is contained in a segment, but not $\widehat{X_u}(1)$, the magnitude $|\widehat{X_t}(1)|$ will pass through the maximum, and satisfies $|\widehat{X_t}(1)| \ge \min(|\widehat{X_r}(1)|, |\widehat{X_s}(1))|$ throughout.

In the light of Corollary 1, we have to study the remaining case where the location of the 1st Fourier coefficient $\widehat{X_u}(1)$ of the most uneven chord X_u in the family X_t is actually between $\widehat{X_r}(1)$ and $\widehat{X_s}(1)$ (in counter-clockwise direction). We will first show in the case $e = 3$ that such chords have one very large step interval, which implies that this situation can not occur with second order Clough-Myerson chords (scales).

Proposition 3. *Consider a 3-chord with a sliding voice $X_t : [0,1) \to \mathbb{T}^e$, such that the pair $X = (\exp(2\pi i t_1), \exp(2\pi i t_2)) \in \mathbb{T}^2$ of its fixed tones satisfies $0 \le t_1 < t_2 < 1$. Further suppose that $X_u = X_0$ is the most uneven chord. Then the fixed tones satisfy $t_1 < \frac{1}{12}$ and $t_2 > \frac{11}{12}$. i.e. the three tones of X_u are located within a segment of the angle $\frac{2\pi}{6}$.*

Proof. As $u = 0$ is the parameter of the most uneven chord, it follows from Proposition 2 that $\arg(\widehat{X^0}(1)) = 2\pi v = \pi$, since $v = u - \frac{1}{2}$. In other words: the 1st Fourier coefficients $\widehat{X^0}(1)$ and $\widehat{X}_0(1) = \widehat{X}^0(1) + \frac{1}{3}$ are both located on the real line and the coefficient $\widehat{X^0}(1)$ must be negative. This implies that the two (non-zero) summands of $3\widehat{X^0}(1) = (\exp(2\pi i(t_1 - \frac{1}{3}) + \exp(2\pi i(t_2 - \frac{2}{3}))$ must be conjugated with a shared negative real part. Hence, $t_2 = 1 - t_1$ and this implies $t_1 \leq \frac{1}{2}$, because $t1 \leq t_2$. We have $Re(\exp(2\pi i(t_1 - \frac{1}{3})) < 0$ iff $\frac{1}{4} < (t_1 + \frac{2}{3}) \bmod 1 < \frac{3}{4}$ iff $0 < t_1 < \frac{1}{12}$ or $\frac{7}{12} < t_1 < 1$. The latter inequality is out of question, because $t_1 < \frac{1}{2}$, Hence we obtain $t_1 < \frac{1}{12}$ along with $t_2 > \frac{11}{12}$.

An analogous statement is most likely true also for chords of cardinality $e > 3$ and should be proven with the help of an estimate for $|\widehat{X_0}(1)|$ in dependence of the size of circular segment around 1 covered by the points of X_0. An estimation for the size of the largest step interval follows also from the following conjecture about the distribution of the summands of the 1st Fourier-Coefficient for a consecutive sequence of points on the unit circle.

Conjecture 1. Consider a sequence $X = (\exp(2\pi it_1), \ldots, \exp(2\pi it_{e-1})) \in \mathbb{T}^{e-1}$ of consecutive points on the unit circle in counter-clockwise order with the property that the first Fourier-Coefficient of $X^0 = (0, \exp(2\pi it_1), \ldots, \exp(2\pi it_{e-1})) \in \mathbb{C}^e$ is a negative real number, more precisely $\widehat{X^0}(1) \in (-1, 0)$. Then the sequence $Y^0 = (\exp(2\pi i(t_1 - 1/e)), \ldots, \exp(2\pi i(t_k - k/e)))$ of the $e - 1$ (non-zero) summands of $e \cdot \widehat{X^0}(1)$ is formed by (clockwise) consecutive points on the unit circle.

According to this conjecture the parameters $t_1, \ldots, t_{e-1}$ must fit into an interval of size $(1 - \frac{e-1}{e}) - \frac{1}{e} = 1 - \frac{e-2}{e} = \frac{2}{e}$. For $e \geq 4$ this excludes all instances of second order Clough-Myerson Chords.

7 Conclusion and Future Research

This paper studied the "integration" problem by proposing a generalization of diatonic triads using second-order Clough-Myerson scales in the role of the triads. It further compared the resulting chord connections to the globally defined voice-leading distance utilizing the novel concepts of distant neighbors and interscalar contiguities. The property of diatonic contiguity does not hold in this general case, as a counterexample demonstrates. The paper particularly presented a generalization of hexatonic cycles that is integrated into the diatonic system.

There are three main directions in which future work will build upon this paper. In order to further understand the relationship between Douthett graphs and betweenness graphs, one utilizes the evenness measure of the first Fourier coefficient to define, study and compare the saturations of the Douthett graphs and the betweenness graphs with passing chords. The second direction brings transformations back into play and aims to interpret suitable subgraphs of these voice-leading graphs as Cayley-graphs of group actions with respect to musically meaningful generators. The third direction interprets the findings of this paper as a potential integrationist enrichment of *Tonfeld* analysis [10,16] and aims to bring this area in closer contact with the discourse on neo-Riemannian approaches.

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