Path Category for Free

Open Morphisms from Coalgebras with Non-deterministic Branching

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Abstract. There are different categorical approaches to variations of transition systems and their bisimulations. One is coalgebra for a functor G, where a bisimulation is defined as a span of G-coalgebra homomorphism. Another one is in terms of path categories and open morphisms, where a bisimulation is defined as a span of open morphisms. This similarity is no coincidence: given a functor G, fulfilling certain conditions, we derive a path-category for pointed G-coalgebras and lax homomorphisms, such that the open morphisms turn out to be precisely the G-coalgebra homomorphisms. The above construction provides path-categories and trace semantics for free for different flavours of transition systems: (1) non-deterministic tree automata (2) regular nondeterministic nominal automata (RNNA), an expressive automata notion living in nominal sets (3) multisorted transition systems. This last instance relates to Lasota's construction, which is in the converse direction.

Keywords: Coalgebra · Open maps · Categories · Nominal sets

1 Introduction

Coalgebras [25] and *open maps* [16] are two main categorical approaches to transition systems and bisimulations. The former describes the branching type of systems as an endofunctor, a system becoming a coalgebra and bisimulations being spans of coalgebra homomorphisms. Coalgebra theory makes it easy to consider state space types in different settings, e.g. nominal sets [17,18] or algebraic categories [5,11,20]. The latter, open maps, describes systems as objects of

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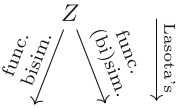
Table 1. Two approaches to categorical (bi)simulations

	worlds	data	systems	func. sim.	func. bisim.	(bi)simulation
this paper ↑	open maps	$J\colon \mathbb{P} \longrightarrow \mathbb{M}$ Def. 2.4	$\mathbf{obj}\,(\mathbb{M})$	$\mathbf{mor}\,(\mathbb{M})$	open maps Def. 2.5	
	coalgebra	$G\colon \mathbb{C} \longrightarrow \mathbb{C}$ Def. 2.7	pointed G-coalg. Sec. 2.2	lax hom. Def. 2.8	coalg. hom. Def. 2.6	



a category and the execution types as particular objects called paths. In this case, bisimulations are spans of open morphisms. Open maps are particularly adapted to extend bisimilarity to history dependent behaviors, e.g. true concurrency [7,8], timed systems [22] and weak (bi)similarity [9]. Coalgebra homomorphisms and open maps are then key concepts to describe bisimilarity categorically. They intuitively correspond to functional bisimulations, that is, those maps between states whose graph is a bisimulation.

We are naturally interested in the relationship between those two categorical approaches to transition systems and bisimulations. A reduction of open maps situations to coalgebra was given by Lasota using multi-sorted transition systems [19]. In this paper, we give the reduction in the other direction: from the category $\mathsf{Coalg}_l(TF)$ of pointed TF-coalgebras and lax homomorphisms, we construct the path-category Path and a functor $J : \mathsf{Path} \longrightarrow \mathsf{Coalg}_l(TF)$ such that Path-open morphisms coincide with strict homomorphisms, hence functional bisimulations. Here, T is a functor describing the branching behaviour and F describes the input type, i.e. the type of data that is processed (e.g. words or trees). This development is carried out with the case where T is a powerset-like functor, and covers transition systems allowing non-deterministic branching.

The key concept in the construction of Path are *F-precise maps*. Roughly speaking in set, a map $f\colon X \longrightarrow FY$ is F-precise if every $y \in Y$ is used precisely once in f, i.e. there is a unique x such that y appears in $f(x)$ and additionally y appears precisely once in $f(x)$. Such an F-precise map represents one deterministic step (of shape F). Then a path $P \in \mathsf{Path}$ is a finite sequence of deterministic steps, i.e. finitely many precise maps. J converts such a data into a pointed TF-coalgebra. There are many existing notions of paths and traces in coalgebra [4,12,13,21], which lack the notion of *precise* map, which is crucial for the present work.

Once we set up the situation $J\colon \mathsf{Path} \longrightarrow \mathsf{Coalg}_l(TF)$, we are on the framework of open map bisimulations. Our construction of Path using precise maps is justified by the characterisation theorem: Path-open morphisms and strict coalgebra homomorphisms coincide (Theorems 3.20 and 3.24). This coincidence relies on the concept of path-reachable coalgebras, namely, coalgebras such that every state can be reached by a path. Under mild conditions, path-reachability is equivalent to an existing notion in coalgebra, defined as the non-existence of a proper sub-coalgebra (Sect. 3.5). Additionally, this characterization produces a canonical trace semantics for free, given in terms of paths (Sect. 3.6).

We illustrate our reduction with several concrete situations: different classes of non-deterministic top-down tree automata using analytic functors (Sect. 4.1), Regular Nondeterministic Nominal Automata (RNNA), an expressive automata notion living in nominal sets (Sect. 4.2), multisorted transition systems, used in Lasota's work to construct a coalgebra situation from an open map situation (Sect. 4.3).

Notation. We assume basic categorical knowledge and notation (see e.g. [1,3]). The cotupling of morphisms $f\colon A \to C$, $g\colon B \to C$ is denoted by $[f,g]\colon A+B \to C$, and the unique morphsim to the terminal object is $!\colon X \to 1$ for every X.

2 Two Categorical Approaches for Bisimulations

We introduce the two formalisms involved in the present paper: the open maps (Sect. 2.1) and the coalgebras (Sect. 2.2). Those formalisms will be illustrated on the classic example of Labelled Transition Systems (LTSs).

Definition 2.1. *Fix a set A, called the alphabet. A* labelled transition system *is a triple (S,i,Δ) with S a set of* states*, $i \in S$ the* initial state*, and $\Delta \subseteq S \times A \times S$ the* transition relation*. When Δ is obvious from the context, we write $s \xrightarrow{a} s'$ to mean $(s,a,s') \in \Delta$.*

For instance, the tuple $(\{0,\cdots,n\}, 0, \{(k-1,a_k,k) \mid 1 \leq k \leq n\})$ is an LTS, and called the *linear system* over the word $a_1 \cdots a_n \in A^\star$. To relate LTSs, one considers functions that preserves the structure of LTSs:

Definition 2.2. *A* morphism of LTSs *from (S,i,Δ) to (S',i',Δ') is a function $f\colon S \longrightarrow S'$ such that $f(i) = i'$ and for every $(s,a,s') \in \Delta$, $(f(s),a,f(s')) \in \Delta'$. LTSs and morphisms of LTSs form a category, which we denote by LTS_A.*

Some authors choose other notions of morphisms (e.g. [16]), allowing them to operate between LTSs with different alphabets for example. The usual way of comparing LTSs is by using simulations and bisimulations [23]. The former describes what it means for a system to have at least the behaviours of another, the latter describes that two systems have exactly the same behaviours. Concretely:

Definition 2.3. *A* simulation *from (S,i,Δ) to (S',i',Δ') is a relation $R \subseteq S \times S'$ such that (1) $(i,i') \in R$, and (2) for every $s \xrightarrow{a} t$ and $(s,s') \in R$, there is $t' \in S'$ such that $s' \xrightarrow{a} t'$ and $(t,t') \in R$. Such a relation R is a* bisimulation *if $R^{-1} = \{(s',s) \mid (s,s') \in R\}$ is also a simulation.*

Morphisms of LTSs are functional simulations, i.e. functions between states whose graph is a simulation. So how to model (1) systems, (2) functional simulations and (3) functional bisimulations categorically? In the next two sections, we will describe known answers to this question, with open maps and coalgebra. In both cases, it is possible to capture similarity and bisimilarity of two LTSs T

and T'. Generally, a simulation is a (jointly monic) span of a functional bisimulation and a functional simulation, and a bisimulation is a simulation whose converse is also a simulation, as depicted in Table 1. Consequently, to understand similarity and bisimilarity on a general level, it is enough to understand functional simulations and bisimulations.

2.1 Open Maps

The categorical framework of open maps [16] assumes functional simulations to be already modeled as a category $\mathbb{M}$. For example, for $\mathbb{M} := \mathsf{LTS}_A$, objects are LTSs, and morphisms are functional simulations. Furthermore, the open maps framework assumes another category $\mathbb{P}$ of 'paths' or 'linear systems', together with a functor J that tells how a 'path' is to be understood as a system:

Definition 2.4 [16]**.** *An* open map situation *is given by categories* $\mathbb{M}$ *('systems' with 'functional simulations') and* $\mathbb{P}$ *('paths') together with a functor* $J\colon \mathbb{P} \to \mathbb{M}$*.*

For example with $\mathbb{M} := \mathsf{LTS}_A$, we pick $\mathbb{P} := (A^\star, \leq)$ to be the poset of words over A with prefix order. Here, the functor J maps a word $w \in A^\star$ to the linear system over w, and $w \leq v$ to the evident functional simulation $J(w \leq v)\colon Jw \longrightarrow Jv$.

In an open map situation $J\colon \mathbb{P} \longrightarrow \mathbb{M}$, we can abstractly represent the concept of a *run* in a system. A run of a path $w \in \mathbb{P}$ in a system $T \in \mathbb{M}$ is simply defined to be an $\mathbb{M}$-morphism of type $Jw \longrightarrow T$. With this definition, each $\mathbb{M}$-morphism $h\colon T \longrightarrow T'$ (i.e. functional simulation) inherently transfers runs: given a run $x\colon Jw \longrightarrow T$, the morphism $h \cdot x\colon Jw \longrightarrow T'$ is a run of w in T'. In the example open map situation $J\colon (A^\star, \leq) \longrightarrow \mathsf{LTS}_A$, a run of a path $w = a_1 \cdots a_n \in A^\star$ in an LTS $T = (S, i, \Delta)$ is nothing but a sequence of states $x_0, \ldots, x_n \in S$ such that $x_0 = i$ and $x_{k-1} \xrightarrow{a_k} x_k$ holds for all $1 \leq k \leq n$.

We introduce the concept of open map [16]. This is an abstraction of the property posessed by *functional bisimulations*. For LTSs $T = (S, i, \Delta)$ and $T' = (S', i', \Delta')$, an LTS_A-morphism $h\colon T \longrightarrow T'$ is a functional bisimulation if the graph of h is a bisimulation. This implies the following relationship between runs in T and runs in T'. Suppose that $w \leq w'$ holds in $A^\star$, and a run x of w in T is given as in (1); here n, m are lengths of w, w' respectively. Then for any run y' of w' in T' extending $h \cdot x$ as in (2), there is a run x' of w' extending x, and moreover its image by h coincides with y' (that is, $h \cdot x' = y'$). Such x' is obtained by repetitively applying the condition of functional bisimulation.

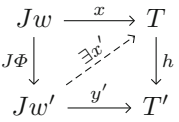
$$\to \overbrace{i \xrightarrow{w_1} x_1 \xrightarrow{w_2} \cdots \xrightarrow{w_n} x_n}^{x} \underbrace{\xrightarrow{w'_{n+1}} x'_{n+1} \xrightarrow{w'_{n+2}} \cdots \xrightarrow{w'_m} x'_m}_{x'} \quad (\text{in } T) \quad (1)$$

$$\to \underbrace{i' \xrightarrow{w_1} h(x_1) \xrightarrow{w_2} \cdots \xrightarrow{w_n} h(x_n) \xrightarrow{w'_{n+1}} y'_{n+1} \xrightarrow{w_{n+2}} \cdots \xrightarrow{w'_m} y'_m}_{y'} \quad (\text{in } T') \quad (2)$$

Observe that y' extending $h \cdot x$ can be represented as $y' \cdot J(w \leq w') = h \cdot x$, and x' extending x as $x' \cdot J(w \leq w') = x$. From these, we conclude that if an

LTS_A-morphism $h\colon T \longrightarrow T'$ is a functional bisimulation, then for any $w \leq w'$ in $A^\star$ and run $x\colon Jw \longrightarrow T$ and $y'\colon Jw' \longrightarrow T'$ such that $y' \cdot J(w \leq w') = h \cdot x$, there is a run $x'\colon Jw' \longrightarrow T$ such that $x' \cdot J(w \leq w') = x$ and $h \cdot x' = y'$ (the converse also holds if all states of T are reachable). This necessary condition of functional bisimulation can be rephrased in any open map situation, leading us to the definition of open map.

Definition 2.5 [16]**.** *Let $J\colon \mathbb{P} \longrightarrow \mathbb{M}$ be an open map situation. An $\mathbb{M}$-morphism $h\colon T \longrightarrow T'$ is said to be* open *if for every morphism $\Phi\colon w \longrightarrow w' \in \mathbb{P}$ making the square on the right commute, there is x' making the two triangles commute.*



Open maps are closed under composition and stable under pullback [16].

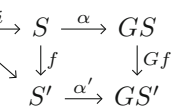
2.2 Coalgebras

The theory of G-coalgebras is another categorical framework to study bisimulations. The type of systems is modelled using an endofunctor $G\colon \mathbb{C} \longrightarrow \mathbb{C}$ and a system is then a coalgebra for this functor, that is, a pair of an object S of $\mathbb{C}$ (modeling the state space), and of a morphism of type $S \longrightarrow GS$ (modeling the transitions). For example for LTSs, the transition relation is of type $\Delta \subseteq S \times A \times S$. Equivalently, this can be defined as a function $\Delta\colon S \longrightarrow \mathcal{P}(A \times S)$, where $\mathcal{P}$ is the powerset. In other words, the transition relation is a coalgebra for the Set-functor $\mathcal{P}(A \times \_)$. Intuitively, this coalgebra gives the one-step behaviour of an LTS: S describes the state space of the system, $\mathcal{P}$ describes the 'branching type' as being non-deterministic, $A \times S$ describe the 'computation type' as being linear, and the function itself lists all possible futures after one-step of computation of the system. Now, changing the underlying category or the endofunctor allows to model different types of systems. This is the usual framework of coalgebra, as described for example in [25].

Initial states are modelled coalgebraically by a pointing to the carrier $i\colon I \longrightarrow S$ for a fixed object I in $\mathbb{C}$, describing the 'type of initial states' (see e.g. [2, Sec. 3B]). For example, an initial state of an LTS is the same as a function from the singleton set $I := \{*\}$ to the state space S. This object I will often be the final object of $\mathbb{C}$, but we will see other examples later. In total, an *I-pointed G-coalgebra* is a $\mathbb{C}$-object S together with morphisms $\alpha\colon S \longrightarrow GS$ and $i\colon I \longrightarrow S$. E.g. an LTS is an I-pointed G-coalgebra for $I = \{*\}$ and $GX = \mathcal{P}(A \times X)$.

In coalgebra, functional bisimulations are the first class citizens to be modelled as homomorphisms. The intuition is that those preserve the initial state, and preserve and reflect the one-step relation.

Definition 2.6. *An* I-pointed G-coalgebra homomorphism *from $I \xrightarrow{i} S \xrightarrow{\alpha} GS$ to $I \xrightarrow{i'} S' \xrightarrow{\alpha'} GS'$ is a morphism $f\colon S \longrightarrow S'$ making the right-hand diagram commute.*



For instance, when $G = \mathcal{P}(A \times \_)$, one can easily see that a function f is a G-coalgebra homomorphism iff it is a functional bisimulation. Thus, if we want to capture functional simulations in LTSs, we need to weaken the condition of homomorphism to the inequality $Gf(\alpha(s)) \subseteq \alpha'(f(s))$ (instead of equality). To express this condition for general G-coalgebras, we introduce a partial order $\sqsubseteq_{X,Y}$ on each homset $\mathbb{C}(X, GY)$ in a functorial manner.

Definition 2.7. *A* partial order on G-homsets *is a functor* $\sqsubseteq\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \longrightarrow \mathsf{Pos}$ *such that* $U \cdot \sqsubseteq = \mathbb{C}(\_, G\_)$*; here,* $U\colon \mathsf{Pos} \longrightarrow \mathsf{Set}$ *is the forgetful functor from the category* Pos *of posets and monotone functions.*

The functoriality of $\sqsubseteq$ amounts to that $f_1 \sqsubseteq f_2$ implies $Gh \cdot f_1 \cdot g \sqsubseteq Gh \cdot f_2 \cdot g$.

Definition 2.8. *Given a partial order on G-homsets, an I-pointed lax G-coalgebra homomorphism $f\colon (S, \alpha, i) \longrightarrow (S', \alpha', i')$ is a morphism $f\colon S \longrightarrow S'$ making the right-hand diagram commute. The I-pointed G-coalgebras and lax homomorphisms form a category, denoted by* $\mathsf{Coalg}_l(I, G)$.

$$\begin{array}{ccccc} I & \xrightarrow{i} & S & \xrightarrow{\alpha} & GS \\ & \searrow^{i'} & \downarrow f & \sqcap & \downarrow Gf \\ & & S' & \xrightarrow[\alpha']{} & GS' \end{array}$$

Conclusion 2.9. In Set, with $I = \{*\}$, $G = \mathcal{P}(A \times \_)$, define the order $f \sqsubseteq g$ in $\mathsf{Set}(X, \mathcal{P}(A \times Y))$ iff for every $x \in X$, $f(x) \subseteq g(x)$. Then $\mathsf{Coalg}_l(\{*\}, \mathcal{P}(A \times \_)) = \mathsf{LTS}_A$. In particular, we have an open map situation

$$\mathbb{P} = (A^\star, \leq) \quad \xrightarrow{J} \quad \mathbb{M} = \mathsf{LTS}_A = \mathsf{Coalg}_l(\{*\}, \mathcal{P}(A \times \_))$$

and the open maps are precisely the coalgebra homomorphisms (for reachable LTSs). In this paper, we will construct a path category $\mathbb{P}$ for more general I and G, such that the open morphisms are precisely the coalgebra homomorphisms.

3 The Open Map Situation in Coalgebras

Lasota's construction [19] transforms an open map situation $J\colon \mathbb{P} \longrightarrow \mathbb{M}$ into a functor G (with a partial order on G-homsets), together with a functor $\mathsf{Beh}\colon \mathbb{M} \longrightarrow \mathsf{Coalg}_l(I, G)$ that sends open maps to G-coalgebra homomorphisms (see Sect. 4.3 for details). In this paper, we provide a construction in the converse direction for functors G of a certain shape.

As exemplified by LTSs, it is a common pattern that G is the composition $G = TF$ of two functors [12], where T is the branching type (e.g. partial, or non-deterministic) and F is the data type, or the 'linear behaviour' (words, trees, words modulo α-equivalence). If we instantiate our path-construction to $T = \mathcal{P}$ and $F = A \times \_$, we obtain the known open map situation for LTSs (Conclusion 2.9).

Fix a category $\mathbb{C}$ with pullbacks, functors $T, F : \mathbb{C} \longrightarrow \mathbb{C}$, an object $I \in \mathbb{C}$ and a partial order $\sqsubseteq^T$ on T-homsets. They determine a coalgebra situation $(\mathbb{C}, I, TF, \sqsubseteq)$ where $\sqsubseteq$ is the partial order on TF-homsets defined by $\sqsubseteq_{X,Y} = \sqsubseteq^T_{X,FY}$. Under some conditions on T and F, we construct a path-category $\mathsf{Path}(I, F+1)$ and an open map situation $\mathsf{Path}(I, F+1) \hookrightarrow \mathsf{Coalg}_l(I, TF)$ where TF-coalgebra homomorphisms and $\mathsf{Path}(I, F+1)$-open morphisms coincide.

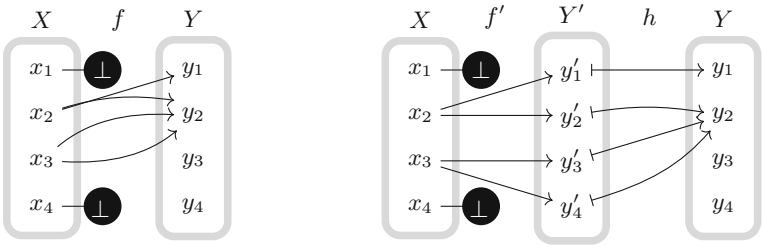


Fig. 1. A non-precise map f that factors through the F-precise $f' \colon X \longrightarrow Y' \times Y' + \{\bot\}$

3.1 Precise Morphisms

While the path category is intuitively clear for $FX = A \times X$, it is not for inner functors F that model tree languages. For example for $FX = A + X \times X$, a $\mathcal{P}F$-coalgebra models transition systems over binary trees with leaves labelled in A, instead of over words. Hence, the paths should be these kind of binary trees. We capture the notion of tree like shape ("every node in a tree has precisely one route to the root") by the following abstract definition:

Definition 3.1. *For a functor $F \colon \mathbb{C} \longrightarrow \mathbb{C}$, a morphism $s \colon S \longrightarrow FR$ is called* F-precise *if for all f, g, h the following implication holds:*

$$\begin{array}{ccc} S & \xrightarrow{f} & FC \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Fh} \\ FR & \xrightarrow{Fg} & FD \end{array} \quad \overset{\exists d}{\Longrightarrow} \quad \begin{array}{ccc} S & \xrightarrow{f} & FC \\ {\scriptstyle s}\downarrow & \nearrow_{Fd} & \\ FR & & \end{array} \quad \& \quad \begin{array}{ccc} & & C \\ & \overset{d}{\nearrow} & \downarrow{\scriptstyle h} \\ R & \xrightarrow{g} & D \end{array}$$

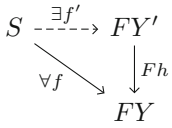
Remark 3.2. If F preserves weak pullbacks, then a morphism s is F-precise iff it fulfils the above definition for $g = \mathrm{id}$.

Example 3.3. Intuitively speaking, for a polynomial Set-functor F, a map $s \colon S \to FR$ is F-precise iff every element of R is mentioned precisely once in the definition of the map f. For example, for $FX = A \times X + \{\bot\}$, the case needed later for LTSs, a map $f \colon X \longrightarrow FY$ is precise iff for every $y \in Y$, there is a unique pair $(x, a) \in X \times A$ such that $f(x) = (a, y)$. For $FX = X \times X + \{\bot\}$ on Set, the map $f \colon X \longrightarrow FY$ in Fig. 1 is not F-precise, because y_2 is used three times (once in $f(x_2)$ and twice in $f(x_3)$), and y_3 and y_4 do not occur in f at all. However, $f' \colon X \longrightarrow FY'$ is F-precise because every element of Y' is used precisely once in f', and we have that $Fh \cdot f' = f$. Also note that f' defines a forest where X is the set of roots, which is closely connected to the intuition that, in the F-precise map f', from every element of Y', there is precisely one edge up to a root in X.

So when transforming a non-precise map into a precise map, one duplicates elements that are used multiple times and drops elements that are not used. We will cover functors F for which this factorization pattern provides F-precise

maps. If F involves unordered structure, this factorization needs to make choices, and so we restrict the factorization to a class $\mathcal{S}$ of objects that have that choice-principle (see Example 4.5 later):

Definition 3.4. *Fix a class of objects $\mathcal{S} \subseteq \mathbf{obj}\,\mathbb{C}$ closed under isomorphism. We say that F* admits precise factorizations w.r.t. $\mathcal{S}$ *if for every $f\colon S \to FY$ with $S \in \mathcal{S}$, there exist $Y' \in \mathcal{S}$, $h\colon Y' \to Y$ and $f'\colon S \to FY'$ F-precise with $Fh \cdot f' = f$.*



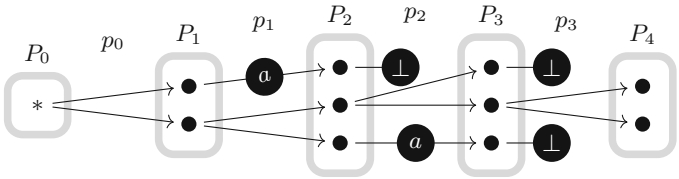


Fig. 2. A path of length 4 for $FX = \{a\} \times X + X \times X + \{\bot\}$ with $I = \{*\}$.

For $\mathbb{C} = \mathsf{Set}$, $\mathcal{S}$ contains all sets. However for the category of nominal sets, $\mathcal{S}$ will only contain the strong nominal sets (see details in Subsect. 4.2).

Remark 3.5. Precise morphisms are essentially unique. If $f_1\colon X \longrightarrow FY_1$ and $f_2\colon X \longrightarrow FY_2$ are F-precise and if there is some $h\colon Y_1 \longrightarrow Y_2$ with $Fh \cdot f_1 = f_2$, then h is an isomorphism. Consequently, if $f\colon S \longrightarrow FY$ with $S \in \mathcal{S}$ is F-precise and F-admits precise factorizations, then $Y \in \mathcal{S}$.

Functors admitting precise factorizations are closed under basic constructions:

Proposition 3.6. *The following functors admit precise factorizations w.r.t. $\mathcal{S}$:*

1. *Constant functors, if $\mathbb{C}$ has an initial object 0 and $0 \in \mathcal{S}$.*
2. *$F \cdot F'$ if $F\colon \mathbb{C} \longrightarrow \mathbb{C}$ and $F'\colon \mathbb{C} \longrightarrow \mathbb{C}$ do so.*
3. *$\prod_{i \in I} F_i$, if all $(F_i)_{i \in I}$ do so and $\mathcal{S}$ is closed under I-coproducts.*
4. *$\coprod_{i \in I} F_i$, if all $(F_i)_{i \in I}$ do so, $\mathbb{C}$ is I-extensive and $\mathcal{S}$ is closed under I-coproducts.*
5. *Right-adjoint functors, if and only if its left-adjoint preserves $\mathcal{S}$-objects.*

Example 3.7. When $\mathbb{C}$ is infinitary extensive and $\mathcal{S}$ is closed under coproducts, every polynomial endofunctor $F\colon \mathbb{C} \longrightarrow \mathbb{C}$ admits precise factorizations w.r.t. $\mathcal{S}$. This is in particular the case for $\mathbb{C} = \mathcal{S} = \mathsf{Set}$. In this case, we shall see later (Sect. 4.1) that many other Set-functors, e.g. the bag functor $\mathcal{B}$, where $\mathcal{B}(X)$ is the set of finite multisets, have precise factorizations. In contrast, $F = \mathcal{P}$ does not admit precise factorizations, and if $f\colon X \longrightarrow \mathcal{P}Y$ is $\mathcal{P}$-precise, then $f(x) = \emptyset$ for all $x \in X$.

3.2 Path Categories in Pointed Coalgebras

We define a path for I-pointed TF-coalgebras as a tree according to F. Following the observation in Example 3.3, one layer of the tree is modelled by a F-precise morphism and hence a path in a TF-coalgebra is defined to be a finite sequence of $(F+1)$-precise maps, where the $\_+1$ comes from the dead states w.r.t. T; the argument is given later in Remark 3.23 when reachability is discussed. Since the $\_+1$ is not relevant yet, we define $\mathsf{Path}(I,F)$ in the following and will use $\mathsf{Path}(I,F+1)$ later. For simplicity, we write $\boldsymbol{X}_n$ for finite families $(X_k)_{0\le k<n}$.

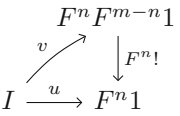
Definition 3.8. *The category* $\mathsf{Path}(I,F)$ *consists of the following. An object is* $(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)$ *for an* $n\in\mathbb{N}$ *with* $P_0=I$ *and* $\boldsymbol{p}_n$ *a family of* F*-precise maps* $(p_k\colon P_k\longrightarrow FP_{k+1})_{k<n}$. *We say that* $(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)$ *is a* path of length n. *A morphism* $\boldsymbol{\phi}_{n+1}\colon(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)\longrightarrow(\boldsymbol{Q}_{m+1},\boldsymbol{q}_m)$, $m\ge n$, *is a family* $(\phi_k\colon P_k\longrightarrow Q_k)_{k\le n}$ *with* $\phi_0=\mathrm{id}_I$ *and* $q_k\cdot\phi_k=F\phi_{k+1}\cdot p_k$ *for all* $0\le k\le n$.

Example 3.9. Paths for $FX=A\times X+1$ and $I=\{*\}$ singleton are as follows. First, a map $f\colon I\longrightarrow FX$ is precise iff (up-to isomorphism) either $X=I$ and $f(*)=(a,*)$ for some $a\in A$; or $X=\varnothing$ and $f(*)=\bot$. Then a path is isomorphic to an object of the form: $P_i=I$ for $i\le k$, $P_i=\varnothing$ for $i>k$, $p_i(*)=(a_i,*)$ for $i<k$, and $p_k(*)=\bot$. A path is the same as a word, plus some "junk", concretely, a word in $A^\star.\bot^\star$. For LTSs, an object in $\mathsf{Path}(I,F)$ with $FX=A\times X$ is simply a word in $A^\star$. For a more complicated functor, Fig. 2 depicts a path of length 4, which is a tree for the signature with one unary, one binary symbol, and a constant. The layers of the tree are the sets $\boldsymbol{P}_4$. Also note that since every p_i is F-precise, there is precisely one route to go from every element of a P_k to $*$.

Remark 3.10. The inductive continuation of Remark 3.5 is as follows. Given a morphism $\boldsymbol{\phi}_{n+1}$ in $\mathsf{Path}(I,F)$, since ϕ_0 is an isomorphism, then ϕ_k is an isomorphism for all $0\le k\le n$. If F admits precise factorizations and if $I\in\mathcal{S}$, then for every path $(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)$, all P_k, $0\le k\le n$, are in $\mathcal{S}$.

Remark 3.11. If in Definition 3.4, the connecting morphism $h\colon Y'\longrightarrow Y$ uniquely exists, then it follows by induction that the hom-sets of $\mathsf{Path}(I,F)$ are at most singleton. This is the case for all polynomial functors, but not the case for the bag functor on sets (discussed in Subsect. 4.1).

Definition 3.12. *The* path poset $\mathsf{PathOrd}(I,F)$ *is the set* $\coprod_{0\le n}\mathbb{C}(I,F^n1)$ *equipped with the order: for* $u\colon I\longrightarrow F^n1$ *and* $v\colon I\longrightarrow F^m1$, *we define* $u\le v$ *if* $n\le m$ *and* $F^n(!)\cdot v=u$.



So $u\le v$ if u is the truncation of v to n levels. This matches the morphisms in $\mathsf{Path}(I,F)$ that witnesses that one path is prefix of another:

Proposition 3.13. *1. The functor* $\mathsf{Comp}\colon\mathsf{Path}(I,F)\longrightarrow\mathsf{PathOrd}(I,F)$ *defined by* $I=P_0\xrightarrow{p_0}FP_1\cdots\to F^nP_n\xrightarrow{F^n!}F^n1$ *on* $(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)$ *is full, and reflects isos.*
2. If F *admits precise factorizations w.r.t.* $\mathcal{S}$ *and* $I\in\mathcal{S}$, *then* Comp *is sujective.*
3. If additionally h *in Definition 3.4 is unique, then* Comp *has a right-inverse.*

In particular, $\mathsf{PathOrd}(I, F)$ is $\mathsf{Path}(I, F)$ up to isomorphism. In the instances, it is often easier to characterize $\mathsf{PathOrd}(I, F)$. This also shows that $\mathsf{Path}(I, F)$ contains the elements – understood as morphisms from I – of the finite start of the final chain of F: $1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} F^3 1 \leftarrow \cdots$.

Example 3.14. When $FX = A \times X + 1$, $F^n 1$ is isomorphic to the set of words in $A^\star.\bot^\star$ of length n. Consequently, $\mathsf{PathOrd}(I, F)$ is the set of words in $A^\star.\bot^\star$, equipped with the prefix order. In this case, Comp is an equivalence of categories.

3.3 Embedding Paths into Pointed Coalgebras

The paths $(\boldsymbol{P}_{n+1}, \boldsymbol{p}_n)$ embed into $\mathsf{Coalg}_l(I, TF)$ as one expects it for examples like Fig. 2: one takes the disjoint union of the P_k, one has the pointing $I = P_0$ and the linear structure of F is embedded into the branching type T.

During the presentation of the results, we require T, F, and I to have certain properties, which will be introduced one after the other. The full list of assumptions is summarized in Table 2:

(Ax1) – The main theorem will show that coalgebra homomorphisms in $\mathsf{Coalg}_l(I, TF)$ are the open maps for the path category $\mathsf{Path}(I, F + 1)$. So from now on, we assume that $\mathbb{C}$ has finite coproducts and to use the results from the previous sections, we fix a class $\mathcal{S} \subseteq \mathbf{obj}\,\mathbb{C}$ such that $F + 1$ admits precise factorizations w.r.t. $\mathcal{S}$ and that $I \in \mathcal{S}$.

(Ax2) – Recall, that a family of morphisms $(e_i \colon X_i \longrightarrow Y)_{i \in I}$ with common codomain is called jointly epic if for $f, g \colon Y \longrightarrow Z$ we have that $f \cdot e_i = g \cdot e_i \; \forall i \in I$ implies $f = g$. For Set, this means, that every element $y \in Y$ is in the image of some e_i. Since we work with partial orders on T-homsets, we also need the generalization of this property if $f \sqsubseteq g$ are of the form $Y \longrightarrow TZ'$.

(Ax3) – In this section, we encode paths as a pointed coalgebra by constructing a functor $J \colon \mathsf{Path}(I, F+1) \hookrightarrow \mathsf{Coalg}_l(I, TF)$. For that we need to embed the linear behaviour $FX + 1$ into TFX. This is done by a natural transformation $[\eta, \bot] \colon \mathrm{Id} + 1 \longrightarrow T$, and we require that $\bot \colon 1 \longrightarrow T$ is a bottom element for $\sqsubseteq$.

Example 3.15. For the case where T is the powerset functor $\mathcal{P}$, η is given by the unit $\eta_X(x) = \{x\}$, and $\bot$ is given by empty sets $\bot_X(*) = \varnothing$.

Definition 3.16. *We have an inclusion functor* $J \colon \mathsf{Path}(I, F + 1) \hookrightarrow \mathsf{Coalg}_l(I, TF)$ *that maps a path* $(\boldsymbol{P}_{n+1}, \boldsymbol{p}_n)$ *to an* I*-pointed* TF*-coalgebra on* $\coprod \boldsymbol{P}_{n+1} := \coprod_{0 \le k \le n} P_k$*. The pointing is given by* $\mathsf{in}_0 \colon I = P_0 \longrightarrow \coprod \boldsymbol{P}_{n+1}$ *and the structure by:*

$$\coprod_{0 \le k < n} P_k + P_n \xrightarrow{[(F\mathsf{in}_{k+1} + 1) \cdot p_k]_{0 \le k < n} + !} F \coprod \boldsymbol{P}_{n+1} + 1 \xrightarrow{[\eta, \bot]} TF \coprod \boldsymbol{P}_{n+1}.$$

Example 3.17. In the case of LTSs, a path, or equivalently a word $a_1 ... a_k.\bot ... \bot \in A^\star.\bot^\star$, is mapped to the finite linear system over $a_1 ... a_k$ (see Sect. 2.1), seen as a coalgebra (see Sect. 2.2).

Proposition 3.18. *Given a morphism* $[x_k]_{k\leq n}\colon \coprod \boldsymbol{P}_{n+1} \longrightarrow X$ *for some system* (X,ξ,x_0) *and a path* $(\boldsymbol{P}_{n+1},\boldsymbol{p}_n)$*, we have*

$$J(\boldsymbol{P}_{n+1},\boldsymbol{p}_n) \xrightarrow{[x_k]_{k\leq n}} (X,\xi,x_0) \text{ a run in } \mathsf{Coalg}_l(I,TF) \iff \forall k<n\colon \begin{array}{ccc} P_k & \xrightarrow{\quad x_k \quad} & X \\ {\scriptstyle p_k}\downarrow & \sqsubseteq & \downarrow{\scriptstyle \xi} \\ FP_{k+1}+1 & \longrightarrow FX+1 \longrightarrow & TFX. \end{array}$$

Also note that the pointing x_0 of the coalgebra is necessarily the first component of any run in it. In a run $[x_k]_{k\leq n}$, p_k corresponds to an edge from x_k to x_{k+1}.

Example 3.19. For LTSs, since the P_k are singletons, x_k just picks the kth state of the run. The right-hand side of this lemma describes that this is a run iff there is a transition from the kth state and the $(k+1)-$th state.

3.4 Open Morphisms Are Exactly Coalgebra Homomorphisms

In this section, we prove our main contribution, namely that $\mathsf{Path}(I,F+1)$-open maps in $\mathsf{Coalg}_l(I,TF)$ are exactly coalgebra homomorphisms. For the first direction of the main theorem, that is, that coalgebra homomorphisms are open, we need two extra axioms:

(Ax4) – describing that the order on $\mathbb{C}(X,TY)$ is point-wise. This holds for the powerset because every set is the union of its singleton subsets.

(Ax5) – describing that $\mathbb{C}(X,TY)$ admits a choice-principle. This holds for the powerset because whenever $y\in h[x]$ for a map $h\colon X\longrightarrow Y$ and $x\subseteq X$, then there is some $\{x'\}\subseteq x$ with $h(x')=y$.

Theorem 3.20. *Under the assumptions of Table 2, a coalgebra homomorphism in* $\mathsf{Coalg}_l(I,TF)$ *is* $\mathsf{Path}(I,F+1)$*-open.*

Table 2. Main assumptions on $F,T\colon \mathbb{C}\longrightarrow\mathbb{C}$, $\sqsubseteq^T$, $\mathcal{S}\subseteq \mathbf{obj}\,\mathbb{C}$

F	(Ax1)	$F+1$ admits precise factorizations, w.r.t. $\mathcal{S}$ and $I\in\mathcal{S}$
T	(Ax2)	If $(e_i\colon X_i\longrightarrow Y)_{i\in I}$ jointly epic, then $f\cdot e_i\sqsubseteq g\cdot e_i$ for all $i\in I\Rightarrow f\sqsubseteq g$.
	(Ax3)	$[\eta,\bot]\colon \mathrm{Id}+1\longrightarrow T$, with $\bot_Y\cdot !_X\sqsubseteq f$ for all $f\colon X\longrightarrow TY$
	(Ax4)	For every $f\colon X\longrightarrow TY$, $X\in\mathcal{S}$, $f=\bigsqcup\{[\eta,\bot]_Y\cdot f'\sqsubseteq f \mid f'\colon X\longrightarrow Y+1\}$
	(Ax5)	$\forall A\in\mathcal{S}$ $\begin{array}{ccc} A & \xrightarrow{x} & TX \\ {\scriptstyle y}\downarrow & \swarrow_{/\!/} & \downarrow{\scriptstyle Th} \\ Y+1 & \xrightarrow{[\eta,\bot]_Y} & TY \end{array}$ $\overset{\exists x'}{\Longrightarrow}$ $\begin{array}{ccc} A & \xrightarrow{x} & TX \\ & {\scriptstyle x'}\searrow \; \sqcup\!\!\mid \; \nearrow_{[\eta,\bot]_X} & \\ & X+1 & \downarrow{\scriptstyle Th} \\ {\scriptstyle y}\searrow & \downarrow{\scriptstyle h+1} & \\ & Y+1 \xrightarrow{[\eta,\bot]_Y} & TY \end{array}$

The converse is not true in general, because intuitively, open maps reflect runs, and thus only reflect edges of reachable states, as we have seen in Sect. 2.1. The notion of a state being reached by a path is the following:

Definition 3.21. *A system* (X, ξ, x_0) *is* path-reachable *if the family of runs* $[x_k]_{k\le n} \colon J(\boldsymbol{P}_{n+1}, \boldsymbol{p}_n) \longrightarrow (X, \xi, x_0)$ *(of paths from* $\mathsf{Path}(I, F+1)$*) is jointly epic.*

Example 3.22. For LTSs, this means that every state in X is reached by a run, that is, there is a path from the initial state to every state of X.

Remark 3.23. In Definition 3.21, it is crucial that we consider $\mathsf{Path}(I, F+1)$ and not $\mathsf{Path}(I, F)$ for functors incorporating 'arities ≥ 2'. This does not affect the example of LTSs, but for $I = 1$, $FX = X \times X$ and $T = \mathcal{P}$ in Set, the coalgebra (X, ξ, x_0) on $X = \{x_0, y_1, y_2, z_1, z_2\}$ given by $\xi(x_0) = \{(y_1, y_2)\}$, $\xi(y_1) = \{(z_1, z_2)\}$, $\xi(y_2) = \xi(z_1) = \xi(z_2) = \emptyset$ is path-reachable for $\mathsf{Path}(I, F+1)$. There is no run of a length 2 path from $\mathsf{Path}(I, F)$, because y_2 has no successors, and so there is no path to z_1 or to z_2.

Theorem 3.24. *Under the assumptions of Table 2, if* (X, ξ, x_0) *is path-reachable, then an open morphism* $h \colon (X, \xi, x_0) \longrightarrow (Y, \zeta, y_0)$ *is a coalgebra homomorphism.*

3.5 Connection to Other Notions of Reachability

There is another concise notion for reachability in the coalgebraic literature [2].

Definition 3.25. *A* subcoalgebra *of* (X, ξ, x_0) *is a coalgebra homomorphism* $h \colon (Y, \zeta, y_0) \longrightarrow (X, \xi, x_0)$ *that is carried by a monomorphism* $h \colon X \rightarrowtail Y$*. Furthermore* (X, ξ, x_0) *is called* reachable *if it has no proper subcoalgebra, i.e. if any subcoalgebra* h *is an isomorphism.*

Under the following assumptions, this notion coincides with the path-based definition of reachability (Definition 3.21).

Assumption 3.26. For the present Subsect. 3.5, let $\mathbb{C}$ be cocomplete, have (epi,mono)-factorizations and wide pullbacks of monomorphisms.

The first direction follows directly from Theorem 3.20:

Proposition 3.27. *Every path-reachable* (X, ξ, x_0) *has no proper subcoalgebra.*

For the other direction it is needed that TF preserves arbitrary intersections, that is, wide pullbacks of monomorphisms. In Set, this means that for a family $(X_i \subseteq Y)_{i\in I}$ of subsets we have $\bigcap_{i\in I} TFX_i = TF \bigcap_{i\in I} X_i$ as subsets of TFY.

Proposition 3.28. *If, furthermore, for every monomorphism* $m \colon Y \longrightarrow Z$*, the function* $\mathbb{C}(-, Tm) \colon \mathbb{C}(X, TY) \longrightarrow \mathbb{C}(X, TZ)$ *reflects joins and if* TF *preserves arbitrary intersections, then a reachable coalgebra* (X, ξ, x_0) *is also path-reachable.*

All those technical assumptions are satisfied in the case of LTSs, and will also be satisfied in all our instances in Sect. 4.

3.6 Trace Semantics for Pointed Coalgebras

The characterization from Theorems 3.20 and 3.24 points out a natural way of defining a trace semantics for pointed coalgebras. Indeed, the paths category $\mathsf{Path}(I, F+1)$ provides a natural way of defining the runs of a system. A possible way to go from runs to trace semantics is to describe accepting runs as the subcategory $J' \colon \mathsf{Path}(I,F) \hookrightarrow \mathsf{Path}(I, F+1)$. We can define the *trace semantics* of a system (X, ξ, x_o) as the set:

$$\mathsf{tr}(X,\xi,x_0) = \{\mathsf{Comp}(\boldsymbol{P}_{n+1}, \boldsymbol{p}_n) \mid \exists \text{ run } [x_k]_{k\leq n} \colon JJ'(\boldsymbol{P}_{n+1}, \boldsymbol{p}_n) \longrightarrow (X,\xi,x_0) \\ \text{with } (\boldsymbol{P}_{n+1}, \boldsymbol{p}_n) \in \mathsf{Path}(I,F)\}$$

Since $\mathsf{Path}(I,F)$-open maps preserve and reflect runs, we have the following:

Corollary 3.29. $\mathsf{tr} \colon \mathsf{Coalg}_l(I, TF) \longrightarrow (\mathcal{P}(\mathsf{PathOrd}(I,F)), \subseteq)$ *is a functor and if* $f \colon (X,\xi,x_0) \longrightarrow (Y,\zeta,y_0)$ *is* $\mathsf{Path}(I,F+1)$*-open, then* $\mathsf{tr}(X,\xi,x_0) = \mathsf{tr}(Y,\zeta,y_0)$.

Let us look at two LTS-related examples (we will describe some others in the next section). First, for $FX = A \times X$. The usual trace semantics is given by all the words in $A^\star$ that are labelled of a run of a system. This trace semantics is obtained because $\mathsf{PathOrd}(I,F) = \coprod_{n\geq 0} A^n$ and because Comp maps every path to its underlying word. Another example is given for $FX = A \times X + \{\checkmark\}$, where $\checkmark$ marks final states. In this case, a path in $\mathsf{Path}(I,F)$ of length n is either a path that can still be extended or encodes less than n steps to an accepting state $\checkmark$. This obtains the trace semantics containing the set of accepted words, as in automata theory, plus the set of possibly infinite runs.

4 Instances

4.1 Analytic Functors and Tree Automata

In Example 3.7, we have seen that every polynomial Set-functors, in particular the functor $X \mapsto A \times X$, has precise factorizations with respect to all sets. This allowed us to see LTSs, modelled as $\{*\}$-pointed $\mathcal{P}(A \times \_)$-coalgebra, as an instance of our theory. This allowed us in particular to describe their trace semantics using our path category in Sect. 3.6. This can be extended to tree automata as follows. Assume given a signature Σ, that is, a collection $(\Sigma_n)_{n\in\mathbb{N}}$ of disjoint sets. When σ belongs to Σ_n, we say that n is the *arity of* σ or that σ is a *symbol of arity* n. A top-down non-deterministic tree automata as defined in [6] is then the same as a $\{*\}$-pointed $\mathcal{P}F$-coalgebra where F is the polynomial functor $X \mapsto \coprod_{\sigma\in\Sigma_n} X^n$. For this functor, $F^n(1)$ is the set of trees over $\Sigma \sqcup \{*(0)\}$ of depth at most $n+1$ such that a leaf is labelled by $*$ if and only if it is at depth $n+1$. Intuitively, elements of $F^n(1)$ are partial runs of length n that can possibly be extended. Then, the trace semantics of a tree automata, seen as a pointed coalgebra, is given by the set of partial runs of the automata. In particular, this contains the set of accepted finite trees as those partial runs

without any $*$, and the set of accepted infinite trees, encoded as the sequence of their truncations of depth n, for every n.

In the following, we would like to extend this to other kinds of tree automata by allowing some symmetries. For example, in a tree, we may not care about the order of the children. This boils down to quotient the set X^n of n-tuples, by some permutations of the indices. This can be done generally given a subgroup G of the permutation group $\mathfrak{S}_n$ on n elements by defining X^n/G as the quotient of X^n under the equivalence relation: $(x_1, \ldots, x_n) \equiv_G (y_1, \ldots, y_n)$ iff there is $\pi \in G$ such that for all i, $x_i = y_{\pi(i)}$. Concretely, this means that we replace the polynomial functor F by a so-called *analytic functor*:

Definition 4.1 [14,15]**.** *An* analytic Set*-functor is a functor of the form* $FX = \coprod_{\sigma \in \Sigma_n} X^n/G_\sigma$ *where for every* $\sigma \in \Sigma_n$*, we have a subgroup* G_σ *of the permutation group* $\mathfrak{S}_n$ *on* n *elements.*

Example 4.2. Every polynomial functor is analytic. The bag-functor is analytic, with $\Sigma = (\{*\})_{n \in \mathbb{N}}$ has one operation symbol per arity and $G_\sigma = \mathfrak{S}_{\mathsf{ar}(\sigma)}$ is the full permutation group on $\mathsf{ar}(\sigma)$ elements. It is the archetype of an analytic functor, in the sense that for every analytic functor $F\colon \mathsf{Set} \longrightarrow \mathsf{Set}$, there is a natural transformation into the bag functor $\alpha\colon F \longrightarrow \mathcal{B}$. If F is given by Σ and G_σ as above, then α_X is given by

$$FX = \coprod_{\sigma \in \Sigma_n} X^n/G_\sigma \twoheadrightarrow \coprod_{\sigma \in \Sigma_n} X^n/\mathfrak{S}_n \to \coprod_{n \in \mathbb{N}} X^n/\mathfrak{S}_n = \mathcal{B}X.$$

Proposition 4.3. *For an analytic* Set*-functor* F*, the following are equivalent (1) a map* $f\colon X \longrightarrow FY$ *is* F*-precise, (2)* $\alpha_Y \cdot f$ *is* $\mathcal{B}$*-precise, (3) every element of* Y *appears precisely once in the definition of* f*, i.e. for every* $y \in Y$*, there is exactly one* x *in* X*, such that* $f(x)$ *is the equivalence class of a tuple* $(y_1, \ldots, y_n)$ *where there is an index* i*, such that* $y_i = y$*; and furthermore this index is unique. So every analytic functor has precise factorizations w.r.t.* Set*.*

4.2 Nominal Sets: Regular Nondeterministic Nominal Automata

We derive an open map situation from the coalgebraic situation for *regular nondeterministic nominal automata* (*RNNAs*) [26]. They are an extension of automata to accept *words with binders*, consisting of literals $a \in \mathbb{A}$ and binders $|_a$ for $a \in \mathbb{A}$; the latter is counted as length 1. An example of such a word of length 4 is $a|_c bc$, where the last c is bound by $|_c$. The order of binders makes difference: $|_a|_b ab \neq |_a|_b ba$. RNNAs are coalgebraically represented in the category of nominal sets [10], a formalism about atoms (e.g. variables) that sit in more complex structures (e.g. lambda terms), and gives a notion of *binding*. Because the choice principles (Ax4) and (Ax5) are not satisfied by every nominal sets, we instead use the class of *strong nominal sets* for the precise factorization (Definition 3.4).

Definition 4.4 [10,24]**.** *Fix a countably infinite set* $\mathbb{A}$*, called the set of* atoms*. For the group* $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ *of finite permutations on the set* $\mathbb{A}$*, a* group action $(X, \cdot)$ *is a set* X *together with a group homomorphism* $\cdot\colon \mathfrak{S}_{\mathsf{f}}(\mathbb{A}) \longrightarrow \mathfrak{S}_{\mathsf{f}}(X)$*, written in*

infix notation. An element $x \in X$ *is* supported by $S \subseteq \mathbb{A}$, *if for all* $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ *with* $\pi(a) = a\ \forall a \in S$ *we have* $\pi \cdot x = x$. *A* nominal set *is a group action for* $\mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ *such that every* $x \in X$ *is finitely supported, i.e. supported by a finite* $S \subseteq \mathbb{A}$. *A map* $f\colon (X,\cdot) \longrightarrow (Y,\star)$ *is* equivariant *if for all* $x \in X$ *and* $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ *we have* $f(\pi \cdot x) = \pi \star f(x)$. *The category of nominal sets and equivariant maps is denoted by* Nom. *A nominal set* $(X,\cdot)$ *is called* strong *if for all* $x \in X$ *and* $\pi \in \mathfrak{S}_{\mathsf{f}}(\mathbb{A})$ *with* $\pi \cdot x = x$ *we have* $\pi(a) = a$ *for all* $a \in \mathsf{supp}(x)$.

Intuitively, the support of an element is the set of free literals. An equivariant map can forget some of the support of an element, but can never introduce new atoms, i.e. $\mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x)$. The intuition behind strong nominal sets is that all atoms appear in a fixed order, that is, $\mathbb{A}^n$ is strong, but $\mathcal{P}_{\mathsf{f}}(\mathbb{A})$ (the finite powerset) is not. We set $\mathcal{S}$ to be the class of strong nominal sets:

Example 4.5. The Nom-functor of unordered pairs admits precise factorizations w.r.t. strong nominal sets, but not w.r.t. all nominal sets.

In the application, we fix the set $I = \mathbb{A}^{\#n}$ of distinct n-tuples of atoms ($n \geq 0$) as the pointing. The hom-sets $\mathsf{Nom}(X, \mathcal{P}_{\mathsf{ufs}}Y)$ are ordered point-wise.

Proposition 4.6. *Uniformly finitely supported powerset* $\mathcal{P}_{\mathsf{ufs}}(X) = \{Y \subseteq X \mid \bigcup_{y\in Y} \mathsf{supp}(y)$ *finite*$\}$ *satisfies* (Ax2-5) *w.r.t.* $\mathcal{S}$ *the class of strong nominal sets.*[1]

As for F, we study an LTS-like functor, extended with the *binding functor* [10]:

Definition 4.7. *For a nominal set* X, *define the* α*-equivalence relation* $\sim_\alpha$ *on* $\mathbb{A} \times X$ *by:* $(a,x) \sim_\alpha (b,y) \Leftrightarrow \exists c \in \mathbb{A} \setminus \mathsf{supp}(x) \setminus \mathsf{supp}(y)$ *with* $(a\,c)\cdot x = (b\,c)\cdot y$. *Denote the quotient by* $[\mathbb{A}]X := \mathbb{A} \times X/{\sim_\alpha}$. *The assignment* $X \mapsto [\mathbb{A}]X$ *extends to a functor, called the* binding functor $[\mathbb{A}]\colon \mathsf{Nom} \longrightarrow \mathsf{Nom}$.

RNNA are precisely $\mathcal{P}_{\mathsf{ufs}}F$-coalgebras for $FX = \{\checkmark\} + [\mathbb{A}]X + \mathbb{A} \times X$ [26]. In this paper we additionally consider initial states for RNNAs.

Proposition 4.8. *The binding functor* $[\mathbb{A}]$ *admits precise factorizations w.r.t. strong nominal sets and so does* $FX = \{\checkmark\} + [\mathbb{A}]X + \mathbb{A} \times X$.

An element in $\mathsf{PathOrd}(\mathbb{A}^{\#n}, F)$ may be regarded as a word with binders under a context $\boldsymbol{a} \vdash w$, where $\boldsymbol{a} \in \mathbb{A}^{\#n}$, all literals in w are bound or in $\boldsymbol{a}$, and w may end with $\checkmark$. Moreover, two word-in-contexts $\boldsymbol{a} \vdash w$ and $\boldsymbol{a}' \vdash w'$ are identified if their closures are α-equivalent, that is, $|_{a_1} \cdots |_{a_n} w = |_{a'_1} \cdots |_{a'_n} w'$. The trace semantics of a RNNA T contains all the word-in-contexts corresponding to runs in T. This trace semantics distinguishes whether words are concluded by $\checkmark$.

4.3 Subsuming Arbitrary Open Morphism Situations

Lasota [19] provides a translation of a small path-category $\mathbb{P} \hookrightarrow \mathbb{M}$ into a functor $\mathbb{F}\colon \mathsf{Set}^{\mathbf{obj}\,\mathbb{P}} \longrightarrow \mathsf{Set}^{\mathbf{obj}\,\mathbb{P}}$ defined by $\mathbb{F}\big(X_P\big)_P = \big(\prod_{Q\in\mathbb{P}} \big(\mathcal{P}(X_Q)\big)^{\mathbb{P}(P,Q)}\big)_{P\in\mathbb{P}}$.

[1] There are two variants of powersets discussed in [26]. The finite powerset $\mathcal{P}_{\mathsf{f}}$ also fulfils the axioms. However, *finitely supported* powerset $\mathcal{P}_{\mathsf{fs}}$ does not fulfil (Ax5).

So the hom-sets $\mathsf{Set}^{\mathbf{obj}\,\mathbb{P}}(X, \mathbb{F}Y)$ have a canonical order, namely the point-wise inclusion. This admits a functor Beh from $\mathbb{M}$ to $\mathbb{F}$-coalgebras and lax coalgebra homomorphisms, and Lasota shows that $f \in \mathbb{M}(X, Y)$ is $\mathbb{P}$-open iff $\mathsf{Beh}(f)$ is a coalgebra homomorphism. In the following, we show that we can apply our framework to $\mathbb{F}$ by a suitable decomposition $\mathbb{F} = TF$ and a suitable object I for the initial state pointing. As usual in open map papers, we require that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object $0_{\mathbb{P}}$. Observe that we have $\mathbb{F} = T \cdot F$ where

$$T(X_P)_{P\in\mathbb{P}} = \big(\mathcal{P}(X_P)\big)_{P\in\mathbb{P}} \quad \text{and} \quad F(X_P)_{P\in\mathbb{P}} = \big(\coprod_{Q\in\mathbb{P}} \mathbb{P}(P, Q) \times X_Q\big)_{P\in\mathbb{P}}.$$

Lasota considers coalgebras without pointing, but one indeed has a canonical pointing as follows. For $P \in \mathbb{P}$, define the characteristic family $\chi^P \in \mathsf{Set}^{\mathbf{obj}\,\mathbb{P}}$ by $\chi^P_Q = 1$ if $P = Q$ and $\chi^P_Q = \emptyset$ if $P \neq Q$. With this, we fix the pointing $I = \chi^{0_{\mathbb{P}}}$.

Proposition 4.9. *T, F and I satisfy the axioms from Table 2, with $\mathcal{S} = \mathsf{Set}^{\mathbf{obj}\,\mathbb{P}}$.*

The path category in $\mathsf{Coalg}_l(I, TF)$ from our theory can be described as follows.

Proposition 4.10. *An object of $\mathsf{Path}(I, F)$ is a sequence of composable $\mathbb{P}$-morphisms $0_{\mathbb{P}} \xrightarrow{m_1} P_1 \xrightarrow{m_2} P_2 \cdots \xrightarrow{m_n} P_n$.*

5 Conclusions and Further Work

We proved that coalgebra homomorphisms for systems with non-deterministic branching can be seen as open maps for a canonical path-category, constructed from the computation type F. This limitation to non-deterministic systems is unsurprising: as we have proved in Sect. 4.3 on Lasota's work [19], every open map situation can been encoded as a coalgebra situation with a powerset-like functor, so with non-deterministic branching. As a future work, we would like to extend this theory of path-categories to coalgebras for further kinds of branching, especially probabilistic and weighted. This will require (1) to adapt open maps to allow those kinds of branching (2) adapt the axioms from Table 2, by replacing the "+1" part of (Ax1) to something depending on the branching type.

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